INTERACTIVE WORKSHEETS FOR CONNECTING SYMBOLIC AND VISUAL REPRESENTATIONS OF 3D VECTOR EQUATIONS

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Learning the close relation between symbolic and visual representations is a key to conceptual understanding of 3-dimensional (3D) vector equations. For learning such a relation, it is valuable that students manipulate and transform the graphic objects directly with observing the simultaneous change of related symbolic equations. The interactive change of graphic and symbolic objects provides the students with opportunities to recognize their relations experimentally. This paper describes how such interactivity is designed as electrical worksheets, implemented into our learning-support www-system, and what reflections they received from students and teachers.

INTRODUCTION

It has been and still is difficult to improve students' conceptual understanding in vector equations of lines or planes in 3D space although the concept is foundation to mathematics, science and engineering. The annual INCT (Institute of National Colleges of Technology, Japan) achievement tests repeatedly show the field as the weakest of our students (INCT, 2012). Some of the students could not even tell if an equation represented a plane or a line in 3D space. Those students tended to memorize a series of formulas without examining their features or understanding the relations of the formulas with the features of graphic objects. In another words, they cannot use visual reasoning and their analytic reasoning is very shallow.

Analytic reasoning is based on the use of symbolic representations and construction of logical inference chains. Instead, visual reasoning is based on visual interpretations of mathematical concepts. Visualization makes it possible to perceive abstract mathematical objects through senses. Visual representations can be considered more concrete than analytic ones, because they are based on external objects. Analytic reasoning is often exact and detailed, but visual reasoning is needed to reveal wider trends of the whole problem solving process and holistic features of the problem situation (Viholainen, 2008). But many students have difficulties in analysing visual representations, and therefore, they cannot utilize them in problem solving (Stylianou & Dubinsky, 1999). In the case of our students, we think that their shallow analytic reasoning and no visual reasoning have been causing their poor performance in 3D linear algebra at the INCT achievement tests.

So we made a lesson reform following the MODEM approach (Haapasalo, 2003), in which he stresses the necessity of the link between conceptual understanding and procedural knowledge, and proposes to include concept-building steps (concept orientation, definition, identification, production, and reinforcement) into learning activities. According to this approach, our lesson plan is directed from concrete applications to abstract mathematical ideas, from handling graphic objects and identifying their characteristics toward defining symbolic expressions and rewriting them (Nishizawa, Yamada, & Yoshioka, 2009). Interactively changing graphics are to provide students with the opportunity to make their own experiments in the concept-building steps by connecting visual and symbolic representations closely together.

The paper explains how we designed the interactive worksheets for learning the relation of planes in 3D space and their vector equations using interactive graphic interface and built-in programming language of a CAS: Mathematica. Attractive features and limitations of those interactive worksheets are to be demonstrated and discussed at the presentation.

RESEARCH METHOD

Our research question is if the students are given the opportunity to manipulate 3D graphic objects in virtual space by dragging specific points of the graphic objects and observe how the vector equations change their parameters in a lesson, do they recognizes more easily the relations of the parameters and the graphic objects through their experiments? To ask this question, we have designed the following interactive worksheets and let our students use the worksheets in a lesson at the computer laboratory on February 2013. We made interviews to some students and gave all the students paper-tests on April and May 2013. The first test was taken two weeks after they started learning the 3D vector equations in traditional lessons. The test scores and their answer sheets were compared to the ones of former students, who had learnt without the worksheets.

INTERACTIVE WORKSHEETS FOR LEARNING VECTOR EQUATIONS

Mathematica's built-in functions offer us a powerful interface, on which we could directly manipulate a graphic object and observe the simultaneous change of related symbolic expression. Although computer screens are 2D, we can locate the position in a 3D virtual space as the cross-point of a plane or a line in the virtual space and the view-line to the 3D space. Every point on the view-line looks like a single point on the screen and cannot be separated from each other. But it indicates a specific location as a cross-point with another graphic object in the space. Drugging the cross-point on the screen with a mouse is the foundation of handling 3D graphic objects in the virtual 3D space.

