Ultraelliptic integrals and two-dimensional sigma-functions*

Takanori Ayano† and Victor M. Buchstaber‡

Abstract

This article is devoted to the classical problem of the inversion of the ultra-elliptic integrals given by basic holomorphic differentials on the curve of genus 2. Here we obtained the basic solutions F and G of this problem, which are obtained from a single-valued 4-periodic meromorphic function on the Abelian covering W of the universal hyperelliptic curve of genus 2. As W we use a non-singular analytic curve $W = \{\mathbf{u} = (u_1, u_3) \in \mathbb{C}^2 : \sigma(\mathbf{u}) = 0\}$, where $\sigma(\mathbf{u})$ is the two-dimensional sigma function. It is shown that $G(z) = F(\xi(z))$, where z is a local coordinate in a neighborhood of a point of the smooth curve W, and $\xi(z)$ is a smooth function in this neighborhood, given by the equation $\sigma\big(u_1, \xi(u_1)\big) = 0$. It was obtained: the differential equations for the functions $F(z)$, $G(z)$ and $\xi(z)$; recurrent formulas for the coefficients of expansion in the series of these functions; transformation of function $G(z)$ into Weierstrass elliptic function $\wp$ under deformation of a curve of genus 2 into an elliptic curve.

1 Introduction.

In 1835, Krellé's periodic journal, Volume 13, published Jacobi's work "*On quadruply periodic functions of two variables, which arise in the theory of Abelian functions*". This article laid the foundations for the theory of ultra-elliptic integrals (see [24]), which is still relevant today due to problems in the theory of functions and mathematical physics (see a brief review below).

Let V be a nonsingular hyperelliptic curve of genus 2 defined by equation

$$Y^2 = X^5 + \lambda_4 X^3 + \lambda_6 X^2 + \lambda_8 X + \lambda_{10}, \quad \lambda_{2k} \in \mathbb{C}.$$

---

*This is a post-peer-review, pre-copyedit version of an article published in Functional Analysis and Its Applications.
The final authenticated version is available online at: https://doi.org/10.1134/S0016266319030018

†Osaka City University, Advanced Mathematical Institute, Osaka, Japan.
Email: ayano@sci.osaka-cu.ac.jp

‡Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia.
Email: buchstab@mi-ras.ru

MSC classes: 14K25, 14H40, 14H42, 14H70.

Let

$$du_1 = -\frac{X}{2Y}dX, \qquad du_3 = -\frac{1}{2Y}dX \tag{1}$$

be a basis of the vector space of holomorphic 1-forms on V, and let $d\mathbf{u} = {}^t(du_1, du_3)$. The sigma function $\sigma(\mathbf{u})$ associated with the curve V is an entire odd function on $\mathbb{C}^2$ (see [7, 6]).

An important specificity of sigma functions is the ability to introduce a grading (see [11]). In this case, we have

$$\deg X = 2,\ \deg Y = 5,\ \deg \lambda_{2k} = 2k,\ k = 2, \dots, 5,\ \deg u_{2q-1} = 1 - 2q,\ q = 1, 2.$$

In this grading, the equation of the curve V is given by a homogeneous polynomial of degree 10, and the sigma function $\sigma(\mathbf{u})$ is given by a homogeneous series of degree -3 whose coefficient under the monomial $u_1^i u_3^j$ (if it is nonzero) is given by a homogeneous polynomial in $\lambda_4, \dots, \lambda_{10}$ of degree $i + 3j - 3$.

The initial segment of the series has the form

$$\sigma(\mathbf{u}; \lambda) = \frac{1}{3}\, u_1^3 - u_3 - \frac{1}{6}\, \lambda_6 u_3^3 + \frac{1}{12}\, \lambda_4 u_1^4 u_3 + \frac{1}{6}\, \lambda_6 u_1^3 u_3^2 + \\ + \frac{1}{6}\, \lambda_8 u_1^2 u_3^3 + \frac{1}{3}\, \lambda_{10} u_1 u_3^4 - \left(\frac{1}{60}\, \lambda_4 \lambda_8 + \frac{1}{120}\, \lambda_6^2 \right) u_3^5 + \dots \tag{2}$$

The choice of the sign “ $-$ ” in formulas (1) leads to the normalization $\sigma(\mathbf{u}, 0) = \frac{1}{3}\, u_1^3 - u_3$. This corresponds to the writing of the function $\sigma(\mathbf{u}, 0)$ in the form of the Adler–Moser polynomial (see [1]) and the Schur–Weierstrass polynomial (see [8]).

In the present paper, we obtain solutions of inversion problems for ultra-elliptic integrals, given by equations and series that are homogeneous in this grading.

Consider the analytic curve $W = \{\mathbf{u} = (u_1, u_3) \in \mathbb{C}^2 \mid \sigma(\mathbf{u}) = 0\}$. The gradient $\nabla\sigma(\mathbf{u}) = \big(\sigma_1(\mathbf{u}), \sigma_3(\mathbf{u})\big)$, where $\sigma_i(\mathbf{u}) = \frac{\partial}{\partial u_i}\sigma(\mathbf{u})$, does not vanish at any point of the curve W, therefore W does not have singular points.

The nonsingular curve on the Jacobian of the curve V, given by equation $\sigma(\mathbf{u}) = 0$, is called the sigma divisor and denoted by (σ).

The Jacobi variety $\mathrm{Jac}(V)$ of the curve V of genus 2 is the base of the universal covering $\mathbb{C}^2 \to \mathrm{Jac}(V)$, whose layer can be identified with the lattice $\Gamma \subset \mathbb{Z}^2$ of periods of holomorphic differentials on V. The image of the Abel–Jacobi map

$$I\colon V \longrightarrow \mathrm{Jac}(V), \quad P \longrightarrow \int_\infty^P d\mathbf{u}$$

is the sigma divisor (σ), therefore the mapping I defines an induced Abelian covering with a base V, whose space can be identified with W. Thus, we obtain the covering $W \to V$ with the layer Γ and the possibility to construct solutions of the inversion problem of the ultra-elliptic integrals in the class of single-valued 4-periodic meromorphic functions on the curve $W \subset \mathbb{C}^2$.

We denote by $\mathcal{H}(\mathbb{C}^2)$ the ring of holomorphic functions on $\mathbb{C}^2$ with coordinates $\mathbf{u} = (u_1, u_3)$ and by $\mathcal{F}(\mathbb{C}^2)$ the corresponding field of meromorphic functions. The field of functions $\mathcal{F}(\mathrm{Jac}(V))$, which are meromorphic on the Jacobian of the curve V, can be identified with the field of functions $f(\mathbf{u}) \in \mathcal{F}(\mathbb{C}^2)$ such that $f(\mathbf{u}+\Omega) \equiv f(\mathbf{u})$ for any period $\Omega \in \Gamma$. Using the classical results of the theory of functions on many complex variables, it is easy to show that any function $f(\mathbf{u}) \in \mathcal{F}(\mathbb{C}^2)$ can be written in the form

$$f(\mathbf{u}) = \frac{1}{h(\mathbf{u})}\big(g_1(\mathbf{u}) + \sigma(\mathbf{u})g_2(\mathbf{u})\big),$$

where $h(\mathbf{u})$, $g_1(\mathbf{u})$ and $g_2(\mathbf{u})$ belong to the ring $\mathcal{H}(\mathbb{C}^2)$. Here, if $g_1(\mathbf{u})$ is not equal identically to zero on $\mathbb{C}^2$, then the restriction of $g_1(\mathbf{u})$ to the curve $W \in \mathbb{C}^2$ is not equal identically to zero. The quotient ring $\mathcal{H}(\mathbb{C}^2)/J$, where J is the ideal generated by function $\sigma(\mathbf{u})$, is an integral domain. The fraction field of the ring $\mathcal{H}(\mathbb{C}^2)/J$ can be identified with the field $\mathcal{F}(W)$ of meromorphic functions on W.

Let us denote by $\mathcal{F}(\mathbb{C}^2, W)$ the field of functions $f(\mathbf{u}) \in \mathcal{F}(\mathbb{C}^2)$ such that $f(\mathbf{u}+\Omega) \equiv f(\mathbf{u})$ for any point $\mathbf{u} \in W$ and any period $\Omega \in \Gamma$. Consider the field $\mathcal{F}\big((\sigma)\big)$ of meromorphic functions on sigma-divisor $(\sigma) \subset \mathrm{Jac}(V)$ and canonical projection $\pi\colon W \to (\sigma)$. The image of the homomorphism $\pi^*\colon \mathcal{F}\big((\sigma)\big) \to \mathcal{F}(W)$ can be identified with the field of functions of the form $\varepsilon\frac{g(\mathbf{u})}{h(\mathbf{u})} \in \mathcal{F}(\mathbb{C}^2, W)$, where ε is constant, and restrictions of the holomorphic functions $g(\mathbf{u})$ and $h(\mathbf{u})$ to the curve W are not equal identically to zero.

Our article is devoted to solving the following problems.

Problem I. Describe the field $\mathcal{F}\big((\sigma)\big)$ in terms of the two-dimensional sigma function $\sigma(\mathbf{u})$.

The field of functions $\mathcal{F}(V)$, that are meromorphic on the curve V, can be identified with the fraction field of polynomial ring $\mathbb{C}[X, Y]$ by the ideal generated by the polynomial

$$Y^2 - X^5 - \lambda_4 X^3 - \lambda_6 X^2 - \lambda_8 X - \lambda_{10}.$$

The mapping I induces the homomorphism

$$\overline{I^*}\ :\ \mathcal{F}\big((\sigma)\big) \to \mathcal{F}(V).$$

Problem II. Describe the homomorphism $\overline{I^*}$.

