Geometric Quantization of Wess-Zumino-Witten model and Virtual Localization Formula

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Based on **[arXiv:0712.2318v3]** updated by yesterday's replacement: **Mixed anomalies of chiral algebras compactified to smooth quasi-projective surfaces**
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Ill-definition of Wess-Zumino-Witten model(closed string S^1)

<u>Definition 0</u>(Physicists' native definition) $M = G$ ((non-)compact real Lie group), G/B (flag variety quotient by upper triangle matrices), or $G/G \cong \{e\}$ will be the "inner space" of superstring theory of SCFT (super-conformal field theory) with boundary. [Modify] I would like to regard M as a total space of space-time of superstring theory, so M is not always ADE type simply-laced compact Lie group or $SL(2, \mathbb{R})(\otimes\mathbb{C})$.

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Remark 1 Historically, "Geometric Quantization" came from the quantization of Chern-Simons 3-form, which string theorists claimed to be "open string" (strip) whose ends have the special Lagrangian submanifolds structure of the total 10-dimensional Euclidean space $M = \mathbb{R}^{9,1} \sim \mathbb{R}^{10}$ (flat); or, rather, its "compactification" of several dimensions with (compact) Calabi-Yau 3-folds (inner space) times Lorentzian space-time $\mathbb{R}^{3,1}$.
$CY3 \times \mathbb{R}^{3,1}$ will not be my target space of topological sigma model; since, the real 10-dimensionality is not confirmed for non-Cartesian space-time, which is not "flat"; $c_1(TM) \neq 0$.

Digression for mathematical physicists

Remark 2(complex differential geometry) I will abbreviate some of the above-mentioned stack singularity (Chan-Paton factor) and its crepant resolution theory, since my main focus in this report is not in the homological mirror symmetry conjecture (physicists' T-duality); rather, I would like to eliminate the divergence coming from the (non-)linear sigma model for summation over embeddings of (compact) Riemann surfaces to, say, a quotient of algebraic group with some generalized (almost) Kähler structures with (un-)broken integrability conditions. (Anomaly 2-form $\psi_{\alpha\beta\gamma}$ for the obstruction against deformation of complex structures.)

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Background "Physics" 3 [S-duality conjecture for superstring theory], which is neither $24+2$-dimensional nor S-duality of gauge theory for Euler number of moduli space of instantons of Uhlenbeck compactification or that of monopoles of Seiberg-Witten theory. The focus is on the asymptotic series of Hilbert-Poincare polynomial of (mixed) Hodge structure (sometimes called H-polynomial), rather than Euler characteristics.

Sound definition and Formulation of Generalized Complex Structures

Definition 4(Generalized Complex Structures) Let T_X be the tanget bundle of smooth quasi-projective complex manifold X, and Ω^1_X be the co-tangent bundle (normal bundle). Then I will define the (almost) generalized complex structure B (anti-symmetric 2-form, so-called B-field, which will be defined at the next slide) on the section v of fibre direct sum

$$v = V \oplus \xi, \tag{1}$$

$$V \in T_X, \xi \in \Omega^1_X. \tag{2}$$

Likewise, let us denote $w = W \oplus \eta \in T_X \oplus \Omega^1_X$ anther vector.

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Definition 5(Courant-Dirac bracket of constraint dynamical system) Under the above notation and definition, let $\mathcal{L}$ be the Lie derivative and ι be the inner product with respect to tangent vector bundle T_X. Then let us define the Courant-Dirac bracket $[[,]]$ by

$$[[v, w]] := [V, W] + \mathcal{L}_V \eta - \mathcal{L}_W \xi - \frac{1}{2} d(\iota_V \eta - \iota_W \xi), \tag{3}$$

where d is the usual exterior derivative.

Operator-Product-Expansion (OPE) preservation and $\beta\gamma$-CFT (Conformal Field Theory) by B-field

<u>Definition 6</u> I will decompose the (half-twisted) $\beta\gamma$-CFT on X w.r.t. a source Riemann surface $\Sigma \ni z, w$ as follows:

$$\beta_i(z)\gamma^j(w) \quad \sim \quad \frac{\delta_i^j}{z-w}, \textit{(up to regular terms)} \tag{4}$$

where $\delta_j^i = 1(i=j), = 0(i \neq j)$ is the Kronecker delta. Let us regard β as conformal dimension 1 (vector field of X) and γ as 0 (scalar field or the target coordinate of X).

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Definition and Lemma 7(Base change and B-field) For a direct computation of coordinate change (base change) $U_0 \to U_1 \subset X$, let us compute the OPE for

$$\tilde{\beta}_a \quad := \quad \beta_i g_a^i + B_{ai}\partial_z \gamma^i, \tag{5}$$

$$B_a \in \Omega^1_{U_1}, \qquad g_a \in T_{U_1}, \tag{6}$$

where summation for $i = 1, \cdots, dim_{\mathbb{C}} X$ is abbreviated when no confusion occurs and the tilde notation $\tilde{}$ stands for U_1.