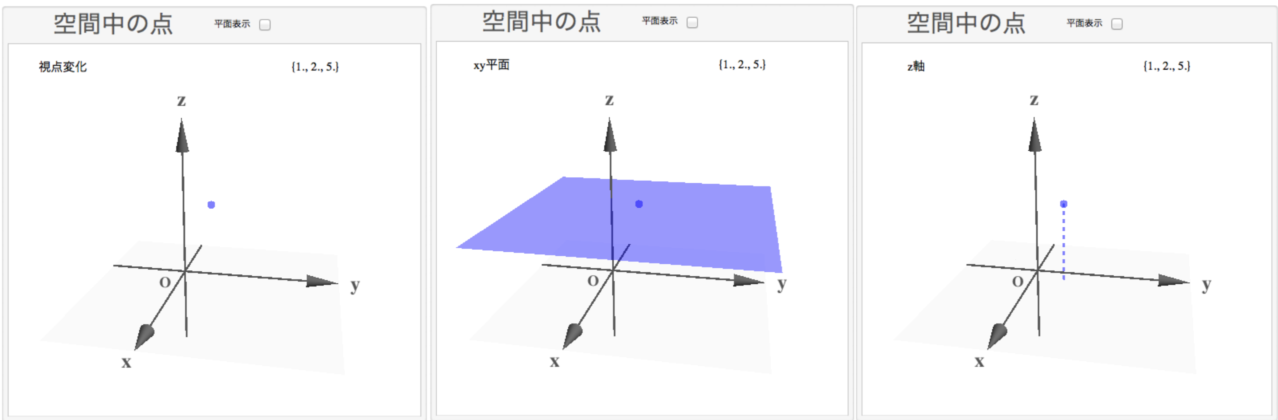

Figure 1: Handling a point in a virtual 3D space

Figure 1 shows how we move a target point in the 3D virtual space, which is shown as a blue dot on the screen, in our worksheets. The left picture is the default mode, where a student can change the viewpoint to the space by clicking any place other than the target point and dragging the whole picture. When she changes the viewpoint, the graphic objects including the target point and three axes on the screen rotate according to the movement of her mouse. The middle picture shows the *xy-plane mode*, which appears when she clicks once on the target point. In this mode, she can move the target point in the plane, which is parallel to the xy-plane, with her dragging motion. When she clicks again on the target point, the *z-axis mode* appears as shown in the right picture. In this mode, she can move the target point vertically, in parallel to the z-axis. If she clicks more times on the point, the mode changes between the *xy-plane mode* and *z-axis mode* in tern. She needs to click on the place other than the target point to return to the default mode.

Figure 2 shows the first interactive worksheet, which uses the handling technique shown in Figure 1. In this worksheet, a student is asked to find the straight line which crosses the two given points A and B in the 3D space. She can move the base and tip of the arrow as shown in the left picture. If she rotates the viewpoint, and observes the 3D space from the top (from a positively distant point on the z axis) or from the side (for example, from a positively distant point on the x axis) in tern, she can easily recognize the locations of the points, and adjust the position of the arrow's base to point A. When she adjusts the arrow's base exactly on point A, the point expands the diameter for her to confirm her success as shown in the middle picture. When she moves the arrow's base, she may recognize that the vector equation displayed on the top right position also changes its parameters along with the movement of the arrow, which represents the direction vector of the line. A careful student also recognizes that the direction vector in the equation does not change the parameters and only the position vector to the base change the parameters when she moves the arrow's base. The direction vector only changes its parameters when she moves the tip of the arrow. When she adjusts the line defined by the arrow to cross both points A and B, the displayed vector equation becomes the right answer to this worksheet. Thus she learns the relation of symbolic and visual representations experimentally.