The solution of problems I and II is given in Theorem 3.4. In this theorem we give an explicit form of the functions f_2 and f_5 from $\mathcal{F}(\mathbb{C}^2, W)$, which define the functions $\tilde{f}_2$ and $\tilde{f}_5$ on $\mathcal{F}\big((\sigma)\big)$. By the same token $\overline{I^*}(\tilde{f}_2) = X$ and $\overline{I^*}(\tilde{f}_5) = Y$. The functions f_2 and f_5 are written as differential expressions in the sigma function $\sigma(\mathbf{u})$ and are not 4-periodic on whole $\mathbb{C}^2$.

In the papers [25], [17] and [27] one constructed the functions g_2 and g_5 from $\mathcal{F}(\mathbb{C}^2, W)$. These functions admit constraints $\tilde{g}_2$ and $\tilde{g}_5$ on (σ). By the same token $\overline{I^*}(\tilde{g}_2) = X$ and $\overline{I^*}(\tilde{g}_5) = Y$. The functions g_2 and g_5 are written as differential expressions in the sigma functions $\sigma(\mathbf{u})$ and $\sigma(2\mathbf{u})$.

Problem III. Describe the functions $f_2 - g_2$ and $f_5 - g_5$.

Section 7 of our work is devoted to this problem.

We have two ultra-elliptic integrals $\int_\infty^P du_1$ and $\int_\infty^P du_3$ obtained with the help of two holomorphic differentials du_1 and du_3. We take a point $P_* \in V$ such that $P_* \neq \infty$ and an open neighborhood U_* of this point P_* such that U_* is homeomorphic to an open disk in $\mathbb{C}$. Let us fix a path γ_* on the curve V from ∞ to the point P_*. We consider the holomorphic mapping defined by an ultra-elliptic integral

$$I_3 \ : \ U_* \to \mathbb{C}, \quad P = (X, Y) \mapsto \int_\infty^P du_3,$$

where as the path of integration we choose the composition of the fixed path γ_* from ∞ to the point P_* and some path in the neighborhood U_* from P_* to the point P. Let us put $u_3^* = I_3(P_*)$. Since $P_* \neq \infty$, then the equation $\sigma(\varphi(u_3), u_3) = 0$ defines a single-valued implicit function $\varphi(u_3)$ in the neighborhood of the point u_3^*.

Let us denote by $\mathcal{F}(U_*)$ the field of meromorphic functions on U_*. For $f \in \mathcal{F}(\mathbb{C}^2, W)$, we can consider the meromorphic function $f\big(\varphi(I_3(P)), I_3(P)\big)$ on U_*. The mapping I_3 induces a ring homomorphism

$$I_3^* \ : \ \mathcal{F}(\mathbb{C}^2, W) \to \mathcal{F}(U_*), \ \ f \mapsto f\big(\varphi(I_3(P)), I_3(P)\big),$$

which defines a homomorphism

$$\overline{I_3^*} \ : \ \mathcal{F}\big((\sigma)\big) \to \mathcal{F}(U_*).$$

Problem IV. Describe the homomorphism $\overline{I_3^*}$.

The solution of this problem is given in Theorem 4.2.

Let us take a point $P_* \in V$ such that $P_* \neq (0, \pm\sqrt{\lambda_{10}})$ and an open neighborhood U_* of the point P_* such that U_* is homeomorphic to an open disk in $\mathbb{C}$. Let us fix a path γ_* on the curve V from ∞ to the point P_*. Let us consider the holomorphic mapping defined by the ultra-elliptic integral

$$I_1 \ : \ U_* \to \mathbb{C}, \quad P = (X, Y) \mapsto \int_\infty^P du_1,$$

where as the path of integration we choose, as above, the composition of the fixed path γ_* from ∞ to the point P_* and some path in U_* from P_* to P.

Let $u_1^* = I_1(P_*)$. Since $P_* \neq (0, \pm\sqrt{\lambda_{10}})$, one can take the unique implicit function $\xi(u_1)$, defined in a neighborhood of u_1^* by the equation $\sigma(u_1, \xi(u_1)) = 0$. Let us denote by $\mathcal{F}(U_*)$ the field of meromorphic functions on U_*. For $g \in \mathcal{F}(\mathbb{C}^2, W)$, we can consider the meromorphic function $g\big(I_1(P), \xi(I_1(P))\big)$ on U_*. The mapping I_1 induces a homomorphism

$$I_1^* \ : \ \mathcal{F}(\mathbb{C}^2, W) \to \mathcal{F}(U_*), \ \ g \mapsto g\big(I_1(P), \xi(I_1(P))\big),$$

which defines a homomorphism

$$\overline{I_1^*} \ : \ \mathcal{F}\big((\sigma)\big) \to \mathcal{F}(U_*).$$

Problem V. Describe the homomorphism $\overline{I_1^*}$.

The solution of this problem is given in Theorem 5.2.

Let $\mathrm{Sym}^2(V)$ be the symmetric square of the non-singular curve V of genus 2. Let us consider the Abel-Jacobi map

$$\mathrm{Sym}^2(V) \to \mathrm{Jac}(V), \quad (P_1, P_2) \mapsto \int_\infty^{P_1} d\mathbf{u} + \int_\infty^{P_2} d\mathbf{u}.$$

Problem VI. Describe the connection between the solution of the problem of the inversion of the Abel–Jacobi map in genus 2 and the solutions of the problems of inversion of the ultra-elliptic integrals I_1 and I_3.

The solution of this problem is given in Theorems 4.9 and 5.10.

Let $f_2(\mathbf{u}) = -\sigma_3(\mathbf{u})/\sigma_1(\mathbf{u})$, where $\sigma_i(\mathbf{u}) = \partial_{u_i}\sigma(\mathbf{u})$.

For $P = (X, Y) \in U_*$ and $z = I_3(P)$, set $F(z) = f_2(\varphi(z), z)$. Then we have (see Proposition 4.4)

$$X = F(z), \quad Y = -F'(z)/2, \tag{3}$$

where F' denotes the derivative of F with respect to z. Using this solution to the inversion problem, it is easy to obtain differential equations (see Theorem 4.5)

$$(F'/2)^2 = F^5 + \lambda_4 F^3 + \lambda_6 F^2 + \lambda_8 F + \lambda_{10}, \tag{4}$$

$$F'' = 10F^4 + 6\lambda_4 F^2 + 4\lambda_6 F + 2\lambda_8.$$

Under given initial conditions, these equations allow us to obtain in an explicit form the series expansion of the function $F(z)$ (see Proposition 4.7 and 4.8).

For $P = (X, Y) \in U_*$ and $z = I_1(P)$, set $G(z) = f_2(z, \xi(z))$. Then we have (see Proposition 5.4)

$$X = G(z), \ \ Y = -G(z)G'(z)/2, \tag{5}$$

where G' denotes the derivative of G with respect to z. Using this solution to the inversion problem, it is easy to obtain differential equations (see Theorem 5.5)

$$G^2\{(G'/2)^2 - G^3 - \lambda_4 G - \lambda_6\} - \lambda_8 G - \lambda_{10} = 0, \tag{6}$$

$$G^4(G''' - 12GG') - 4\lambda_8 GG' - 12\lambda_{10}G' = 0. \tag{7}$$

Under given initial conditions, these equations allow us to obtain in an explicit form the series expansion of the function $G(z)$ (see Propositions 5.7, 5.8, 6.2). If $\lambda_8 = \lambda_{10} = 0$, then the function $G(z)$ coincides with the Weierstrass elliptic function $\wp(z)$ (see Corollary 6.5). The functions $F(z)$ and $G(z)$ are related by $G(z) = F(\xi(z))$ (see Proposition 5.9).

In [18, 19, 20, 21, 22], the inversion of the ultra-elliptic integrals is applied to the construction of the analytic solutions of the geodesic equations in physics. These papers state that the functions $F(z)$ and $G(z)$ are solutions of differential equations (4) and (6) in our notation, respectively.

In [14], the inversion of the ultra-elliptic integrals for differentials of the first and second kind is applied in the problem of the motion of the double pendulum.

In the papers [12] and [21] one described the following procedure for calculating the functions $F(z)$ and $G(z)$. First, the basis in homology is fixed on the curve V and the period matrices that are necessary for constructing the theta function are calculated. Then, using the Newton method, implicit functions φ and ξ (in our notation) are calculated. Finally, the functions $F(z)$ and $G(z)$ are calculated based on the expression of the sigma function in terms of theta function.

In [4, 5], the authors consider the ultra-elliptic integrals in terms of the two-dimensional theta function, i.e. the ultra-elliptic integrals of the normalized holomorphic differentials. In these papers, the inversion problem of the ultra-elliptic integrals is solved with the using of the theta functions of genus 2 (see [4], p.1729). The period matrix of holomorphic differentials of the curve, which is necessary for constructing theta functions, is calculated with the using of some linear and non-linear equations that are solved by the Newton method. In these papers, the inversion of ultra-elliptic integrals is applied to calculate the conformal mappings of the upper half-plane into rectangular polygons.

In [5], this approach is applied to some physical problems such as the calculating the 2D-flow of ideal fluid over a rectangular surface and the evaluation of the capacity of multi-component rectangular capacitors with axial symmetry.

Finally, we note that the hyperelliptic integrals of the third kind are expressed in terms of the sigma functions in [22] for $g = 2$ and in [13] for any genus. In [22], in the case $g = 2$, these expressions are used to solve the geodesic equations in the Kerr-de Sitter space-time. In [13], in the case $g = 3$, these expressions are used to solve the geodesic equations for the massive test particles in the Hořava-Lifshitz black hole space-times.

