MATHIAS-PRIKRY AND LAVER-PRIKRY TYPE FORCING

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Abstract. We study the Mathias-Prikry and Laver-Prikry forcings associated with filters on ω. We give a combinatorial characterization of Martin's number for these forcing notions and present a general scheme for analyzing preservation properties for them. In particular, we give a combinatorial characterization of those filters for which the Mathias-Prikry forcing does not add any dominating reals.

Introduction

In recent years, a variety of consistency results have been given using the Mathias-Prikry and the Laver-Prikry forcing associated with filters.

Let $\mathcal{F}$ be a filter on ω. The *Mathias-Prikry* forcing associated with $\mathcal{F}$, denoted by $\mathbb{M}_{\mathcal{F}}$ consists of pairs $\langle s, A\rangle$ such that $s \in [\omega]^{<\omega}$, $A \in \mathcal{F}$ and $s \cap A = \emptyset$. The ordering $\langle s, A\rangle \leq \langle t, B\rangle$ if $s \supset t$, $A \subset B$ and $s \setminus t \subset B$.

We will refer to the union of the first coordinates of conditions in the generic filter as the generic subset of ω, and denote it by $\dot{a}_{gen}$.

The *Laver-Prikry* forcing associated with $\mathcal{F}$, denoted by $\mathbb{L}_{\mathcal{F}}$ consists of subtrees $T \subset \omega^{<\omega}$ which have a stem $s \in T$ (denoted by $\mathsf{stem}(T)$) such that for every $t \in T$ either $t \subset s$ or $s \subset t$ and for every $t \in T$ extending s the set

$$\mathsf{Succ}_T(t) = \{n \in \omega : t^\frown\langle n\rangle \in T\} \in \mathcal{F}.$$

The order on $\mathbb{L}_{\mathcal{F}}$ is given by inclusion.

These forcing notions play a significant role in the use of the matrix iteration introduced by Blass and Shelah [5] and further developed and used by Shelah [20], Brendle [10] and Brendle and Fischer [11].

The Laver-Prikry forcing was used to separate variants of the groupwise density number and the distributivity numbers by Brendle in [7, 8, 9] and by Brendle and Hrušák to show it is relatively consistent that every countable FU_{fin} space of weight $\aleph_1$ is metrizable [12].

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In [19], Raghavan constructs a model of ZFC without strongly separable almost disjoint families by using a similar technique.

In this paper, we shall study the relation between combinatorial properties of an ideal $\mathcal{I}$ and the forcing properties of the Mathias-Prikry and the Laver-Prikry type forcings associated with the dual-filter $\mathcal{I}^*$ (denoted by $\mathbb{M}_{\mathcal{I}^*}$ and $\mathbb{L}_{\mathcal{I}^*}$ respectively) often expressed in terms of the Katětov order, paying special attention to definable (Borel, analytic) ideals.

Both forcing notions are clearly c.c.c, in fact, σ-centered. Also $\mathbb{L}_{\mathcal{I}^*}$ adds a dominating real (the generic function $\dot{f}_{gen}$ is dominating).

In section 1, we give a combinatorial characterization of the Martin number of $\mathbb{M}_{\mathcal{I}^*}$ and $\mathbb{L}_{\mathcal{I}^*}$ and introduce the separating number of the corresponding ideal. In section 2, we investigate the relationship between preservation statements for $\mathbb{M}_{\mathcal{I}^*}$ and $\mathbb{L}_{\mathcal{I}^*}$ and combinatorial properties of $\mathcal{I}$. Finally, in section 3, we give a characterization of those ideals $\mathcal{I}$ such that $\mathbb{M}_{\mathcal{I}^*}$ does not add any dominating reals.

For a set X, we call $\mathcal{I} \subset \mathcal{P}(X)$ an *ideal on* X if

(1) for $A, B \in \mathcal{I}$, $A \cup B \in \mathcal{I}$,
(2) for $A, B \subset X$, $A \subset B$ and $B \in \mathcal{I}$ implies $A \in \mathcal{I}$ and
(3) $X \notin \mathcal{I}$.

We assume that all ideals on X contain $[X]^{<\omega}$, all finite subsets of X. If $\mathcal{I}$ is an ideal on X, $\mathcal{I}^*$ is the dual filter, consisting of complements of the sets in $\mathcal{I}$. $\mathcal{I}^+$ denotes the collection of $\mathcal{I}$-positive set, i.e., subsets of X which are not in $\mathcal{I}$. We say that an ideal $\mathcal{I}$ on the set of all natural numbers ω is *tall* if for each $A \in [\omega]^\omega$ there is a $I \in \mathcal{I}$ such that $I \cap A$ is infinite. If $\mathcal{I}$ is an ideal on ω and $Y \in \mathcal{I}^+$, we denote by $\mathcal{I} \restriction Y$ the ideal $\{I \cap Y : I \in \mathcal{I}\}$ on Y.

The topology of $\mathcal{P}(\omega)$ is induced by identifying each subset of ω with its characteristic function, where 2^ω is equipped with the product topology. We call an ideal $\mathcal{I}$ on ω a *Borel idea*l if $\mathcal{I}$ is Borel in this topology.

Given a tall ideal $\mathcal{I}$ on ω and a forcing notion $\mathbb{P}$, we say that the forcing $\mathbb{P}$ *destroys* $\mathcal{I}$ if there is a $\mathbb{P}$-name $\dot{x}$ for en element of $[\omega]^\omega$ such that

$$\Vdash_{\mathbb{P}} \forall I \in \mathcal{I} \cap V(|I \cap \dot{x}| < \aleph_0).$$

We say that a family $\mathcal{K} \subset [\omega]^\omega$ is *countably tall (or ω-hitting)* if given $(A_n : n \in \omega) \subset [\omega]^\omega$ there is an $K \in \mathcal{K}$ such that for $n \in \omega$, $|K \cap A_n| = \aleph_0$.

The Katětov order on ideals is defined as follows: Suppose $\mathcal{I}$ and $\mathcal{J}$ are ideals on countable sets X and Y respectively. Then $\mathcal{I} \leq_K \mathcal{J}$ if there exists a function $f : Y \to X$ such that for each $I \in \mathcal{I}$, $f^{-1}[I] \in \mathcal{J}$.

When dealing with ideals on countable sets, we often use the following cardinal invariants [15]:

$$\begin{aligned}
\mathsf{add}^*(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{I} \wedge (\forall X \in \mathcal{I})(\exists A \in \mathcal{A})(A \not\subset^* X)\}.\\
\mathsf{cov}^*(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{I} \wedge (\forall X \in [\omega]^\omega)(\exists A \in \mathcal{A})(|A \cap X| = \aleph_0)\}.\\
\mathsf{non}^*(\mathcal{I}) &= \min\{|\mathcal{X}| : \mathcal{X} \subset [\omega]^\omega \wedge (\forall I \in \mathcal{I})(\exists X \in \mathcal{X})(|I \cap X| < \aleph_0)\}.\\
\mathsf{cof}^*(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{I} \wedge (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(I \subset^* A)\}^1.
\end{aligned}$$

1. Martin numbers of $\mathbb{M}_{\mathcal{I}^*}$ and $\mathbb{L}_{\mathcal{I}^*}$

Recall that the *Martin number* $\mathfrak{m}(\mathbb{P})$ of a partial order $\mathbb{P}$ is the minimal size of a family of dense open subsets of $\mathbb{P}$ such that no filter on $\mathbb{P}$ intersects with them all.

In this section, we shall give a combinatorial characterization of the cardinal invariants $\mathfrak{m}(\mathbb{L}_{\mathcal{I}^*})$ and $\mathfrak{m}(\mathbb{M}_{\mathcal{I}^*})$.

Both forcings $\mathbb{M}_{\mathcal{I}^*}$, and $\mathbb{L}_{\mathcal{I}^*}$ destroy the ideal $\mathcal{I}$. In fact, they do more than that. $\mathbb{M}_{\mathcal{I}^*}$ and $\mathbb{L}_{\mathcal{I}^*}$ *separate* $\mathcal{I}$ and $\mathcal{I}^+$, that is, they add a set $a_{gen} \subset \omega$ which is almost disjoint from every $I \in \mathcal{I}$, and have infinite intersection with every $X \in \mathcal{I}^+ \cap V$. It is useful to introduce the corresponding cardinal invariant, the separating number of an ideal $\mathcal{I}$.