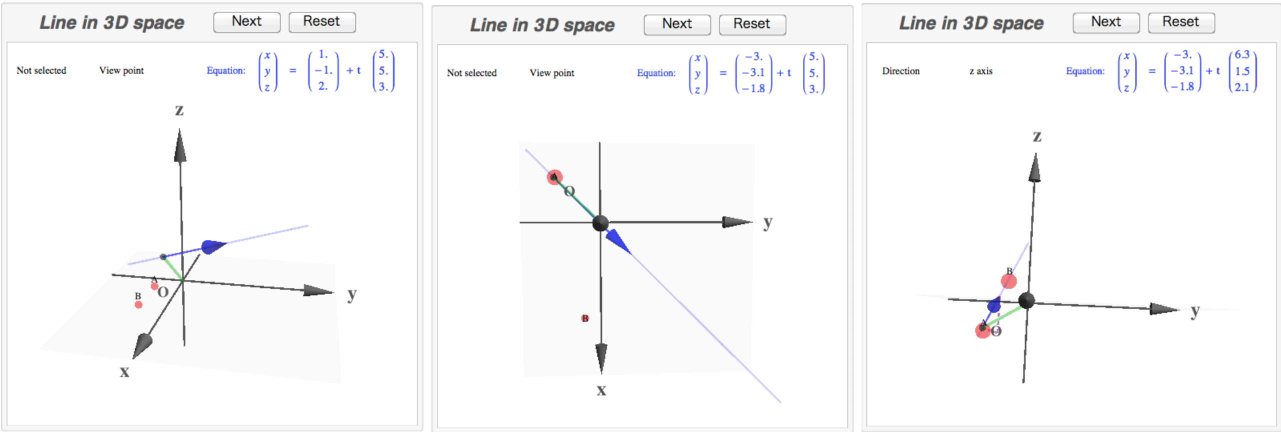

Figure 2: Fixing a straight line in the 3D space

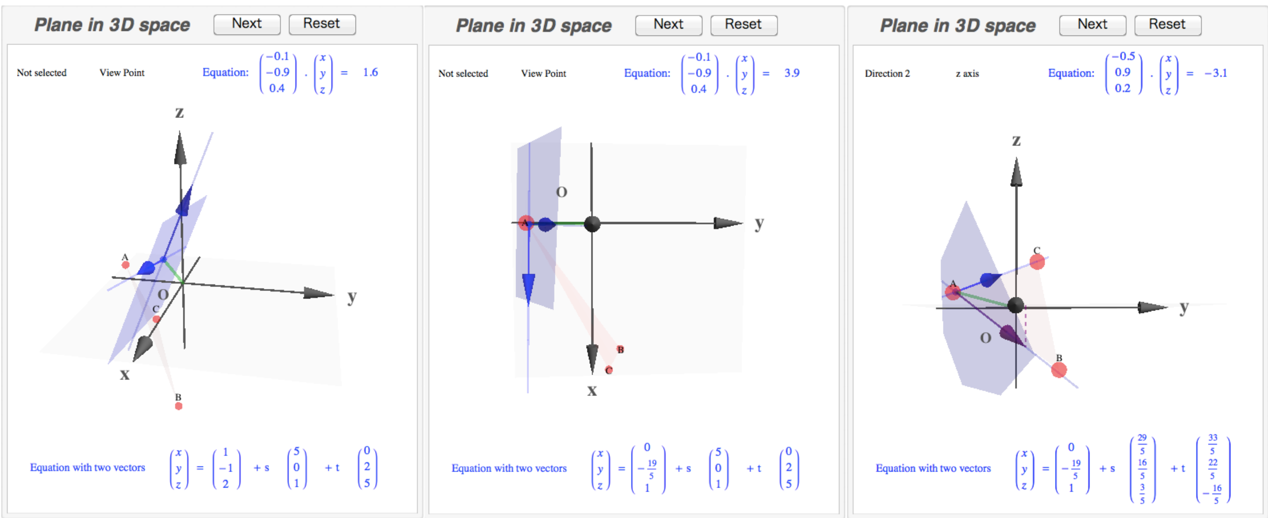

Figure 3: Fixing a plane in the 3D space

In the worksheet shown in Figure 3, a student is asked to find the plane which crosses the given three points A, B, and C in the 3D space by adjusting the base point of two arrows and their tip points. The two arrows represent the direction vectors of the vector equation displayed at the bottom of the window. The equation is related to the arrows in the 3D space and changes its parameters simultaneously with the graphic objects. As in the worksheet of finding lines, the worksheet has two activities; matching the plane to the target graphically and finding the values of parameters. Although the worksheet also shows another vector equation at the top right corner of the window, which is expressed with the normal vector to the blue plane, the relation between this equation and the graphical representation is not clear in this worksheet.

The plane's vector equation that uses the inner product with the normal vector is easier to relate to the visual representation in the worksheet shown in Figure 4. In this worksheet, a student is asked to overlay the adjustable plane to the target plane by dragging the target point. The adjustable plane is the tangent plane to the sphere centered at the origin, and the target point is the touching point of the sphere and the adjustable plane. In the *sphere mode*, a student can drag the target point on the surface of the sphere, so the adjustable plane changes its direction but not the distance to the origin. If she switches the mode to the *radius mode*, the target point moves on the line starting from the origin to the target point and far beyond. In the radius *mode*, the adjustable plane changes its distance to the origin without change its direction. In this way, the target point controls the adjustable plane's direction and the distance to the origin.