In our paper, we realised a new approach to the inversion problem of ultra-elliptic integrals. We obtain the series expansion of the functions $F(z)$ and $G(z)$ directly from the differential equations which they satisfy and show that the coefficients of the series expansions are homogeneous polynomials in the parameters $\lambda_4, \dots, \lambda_{10}$.

Note that in the approach based on sigma functions, it becomes possible to obtain results under continuous deformation of the parameters of the curves until the curve's degeneration (see, for example, [2]). As is well known, approach based on the theta functions does not provide such an opportunity.

2 The sigma function.

Let polynomial

$$Q(X) = X^5 + \lambda_4 X^3 + \lambda_6 X^2 + \lambda_8 X + \lambda_{10}, \quad \lambda_i \in \mathbb{C},$$

has no multiple roots. Let us consider a nonsingular hyperelliptic curve of genus 2

$$V = \{(X, Y) \in \mathbb{C}^2 \mid Y^2 = Q(X)\}. \tag{8}$$

In this section, we recall the construction of the sigma function for the curve V in terms of the theta function for this curve (see [9]) and give the facts that we will need later.

In the coordinates $(X, Y) \in V$ we choose

$$du_1 = -\frac{X}{2Y}dX, \quad du_3 = -\frac{1}{2Y}dX$$

as the basis of the vector space of holomorphic 1-forms on V and set $d\mathbf{u} = {}^t(du_1, du_3)$. Next, choose

$$dr_1 = -\frac{X^2}{2Y}dX, \quad dr_3 = \frac{-\lambda_4 X - 3X^3}{2Y}dX$$

as the meromorphic 1-forms on V.

Let $\{\alpha_i, \beta_i\}_{i=1}^2$ be the canonical basis in the one-dimensional homology group of the curve V. Let us define the matrices of periods by

$$2\omega_1 = \left(\int_{\alpha_j} du_i\right), \ 2\omega_2 = \left(\int_{\beta_j} du_i\right), \ -2\eta_1 = \left(\int_{\alpha_j} dr_i\right), \ -2\eta_2 = \left(\int_{\beta_j} dr_i\right).$$

Let us introduce the matrix of normalized periods $\tau = \omega_1^{-1}\omega_2$.

Let $\delta = \tau\delta' + \delta''$, $\delta', \delta'' \in \mathbb{R}^2$, be the vectors of Riemann's constants with respect to $(\{\alpha_i, \beta_i\}, \infty)$ and $\delta := {}^t({}^t\delta', {}^t\delta'')$. Then we have $\delta' = {}^t(\frac{1}{2}, \frac{1}{2})$ and $\delta'' = {}^t(1, \frac{1}{2})$. The sigma function $\sigma(\mathbf{u})$, where $\mathbf{u} = {}^t(u_1, u_3) \in \mathbb{C}^2$, is defined by

$$\sigma(\mathbf{u}) = C \exp\left({}^t\mathbf{u}\varkappa\mathbf{u}\right) \theta[\delta](\mathbf{w}, \tau),$$

where C is a constant, $\varkappa = \frac{1}{2}\eta_1\omega_1^{-1}$, $\mathbf{w} = \frac{1}{2}\omega_1^{-1}\mathbf{u}$ and $\theta[\delta](\mathbf{w})$ is the Riemann's theta function with characteristics δ, which is defined by

$$\theta[\delta](\mathbf{w}) = \sum_{n \in \mathbb{Z}^2} \exp\{\pi\sqrt{-1}\, {}^t(n + \delta')\tau(n + \delta') + 2\pi\sqrt{-1}\, {}^t(n + \delta')(\mathbf{w} + \delta'')\}.$$

Note that the transition from the period matrices $2\omega_1$ and $2\omega_2$ to the normalized period matrix $\tau = \omega_1^{-1}\omega_2$ deprives the coordinates of the vector $\mathbf{w}$ of grading.

We set

$$\sigma_i = \partial_i\sigma(\mathbf{u}), \quad \sigma_{i,j} = \partial_i\partial_j\sigma(\mathbf{u}), \quad \wp_{i,j}(\mathbf{u}) = -\partial_i\partial_j \log\sigma(\mathbf{u}), \quad \wp_{i,j,k}(\mathbf{u}) = -\partial_i\partial_j\partial_k \log\sigma(\mathbf{u}),$$

where $\partial_i = \frac{\partial}{\partial u_i}$, $i = 1, 3$. Let us define the period lattice $\Lambda = \{2\omega_1 m_1 + 2\omega_2 m_2 \mid m_1, m_2 \in \mathbb{Z}^2\}$ and let

$$W = \{\mathbf{u} \in \mathbb{C}^2 \mid \sigma(\mathbf{u}) = 0\}.$$

Proposition 2.1. (see [9], Theorem 1.1, and [26], p. 193) *For $m_1, m_2 \in \mathbb{Z}^2$, we set $\Omega = 2\omega_1 m_1 + 2\omega_2 m_2$, and let*

$$A = (-1)^{2({}^t\delta' m_1 - {}^t\delta'' m_2) + {}^t m_1 m_2} \exp({}^t(2\eta_1 m_1 + 2\eta_2 m_2)(\mathbf{u} + \omega_1 m_1 + \omega_2 m_2)).$$

Then

(i) $\sigma(\mathbf{u} + \Omega) = A\sigma(\mathbf{u})$, *where* $\mathbf{u} \in \mathbb{C}^2$.

(ii) $\sigma_i(\mathbf{u} + \Omega) = A\sigma_i(\mathbf{u})$, $i = 1, 3$, *where* $\mathbf{u} \in W$.

Proposition 2.1 (i) implies that $\mathbf{u} + \Omega \in W$ д л я for any $\mathbf{u} \in W$ and $\Omega \in \Lambda$. The surface

$$(\sigma) := \{\mathbf{u} \in \mathbb{C}^2/\Lambda \mid \sigma(\mathbf{u}) = 0\}$$

is called the sigma divisor.

Theorem 2.2. (see [8], Theorem 6.3, [9], Theorem 7.7, [11], [26], Theorem 3) *The sigma function $\sigma(\mathbf{u}) = \sigma(u_1, u_3)$ is an entire odd function on $\mathbb{C}^2$, and it is given by the series*

$$\sigma(\mathbf{u}) = \frac{1}{3}u_1^3 - u_3 + \sum_{i+3j \geqslant 7} \alpha_{i,j} u_1^i u_3^j, \tag{9}$$

where the coefficients $\alpha_{i,j} \in \mathbb{Q}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ are homogeneous polynomials of degree $i + 3j - 3$, if $\alpha_{i,j} \neq 0$.

3 The field of meromorphic functions on the sigma divisor.

The Abel-Jacobi mapping

$$I \colon V \to \mathrm{Jac}(V) = \mathbb{C}^2/\Lambda, \qquad P \mapsto \int_\infty^P d\mathbf{u},$$

is an embedding, and its image is equal to the sigma divisor (σ). Let us consider the following meromorphic functions on $\mathbb{C}^2$

$$f_2 = -\frac{1}{\sigma_1}\sigma_3, \qquad f_5 = \frac{1}{2\sigma_1^3}(\sigma_1^2\sigma_{3,3} - 2\sigma_1\sigma_3\sigma_{1,3} + \sigma_3^2\sigma_{1,1}).$$

Note that $f_2 = \frac{1}{u_1^2} + \ldots$ and $f_5 = \frac{1}{u_1^5} + \ldots$. Let us set $P = (X, Y) \in V$ and

$$\mathbf{u} = \int_\infty^P d\mathbf{u}.$$

Lemma 3.1. (see [16], p. 129, and [23], Theorem 1) *The function $\sigma_1(\mathbf{u})$ treated as a function of the point P, does not identically vanish. There is the equality of the meromorphic functions on V*

$$X = f_2(\mathbf{u}). \tag{10}$$

Proof. From the well-known solution of the Abel–Jacobi inversion problem (see [3] and [9] Theorem 2.4), we have

$$X_1 + X_2 = \wp_{1,1}\left(\int_\infty^{P_1} d\mathbf{u} + \int_\infty^{P_2} d\mathbf{u}\right), \quad -X_1X_2 = \wp_{1,3}\left(\int_\infty^{P_1} d\mathbf{u} + \int_\infty^{P_2} d\mathbf{u}\right),$$

where $P_i = (X_i, Y_i) \in V,\ i = 1, 2$. Since by definition

$$\wp_{1,1} = \frac{\sigma_1^2 - \sigma\sigma_{1,1}}{\sigma^2}, \quad \wp_{1,3} = \frac{\sigma_1\sigma_3 - \sigma\sigma_{1,3}}{\sigma^2},$$

we have

$$-\frac{X_1X_2}{X_1 + X_2} = \frac{\sigma_1\sigma_3 - \sigma\sigma_{1,3}}{\sigma_1^2 - \sigma\sigma_{1,1}}.$$

When $P_2 = (X_2, Y_2) \to (\infty, \infty)$, the left side of the equation tends to $-X_1$ and $\sigma(\mathbf{u}) \to 0$ so the right side of equality tends to $-f_2(\mathbf{u})$. □

Lemma 3.2. ([9] p. 116) *There is the equality of meromorphic functions on V*

$$Y = f_5(\mathbf{u}). \tag{11}$$

Proof. Differentiating equation (10) with respect to X, we obtain

$$1 = f_{2,1}(\mathbf{u})\left(-\frac{X}{2Y}\right) + f_{2,3}(\mathbf{u})\left(-\frac{1}{2Y}\right),$$

where $f_{2,i} = \partial_{u_i} f_2$. Therefore we have

$$2Y = -f_{2,1}(\mathbf{u})X - f_{2,3}(\mathbf{u}) = -f_{2,1}(\mathbf{u})f_2(\mathbf{u}) - f_{2,3}(\mathbf{u}) = 2f_5(\mathbf{u}). \tag{12}$$