Let $\mathcal{I}$ be an ideal on ω. Let $\mathcal{G} \subset \mathcal{I}$, $\mathcal{H} \subset \mathcal{I}^+$ and $A \subset \omega$. We say A *separates* $\mathcal{G}$ from $\mathcal{H}$ if

(1) $|A \cap I| < \aleph_0$ for $I \in \mathcal{G}$ and
(2) $|A \cap X| = \aleph_0$ for $X \in \mathcal{H}$.

For an ideal $\mathcal{I}$, the *separating number* $\mathsf{sep}(\mathcal{I})$ is

$$\begin{aligned}
\mathsf{sep}(\mathcal{I}) = \min\{|\mathcal{G}| + |\mathcal{H}| : \mathcal{G} \subset \mathcal{I} \wedge \mathcal{H} \subset \mathcal{I}^+ \wedge \\
\forall A \subset \omega \exists I \in \mathcal{G} \exists X \in \mathcal{H}\, (|A \cap I| = \omega \text{ or } |A \cap X| < \omega)\}.
\end{aligned}$$

It is clear from the definition that $\mathsf{add}^*(\mathcal{I}) \leq \mathsf{sep}(\mathcal{I}) \leq \mathsf{cov}^*(\mathcal{I})$ for every tall ideal $\mathcal{I}$ and that $\mathsf{sep}(\mathcal{I}) = \mathsf{cov}^*(\mathcal{I})$ if $\mathcal{I}$ is a maximal ideal.

Proposition 1.1. [15] *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω. Suppose $\mathcal{I}$ is below $\mathcal{J}$ in the Rudin-Kiesler order, that is, there exists $f : \omega \to \omega$ such that for every $A \subset \omega$, $A \in \mathcal{I}$ if and only if $f^{-1}[A] \in \mathcal{J}$. Then* $\mathsf{sep}(\mathcal{I}) \leq \mathsf{sep}(\mathcal{J})$.

[1] In [13], Brendle and Shelah introduced cardinal invariants $\mathfrak{p}(\mathcal{F})$ and $\pi\mathfrak{p}(\mathcal{F})$ associated with an ultrafilter $\mathcal{F}$. For all tall ideals $\mathcal{I}$, $\mathsf{add}^*(\mathcal{I}) = \mathfrak{p}(\mathcal{I}^*)$, $\mathsf{cov}^*(\mathcal{I}) = \pi\mathfrak{p}(\mathcal{I}^*)$, $\mathsf{non}^*(\mathcal{I}) = \pi_\chi(\mathcal{I}^*)$ and $\mathsf{cof}^*(\mathcal{I}) = \chi(\mathcal{I}^*)$.

Brendle and Shelah characterized the Martin number of the Mathias-Prikry and Laver-Prikry type for ultrafilters in [13].

Theorem 1.2. [13] *Let* $\mathcal{U}$ *be an ultrafilter. Then*

(1) $\mathfrak{m}(\mathbb{M}_{\mathcal{U}}) = \mathsf{cov}^*(\mathcal{U}^*)$.
(2) $\mathfrak{m}(\mathbb{L}_{\mathcal{U}}) = \min\{\mathfrak{b}, \mathsf{cov}^*(\mathcal{U}^*)\}^2$.

We will prove analogous results for arbitrary filter/ideal.

1.1. **Martin number of** $\mathbb{L}_{\mathcal{I}^*}$**.** Recall that an ultrafilter $\mathcal{U}$ on ω is *nowhere dense* if for every function $f : \omega \to \mathbb{R}$ there is a $U \in \mathcal{U}$ such that $f[U]$ is a nowhere dense subset of $\mathbb{R}$. It is known (see [3]) that the Laver-Prikry forcing with $\mathcal{U}$ adds a Cohen real if and only if $\mathcal{U}$ is not a nowhere dense ultrafilter.

The following was announced in [15].

Theorem 1.3. *For every ideal* $\mathcal{I}$ *on* ω*,*

$$\mathfrak{m}(\mathbb{L}_{\mathcal{I}^*}) = \begin{cases} \min\{\mathfrak{b}, \mathsf{sep}(\mathcal{I})\} & \text{if } \mathcal{I}^* \text{ is a nowhere dense ultrafilter,} \\ \min\{\mathsf{add}(\mathcal{M}), \mathsf{sep}(\mathcal{I})\} & \text{otherwise.} \end{cases}$$

Proof. (i) If $\mathcal{I}^*$ is a nowhere dense ultrafilter, then the required statement holds by Theorem 1.2 (2) as $\mathsf{sep}(\mathcal{U}^*) = \mathsf{cov}^*(\mathcal{U}^*)$ for every ultrafilter $\mathcal{U}$.

Suppose that $\mathcal{I}^*$ is not a nowhere dense ultrafilter. First we shall show $\mathfrak{m}(\mathbb{L}_{\mathcal{I}^*}) \leq \min\{\mathsf{add}(\mathcal{M}), \mathsf{sep}(\mathcal{I})\}$.

Since $\mathbb{L}_{\mathcal{I}^*}$ adds a dominating real, $\mathfrak{m}(\mathbb{L}_{\mathcal{I}^*}) \leq \mathfrak{b}$. Since $\mathcal{I}^*$ is not a nowhere dense ultrafilter, $\mathbb{L}_{\mathcal{I}^*}$ adds a Cohen real (see [3]). So $\mathfrak{m}(\mathbb{L}_{\mathcal{I}^*}) \leq \mathsf{cov}(\mathcal{M})$. Since $\mathsf{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathsf{cov}(\mathcal{M})\}$, $\mathfrak{m}(\mathbb{L}_{\mathcal{I}^*}) \leq \mathsf{add}(\mathcal{M})$.

To see that $\mathfrak{m}(\mathbb{L}_{\mathcal{I}^*}) \leq \mathsf{sep}(\mathcal{I})$, suppose that $\kappa < \mathfrak{m}(\mathbb{L}_{\mathcal{I}^*})$ and let $\mathcal{J} \subset \mathcal{I}$ and $\mathcal{H} \subset \mathcal{I}^+$ such that $|\mathcal{J}| + |\mathcal{H}| \leq \kappa$. For $J \in \mathcal{J}$, put

$$D_J = \{T \in \mathbb{L}_{\mathcal{I}^*} : \forall t \in T(\mathsf{stem}(T) \subset t \to \mathsf{Succ}_T(t) \cap J = \emptyset)\}.$$

For $H \in \mathcal{H}$ and $n \in \omega$, define

$$E_{H,n} = \{T \in \mathbb{L}_{\mathcal{I}^*} : \mathsf{rang}(\mathsf{stem}(T)) \cap H \setminus n \neq \emptyset\}.$$

Then D_J and $E_{H,n}$ are dense for $J \in \mathcal{J}$, $H \in \mathcal{H}$ and $n \in \omega$.

Let $G \subset \mathbb{L}_{\mathcal{I}^*}$ be a $\{D_J : J \in \mathcal{J}\} \cup \{E_{H,n} : H \in \mathcal{H} \wedge n \in \omega\}$-generic. Let $f_G = \bigcup\{\mathsf{stem}(T) : T \in G\}$. By genericity, $\mathsf{rang}(f_G) \cap J$ is finite for $J \in \mathcal{J}$ and $\mathsf{rang}(f_G) \cap H$ is infinite for $H \in \mathcal{H}$. So $\mathsf{rang}(f_G)$ separates $\mathcal{J}$ from $\mathcal{H}$. Therefore $\kappa < \mathsf{sep}(\mathcal{I})$.

(ii) $\min\{\mathsf{add}(\mathcal{M}), \mathsf{sep}(\mathcal{I})\} \leq \mathfrak{m}(\mathbb{L}_{\mathcal{I}^*})$.