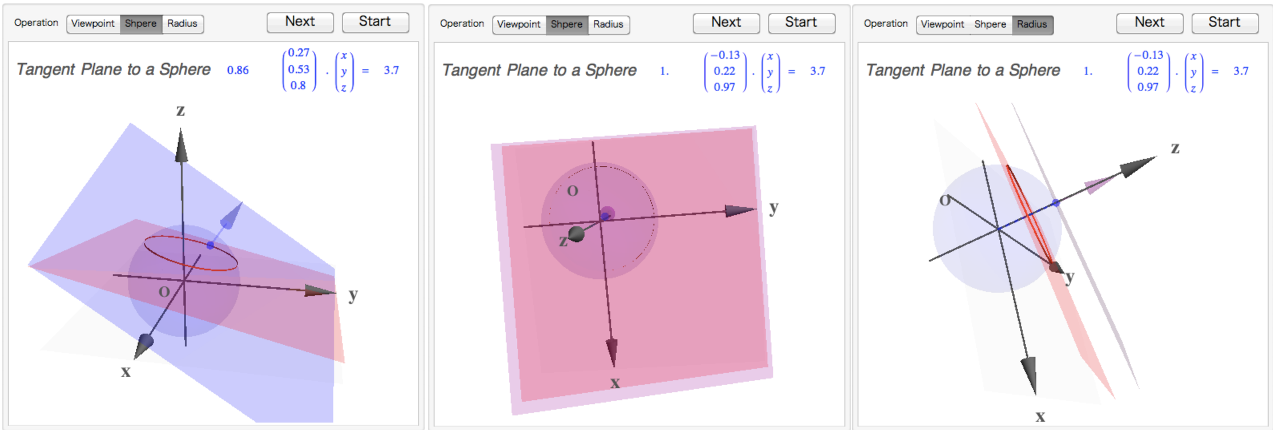

Figure 4: Finding the vector equation based on an inner product for a plane

The cross points of the target plane and the sphere make a circle in the virtual space if the target plane cuts the sphere. This circle helps students to adjust the direction of the adjustable plane to the same direction of target plane as shown in the middle picture. Once the directions of two planes are matched, the resulting task is to adjust the distance to the origin as shown in the right picture. When a student changes the distance of the adjustable plane to the origin, she recognizes that the direction vector in the equation, which is shown at the upper right corner of the window, does not change its parameters and only the number on the right hand side of the equation changes its value.

In this worksheet, we had to choose menu buttons instead of directly clicking on the graphic objects for switching the operating mode because of the slower response of this worksheet than the other ones. The slow response irritated our students and made them switch to the unwilling mode. The menu buttons are located on the upper left corner of the window, and students can change the viewpoint or move the target point either on the sphere or along the radius after selecting a mode from the menu.

TEMPORARY RESULTS

The interviews to some of the students showed that the connections between symbolic and visual representations were easy for lines (Figure 2), and manageable for planes expressed by two vectors (Figure 3). Many students found the direct connections of the position and direction of the arrows and the parameters of the vector equation during their experiments.

However the interviews also revealed two opposite reflections to the worksheet of Figure 4. A student liked this display and thought it deepened his conceptual understanding. He even admired the simplicity of the vector equation. Another student could not find the connection between symbolic and visual representations only with this worksheet. He felt he needed additional explanations, for example the relation of symbolic and visual representations of 2D vector equations, to convince himself of the connection.

The test scores also showed opposite results (Table 1). The 2013 students' average score is significantly higher than the 2011 and 2012 students' averages for the first test of deducing equations of lines, but it is lower for the second test of deducing equations of planes and calculating the planes' distances from the origin. 2013's average is significantly lower than 2011's, and is also lower than 2012's although not significantly. The lower score in the second test made sense with the negative reflection to the worksheet of Figure 4, and implied the limited effectiveness of the worksheets for expressing planes when it was used in a short lesson.