□

Remark 3.3. *The following formula was obtained in [25]*

$$2X = \wp_{1,1}(2\mathbf{u}).$$

The following formula was obtained in [17], p. 128 (see also [27] Lemma 3.2.4)

$$2Y = \frac{\sigma(2\mathbf{u})}{\sigma_1(\mathbf{u})^4}. \tag{13}$$

Let $\mathcal{F}(\mathbb{C}^2)$ be the field of all meromorphic functions on $\mathbb{C}^2$. Let us denote by $\mathcal{F}(\mathbb{C}^2, W)$ the field of functions $f(\mathbf{u}) \in \mathcal{F}(\mathbb{C}^2)$ such that $f(\mathbf{u} + \Omega) \equiv f(\mathbf{u})$ for any point $\mathbf{u} \in W$ and any period $\Omega \in \Gamma$. Let J^* be the set of meromorphic functions $f \in \mathcal{F}(\mathbb{C}^2, W)$, which identically equal to zero on W and $\mathcal{F}\big((\sigma)\big) := \mathcal{F}(\mathbb{C}^2, W)/J^*$. The mapping I induces the composition

$$I^*\colon \mathcal{F}(\mathbb{C}^2, W) \to \mathcal{F}(\mathbb{C}^2, W)/J^* \to \mathcal{F}(V), \quad f \mapsto f \circ I.$$

Theorem 3.4. *The mapping $I^*\colon \mathcal{F}(\mathbb{C}^2, W) \to \mathcal{F}(V)$ induces the isomorphism of fields*

$$\overline{I^*} \;:\; \mathcal{F}\big((\sigma)\big) \to \mathcal{F}(V), \qquad \overline{f_2} \mapsto X, \;\; \overline{f_5} \mapsto Y,$$

where $\overline{f_i}$ is the equivalence class of the function f_i in $\mathcal{F}\big((\sigma)\big)$ for $i = 2$ and 5.

Proof. From the equality $\operatorname{Ker} I^* = J^*$ we obtain that the mapping $\overline{I^*}$ is injective. From the equality $I^*(f_2) = X$ and $I^*(f_5) = Y$ we obtain that the mapping $\overline{I^*}$ is surjective. □

Let us introduce the following differentiation of the field $\mathcal{F}(V)$

$$\mathcal{L}_3 = -2Y\frac{d}{dX}.$$

Let us describe in more detail the action of this operator. Consider the full differential df of the function $f \in \mathcal{F}(V)$ and the full differential dX of the meromorphic function X on V. Then df/dX is the meromorphic function on V which uniquely determined by the formula $df = (df/dX)\cdot dX$. Let us consider the following differentiation of the field $\mathcal{F}(\mathbb{C}^2)$

$$L_3 = \partial_{u_3} + f_2\partial_{u_1}.$$

Proposition 3.5.

(i) *Let $g \in \mathcal{F}(\mathbb{C}^2, W)$, then $L_3 g \in \mathcal{F}(\mathbb{C}^2, W)$.*

(ii) *Let $g \in J^*$, then $L_3 g \in J^*$. Therefore, we can consider the operator L_3 as a differentiation of the field $\mathcal{F}\big((\sigma)\big) = \mathcal{F}(\mathbb{C}^2, W)/J^*$.*

(iii) *There is the formula $\mathcal{L}_3 \circ \overline{I^*} = \overline{I^*} \circ L_3$.*

Proof. Set $g \in \mathcal{F}(\mathbb{C}^2, W)$. Using the formula $X = f_2(\mathbf{u})$, we obtain

$$\mathcal{L}_3 \circ I^*(g) = \mathcal{L}_3 g(\mathbf{u}) = -2Y\left(-\frac{X}{2Y}g_1(\mathbf{u}) - \frac{1}{2Y}g_3(\mathbf{u})\right) = f_2(\mathbf{u})g_1(\mathbf{u}) + g_3(\mathbf{u}),$$

where $g_i = \partial_{u_i} g$. Since the operator $\mathcal{L}_3$ transforms the field $\mathcal{F}(V)$ into itself, we obtain

$$f_2(\mathbf{u})g_1(\mathbf{u}) + g_3(\mathbf{u}) \in \mathcal{F}(V).$$

Consequently, $L_3 g = f_2 g_1 + g_3 \in \mathcal{F}(\mathbb{C}^2, W)$. Therefore we obtain a proof of (i).

Set $g \in J^*$, then the function $f_2(\mathbf{u})g_1(\mathbf{u}) + g_3(\mathbf{u})$ as an element of $\mathcal{F}(V)$ equals to zero. This means that $L_3 g = f_2 g_1 + g_3 \in J^*$. Thus, we obtain a proof of assertion (ii).

We have

$$I^* \circ L_3 g = I^*(f_2 g_1 + g_3) = f_2(\mathbf{u})g_1(\mathbf{u}) + g_3(\mathbf{u}).$$

Therefore, $\mathcal{L}_3 \circ I^*(g) = I^* \circ L_3 g$. This proves statement (iii). □

Proposition 3.6. *There is the formula*

$$Y = -\frac{1}{2}(L_3 f_2)(\mathbf{u}).$$

Proof. The statement follows directly from the formula (12) and the definition of the operator L_3. □

4 The ultra-elliptic integral I_3.

Lemma 4.1. (see [27], Proposition 2.2.1 (2) and [28], Lemma 1.9 (1)) *If* $\mathbf{u} \in W$ *and* $\mathbf{u} \notin \Lambda$*, then* $\sigma_1(\mathbf{u}) \neq 0$*.*

Proof. Let us take a point $\mathbf{u} \in W$ such that $\mathbf{u} \notin \Lambda$. Let $\sigma_1(\mathbf{u}) = 0$. Note that the meromorphic function X on V has a pole only at ∞. From the lemma 3.1 we get $\sigma_3(\mathbf{u}) = 0$. Since the set of critical points $\{\mathbf{u} \in \mathbb{C}^2 \mid \sigma(\mathbf{u}) = \sigma_1(\mathbf{u}) = \sigma_3(\mathbf{u}) = 0\}$ of the function $\sigma(\mathbf{u})$ is the empty set for the curve of genus 2 (see [9], [15], [27], Lemma 1.7.2) we arrive at a contradiction. Therefore we obtain $\sigma_1(\mathbf{u}) \neq 0$. $\square$

Let us take a point $P_* \in V$ such that $P_* \neq \infty$ and an open neighborhood U_* of this point that is homeomorphic to an open disk in $\mathbb{C}$. We fix a path γ_* on the curve V from ∞ to P_*. Let us consider holomorphic mappings

$$I_3 \; : \; U_* \to \mathbb{C}, \quad P = (X, Y) \mapsto \int_\infty^P du_3,$$

$$\widetilde{I}_3 \; : \; U_* \to W, \quad P = (X, Y) \mapsto \int_\infty^P d\mathbf{u},$$

where as the path of integration we choose the composition of the path γ_* from ∞ to the point P_* and any path in the neighborhood U_* from P_* to the point P. Set $\mathbf{u}^* = \widetilde{I}_3(P_*)$, where $\mathbf{u}^* = (u_1^*, u_3^*)$. Since $P_* \neq \infty$, we have $\mathbf{u}^* \in W$ and $\mathbf{u}^* \notin \Lambda$. Then, according to the lemma 4.1, we have $\sigma_1(\mathbf{u}^*) \neq 0$. In a sufficiently small open neighborhood D_* of the point $\mathbf{u}^*$ there exist an open neighborhood E_* of u_3^* and a uniquely determined holomorphic function $\varphi(u_3)$ on E_* such that $D_* \cap W = \{(\varphi(u_3^*), u_3^*) \mid u_3^* \in E_*\}$.

Note that the differential du_3 vanishes only at ∞. Since $P_* \neq \infty$, the total derivative of I_3 at P_* is not equal to zero. Therefore, if one take the open neighborhood U_* sufficiently small, then the image of $\widetilde{I}_3$ will belong to D_*, and the mapping I_3 is injective.

Let U_3 be the image of I_3. Then U_3 is an open set which contains u_3^*, and the mapping $I_3 : U_* \to U_3$ is bijective. Let $\mathcal{F}(U_*)$ be the field of meromorphic functions on U_*. For $f \in \mathcal{F}(\mathbb{C}^2, W)$, we can consider the meromorphic function $f(\varphi(I_3(P)), I_3(P))$ on U_*. The mapping I_3 induces the homomorphism

$$I_3^* \; : \; \mathcal{F}(\mathbb{C}^2, W) \to \mathcal{F}(U_*), \quad f \mapsto f(\varphi(I_3(P)), I_3(P)).$$

Theorem 4.2. *The mapping* I_3^* *induces the injective homomorphism*

$$\overline{I_3^*} \; : \; \mathcal{F}\big((\sigma)\big) \to \mathcal{F}(U_*), \quad \overline{f_2} \mapsto X, \;\; \overline{f_5} \mapsto Y.$$

Proof. For $P = (X, Y) \in U_*$ set $z = I_3(P)$. Since the image of $\widetilde{I}_3$ belongs to D_*, then $\widetilde{I}_3(P) = (\varphi(z), z)$. According to the lemmas 3.1 and 3.2 we have $I_3^*(f_2) = X$ and $I_3^*(f_5) = Y$. If $f \in J^*$, then $I_3^*(f) = 0$. Therefore the mapping I_3^* induces the homomorphism

$$\overline{I_3^*} \; : \; \mathcal{F}\big((\sigma)\big) \to \mathcal{F}(U_*).$$

Since $\mathcal{F}\big((\sigma)\big)$ is a field and $\overline{I_3^*}$ is not zero mapping, then the mapping $\overline{I_3^*}$ is injective. $\square$

Lemma 4.3. *For any meromorphic function $h = h(u_1, u_3)$ on $\mathbb{C}^2$ there is the formula*

$$\frac{\partial}{\partial u_3} h(\varphi(u_3), u_3) = L_3 h.$$

Proof. According to the definition of the function φ we have

$$\frac{\partial \varphi}{\partial u_3} = -\frac{\sigma_3}{\sigma_1}(\varphi(u_3), u_3).$$

Therefore

$$\frac{\partial}{\partial u_3} h(\varphi(u_3), u_3) = -\frac{\sigma_3}{\sigma_1}(\partial_{u_1} h) + \partial_{u_3} h = L_3 h.$$

□

We define the function $F(z) = f_2(\varphi(z), z)$ on E_*.