[0] Brendle and Shelah investigated cardinal invariants of ideals $\ell^0_{\mathcal{U}}$ and $r^0_{\mathcal{U}}$ associated with an ultrafilter $\mathcal{U}$. For all ideals $\mathcal{I}$, $\mathsf{cov}(\ell^0_{\mathcal{I}^*}) = \mathfrak{m}(\mathbb{L}_{\mathcal{I}^*})$ and $\mathsf{cov}(r^0_{\mathcal{I}^*}) = \mathfrak{m}(\mathbb{M}_{\mathcal{I}^*})$.

Suppose $\kappa < \mathsf{add}(\mathcal{M}), \mathsf{sep}(\mathcal{I})$. Let $\{D_\alpha : \alpha < \kappa\}$ be a family of dense open subsets of $\mathbb{L}_{\mathcal{I}^*}$.

For each $\alpha < \kappa$, let $\{T^n_\alpha : n \in \omega\}$ be a maximal antichain of D_α. Let $I^n_{\alpha,t} = \omega \setminus \mathsf{Succ}_{T^n_\alpha}(t) \in \mathcal{I}$ for $\alpha < \kappa$, $n \in \omega$ and $t \in T^n_\alpha$ with $t \supset \mathsf{stem}(T^n_\alpha)$.

Fix $\alpha < \kappa$. Define a rank function $\mathsf{rk}_\alpha : \omega^{<\omega} \to Ord$ by

(1) $\mathsf{rk}_\alpha(t) = 0$ if $\exists n \in \omega\, (t \in T^n_\alpha$ and $t \supset \mathsf{stem}(T^n_\alpha))$.
(2) $\mathsf{rk}_\alpha(t) \leq \beta$ if $H^\alpha_t = \{n \in \omega : \mathsf{rk}_\alpha(t^\frown\langle n\rangle) < \beta\} \in \mathcal{I}^+$.

Claim 1.4. [12, Lemma 4] *For all $t \in \omega^{<\omega}$, $\mathsf{rk}_\alpha(t)$ is defined.*

Since $\kappa < \mathsf{sep}(\mathcal{I})$, there is an $A \in [\omega]^\omega$ such that for every $\alpha < \kappa$, $n \in \omega$, $t \in T^n_\alpha$ and $s \in \omega^{<\omega}$, $\left|A \cap I^n_{\alpha,t}\right| < \aleph_0$, i.e., $A \subset^* \mathsf{Succ}_{T^n_\alpha}(t)$ and $|A \cap H^\alpha_s| = \aleph_0$.

Let $\mathbb{L}_{\mathsf{fin}(A)^*}$ be the Laver-Prikry forcing on A associated with the ideal $\mathsf{fin}(A)$ of finite subsets of A. Let

$$D'_\alpha = \{T \cap A^{<\omega} : T \in D_\alpha \wedge T \cap A^{<\omega} \in \mathbb{L}_{\mathsf{fin}(A)^*}\}.$$

Claim 1.5. *$\{T^n_\alpha \cap A^{<\omega} : n \in \omega$ and $T^n_\alpha \cap A^{<\omega} \in \mathbb{L}_{\mathsf{fin}(A)^*}\}$ is predense in $\mathbb{L}_{\mathsf{fin}(A)^*}$. Therefore, D'_α is dense in $\mathbb{L}_{\mathsf{fin}(A)^*}$.*

Proof of Claim 1.5. Fix $\alpha < \kappa$. Let $S \in \mathbb{L}_{\mathsf{fin}(A)^*}$ and $s = \mathsf{stem}(S)$.

Then $\mathsf{rk}_\alpha(s) < \infty$. Since $\{n \in \omega : s^\frown\langle n\rangle \in T\} \in \mathsf{fin}(A)^*$ and $H^\alpha_s \cap A$ is infinite, $H^\alpha_s \cap \{n \in \omega : t^\frown\langle n\rangle \in T\} \neq \emptyset$.

By induction on rank, there exists $t \in S$ such that $t \supset s = \mathsf{stem}(S)$ and $\mathsf{rk}_\alpha(t) = 0$, that is, $t \in T^n_\alpha$ and $t \supset \mathsf{stem}(T^n_\alpha)$ for some $n \in \omega$. Fix such $n \in \omega$.

Since $t \in A^{<\omega}$, $\mathsf{stem}(T^n_\alpha) \in A^{<\omega}$. For every $u \in T^n_\alpha$ with $u \supset \mathsf{stem}(T^n_\alpha)$, $A \subset^* \{n \in \omega : u^\frown\langle n\rangle \in T^n_\alpha\}$. So $T^n_\alpha \cap A^{<\omega} \in \mathbb{L}_{\mathsf{fin}(A)^*}$ and $T^n_\alpha \cap A^{<\omega}$ is compatible with S. Hence $\{T^n_\alpha \cap A^{<\omega} : n \in \omega \wedge T^n_\alpha \cap A^{<\omega} \in \mathbb{L}_{\mathsf{fin}(A)^*}\}$ is predense. □

Let $T^{n\prime}_\alpha = T^n_\alpha \cap A^{<\omega}$. For each $\alpha < \kappa$ and $n \in \omega$ with $T^{n\prime}_\alpha \in \mathbb{L}_{\mathsf{fin}(A)^*}$, define $g^\alpha_n : A^{<\omega} \to \omega$ by

$$g^\alpha_n(s) = \begin{cases} \min\{n : A \setminus n \subset \mathsf{Succ}_{T^{n\prime}_\alpha}(s)\} & \text{if } s \in T^{n\prime}_\alpha \text{ and } s \supset \mathsf{stem}(T^{n\prime}_\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

Notice that when $\mathsf{Stem}(T^n_\alpha) \notin A^{<\omega}$, $T^{n\prime}_\alpha \notin \mathbb{L}_{\mathsf{fin}(A)^*}$ and g^α_n is undefined.

Since $\kappa < \mathsf{add}(\mathcal{M}) \leq \mathfrak{b}$, there exists $g : A^{<\omega} \to \omega$ such that for $\alpha < \kappa$ and $n \in \omega$, for almost all $t \in A^{<\omega}$, $g^\alpha_n(t) \leq g(t)$. Define $S \in \mathbb{L}_{\mathsf{fin}(A)^*}$ so that

(1) $\emptyset \in S$ and
(2) if $s \in S$, then $s^\frown\langle k\rangle \in S$ if and only if $k > g(s)$ and $k \in A$.

For each $\alpha < \kappa$, put

$$D''_\alpha = \{t \in S : \exists n \in \omega \big(t \in T^{n\prime}_\alpha,\ T^{n\prime}_\alpha \in \mathbb{L}_{\mathsf{fin}(A)^*} \text{ and } \forall s \in S(s \supset t \to g(s) \geq g^n_\alpha(s))\big)\}.$$

Let $M_\alpha = \{f \in [S] : \forall n \in \omega(f \restriction n \notin D''_\alpha)\}$. Then M_α is nowhere dense in $[S]$. Since $\kappa < \mathsf{add}(\mathcal{M}) \leq \mathsf{cov}(\mathcal{M})$, there exists $f \in [S]$ such that $f \notin M_\alpha$ for every $\alpha < \kappa$.

Claim 1.6. *For every $\alpha < \kappa$, there exists $T \in D_\alpha$ such that $f \in [T]$.*

Proof. For $\alpha < \kappa$, let $n \in \omega$ such that $f \restriction n \in D''_\alpha$. Then there exists $m \in \omega$ such that $f \restriction n \in T^{m\prime}_\alpha$ and for $s \in S$ whenever $s \supset f \restriction n$, $g(s) \geq g^m_\alpha(s)$. By definition of g^m_α, $S_{f\restriction n} \subset T^{m\prime}_\alpha$. Hence $f \in [T^{m\prime}_\alpha]$. So $f \in [T]$ for some $T \in D_\alpha$. □

By construction of f, f is a $\{D_\alpha : \alpha < \kappa\}$-generic real, i.e., $\{T : f \in [T]\}$ is a filter intersecting with D_α for all $\alpha < \kappa$. □

Corollary 1.7. *For every ideal $\mathcal{I}$ on ω,*

$$\mathfrak{m}(\mathbb{L}_{\mathcal{I}^*}) = \begin{cases} \min\{\mathfrak{b}, \mathsf{sep}(\mathcal{I})\} & \text{if } \mathcal{I}^* \text{ is ultrafilter} \\ \min\{\mathsf{add}(\mathcal{M}), \mathsf{sep}(\mathcal{I})\} & \text{otherwise.} \end{cases}$$

1.2. **Martin number of $\mathbb{M}_{\mathcal{I}^*}$.** It seems that the rank argument does not work for the Mathias-Prikry type forcings. However, they can be investigated by studying the ideal $\mathcal{I}^{<\omega}$ on $[\omega]^{<\omega} \setminus \{\emptyset\}$ associated to an ideal $\mathcal{I}$ on ω.