<table>
<tr><td colspan="2">Academic year</td><td>2011</td><td>2012</td><td>2013</td></tr>
<tr><td>Test</td><td>Used interactive worksheets</td><td colspan="2">No</td><td>Yes</td></tr>
<tr><td rowspan="2">Deduce equations of lines in 3D space (April)</td><td>Number of students</td><td>42</td><td>40</td><td>39</td></tr>
<tr><td>Average / Full mark (standard deviation)</td><td>1.9 / 4 (1.2)</td><td>1.1 / 4 (1.1)</td><td>3.1 / 4 (1.3)</td></tr>
<tr><td rowspan="2">Deduce equations of planes in 3D and calculate the distances from the origin (May)</td><td>Number of students</td><td>41</td><td>44</td><td>42</td></tr>
<tr><td>Average / Full mark (standard deviation)</td><td>4.7 / 10 (3.1)</td><td>4.0 / 10 (2.9)</td><td>3.1 / 10 (2.2)</td></tr>
</table>

Table 1: Test results of deducing equations of given lines or planes in 3D space

* The test composed of two or three problems, each requested the vector and algebraic equation of a line in 3D space. One of them was perpendicular to an axis, and the algebraic equation for a line became a system of two equations.

DISCUSSIONS

One of the benefits of the 3D interactive worksheets is that they could show the advantage of vector equations over algebraic linear equations in connecting symbolic and visual representations. Novice students tend to ignore the advantage when they are explained in 2D. Algebraic equation is much more familiar for them, and they don't feel the need to select another way of expressing the same graphic object. Either they select to use a newly learned vector equation or the already familiar algebraic equation is irrelevant when they express a straight line in 2D. Vector equation stays to be only one of many selections. It might not be a good approach to start teaching vector equations in 2D for the sake of simplicity in drawing on chalkboards. Maybe we should show 3D graphic objects at first without explaining the detail, then use 2D objects as their special case and start explaining.

The advantage of vector equations is finally obvious in 3D when algebraic linear equations become complex. Algebraic equations change their forms in special cases, for example, where a plane or a line is parallel or perpendicular to one of the axes. They have different forms from the other cases. The changing forms of algebraic equations always annoy the students who tend to memorize formulas without considering the mechanism. By using the 3D interactive worksheets, we could start our lessons from vector equations in 3D. Algebraic equations in 3D may be learnt after vector equations. It could change the learning of vector equations more meaningful to some students if we use the close connection between symbolic and visual representations from the start.

On the other hand, understanding the vector equations that use inner product of the normal vector to the plane may need more students' experience on concrete examples other than our interactive worksheets. A single lesson with the worksheet could not deepen the conceptual understanding. For deeper understanding, students may have to feel confident with the visual meaning of inner product as they already have with the visual meaning of arithmetic product. One of such our approaches may be the use of a virtual game, which uses inner product of 3D vectors to decide the winner of the game (Nishizawa, Shimada, & Yoshioka, 2013).

CONCLUSION

Interactive worksheets presented in this paper connected visual representations of straight lines and planes in 3D virtual space with symbolic vector equations in students' mind. In some cases, such connections lead students to quick understanding of how the vector equations were constructed. In other cases, it was not tight enough to deepen the conceptual understanding. However, they have a potential to improve the learning of vector equations to more meaningful and enjoyable one if they are appropriately incorporated into the lessons and are combined with additional explanations.

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REFERENCES

Haapasalo, L. (2003). The conflict between conceptual and procedural knowledge: Should we need to understand in order to be able to do, or vice versa?, *Proceedings on the IXX Symposium of the Finnish Mathematics and Science Education Research Association*, (pp. 1-20). Joensuu, Finland: University of Joensuu.

INCT (2012). The Results of the 2010-2012 INCT Achievement Tests. Retrieved from INCT website: http://www.kosen-k.go.jp/attainment.html.

Mathematica (Version 8) [Computer software]: Champaign, IL, USA: Wolfram Research.

Nishizawa, H., Yamada, Y. & Yoshioka, T. (2009, December). The Effectiveness of ICT-assisted Approach in Learning Linear Algebra, *Proceedings of the 16th Asian Technology Conference in Mathematics* (pp. 281-290). Blacksburg, VA, USA: Mathematics and Technology, LLC.

Nishizawa, H., Shimada, K. & Yoshioka, T. (2013, February). Learning Linear Algebra by Combining a Virtual Game in Class and Web-based Learning Activities, *Proceedings of the 10th IASTED International Conference on Web-based Education* (#792-016). Calgary, Canada: IASTED.

Stylianou, D. A. & Dubinsky, E. (1999). Determining linearity: the use of visual imagery in problem solving, *Proceedings of the 21th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 245-252). Columbus, USA.

Viholainen, A. (2008). *Prospective mathematics teachers' informal and formal reasoning about the concepts of derivative and differentiability* (Doctoral dissertation, University of Jyvaskyla, Finland).