Proposition 4.4. *Set $z = I_3(P)$, where $P = (X, Y) \in U_*$. Then $X = F(z)$ and $Y = -F'(z)/2$, where F' is the derivative of F with respect to z.*

Proof. According to the theorem 4.2 we have $X = F(z)$. The formula for Y is obtained with using the operator L_3 acting on the field $\mathcal{F}\big((\sigma)\big)$. From Proposition 3.6 and Lemma 4.3 we obtain $Y = -F'(z)/2$. □

Theorem 4.5. *The function $F(z)$ satisfies the following ordinary differential equations:*

$$(F'/2)^2 = F^5 + \lambda_4 F^3 + \lambda_6 F^2 + \lambda_8 F + \lambda_{10}, \tag{14}$$

$$F'' = 10F^4 + 6\lambda_4 F^2 + 4\lambda_6 F + 2\lambda_8. \tag{15}$$

Proof. Using Proposition 4.4 and the relation $Y^2 = X^5 + \lambda_4 X^3 + \lambda_6 X^2 + \lambda_8 X + \lambda_{10}$ we obtain the equation (14). By differentiating the both sides of (14) with respect to z, we obtain the equation (15). □

Remark 4.6. *In [18] there is also the statement that the function $F(z)$ is a solution of the differential equation (14) and therefore the equation (15).*

From Proposition 4.4 and Theorem 4.5, one can obtain the series expansion of $F(z)$. Since the function $F(z)$ is holomorphic in a neighborhood of $z = u_3^*$, then the expansion in the neighborhood of this point has the form

$$F(z) = \sum_{n=0}^{\infty} p_{3n+2}(z - u_3^*)^n, \quad p_{3n+2} \in \mathbb{C}. \tag{16}$$

Proposition 4.7. *Set $P_* = (X_*, Y_*)$. Then in the expansion (16) we have $p_2 = X_*$ and $p_5 = -2Y_*$.*

Proof. Since $p_2 = F(u_3^*)$ and $p_5 = F'(u_3^*)$, then the statement of this proposition follows directly from Proposition 4.4. □

In the graduation introduced above, we have: $\deg p_2 = 2$ and $\deg p_5 = 5$. Denote by $\mathbb{Z}_{\geqslant r}$ the set of integers that are not less than r.

Proposition 4.8. *The coefficients p_{3n+2} in the expansion (16) are determined from the following recurrence relations:*

- $p_8 = 5p_2^4 + 3\lambda_4 p_2^2 + 2\lambda_6 p_2 + \lambda_8$,
- $(n+2)(n+1)p_{3n+8} = 10\sum_{(n_1,n_2,n_3,n_4)\in S_1} p_{3n_1+2}\, p_{3n_2+2}\, p_{3n_3+2}\, p_{3n_4+2}+$
  $+6\lambda_4 \sum_{(n_1,n_2)\in S_2} p_{3n_1+2}\, p_{3n_2+2} + 4\lambda_6 p_{3n+2}, \quad n \geqslant 1,$ *where*

$S_1 = \{(n_1,n_2,n_3,n_4) \in \mathbb{Z}^4_{\geqslant 0} \mid n_1+n_2+n_3+n_4 = n\}$, $S_2 = \{(n_1,n_2) \in \mathbb{Z}^2_{\geqslant 0} \mid n_1+n_2 = n\}$, *and the coefficients $p_{3n+2} \in \mathbb{Q}[p_2, p_5, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ are homogeneous polynomials of degree $3n+2$, if $p_{3n+2} \neq 0$.*

Proof. By substituting (16) into (15) and comparing the coefficients, we obtain the assertion. □

Set $P_i = (X_i, Y_i) \in U_*$, $i = 1,2$. Let us put $\mathbf{u}^{(i)} = \widetilde{I}_3(P_i)$, $\mathbf{u}^{(i)} = {}^t(u_1^{(i)}, u_3^{(i)})$ and $\mathbf{u}^{(0)} = \mathbf{u}^{(1)} + \mathbf{u}^{(2)}$.

Theorem 4.9. *There exist the following relations*

$$F(u_3^{(1)}) + F(u_3^{(2)}) = \wp_{1,1}(\mathbf{u}^{(0)}), \quad F(u_3^{(1)})F(u_3^{(2)}) = -\wp_{1,3}(\mathbf{u}^{(0)}),$$

$$F'(u_3^{(i)}) = F(u_3^{(i)})\wp_{1,1,1}(\mathbf{u}^{(0)}) + \wp_{1,1,3}(\mathbf{u}^{(0)}), \quad i = 1,2.$$

Proof. The well-known solution of the inversion problem of the Abel–Jacobi mapping (see [3] and [9], Theorem 2.4) is given by the formulas

$$X_1 + X_2 = \wp_{1,1}(\mathbf{u}^{(0)}), \quad -X_1X_2 = \wp_{1,3}(\mathbf{u}^{(0)}), \tag{17}$$

$$2Y_i = -X_i\wp_{1,1,1}(\mathbf{u}^{(0)}) - \wp_{1,1,3}(\mathbf{u}^{(0)}), \quad i = 1,2. \tag{18}$$

From Proposition 4.4 we have $X_i = F(u_3^{(i)})$ and $Y_i = -F'(u_3^{(i)})/2$ for $i = 1,2$. From here we obtain the statement of the theorem. □

5 The ultra-elliptic integral I_1.

Consider the following differentiation operator

$$L_1 = \partial_{u_1} + f_2^{-1}\partial_{u_3}.$$

We have $L_3 = f_2 L_1$.

Lemma 5.1. *If $P \neq (0, \pm\sqrt{\lambda_{10}})$, where $\sqrt{\lambda_{10}}$ is a complex number t such that $t^2 = \lambda_{10}$, then $\sigma_3(I(P)) \neq 0$.*

Proof. According to Theorem 2.2 we have $\sigma_3(0,0) = -1$. Therefore, from Proposition 2.1 (ii) we have $\sigma_3(I(\infty)) \neq 0$. Let $P \neq (0, \pm\sqrt{\lambda_{10}})$ and $P \neq \infty$. Then according to Lemma 4.1 we have $\sigma_1(I(P)) \neq 0$. Therefore, according to Lemma 3.1 we obtain $\sigma_3(I(P)) \neq 0$. $\square$

Let us take a point $P_* \in V$ such that $P_* \neq (0, \pm\sqrt{\lambda_{10}})$ and an open neighborhood U_* of the point P_*, that is homeomorphic to an open disk in $\mathbb{C}$. Let us fix a path γ_* on the curve V from ∞ to the point P_*. We consider the holomorphic mappings

$$I_1 \,:\, U_* \to \mathbb{C}, \quad P = (X,Y) \mapsto \int_\infty^P du_1,$$

$$\widetilde{I}_1 \,:\, U_* \to W, \quad P = (X,Y) \mapsto \int_\infty^P d\mathbf{u},$$

where as the path of integration we choose the composition of the fixed path γ_* from ∞ to the point P_* and any path in the neighborhood U_* from P_* to the point P. Set $\mathbf{u}^* = \widetilde{I}_1(P_*)$ and $\mathbf{u}^* = (u_1^*, u_3^*)$. According to Lemma 5.1 we have $\sigma_3(\mathbf{u}^*) \neq 0$. In a sufficiently small open neighborhood D_* of the point $\mathbf{u}^*$ there exist an open neighborhood E_* of the point u_1^* and a uniquely determined holomorphic function $\xi(u_1)$ on E_* such that $D_* \cap W = \{(u_1, \xi(u_1)) \mid u_1 \in E_*\}$. Note that the differential du_1 vanishes only at $(0, \pm\sqrt{\lambda_{10}})$. Since $P_* \neq (0, \pm\sqrt{\lambda_{10}})$ then the total derivative of the function I_1 is not equal to zero at the point P_*. Therefore, if one take the open neighborhood U_* sufficiently small, then the image of $\widetilde{I}_1$ will belong to D_*, and the mapping I_1 is injective. Let $\mathcal{F}(U_*)$ be the field of meromorphic functions on U_*. For $g \in \mathcal{F}(\mathbb{C}^2, W)$, we can consider the meromorphic function $g(I_1(P), \xi(I_1(P)))$ on U_*. The mapping I_1 induces the homomorphism

$$I_1^* \,:\, \mathcal{F}(\mathbb{C}^2, W) \to \mathcal{F}(U_*), \quad g \mapsto g\big(I_1(P), \xi(I_1(P))\big).$$