Definition 2. Given ideal $\mathcal{I}$ on ω, let

$$\mathcal{I}^{<\omega} = \{A \subset [\omega]^{<\omega} \setminus \{\emptyset\} : \exists I \in \mathcal{I} \forall a \in A (a \cap I \neq \emptyset)\}.$$

This ideal was considered by Sirota [21] and Louveau [18] in the construction of an extremely disconnected topological group. Recall that an ultrafilter $\mathcal{U}$ on ω is selective if for every partition $\{I_n : n \in \omega\}$ of ω either there is an n such that $I_n \in \mathcal{U}$ or there is a $U \in \mathcal{U}$ such that $|I_n \cap U| \leq 1$ for every $n \in \omega$.

Theorem 2.1. *For every ideal $\mathcal{I}$ on ω,*

$$\mathfrak{m}(\mathbb{M}_{\mathcal{I}^*}) = \begin{cases} \mathsf{sep}(\mathcal{I}) & \text{if } \mathcal{I}^* \text{ is a selective ultrafilter.} \\ \min\{\mathsf{sep}(\mathcal{I}^{<\omega}), \mathsf{cov}(\mathcal{M})\} & \text{otherwise.} \end{cases}$$

If $\mathcal{I}^*$ is a selective ultrafilter, then $\mathfrak{m}(\mathbb{M}_{\mathcal{I}^*}) = \mathsf{cov}^*(\mathcal{I})$ by Theorem 1.2 (1).

To prove the rest of this theorem, we will first introduce two variations of $\mathsf{sep}(\mathcal{I})$. Define $\widetilde{\mathsf{sep}}(\mathcal{I})$ by $\kappa < \widetilde{\mathsf{sep}}(\mathcal{I})$ if for $\mathcal{J} \subset \mathcal{I}$ and $\mathcal{H} \subset (\mathcal{I}^{<\omega})^+$ with $|\mathcal{J}| + |\mathcal{H}| \leq \kappa$, there exists $A \subset \omega$ such that

(1) $|A \cap J| < \aleph_0$ for $J \in \mathcal{J}$ and
(2) $|[A]^{<\omega} \cap H| = \aleph_0$ for $H \in \mathcal{H}$.

Define $\widetilde{\widetilde{\mathsf{sep}}}(\mathcal{I})$ by $\kappa < \widetilde{\widetilde{\mathsf{sep}}}(\mathcal{I})$ if for $\mathcal{J} \subset \mathcal{I}$ and $\mathcal{H} \subset (\mathcal{I}^{<\omega})^+$ with $|\mathcal{J}| + |\mathcal{H}| \leq \kappa$, there exists $A \subset \omega$ such that

(1) $|A \cap J| < \aleph_0$ for $J \in \mathcal{J}$ and
(2) $|[A \setminus n]^{<\omega} \cap H| = \aleph_0$ for $H \in \mathcal{H}$ and $n \in \omega$.

Claim 2.2. $\widetilde{\mathsf{sep}}(\mathcal{I}) = \widetilde{\widetilde{\mathsf{sep}}}(\mathcal{I})$.

Proof of Claim 2.2. By definition, it is clear that $\widetilde{\widetilde{\mathsf{sep}}}(\mathcal{I}) \leq \widetilde{\mathsf{sep}}(\mathcal{I})$. We shall show $\widetilde{\widetilde{\mathsf{sep}}}(\mathcal{I}) \geq \widetilde{\mathsf{sep}}(\mathcal{I})$.

Let $\mathcal{J} \subset \mathcal{I}$ and $\mathcal{H} \subset (\mathcal{I}^{<\omega})^+$ with $|\mathcal{J}| + |\mathcal{H}| < \widetilde{\mathsf{sep}}(\mathcal{I})$. Let $\mathcal{H}^* = \{H_n : H \in \mathcal{H}, n \in \omega \text{ and } H_n = H \cap [\omega \setminus n]^{<\omega}\}$. Since $|\mathcal{J}| + |\mathcal{H}^*| = |\mathcal{J}| + |\mathcal{H}| < \widetilde{\mathsf{sep}}(\mathcal{I})$, we can pick $A \subset \omega$ so that

(1) $|A \cap J| < \aleph_0$ for $J \in \mathcal{J}$ and
(2) $|[A]^{<\omega} \cap H| = \aleph_0$ for $H \in \mathcal{H}^*$.

Since $[A]^{<\omega} \cap H_n = [A]^{<\omega} \cap [\omega \setminus n]^{<\omega} \cap H = [A \setminus n]^{<\omega} \cap H$ for $H \in \mathcal{H}$ and $n \in \omega$, $|A \cap J| < \aleph_0$ for $J \in \mathcal{J}$ and $|[A \setminus n]^{<\omega} \cap H| = \aleph_0$ for $H \in \mathcal{H}$ and $n \in \omega$. Therefore $\widetilde{\widetilde{\mathsf{sep}}}(\mathcal{I}) \geq \widetilde{\mathsf{sep}}(\mathcal{I})$.

□

Lemma 2.3. *If $\mathcal{I}$ is not a selective ultrafilter, then*

$$\mathfrak{m}(\mathbb{M}_{\mathcal{I}^*}) = \min\{\widetilde{\mathsf{sep}}(\mathcal{I}), \mathsf{cov}(\mathcal{M})\}.$$

Proof of Lemma 2.3. We shall show that $\mathfrak{m}(\mathbb{M}_{\mathcal{I}^*}) = \min\{\widetilde{\widetilde{\mathsf{sep}}}(\mathcal{I}), \mathsf{cov}(\mathcal{M})\}$.
(i) $\mathfrak{m}(\mathbb{M}_{\mathcal{I}^*}) \geq \min\{\widetilde{\widetilde{\mathsf{sep}}}(\mathcal{I}), \mathsf{cov}(\mathcal{M})\}$.

Let $\kappa < \widetilde{\widetilde{\mathsf{sep}}}(\mathcal{I}), \mathsf{cov}(\mathcal{M})$. Let $\{D_\alpha : \alpha < \kappa\}$ be a family of open dense subsets of $\mathbb{M}_{\mathcal{I}^*}$. Let $\{\langle s^n_\alpha, F^n_\alpha\rangle : n \in \omega\}$ be a maximal antichain in D_α. Let $I^n_\alpha = \omega \setminus F^n_\alpha \in \mathcal{I}$ for $n \in \omega$ and $\alpha < \kappa$. Let

$$H^\alpha_s = \{t \in [\omega]^{<\omega} : \exists n \in \omega (s^n_\alpha \subset s \cup t \subset s^n_\alpha \cup F^n_\alpha)\}.$$

Claim 2.4. *$H^\alpha_s \in (\mathcal{I}^{<\omega})^+$ for all $s \in [\omega]^{<\omega}$ and $\alpha < \kappa$.*

Proof of Claim 2.4. Let $s \in [\omega]^{<\omega}$, $\alpha < \kappa$ and $I \in \mathcal{I}$. Then $\langle s, \omega \setminus (I \cup s)\rangle \in \mathbb{M}_{\mathcal{I}^*}$. Since $\{\langle s^n_\alpha, F^n_\alpha\rangle : n \in \omega\}$ is a maximal antichain, $\langle s, \omega \setminus (I \cup s)\rangle$ is compatible with some $\langle s^n_\alpha, F^n_\alpha\rangle$. So there are $n \in \omega$, $t \in [\omega \setminus (I \cup s)]^{<\omega}$ and $F \in \mathcal{I}^*$ such that $\langle s \cup t, F\rangle \leq \langle s^n_\alpha, F^n_\alpha\rangle, \langle s, \omega \setminus (I \cup s)\rangle$.