Computation for OPE preservation under base change $U_0 \to U_1$ (1/2) [details abbreviated]

<u>Lemma 8</u>(sketch of "linear algebra" [To be reconsidered later])
Under the above notation and definition, let us compute the OPE satisfying

$$\tilde{\beta}_a(z)\tilde{\gamma}^b(w) \quad \sim \quad \frac{\delta_a^b}{z-w}. \tag{7}$$

We will obtain the equation $g_a^i = \dfrac{\partial \gamma^i}{\partial \tilde{\gamma}^a}$.

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Note 9(Jacobian matrices g_a^i; not the generalized metric g(v,w))
From the above computation, we have no quantum anomaly correction yet. The transformation law of γ and g (likewise) is the usual one of differential geometry.

OPE preservation under base change $U_0 \to U_1$ (2/2)

Proposition 10(Decomposition of B-field and anti-symmetric 2-form μ) Let us decompose B-field B_a to symmetric part σ_{ab} and anti-symmetric part μ_{ab},

$$B_a = \frac{1}{2}(\sigma_{ab} - \mu_{ab})d\tilde{\gamma}^b. \tag{8}$$

Then, from the other OPE preservation relation,

$$\begin{aligned} \tilde{\beta}_a(z+\epsilon)\tilde{\beta}_b(z) &\sim \frac{1}{\epsilon^2}((A)\textit{generalized "metric" part}) \\ &+ \frac{1}{\epsilon}((B)\textit{terms involving Courant} - \textit{Dirac bracket}) \\ &\sim 0, \end{aligned} \tag{9}$$

we obtain (A) symmetric part

$$\sigma_{ab} = \sigma_{ba} = \sum_{i,j} \partial_i g_a^j \partial_j g_b^i, \tag{10}$$

and (B) anti-symmetric part ("2-form moduli" μ)

$$d\mu = tr(g^{-1}dg)^3, \tag{11}$$

up to the undetermined freedom of exterior derivative d.

Reconsideration of the OPE derivation (1/2)

Physics Historical Note 11(Witten's paper(1983)) After the Bott Periodicity theorem, we modify the classical mechanics (action functional S) of Euler-Lagrange equation by topological term Γ,

$$S \rightarrow S + n\Gamma, \tag{12}$$

$$\Gamma = \int_Q \omega_{ijklm} d\Omega^{ijklm}, \text{ (Unit Cubic Area Element)} \tag{13}$$

$$Q \simeq D^5 \ (Q \cup Q' \simeq S^5 \simeq SU(3)), \tag{14}$$

since $n \in \pi_5(SU(3)) \cong \mathbb{Z}$. This is the historical origin of Wess-Zumino term (real 4-dimensional) and its CFT interpretation by Witten (1983). (Nambu-Goldstone bosons for Spontaneous Breaking of $SU(3) \times SU(3)$ symmetry to diagonal $SU(3)$ and Dynamical Symmetry Breaking by instantons.) I have nothing to say about this physics history of long time ago.

Reconsideration of the OPE derivation (2/2)

Working Hypothesis 12(Nekrasov's Ansatz(2005)) Nekrasov claimed that the papers of Malikov-Gorbounov-Schechtman (1999-2000) can be interpreted as the level 1 WZW term L_{WZW} to the action functional S, whose paper was not submitted unlike proposed.

$$S = \frac{1}{2\pi}\int_{\Sigma} \beta_i \bar{\partial}\gamma^i. \tag{15}$$

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$$S = \frac{1}{2\pi}\int_{\Sigma}\beta_i\bar{\partial}\gamma^i. \tag{15}$$

Just citations of his claim: "pull-back" of local affine chart $\tilde{\gamma} \in U_\alpha$,

$$\frac{1}{2\pi}\tilde{\beta}\bar{\partial}\tilde{\gamma} = \frac{1}{2\pi}\beta\bar{\partial}\gamma + L_{WZW}, \tag{16}$$

$$S \to S + \int_{\Sigma} L_{WZW}, \tag{17}$$

$$dL_{WZW} = \frac{1}{4\pi}d\ tr(g^{-1}\partial g g^{-1}\bar{\partial}g) + \frac{1}{12\pi}tr\left[(g^{-1}dg)^3\right], \tag{18}$$

$$d = \partial' + \partial''. \ (\textit{Double Complex}) \tag{19}$$

The original paper's definition was more ambiguous with the use of d^{-1}. The usual notation for $d = \partial + \bar{\partial}$ was avoided for the use of derivative ∂_z on source Riemann surface $\Sigma \ni z$.