Theorem 5.2. *The mapping I_1^* induces the injective homomorphism*

$$\overline{I_1^*} \,:\, \mathcal{F}\big((\sigma)\big) \to \mathcal{F}(U_*), \quad \overline{f_2} \mapsto X, \;\; \overline{f_5} \mapsto Y.$$

Proof. Let $z = I_1(P)$, where $P = (X,Y) \in U_*$. Since the image of $\widetilde{I}_1$ belongs to D_*, we have $\widetilde{I}_1(P) = (z, \xi(z))$. According to Lemmas 3.1 and 3.2 we have $I_1^*(f_2) = X$ and $I_1^*(f_5) = Y$. If $g \in J^*$, then $I_1^*(g) = 0$. Therefore the mapping I_1^* induces the homomorphism

$$\overline{I_1^*} \,:\, \mathcal{F}\big((\sigma)\big) \to \mathcal{F}(U_*).$$

Since $\mathcal{F}\big((\sigma)\big)$ is a field and $\overline{I_1^*}$ is not zero mapping, then the mapping $\overline{I_1^*}$ is injective. $\square$

Lemma 5.3. *For any meromorphic function $h = h(u_1, u_3)$ on $\mathbb{C}^2$ there exists the formula*

$$\frac{\partial}{\partial u_1} h(u_1, \xi(u_1)) = L_1 h.$$

Proof. The proof is similar to that of Lemma 4.3. $\square$

Let us define the function $G(z) = f_2(z, \xi(z))$ on E_*.

Proposition 5.4. *For $P = (X, Y) \in U_*$ let $z = I_1(P)$. Then we have $X = G(z)$ and $Y = -G(z)G'(z)/2$, where G' is the derivative of G respect to z.*

Proof. According to Theorem 5.2, we have $X = G(z)$. The expression for Y is obtained using the differentiation operator L_1, that acts on the field $\mathcal{F}\big((\sigma)\big)$. According to Proposition 3.6, $L_3 = f_2 L_1$, and Lemma 5.3, we obtain $Y = -G(z)G'(z)/2$. $\square$

Theorem 5.5. *The function $G(z)$ satisfies the following ordinary differential equations:*

$$(GG'/2)^2 = G^5 + \lambda_4 G^3 + \lambda_6 G^2 + \lambda_8 G + \lambda_{10}, \tag{19}$$

$$G^4(G''' - 12GG') - 4\lambda_8 GG' - 12\lambda_{10}G' = 0. \tag{20}$$

Proof. From Proposition 5.4 and the relation $Y^2 = X^5 + \lambda_4 X^3 + \lambda_6 X^2 + \lambda_8 X + \lambda_{10}$, we obtain the equation (19). By dividing the both sides of (19) by G^2, we obtain

$$(G'/2)^2 = G^3 + \lambda_4 G + \lambda_6 + \lambda_8 G^{-1} + \lambda_{10} G^{-2}. \tag{21}$$

By differentiating twice the both sides of (21) with respect to z, we obtain (20). $\square$

Remark 5.6. *In [18, 21, 22] one stated that $G(z)$ is a solution of the differential equation (19). The equation (21) can be considered as a deformation with parameters λ_8 and λ_{10} of the classical equation for Weierstrass $\wp$-function of the elliptic curve $Y^2 = X^3 + \lambda_4 X + \lambda_6$:*

$$(\wp'/2)^2 = \wp^3 + \lambda_4 \wp + \lambda_6. \tag{22}$$

The equation (20) can be considered as a deformation of the stationary KdV-equation (see [10], Corollary 3.1):

$$\wp''' - 12\wp\wp' = 0.$$

Let us assume that $P_* \neq (0, \pm\sqrt{\lambda_{10}})$ and $P_* \neq \infty$. Using Proposition 5.4 and Theorem 5.5, one can obtain the series expansion of the function $G(z)$. Since the function $G(z)$ is holomorphic in the neighborhood of the point u_1^*, then this expansion in the neighborhood of this point has the form

$$G(z) = \sum_{n=0}^{\infty} q_{n+2}(z - u_1^*)^n, \quad q_{n+2} \in \mathbb{C}. \tag{23}$$

Proposition 5.7. *Let $P_* = (X_*, Y_*)$. Then we have $q_2 = X_*$ and $q_3 = -2Y_*/X_*$.*

Proof. We have $q_2 = G(u_1^*)$ and $q_3 = G'(u_1^*)$. Using Proposition 5.4, we obtain the statement. $\square$

Let us set $\deg q_2 = 2$ and $\deg q_3 = 3$.

Proposition 5.8. *The coefficients q_{n+2} are determined from the following recurrence relations:*

- $q_4 = q_2^{-3}(3q_2^5 + \lambda_4 q_2^3 - \lambda_8 q_2 - 2\lambda_{10})$,
- $q_2^3(n+2)(n+1)q_{n+4} = -\sum_{k=0}^{n-1}\left\{(k+2)(k+1)q_{k+4}\sum_{(n_1,n_2,n_3)\in T_1^{(k)}} q_{n_1+2}\, q_{n_2+2}\, q_{n_3+2}\right\}+$

  $+6\sum_{(n_1,n_2,n_3,n_4,n_5)\in T_2} q_{n_1+2}\; q_{n_2+2}\; q_{n_3+2}\; q_{n_4+2}\; q_{n_5+2}+$

  $+2\lambda_4\sum_{(n_1,n_2,n_3)\in T_3} q_{n_1+2}\; q_{n_2+2}\; q_{n_3+2} - 2\lambda_8 q_{n+2}, \quad n \geqslant 1,$ *where*

$T_1^{(k)} = \{(n_1,n_2,n_3) \in \mathbb{Z}^3_{\geqslant 0} \mid n_1+n_2+n_3 = n-k\}$, $T_2 = \{(n_1,n_2,n_3,n_4,n_5) \in \mathbb{Z}^5_{\geqslant 0} \mid n_1 + n_2+n_3+n_4+n_5 = n\}$, $T_3 = \{(n_1,n_2,n_3) \in \mathbb{Z}^3_{\geqslant 0} \mid n_1+n_2+n_3 = n\}$, *and the coefficients* $q_{n+2} \in \mathbb{Q}[q_2, q_3, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ *are homogeneous polynomials of degree* $n+2$ *if* $q_{n+2} \neq 0$.

Proof. Differentiating the both sides of the equation (21) with respect to z, we obtain

$$G^3 G'' = 6G^5 + 2\lambda_4 G^3 - 2\lambda_8 G - 4\lambda_{10}. \tag{24}$$

Substituting (23) into (24) and comparing the coefficients, we obtain the statement. □

Proposition 5.9. *Let* $P_* \neq (0, \pm\sqrt{\lambda_{10}})$ *and* $P_* \neq \infty$. *Then the functions* $F(z)$ *and* $G(z)$ *related by* $G(z) = F(\xi(z))$. *Moreover, the function* $\xi(z)$ *satisfies the following differential equation*

$$(\xi''/2)^2 = \lambda_{10}(\xi')^6 + \lambda_8(\xi')^5 + \lambda_6(\xi')^4 + \lambda_4(\xi')^3 + \xi'.$$

Proof. We have

$$G(z) = f_2(z, \xi(z)) = f_2(\varphi(\xi(z)), \xi(z)) = F(\xi(z)).$$

According to the definition of the function ξ we have

$$\xi'(z) = \frac{1}{G(z)}.$$

The equation (19) allows us to complete the proof. □

For $P_i = (X_i, Y_i) \in U_*$, $i = 1, 2$, let us put $\mathbf{u}^{(i)} = \widetilde{I}_1(P_i)$, $\mathbf{u}^{(i)} = {}^t(u_1^{(i)}, u_3^{(i)})$ and $\mathbf{u}^{(0)} = \mathbf{u}^{(1)} + \mathbf{u}^{(2)}$.

Theorem 5.10. *There exist the following formulas*

$$G(u_1^{(1)}) + G(u_1^{(2)}) = \wp_{1,1}(\mathbf{u}^{(0)}), \quad G(u_1^{(1)})G(u_1^{(2)}) = -\wp_{1,3}(\mathbf{u}^{(0)}),$$

$$G(u_1^{(i)})G'(u_1^{(i)}) = G(u_1^{(i)})\wp_{1,1,1}(\mathbf{u}^{(0)}) + \wp_{1,1,3}(\mathbf{u}^{(0)}), \quad i = 1, 2.$$

Proof. From Proposition 5.4, we have $X_i = G(u_1^{(i)})$ and $Y_i = -G(u_1^{(i)})G'(u_1^{(i)})/2$ for $i = 1, 2$. Thus, from the relations (17) and (18) we obtain the statement. □

6 Relation with a curve of genus 1.

Let us take $P_* = \infty$ and the path γ_* defined by the function $R : [0,1] \to V$ such that $R(r) = \infty$ for any point $r \in [0,1]$. Then we have $\mathbf{u}^* = (0,0)$. It follows from Theorem 2.2 that $\sigma(\mathbf{u}^*) = 0$ and $\sigma_3(\mathbf{u}^*) = -1$. The function $\xi(u_1)$ is holomorphic in a neighborhood of the point $u_1 = 0$.

Proposition 6.1. *In a neighborhood of the point $u_1 = 0$ the function $\xi(u_1)$ is given by a series*

$$\xi(u_1) = \frac{1}{3}u_1^3 + \sum_{n\geqslant 3} a_{2n-2}u_1^{2n+1},$$

where the coefficients $a_{2n-2} \in \mathbb{Q}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ are the homogeneous polynomials of degree $2n-2$ if $a_{2n-2} \neq 0$.