Since $\langle s \cup t, F\rangle \leq \langle s^n_\alpha, F^n_\alpha\rangle$, $s^n_\alpha \subset s \cup t \subset s^n_\alpha \cup F^n_\alpha$. Since $\langle s \cup t, F\rangle \leq \langle s, \omega \setminus (I \cup S)\rangle$, $t \cap I = \emptyset$. Hence for each $I \in \mathcal{I}$, there exists $t \in H^\alpha_s$ such that $t \cap I = \emptyset$. Therefore $H^\alpha_s \in (\mathcal{I}^{<\omega})^+$. □

As $\kappa < \widetilde{\widetilde{\mathsf{sep}}}(\mathcal{I})$, there is an $A \subset \omega$ such that

(1) $|A \cap I_\alpha^n| < \aleph_0$ for every $n \in \omega$ and $\alpha < \kappa$ and
(2) $|[A \setminus n]^{<\omega} \cap H_s^\alpha| = \aleph_0$ for every $n \in \omega$, $s \in [\omega]^{<\omega}$ and $\alpha < \kappa$.

Let $\mathcal{A}_\alpha = \{\langle s_\alpha^n, F_\alpha^n \cap A\rangle : s_\alpha^n \subset A \text{ and } n \in \omega\} \subset \mathbb{M}_{\mathsf{fin}(A)^*}$, where $\mathbb{M}_{\mathsf{fin}(A)^*}$ is the Mathias-Prikry forcing associated with the ideal $\mathsf{fin}(A)$ of finite subsets of A and $\mathbb{M}_{\mathsf{fin}(A)^*}$ consists of pairs $\langle s, B\rangle$ such that $s \in [A]^{<\omega}$, $B \in \mathsf{fin}(A)^*$ and $s \cap B = \emptyset$.

Claim 2.5. *$\mathcal{A}_\alpha$ is predense in $\mathbb{M}_{\mathsf{fin}(A)^*}$.*

Proof of Claim 2.5. Let $\langle s, B\rangle \in \mathbb{M}_{\mathsf{fin}(A)^*}$. Let $n \geq \max(s)$ such that $B \setminus n = A \setminus n$. Since $|[A \setminus n]^{<\omega} \cap H_s^\alpha| = \aleph_0$, pick $t \in [A \setminus n]^{<\omega} \cap H_s^\alpha$. Then there is an $n \in \omega$ such that $s_\alpha^n \subset s \cup t \subset s_\alpha^n \cup (F_\alpha^n \cap A)$. So $\langle s \cup t, (F_\alpha^n \setminus s \cup t) \cap A\rangle \leq \langle s_\alpha^n, F_\alpha^n \cap A\rangle$ and $\langle s \cup t, (F_\alpha^n \setminus s \cup t) \cap A\rangle \in \mathbb{M}_{\mathsf{fin}(A)^*}$. Since $t \in [A \setminus n]^{<\omega} = [B \setminus n]^{<\omega}$, $\langle s \cup t, B \setminus (s \cup t)\rangle \leq \langle s, B\rangle$. So $\langle s, B\rangle$ is compatible with $\langle s_\alpha^n, F_\alpha^n \cap A\rangle$ for some $n \in \omega$. □

Let $D'_\alpha = \{\langle s, F \cap A\rangle : s \subset A, F \cap A \in \mathsf{fin}(A)^* \text{ and } \langle s, F\rangle \in D_\alpha\}$. Then D'_α is dense open subset of $\mathbb{M}_{\mathsf{fin}(A)^*}$. Since $\left|\mathbb{M}_{\mathsf{fin}(A)^*}\right| = \aleph_0$, $\mathbb{M}_{\mathsf{fin}(A)^*} \cong \mathbb{C}$. Since $\kappa < \mathsf{cov}(\mathcal{M})$, there exists A_{gen} such that for every $\alpha < \kappa$ there is $\langle s, F \cap A\rangle \in D'_\alpha$ so that $s \subset A_{gen} \subset s \cup (F \cap A)$. Hence, for every $\alpha < \kappa$ there exists $\langle s, F\rangle \in D_\alpha$ such that $s \subset A_{gen} \subset s \cup F$.

(ii) $\mathfrak{m}(\mathbb{M}_{\mathcal{I}^*}) \leq \min\{\widetilde{\mathsf{sep}}(\mathcal{I}), \mathsf{cov}(\mathcal{M})\}$.

Suppose $\kappa < \mathfrak{m}(\mathbb{M}_{\mathcal{I}^*})$. Let $\mathcal{J} \subset \mathcal{I}$ and $\mathcal{H} \subset (\mathcal{I}^{<\omega})^+$ with $|\mathcal{J}| + |\mathcal{H}| \leq \kappa$. Let $D_J = \{\langle s, F\rangle \in \mathbb{M}_{\mathcal{I}^*} : F \cap J = \emptyset\}$ for $J \in \mathcal{J}$, and let $E_H^n = \{\langle s, F\rangle \in \mathbb{M}_{\mathcal{I}^*} : |[s]^{<\omega} \cap H| \geq n\}$ for $H \in \mathcal{H}$ and $n \in \omega$. Since $\mathcal{J} \subset \mathcal{I}$ and $\mathcal{H} \subset (\mathcal{I}^{<\omega})^+$, D_J and E_H^n are dense subsets of $\mathbb{M}_{\mathcal{I}^*}$ for $J \in \mathcal{J}$, $H \in \mathcal{H}$ and $n \in \omega$. Let $A \subset \omega$ be a $\{D_J : J \in \mathcal{J}\} \cup \{E_H^n : H \in \mathcal{H} \wedge n \in \omega\}$-generic real. Then $|A \cap J| < \aleph_0$ for $J \in \mathcal{J}$ and $|A \cap H| = \aleph_0$ for $H \in \mathcal{H}$. So $\kappa < \widetilde{\mathsf{sep}}(\mathcal{I})$.

Since $\mathcal{I}^*$ is not selective ultrafilter, $\mathbb{M}_{\mathcal{I}^*}$ adds a Cohen real (see [3]). Therefore $\kappa < \mathsf{cov}(\mathcal{M})$. □

Lemma 2.6.

$$\min\{\widetilde{\mathsf{sep}}(\mathcal{I}), \mathsf{cov}(\mathcal{M})\} = \min\{\mathsf{sep}(\mathcal{I}^{<\omega}), \mathsf{cov}(\mathcal{M})\}.$$

Proof. To prove $\min\{\widetilde{\mathsf{sep}}(\mathcal{I}), \mathsf{cov}(\mathcal{M})\} \geq \min\{\mathsf{sep}(\mathcal{I}^{<\omega}), \mathsf{cov}(\mathcal{M})\}$, we shall show that $\widetilde{\mathsf{sep}}(\mathcal{I}) \geq \mathsf{sep}(\mathcal{I}^{<\omega})$.

Claim 2.7. $\mathsf{sep}(\mathcal{I}^{<\omega}) \leq \widetilde{\mathsf{sep}}(\mathcal{I}) \leq \mathsf{sep}(\mathcal{I})$.

Proof of Claim 2.7. Suppose $\kappa < \mathsf{sep}(\mathcal{I}^{<\omega})$. Let $\mathcal{J} \subset \mathcal{I}$ and $\mathcal{H} \subset (\mathcal{I}^{<\omega})^+$ with $|\mathcal{J}| + |\mathcal{H}| \leq \kappa$. For $J \in \mathcal{J}$, put $\hat{J} = \{a \in [\omega]^{<\omega} : a \cap J \neq \emptyset\}$. Then $\hat{J} \in \mathcal{I}^{<\omega}$.

Let $A \subset [\omega]^{<\omega}$ be such that

(1) $\left|A \cap \hat{J}\right| < \aleph_0$ for $J \in \mathcal{J}$ and
(2) $|A \cap H| = \aleph_0$ for $H \in \mathcal{H}$.