My Cases of Complex 2 Dimension $\mathbb{R}^{3,1} \sim \mathbb{C}^2$ [S.2007]

I did not deal with the detailed calculus of Feynman's oscillator integration utilized by Nekrasov[2005] and Witten[2007]. I just tested whether the Wess-Zumino-Witten term for group manifold G or its quotient G/B, G/G can be rewritten for weighted-projective surfaces. I did not make a general algorithm to compute 2nd Chern character $ch_2(TM)$ for non-toric Kähler stacks $X = M^{3,1}$. Rather than real differential geometry for Chern-Simons 3-form with a timeline, I computed the anomalies ψ for toric del Pezzo surfaces, where no technical (complex) 3-dimensional term occurs: $\gamma \in U_\alpha \cap U_\beta \cap U_\gamma$ for inhomogeneous coordinates

$$\begin{aligned} \psi_{\alpha\beta\gamma} &= \mu_{\alpha\beta} + \mu_{\beta\gamma} + \mu_{\gamma\alpha} - tr(g'' dg' \wedge dg) \\ &\in H^2(X, T_X \oplus \Omega^2_X / d\Omega^1_X), \end{aligned} \tag{20}$$

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where

$$d\mu = tr(g^{-1}dg)^3 = 0. \tag{21}$$

I computed the Čech cohomology of holomorphic closed 2-forms, which was in agreement with Hirzebruch-Riemann-Roch and is indeed vanished for 1-point blowup $(1 = n \in \{0, 1, 2, 3\})$

$$ch_2(X) = \frac{3}{2}(n-1). \ (X = \mathbb{CP}^2 \bigcup \cup_{k=1}^{n} \mathbb{CP}^1) \tag{22}$$

Appendix: How to compute anomaly 2−form ψ in more general?

Definition 13(2-cochain Coefficients $c_{\alpha\beta\gamma}$ and Regularity of Operators) Let us decompose the above-mentioned anomaly 2-form $\psi_{\alpha\beta\gamma} \in H^2(X, \Omega^2_X/d\Omega_X)$ as follows:

$$\begin{aligned} \psi_{\alpha\beta\gamma} &= (\mu_{\alpha\beta} + \mu_{\beta\gamma} + \mu_{\gamma\alpha}) + (-tr\ (g'' dg' \wedge dg)), \quad (23) \\ \psi^0_{\alpha\beta\gamma} &:= -tr\ (g'' dg' \wedge dg) \\ &= c_{\alpha\beta\gamma} \frac{dx \wedge dy}{xy}|_{U_\alpha \cap U_\beta \cap U_\gamma}, \quad (24) \end{aligned}$$

where $\Omega_X = \Omega^{cl}$ is the main object of double complex that was defined in this generalized complex geometry. Notice also that, x and y are the inhomogeneous coordinates defined w.r.t. the affine patch $\gamma \in U_\alpha \cap U_\beta \cap U_\gamma \subset X$.
[Technical Detail]$(g, \mu_{\alpha\beta})$,$(g', \mu_{\beta\gamma})$, and $(g'', \mu_{\gamma\alpha})$ are the 3 pairs of symmetric Jacobians and anti-symmetric 2-forms w.r.t. $U_\alpha \cap U_\beta$, $U_\beta \cap U_\gamma$, and $U_\gamma \cap U_\alpha$.

Whether the anomaly 2-from ψ vanishes by anti-symmetry of B-field?

Definition 14(Definition of Coefficients $c_{\alpha\beta}, c_{\beta\gamma}, c_{\gamma\alpha}$ for the former half of anomaly 2-form $\psi_{\alpha\beta\gamma}$) Under the above notation and definition, let us define the coefficient $c_{\alpha\beta}$ w.r.t. $U_\alpha \cap U_\beta$(Regularity of 1-cochain μ) by

$$\mu_{\alpha\beta} = c_{\alpha\beta} \frac{dx \wedge dy}{xy}|_{U_\alpha \cap U_\beta}, \tag{25}$$

and $c_{\beta\gamma}, c_{\gamma\alpha}$ for intersection $U_\beta \cap U_\gamma, U_\gamma \cap U_\alpha$ respectively.

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and $c_{\beta\gamma}, c_{\gamma\alpha}$ for intersection $U_\beta \cap U_\gamma, U_\gamma \cap U_\alpha$ respectively.
Proposition 15(The anomaly 2-form $\psi_{\alpha\beta\gamma}$)The anomaly can be vanished if and only if there is a solution to the equation,

$$c_{\alpha\beta\gamma} = c_{\alpha\beta} + c_{\beta\gamma} + c_{\gamma\alpha}, \tag{26}$$

for every choice of 3 affine patches $U_\alpha, U_\beta, U_\gamma$ of the generalized complex manifold X. This is indeed the case for 1-point blowup, where at least 4 affine patches exist for the base space X of (pre-sheaf $\Omega^2_X/d\Omega_X$ of) holomorphic closed 2-forms (weakly irregular terms of "logarithmic derivative"). Idea of proof Diagram chasing (and triangulation) of $\begin{pmatrix} n+3 \\ 3 \end{pmatrix}$ choices of 3 affine patches of maximally $n+3$ covers of $\bigcup_{\alpha=1}^{n+3} U_\alpha = X$. ($n$: the number of exceptional divisors)