Proof. Since $\xi(0) = 0$, then in the neighborhood of the point $u_1 = 0$ the function $\xi(u_1)$ is given by a series

$$\xi(u_1) = \sum_{k\geqslant 1} a_{k-3}u_1^k,$$

where $a_{k-3} \in \mathbb{C}$. From the condition $\sigma(u_1, \xi(u_1)) = 0$ and Theorem 2.2 we have

$$\frac{1}{3}u_1^3 - \sum_{k\geqslant 1} a_{k-3}u_1^k + \sum_{i+3j\geqslant 7} \alpha_{i,j}u_1^i \left(\sum_{k\geqslant 1} a_{k-3}u_1^k\right)^j = 0. \tag{25}$$

Comparing the coefficients at u_1^k for $1 \leqslant k \leqslant 6$ in (25), we obtain $a_{-2} = a_{-1} = 0$, $a_0 = 1/3$ and $a_1 = a_2 = a_3 = 0$.

Assume $k \geqslant 7$ and the coefficients $a_{\ell-3} \in \mathbb{Q}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ are the homogeneous polynomials of degree $\ell - 3$ if $a_{\ell-3} \neq 0$ for any $\ell < k$. Comparing the coefficients at u_1^k in (25), we obtain

$$a_{k-3} = \alpha_{k,0} + \sum_{(i,j)\in T_1} \alpha_{i,j} \sum_{(\ell_1,\dots,\ell_j)\in T_2^{(i,j)}} a_{\ell_1-3}\cdots a_{\ell_j-3},$$

where $T_1 = \{(i,j) \in \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 1} \mid 7 \leqslant i + 3j \leqslant k\}$ and $T_2^{(i,j)} = \{(\ell_1, \dots, \ell_j) \in \mathbb{Z}_{\geqslant 3}^j \mid \ell_1 + \dots + \ell_j = k - i\}$. Thus, $a_{k-3} \in \mathbb{Q}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ is the homogeneous polynomial of degree $k-3$ if $a_{k-3} \neq 0$. Taking into account graduation of parameters $\lambda_4, \lambda_6, \lambda_8, \lambda_{10}$, we find that $a_{k-3} = 0$ if k is even. □

Proposition 6.2. *In a neighborhood of the point $z = 0$ the function $G(z)$ is given by a series*

$$G(z) = \frac{1}{z^2} - \frac{\lambda_4}{5}z^2 - \frac{\lambda_6}{7}z^4 + \left(\frac{\lambda_4^2}{75} - \frac{\lambda_8}{9}\right)z^6 + \left(\frac{3}{385}\lambda_4\lambda_6 - \frac{\lambda_{10}}{11}\right)z^8 + \sum_{n\geqslant 5} b_{2n+2}z^{2n},$$

where the coefficients $b_{2n+2} \in \mathbb{Q}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ are the homogeneous polynomials of degree $2n+2$ if $b_{2n+2} \neq 0$.

Proof. From Theorem 2.2 and Proposition 6.1, we find that in a neighborhood of the point $z = 0$ the series expansion of the function $G(z)$ has a form

$$G(z) = \frac{1}{z^2} + \sum_{n\geqslant 0} b_{2n+2} z^{2n}, \tag{26}$$

where the coefficients $b_{2n+2} \in \mathbb{Q}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ are the homogeneous polynomials of degree $2n + 2$ if $b_{2n+2} \neq 0$. Substituting (26) into (19) and comparing the coefficient at z^{-8}, we obtain $b_2 = 0$. In a neighborhood of the point $z = 0$ the series expansion of the function $GG'/2$ has a form

$$\frac{G(z)G'(z)}{2} = -\frac{1}{z^5} + \sum_{n\geqslant 1} c_{2n+4} z^{2n-1}, \tag{27}$$

where the coefficients $c_{2n+4} \in \mathbb{Q}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ are the homogeneous polynomials of degree $2n + 4$ if $c_{2n+4} \neq 0$. We have

$$c_{2n+4} = n b_{2n+4} + \sum_{m=1}^{n-1} (n - m) b_{2m+2} b_{2n-2m+2}.$$

Substituting (26) and (27) into (19) and comparing the coefficient at z^{2k} for $-3 \leqslant k \leqslant 0$, we obtain $b_4 = -\lambda_4/5$, $b_6 = -\lambda_6/7$, $b_8 = \lambda_4^2/75 - \lambda_8/9$, and $b_{10} = 3\lambda_4\lambda_6/385 - \lambda_{10}/11$. Similarly, we can determine all the coefficients b_{2n+2} from the equation (19). $\square$

For completeness, we present the following well-known fact.

Lemma 6.3. *Let a formal power series*

$$Z(z) = \frac{1}{z^2} + \sum_{n\geqslant 0} d_{2n+2} z^{2n}, \quad d_{2n+2} \in \mathbb{C}, \tag{28}$$

satisfies the differential equation

$$(Z'/2)^2 = Z^3 + \lambda_4 Z + \lambda_6. \tag{29}$$

Then

$$Z(z) = \frac{1}{z^2} - \frac{\lambda_4}{5} z^2 - \frac{\lambda_6}{7} z^4 + \frac{\lambda_4^2}{75} z^6 + \frac{3\lambda_4\lambda_6}{385} z^8 + \sum_{n\geqslant 5} d_{2n+2} z^{2n},$$

where the coefficients $d_{2n+2} \in \mathbb{Q}[\lambda_4, \lambda_6]$ *are the homogeneous polynomials of degree* $2n+2$ *if* $d_{2n+2} \neq 0$, *and the series* $Z(z)$ *defines the Weierstrass function* $\wp(z)$ *of the elliptic curve* $Y^2 = X^3 + \lambda_4 X + \lambda_6$.

Proof. Substituting (28) into (29) and comparing the coefficient at z^{2k} for $-2 \leqslant k \leqslant 0$, we obtain $d_2 = 0$, $d_4 = -\lambda_4/5$ and $d_6 = -\lambda_6/7$. Differentiating the equation (29) with respect to z, we obtain

$$Z''/2 = 3Z^2 + \lambda_4. \tag{30}$$

Substituting (28) into (30) and comparing the coefficient at z^{2n-2} for $n \geqslant 3$, we obtain

$$(2n^2 - n - 6)d_{2n+2} = 3\sum_{m=1}^{n-2} d_{2m+2}d_{2n-2m}.$$

Consequently, we have $d_8 = \lambda_4^2/75$ and $d_{10} = 3\lambda_4\lambda_6/385$. Since the function $\wp(z)$ satisfies the differential equation (22), we obtain $Z(z) = \wp(z)$. $\square$

From Proposition 6.2 and Lemma 6.3, we obtain:

Corollary 6.4. *There exists the formula*

$$G(z) = \wp(z) + g(z),$$

where $g(z)$ is a holomorphic function that in a neighborhood of the point $z = 0$ is given by a series

$$g(z) = -\frac{\lambda_8}{9}z^6 - \frac{\lambda_{10}}{11}z^8 + \sum_{n\geqslant 5} e_{2n+2}z^{2n}.$$

Here the coefficients $e_{2n+2} \in \mathbb{Q}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ are the homogeneous polynomials of degree $2n+2$ if $e_{2n+2} \neq 0$.

Denote by $G_d(z)$ the function obtained from $G(z)$ by substitution $\lambda_8 = \lambda_{10} = 0$ in the series expansion of this function in a neighborhood of the point $z = 0$.

Corollary 6.5. *We have $G_d(z) = \wp(z)$.*

Proof. From the equation (21) it follows that the function $G_d(z)$ satisfy the differential equation

$$(G_d'/2)^2 = G_d^3 + \lambda_4 G_d + \lambda_6.$$

Using the lemma 6.3, we obtain the statement $\square$

7 Applications of the addition theorem.

For the two-dimensional sigma-function, there exists the addition theorem, first obtained by A. Baker (see [3], [9]). In our graded notation, it has the form

$$\frac{\sigma(\mathbf{u}+\mathbf{v})\sigma(\mathbf{u}-\mathbf{v})}{\sigma^2(\mathbf{u})\sigma^2(\mathbf{v})} = \wp_{1,1}(\mathbf{u})\wp_{1,3}(\mathbf{v}) - \wp_{1,3}(\mathbf{u})\wp_{1,1}(\mathbf{v}) + \wp_{3,3}(\mathbf{v}) - \wp_{3,3}(\mathbf{u}) \tag{31}$$

There exists an equivalent formula

$$\sigma(\mathbf{u}+\mathbf{v})\sigma(\mathbf{u}-\mathbf{v}) = \varphi_{1,1}(\mathbf{u})\varphi_{1,3}(\mathbf{v}) - \varphi_{1,3}(\mathbf{u})\varphi_{1,1}(\mathbf{v}) + \sigma^2(\mathbf{u})\varphi_{3,3}(\mathbf{v}) - \varphi_{3,3}(\mathbf{u})\sigma^2(\mathbf{v}), \tag{32}$$

where $\varphi_{i,j}(\mathbf{u}) = \sigma^2(\mathbf{u})\wp_{i,j}(\mathbf{u}) = \sigma_i(\mathbf{u})\sigma_j(\mathbf{u}) - \sigma(\mathbf{u})\sigma_{i,j}(\mathbf{u})$ are entire functions. From the addition theorem it is not difficult to obtain the formula

$$\frac{\sigma(2\mathbf{u})}{\sigma^4(\mathbf{u})} = \wp_{1,1}(\mathbf{u})\wp_{1,3,3}(\mathbf{u}) - \wp_{1,3}(\mathbf{u})\wp_{1,1,3}(\mathbf{u}) + \wp_{3,3,3}(\mathbf{u}). \tag{33}$$