Then $|\bigcup A \cap J| < \aleph_0$, and since $A \subset [\bigcup A]^{<\omega}$, $\bigcup A$ satisfies

(1) $|\bigcup A \cap J| < \aleph_0$ for $J \in \mathcal{J}$ and
(2) $|[\bigcup A]^{<\omega} \cap H| = \aleph_0$ for $H \in \mathcal{H}$.

Hence $\kappa < \widetilde{\mathsf{sep}}(\mathcal{I})$. Therefore $\mathsf{sep}(\mathcal{I}^{<\omega}) \leq \widetilde{\mathsf{sep}}(\mathcal{I})$.

$\widetilde{\mathsf{sep}}(\mathcal{I}) \leq \mathsf{sep}(\mathcal{I})$ follows from the fact when $H \in \mathcal{I}^+$, $H^* = \{\{n\} : n \in H\} \in (\mathcal{I}^{<\omega})^+$. □

To finish the proof of the theorem, we shall show $\min\{\widetilde{\mathsf{sep}}(\mathcal{I}), \mathsf{cov}(\mathcal{M})\} \leq \min\{\mathsf{sep}(\mathcal{I}^{<\omega}), \mathsf{cov}(\mathcal{M})\}$.

Suppose $\kappa < \widetilde{\mathsf{sep}}(\mathcal{I})$, $\mathsf{cov}(\mathcal{M})$. Let $\mathcal{J} \subset \mathcal{I}^{<\omega}$ and $\mathcal{H} \subset (\mathcal{I}^{<\omega})^+$ with $|\mathcal{J}| + |\mathcal{H}| \leq \kappa$. For $J \in \mathcal{J}$, fix $I_J \in \mathcal{I}$ so that $a \cap I_J \neq \emptyset$ for $a \in J$.

Let $A \subset \omega$ be such that

(1) $|A \cap I_J| < \aleph_0$ for all $J \in \mathcal{J}$.
(2) $|[A \setminus n]^{<\omega} \cap H| = \aleph_0$ for every $H \in \mathcal{H}$ and $n \in \omega$.

We will construct $B \subset [A]^{<\omega}$ so that

(1) $|B \cap J| < \aleph_0$ for $J \in \mathcal{J}$.
(2) $|B \cap H| = \aleph_0$ for $H \in \mathcal{H}$.

In order to do so, define a forcing notion $\mathbb{P}$ by $\langle F, n\rangle \in \mathbb{P}$ if $F \in [[A]^{<\omega}]^{<\omega}$ and $n \in \omega$ ordered by $\langle F, n\rangle \leq \langle G, m\rangle$ if $F \supset G$, $n \geq m$ and $\min(a) \geq m$ for $a \in F \setminus G$. Since $|\mathbb{P}| = \aleph_0$, $\mathbb{C} \cong \mathbb{P}$. Let $D_{H,n}$ and $E_J \subset \mathbb{P}$ for $H \in \mathcal{H}$, $n \in \omega$ and $J \in \mathcal{J}$ be defined by

$$\begin{aligned} D_{H,n} &= \{\langle F, m\rangle : \exists a \in F(\min(a) > n \text{ and } a \in H)\}. \\ E_J &= \{\langle F, m\rangle : m > \max(A \cap I_J)\}. \end{aligned}$$

Then $D_{H,n}$ is dense for $H \in \mathcal{H}$ and $n \in \omega$, and E_J is dense for $J \in \mathcal{J}$.

Since $\kappa < \mathsf{cov}(\mathcal{M})$, there is a $\{D_{H,n} : H \in \mathcal{H} \text{ and } n \in \omega\} \cup \{E_J : J \in \mathcal{J}\}$-generic G. Let $A_G = \cup\{F : \langle F, n\rangle \in G\}$. Then

(1) $|A_G \cap J| < \aleph_0$ for $J \in \mathcal{J}$ and
(2) $|A_G \cap H| = \aleph_0$ for $H \in \mathcal{H}$.

So $\kappa < \mathsf{sep}(\mathcal{I}^{<\omega})$. Therefore

$$\min\{\widetilde{\mathsf{sep}}(\mathcal{I}), \mathsf{cov}(\mathcal{M})\} \leq \min\{\mathsf{sep}(\mathcal{I}^{<\omega}), \mathsf{cov}(\mathcal{M})\}.$$

□

Corollary 2.8. *For every ideal $\mathcal{I}$ on ω,*

$$\mathfrak{m}(\mathbb{M}_{\mathcal{I}^*}) = \begin{cases} \mathsf{sep}(\mathcal{I}) & \textit{if } \mathcal{I}^* \textit{ is an ultrafilter.} \\ \min\{\mathsf{sep}(\mathcal{I}^{<\omega}), \mathsf{cov}(\mathcal{M})\} & \textit{if } \mathcal{I} \textit{ is not an ultrafilter.} \end{cases}$$

3. Preservation properties of $\mathbb{M}_{\mathcal{I}^*}$

The methods for studying properties of the forcing $\mathbb{L}_{\mathcal{I}^*}$ are well known (see [2, 7, 12]). Here we concentrate on the preservation properties of the forcings $\mathbb{M}_{\mathcal{I}^*}$. In [12] it is shown that a useful characterization for when $\mathbb{L}_{\mathcal{I}^*}$ preserves ω-hitting families. An analogous result also holds for $\mathbb{M}_{\mathcal{I}^*}$.[1]

Theorem 3.1. *Let $\mathcal{I}$ be an ideal on ω. The following are equivalent:*

(1) $\forall X \in (\mathcal{I}^{<\omega})^+ \ \forall \mathcal{J} \leq_K \mathcal{I}^{<\omega} \restriction X$ ($\mathcal{J}$ is not ω-hitting,)
(2) $\mathbb{M}_{\mathcal{I}^*}$ strongly preserves ω-hitting families, and
(3) $\mathbb{M}_{\mathcal{I}^*}$ preserves ω-hitting families.

Proof. From (1) to (2).
Suppose (2) doesn't hold. Let $\dot{A}$ be $\mathbb{M}_{\mathcal{I}^*}$-names witnessing the negation of (2), i.e., for every $(B_n : n \in \omega)$ there exists $B \in [\omega]^\omega$ such that $|B_n \cap B| = \aleph_0$ for every $n \in \omega$ and there exist $p_B = \langle s_B, F_B\rangle \in \mathbb{M}_{\mathcal{I}^*}$ and $m_B \in \omega$ such that $p_B \Vdash B \cap \dot{A} \subset m_B$.

Let $\mathcal{B}$ be the family of all such B. By the assumption that $\mathcal{B}$ is ω-hitting, there are $s \in [\omega]^{<\omega}$ and $m \in \omega$ such that $\mathcal{B}_0 = \{B \in \mathcal{B} : s_B = s \text{ and } m_B = m\}$ is ω-hitting (If an ω-hitting family is split into countably pieces, one of them is ω-hitting). Fix such $s \in [\omega]^{<\omega}$ and $m \in \omega$ and let

$$X_s = \{t \in [\omega]^{<\omega} : \exists k > m \exists F \in \mathcal{I}^* \Big(\langle s \cup t, F\rangle \Vdash k \in \dot{A}\Big)\}.$$

Claim 3.2. $X_s \in (\mathcal{I}^{<\omega})^+$.

Proof of Claim 3.2. Given $I \in \mathcal{I}$, there are $t \in [\omega \setminus I]^{<\omega}$, $k > m$ and $F \in \mathcal{I}$ such that $\langle s \cup t, F\rangle \leq \langle s, \omega \setminus I\rangle$ and $\langle s \cup t, F\rangle \Vdash k \in \dot{A}$. Then $t \in X_s$ and $t \cap I = \emptyset$. □

Define $f : X_s \to \omega$ by

$$f(t) = \begin{cases} \max\{k > m : \exists F \in \mathcal{I}^*(\langle s \cup t, F\rangle \Vdash k \in \dot{A})\} \\ \quad \text{if there are finitely many such } k, \\ \\ \min\{k > \max(t \cup \{m\}) : \exists F \in \mathcal{I}^*(\langle s \cup t, F\rangle \Vdash k \in \dot{A})\} \\ \quad \text{otherwise.} \end{cases}$$

Claim 3.3. *For every $k \in \omega$, $f^{-1}[\omega \setminus k] \in (\mathcal{I}^{<\omega})^+$.*

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[1]A forcing $\mathbb{P}$ *strongly preserves ω-hitting families* if given a $\mathbb{P}$-name $\dot{A}$ for an infinite subset of ω there is a countable family $\mathcal{H}$ of infinite subsets of ω such that whenever $B \subseteq \omega$ has an infinite intersection with every element of $\mathcal{H}$ then $\Vdash_{\mathbb{P}}$ "$|\dot{A} \cap B| = \omega$".

So we can assume that for all but finitely many $k > m$, $f^{-1}(\{k\}) \in \mathcal{I}^{<\omega}$ or there exist infinitely many $k \in \omega$ such that $f^{-1}(\{k\}) \in (\mathcal{I}^{<\omega})^+$.

Claim 3.4. *For all but finitely many $k > m$, $f^{-1}(\{k\}) \in \mathcal{I}^{<\omega}$.*

Proof. Assume to the contrary that there are infinitely many $k > m$ such that $f^{-1}(\{k\}) \in (\mathcal{I}^{<\omega})^+$. Let $C = \{k > m : f^{-1}(\{k\}) \in (\mathcal{I}^{<\omega})^+\}$. Since $\mathcal{B}_0$ is ω-hitting, there exists $B \in \mathcal{B}_0$ such that $B \cap C$ is infinite. Let $k > m$ such that $k \in C \cap B$ and $f^{-1}(\{k\}) \in (\mathcal{I}^{<\omega})^+$. Then $|[F_B]^{<\omega} \cap f^{-1}(\{k\})| = \aleph_0$. Let $t \in [F_B]^{<\omega} \cap f^{-1}(\{k\})$. Then there exists $F \in \mathcal{I}^*$ such that $\langle s \cup t, F\rangle \Vdash k \in \dot{A} \cap B$. Since $t \in [F_B]^{<\omega}$, $\langle s, F_B\rangle$ is compatible with $\langle s \cup t, F\rangle$. However $\langle s, F_B\rangle \Vdash \dot{A} \cap B \subset m$, which is a contradiction. ☐

Claim 3.5. *f witnesses that $\langle \mathcal{B}_0\rangle \leq_K \mathcal{I}^{<\omega} \restriction X$.*

Proof. Assume to the contrary that there is a $B \in \mathcal{B}_0$ such that $f^{-1}[B] \in (\mathcal{I}^{<\omega})^+$. Since $F_B \in \mathcal{I}^*$ and $|[F_B]^{<\omega} \cap f^{-1}[B]| = \aleph_0$, there is a $t \in [F_B]^{<\omega} \cap f^{-1}[B]$ such that for some $F \in \mathcal{I}^*$ and $k > m$ $\langle s \cup t, F\rangle \Vdash k \in \dot{A} \cap B$.

Since $t \in [F_B]^{<\omega}$, $s \subset s \cup t \subset s \cup F_B$. So $\langle s, F_B\rangle$ is compatible with $\langle s \cup t, F\rangle$. However, $\langle s, F_B\rangle \Vdash \dot{A} \cap B \subset m$, which is a contradiction. ☐

Since $\langle \mathcal{B}_0\rangle$ is ω-hitting, (1) doesn't hold.

Obviously (2) implies (3).

We shall prove (3) implies (1). Assume to the contrary that there exists an ideal $\mathcal{J}$ on ω such that there exist $X \in (\mathcal{I}^{<\omega})^+$ and $f : X \to \omega$ so that

(1) For every $J \in \mathcal{J}$, $f^{-1}[J] \in \mathcal{I}^{<\omega}$ and
(2) $\mathcal{J}$ is ω-hitting.

Let $\dot{a}_{gen}$ be the canonical name for the $\mathbb{M}_{\mathcal{I}^*}$-generic subset of ω. We shall show that $\Vdash \mathcal{J}$ is not ω-hitting. Let $\dot{X}_n$ be a $\mathbb{M}_{\mathcal{I}^*}$-name such that

$$\Vdash \dot{X}_n = f\left[[\dot{a}_{gen} \setminus n]^{<\omega}\right].$$

Claim 3.6. $\Vdash \dot{X}_n$ *is infinite.*

Proof of the Claim. If $\Vdash \dot{X}_n$ were finite, then

$$\Vdash [\dot{a}_{gen} \setminus n]^{<\omega} \cap X \subset f^{-1}[\dot{X}_n] \in \mathcal{I}^{<\omega}.$$

However, $\Vdash \forall I \in \mathcal{I}(|\dot{a}_{gen} \cap I| < \aleph_0)$ and $\Vdash |[\dot{a}_{gen} \setminus n]^{<\omega} \cap X| = \aleph_0$ by genericity. So $\Vdash \forall I \in \mathcal{I} \forall n \in \omega \exists a \in [\dot{a}_{gen} \setminus n]^{<\omega}(a \cap I = \emptyset)$, which is a contradiction. ☐

Claim 3.7. $\Vdash \forall J \in \mathcal{J} \exists n \in \omega (J \cap \dot{X}_n = \emptyset)$.

Proof. For every $J \in \mathcal{J}$ and $\langle s, F\rangle \in \mathbb{M}_{\mathcal{I}^*}$, pick $I \in \mathcal{I}$ such that $a \cap I \neq \emptyset$ for $a \in f^{-}[J]$, $G = F \cap (\omega \setminus I)$ and $n = \max(s)$. Then $\langle s, G\rangle \Vdash [\dot{a}_{gen} \setminus n]^{<\omega} \cap f^{-1}[J] = \emptyset$. So $\langle s, G\rangle \Vdash \dot{X}_n \cap J = \emptyset$. □

So $\mathcal{J}$ is not ω-hitting in the extension, contradiction. □

Now, we turn our attention to the question of when does the forcing $\mathbb{M}_{\mathcal{I}^*}$ add a dominating real. This line of investigation was started by M. Canjar in [14], where he assuming $\mathfrak{d} = \mathfrak{c}$ constructed an ultrafilter $\mathcal{U}$ such that $\mathbb{M}_{\mathcal{U}}$ doesn't add any dominating reals. He also noticed that such an ultrafilter has to be a P-point without rapid Rudin-Keisler predecessors (see e.g. [1] for definitions and more information) and asked whether the converse is also true. Here we give a simple combinatorial characterizations of ideals $\mathcal{I}$ such that $\mathbb{M}_{\mathcal{I}^*}$ doesn't add any dominating reals.

We call an ideal $\mathcal{J}$ a *P^+-ideal* if for every decreasing sequence $\{X_n : n \in \omega\}$ of $\mathcal{J}$ positive sets, there is an $X \in \mathcal{J}^+$ such that $X \subset^* X_n$ for all $n \in \omega$.

Theorem 3.8. *The following are equivalent.*

(1) $\mathbb{M}_{\mathcal{I}^*}$ adds a dominating real.
(2) $\mathcal{I}^{<\omega}$ is not a P^+-ideal.

Proof. (1) implies (2).

Let $\dot{g}$ be a $\mathbb{M}_{\mathcal{I}^*}$-name for a dominating real, i.e., $\forall f \in \omega^\omega \cap V (\Vdash f <^* \dot{g})$. In particular, for every $f \in \omega^\omega \cap V$, there are $s_f \in [\omega]^{<\omega}$, $F_f \in \mathcal{I}^*$ and $n_f \in \omega$ such that

$$\langle s_f, F_f\rangle \Vdash \forall n \geq n_f (f(n) < \dot{g}(n)).$$

If one partitions a dominating family into countably many pieces, one of the pieces is also dominating. So there are $s \in [\omega]^{<\omega}$ and $n \in \omega$ such that

$$\mathcal{F} = \{f \in \omega^\omega : s_f = s \wedge n_f = n\}$$

is a dominating family. Fix such $s \in [\omega]^{<\omega}$ and $n \in \omega$ and let

$$X_s = \{t \in [\omega \setminus \max(s)]^{<\omega} : \exists F \in \mathcal{I}^* \exists m \geq n \, (\langle s \cup t, F\rangle \text{ decides } \dot{g}(m))\}.$$

Claim 3.9. *$X_s \in (\mathcal{I}^{<\omega})^+$.*

For each $t \in X_s$, let $z_t = \{m \geq n : \exists F \in \mathcal{I}^*(\langle s \cup t, F\rangle \text{ decides } \dot{g}(m))\}$. Put $k_t, l_t \in \omega$ so that

$$k_t = \begin{cases} \max(z_t) & \text{if } |z_t| < \omega \\ \min(z_t \setminus \max(t)) & \text{otherwise.} \end{cases}$$

and there is an $F \in \mathcal{I}^*$ such that $\langle s \cup t, F\rangle \Vdash \dot{g}(k_t) = l_t$.