Consequently,

$$\sigma(2\mathbf{u}) = (-\sigma^3\sigma_{3,3,3} + 3\sigma^2\sigma_3\sigma_{3,3} - \sigma\sigma_1\sigma_{1,1}\sigma_{3,3} + \sigma_1^3\sigma_{3,3} - 2\sigma\sigma_3^3 + \sigma_1\sigma_3^2\sigma_{1,1} - \sigma\sigma_3\sigma_{1,1}\sigma_{1,3} - \\ - 2\sigma_1^2\sigma_3\sigma_{1,3} + \sigma\sigma_1\sigma_3\sigma_{1,1,3} + \sigma^2\sigma_{1,1}\sigma_{1,3,3} - \sigma\sigma_1^2\sigma_{1,3,3} + 2\sigma\sigma_1\sigma_{1,3}^2 - \sigma^2\sigma_{1,3}\sigma_{1,1,3})(\mathbf{u}). \quad (34)$$

Let us set, as above,

$$f_5(\mathbf{u}) = \frac{\sigma_1^2\sigma_{3,3} - 2\sigma_1\sigma_3\sigma_{1,3} + \sigma_3^2\sigma_{1,1}}{2\sigma_1^3}(\mathbf{u}) \quad (35)$$

and

$$g_5(\mathbf{u}) = \frac{\sigma(2\mathbf{u})}{\sigma_1^4(\mathbf{u})}. \quad (36)$$

Then according to the formulas (11) and (13) the restrictions of the functions $2f_5(\mathbf{u})$ and $g_5(\mathbf{u})$ to the curve $W \subset \mathbb{C}^2$ coincide.

Lemma 7.1. *On $\mathbb{C}^2$, there exists the formula*

$$2f_5 - g_5 = \sigma h, \quad (37)$$

where

$$h = \sigma_1^{-4}(\sigma^2\sigma_{3,3,3} - 3\sigma\sigma_3\sigma_{3,3} + \sigma_1\sigma_{1,1}\sigma_{3,3} + 2\sigma_3^3 + \sigma_3\sigma_{1,1}\sigma_{1,3} - \sigma_1\sigma_3\sigma_{1,1,3} - \\ - \sigma\sigma_{1,1}\sigma_{1,3,3} + \sigma_1^2\sigma_{1,3,3} - 2\sigma_1\sigma_{1,3}^2 + \sigma\sigma_{1,3}\sigma_{1,1,3}). \quad (38)$$

In the rational limit we obtain $\sigma = \frac{1}{3}u_1^3 - u_3$. Therefore, in this case

$$h = -\frac{2}{u_1^8} \quad (39)$$

Using the formula (34), we obtain the expression for the Abelian function

$$\wp_{1,1}(2\mathbf{u}) = \left(\frac{\sigma_1(2\mathbf{u})}{\sigma(2\mathbf{u})}\right)^2 - \frac{\sigma_{1,1}(2\mathbf{u})}{\sigma(2\mathbf{u})}$$

in the form of a differential polynomial of the function $\sigma(\mathbf{u})$.

Lemma 7.2. *The function*

$$\frac{1}{2}\wp_{1,1}(2\mathbf{u}) - \left(-\frac{\sigma_3(\mathbf{u})}{\sigma_1(\mathbf{u})}\right)$$

in the rational limit has the form σk, where

$$k = \frac{3(-10u_1^3 + 3u_3)}{u_1^2(4u_1^3 - 3u_3)^2}.$$

Acknowledgements. The authors are grateful to A. B. Bogatyrev and V.Z. Enolskii for the stimulating discussions and literary references.

References

[1] M. Adler, J. Moser, *On a Class of Polynomials Connected with the Korteweg-de Vries Equation.*, Commun. Math. Phys. 61, 1978, 1–30.

[2] T. Ayano, V. M. Buchstaber, *Construction of Two Parametric Deformation of KdV-Hierarchy and Solution in Terms of Meromorphic Functions on the Sigma Divisor of a Hyperelliptic Curve of Genus 3.*, SIGMA 15 (2019), 032, 15 pages; arXiv: 1811.07138.

[3] H. F. Baker, *An introduction to the theory of multiply periodic functions.*, Univ. Press, Cambridge, 1907.

[4] A. B. Bogatyrev, *Conformal mapping of rectangular heptagons.*, Sb. Math., 203:12 (2012), 1715–1735.

[5] A. B. Bogatyrev, O. A. Grigor'ev, *Conformal Mapping of Rectangular Heptagons II.*, Computational Methods and Function Theory, 18:2 (2018), 221–238.

[6] V. M. Buchstaber, V. Z. Enolskii, D. V. Leikin, *Kleinian functions, hyperelliptic Jacobians and applications.*, Reviews in Mathematics and Math. Physics, I. M. Krichever, S. P. Novikov Editors, v. 10, part 2, Gordon and Breach, London, 1997, 3–120.

[7] V. M. Buchstaber, V. Z. Enolskii, D. V. Leikin, *Hyperelliptic Kleinian functions and applications.*, Solitons, Geometry and Topology: On the Crossroad, V. M. Buchstaber, S. P. Novikov Editors, AMS Trans., 179:2, 1997, 1–33.

[8] V. M. Buchstaber, V. Z. Enolskii, D. V. Leikin, *Rational analogues of Abelian functions.*, Funct. Anal. Appl., 33:2, 1999, 83–94.

[9] V. M. Buchstaber, V. Z. Enolski, D. V. Leykin, *Multi-dimensional sigma-functions.*, arXiv:1208.0990, (2012).

[10] V. M. Buchstaber, *Polynomial dynamical systems and the Korteweg-de Vries equation.*, Proc. Steklov Inst. Math., 294, 2016, 176-–200; arXiv:1605.04061.

[11] V. M. Buchstaber, V. Z. Enolski, D. V. Leykin, *Sigma-functions: old and new results.*, in "Integrable Systems and Algebraic Geometry", Cambridge Univ. Press; LMS Lecture Notes, v. 2, 2019; arXiv:1810.11079, (2018).

[12] V. Z. Enolski, E. Hackmann, V. Kagramanova, J. Kunz, C. Lämmerzahl, *Inversion of hyperelliptic integrals of arbitrary genus with application to particle motion in general relativity.*, Journal of Geometry and Physics, Vol. 61, Issue 5, (2011), 899–921.

[13] V. Enolski, B. Hartmann, V. Kagramanova, J. Kunz, C. Lämmerzahl, P. Sirimachan, *Inversion of a general hyperelliptic integral and particle motion in Hořava-Lifshitz black hole space-times.*, Journal of Mathematical Physics, Vol. 53, Issue 1, 012504, (2012).

[14] V. Z. Enolskii, M. Pronine, P. H. Richter, *Double Pendulum and θ-Divisor.*, Journal of Nonlinear Science, Vol. 13, Issue 2, (2003), 157–174.

[15] J. D. Fay, *Theta functions on Riemann surfaces.*, Lecture Notes in Math., Vol. 352, Springer–Verlag, Berlin–New York, 1973.

[16] D. Grant, *A generalization of Jacobi's derivative formula to dimension two.*, J. Reine Angew. Math., Vol. 392, (1988), 125–136.

[17] D. Grant, *A generalization of a formula of Eisenstein.*, Proc. London Math. Soc. Vol. 62, Issue 1, (1991), 121–132.

[18] E. Hackmann, V. Kagramanova, J. Kunz, C. Lämmerzahl, *Analytic solutions of the geodesic equation in higher dimensional static spherically symmetric spacetimes.*, Phys. Rev. D, 78, 124018, (2008).

[19] E. Hackmann, V. Kagramanova, J. Kunz, C. Lämmerzahl, *Analytic solutions of the geodesic equation in axially symmetric space-times.*, EPL (Europhysics Letters), Vol. 88, No. 3, 30008, (2009).

[20] E. Hackmann, C. Lämmerzahl, *Complete Analytic Solution of the Geodesic Equation in Schwarzschild-(Anti-)de Sitter Spacetimes.*, Phys. Rev. Lett., 100, 171101, (2008).

[21] E. Hackmann, C. Lämmerzahl, *Geodesic equation in Schwarzschild-(anti-)de Sitter space-times: Analytical solutions and applications.*, Phys. Rev. D, 78, 024035, (2008).

[22] E. Hackmann, C. Lämmerzahl, V. Kagramanova, J. Kunz, *Analytical solution of the geodesic equation in Kerr-(anti-) de Sitter space-times.*, Phys. Rev. D, 81, 044020, (2010).

[23] J. Jorgenson, *On directional derivatives of the theta function along its divisor.*, Israel J. Math., Vol. 77, Issue 3, (1992), 273–284.

[24] A. I. Markushevich, *Introduction to the Classical Theory of Abelian Functions.*, Translations of Mathematical Monographs, v. 96, AMS Providence, Rhole Iailand, 1992.

[25] S. Matsutani, *Recursion relation of hyperelliptic PSI-functions of genus two.*, Integral Transforms and Special Functions, Vol. 14, Issue 6, (2003), 517–527.

[26] A. Nakayashiki, *On algebraic expressions of sigma functions for* (n, s) *curves.*, Asian J. Math., Vol. 14, No. 2, (2010), 175–212, arXiv:0803.2083.

[27] Y. Onishi, *Complex multiplication formulae for hyperelliptic curves of genus three.*, Tokyo J. Math., Vol. 21, No. 2, (1998), 381–431.

[28] Y. Onishi, *Determinant expressions for Abelian functions in genus two.*, Glasgow Math. J., Vol. 44, Issue 3, (2002), 353–364.