Then define $H : X_s \to \omega \times \omega$ by $H(t) = \langle k_t, l_t\rangle$.

Claim 3.10. *For every $m \in \omega$, $H^{-1}[(\omega \setminus m) \times \omega] \in (\mathcal{I}^{<\omega})^+$.*

Let $K = \{k_t : t \in X_s\}$. Then K is infinite and let $\{k_i : i \in \omega\}$ be its increasing enumeration. Put $K_i = \{l \in \omega : \langle k_i, l\rangle \in H[X_s]\}$.

Claim 3.11. *There are infinitely many $i \in \omega$ such that K_i is infinite.*

Proof of Claim 3.11. Assume to the contrary that for all but finitely many $i \in \omega$, K_i is finite. Then define $g : \omega \to \omega$ by

$$g(m) = \begin{cases} \max(K_i) & \text{if } m = k_i \text{ and } |K_i| < \aleph_0 \text{ for some } i \in \omega, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathcal{F}$ is a dominating family, there is an $f \in \mathcal{F}$ such that $g \leq^* f$. Let $m_0 \geq n$ be such that $g(m) \leq f(m)$ for $m \geq m_0$ and $k_i \geq m_0$ implies $|K_i| < \aleph_0$.

By Claim 3.10, $H^{-1}[(\omega \setminus m) \times \omega] \cap [F]^{<\omega} \in (\mathcal{I}^{<\omega})^+$ for $F \in \mathcal{I}^*$ and $m \in \omega$.

Let $t \in [F_f]^{<\omega} \cap H^{-1}[(\omega \setminus m_0) \times \omega]$. Then there is an $F \in \mathcal{I}^*$ such that $\langle s \cup t, F\rangle \Vdash \dot{g}(k_t) = l_t \leq g(k_t) \leq f(k_t)$. However, $\langle s, F_f\rangle$ is compatible with $\langle s \cup t, F\rangle$ and $\langle s, F_f\rangle \Vdash f(k_t) < \dot{g}(k_t)$, which is a contradiction. □

Without loss of generality, we can assume that for every $i \in \omega$, K_i is infinite.

Let $Y_m = H^{-1}[\bigcup_{i>m} K_i]$ for $m \in \omega$. Then $Y_{m+1} \subset Y_m$. As Claim 3.10, we can prove the following.

Claim 3.12. *$Y_m \in (\mathcal{I}^{<\omega})^+$ for $m \in \omega$.*

Proof of Claim 3.12. Let $I \in \mathcal{I}$ and $m \in \omega$. We shall show that there exists $t \in Y_m$ such that $t \cap I = \emptyset$. Let $t \in [\omega \setminus I]^{<\omega}$ such that $\langle s \cup t, F\rangle \leq \langle s, \omega \setminus I\rangle$ decides $\dot{g}(k)$ for some $k > m$ and $\max(t) > m$. By definition of k_t, $k_t \geq k > m$. Then $H(t) \in \bigcup_{i>m} K_i$. So $t \in Y_m$ and $t \cap I = \emptyset$. □

Let $Y \subset^* Y_m$ for $m \geq n$. We shall show that $Y \in \mathcal{I}^{<\omega}$.

Assume to the contrary that $Y \in (\mathcal{I}^{<\omega})^+$.

Since $Y \subset^* Y_m$, $K_m \cap H[Y]$ is finite for every $m \in \omega$. Define a function h from ω to ω by

$$h(m) = \begin{cases} \max\{l_t : \exists t \in Y \cap Y_m\} \text{ if } Y \cap Y_m \neq \emptyset, \\ \\ 0 \text{ otherwise.} \end{cases}$$

Then there are infinitely many m such that $h(m) > 0$.

Since $\mathcal{F}$ is a dominating family, there is an $f \in \mathcal{F}$ such that $h \leq^* f$. Let $m_0 \geq n$ be such that $h(m) \leq f(m)$ for $m \geq m_0$. Since $Y \subset^* Y_m$, $F_f \in \mathcal{I}^*$ and $Y \in (\mathcal{I}^{<\omega})^+$, there is an $m \geq m_0$ such that $Y \cap Y_m \cap$

$[F_f]^{<\omega} \neq \emptyset$. Let $t \in Y \cap Y_m \cap [F_f]^{<\omega}$. Since $t \in Y$ there is an $F \in \mathcal{I}^*$ such that $\langle s \cup t, F\rangle \Vdash \dot{g}(m) \leq h(m)$. However, $\langle s, F_f\rangle \Vdash$ "$\forall m \geq n(f(m) < \dot{g}(m))$" and $\langle s \cup t, F\rangle$ is compatible with $\langle s, F_f\rangle$, which is a contradiction. Therefore $Y \in \mathcal{I}^{<\omega}$. So $\mathcal{I}^{<\omega}$ is not P^+-ideal.

(2) implies (1).

Let $\langle X_n : n \in \omega\rangle$ be a decreasing sequence of $\mathcal{I}^{<\omega}$-positive sets without a pseudointersection in $(\mathcal{I}^{<\omega})^+$. Let $\langle a_k : k \in \omega\rangle$ be an enumeration of $[\omega]^{<\omega} \setminus \{\emptyset\}$ and let $\dot{a}_{gen}$ be the canonical name for the $\mathbb{M}_{\mathcal{I}^*}$-generic real. Define a $\mathbb{M}_{\mathcal{I}^*}$-name $\dot{g}$ for a function from ω to ω by

$$\Vdash \dot{g}(n) = \min\{k : a_k \subset [\dot{a}_{gen}]^{<\omega} \cap X_n \wedge \max(\bigcup\{a_m : l < n \wedge m = \dot{g}(l)\}) < \min(a_k)\}.$$

We shall show that $\dot{g}$ is a dominating real. Let $f \in \omega^\omega \cap V$ and $\langle s, F\rangle \in \mathbb{M}_{\mathcal{I}^*}$. Let $I_f = \{a_k \in [\omega]^{<\omega} \setminus \{\emptyset\} : \exists n \in \omega(a_k \in X_n \wedge k \leq f(n))\}$. Then $I_f \subset^* X_n$ for every $n \in \omega$. Therefore $I_f \in \mathcal{I}^{<\omega}$ by definition of X_n. Let $I \in \mathcal{I}$ such that $\forall a \in I_f(a \cap I \neq \emptyset)$. Then $F \setminus I \in \mathcal{I}^*$ and $[F \setminus I]^{<\omega} \cap I_f = \emptyset$.

Claim 3.13. *Let $\langle t_n : n < \alpha\rangle$ be a sequence of finite subsets of ω such that*

(1) $t_n \in [s \cup (F \setminus I)]^{<\omega} \cap X_n$
(2) $\max(t_n) < \min(t_{n+1})$
(3) $\exists k \in \omega(t_n = a_k \wedge k \leq f(n))$

Then $\alpha \leq |s|$.

Proof of Claim. If $t \in [F \setminus I]^{<\omega}$, then $t = a_k$ and $t \in X_n$ implies $k > f(n)$ by $[F \setminus I]^{<\omega} \cap I_f = \emptyset$. So by (2), $\alpha \leq |s|$. □

Put $|s|$=m. Then $\langle s, F \setminus I\rangle \leq \langle s, F\rangle$ and

$$\langle s, F \setminus I\rangle \Vdash \forall n > m(f(n) < \dot{g}(n)).$$

□

Recently, using our characterizations, M. Hrušák and J. Verner showed that if $\mathcal{I}$ is an F_σ P-ideal, then $\mathcal{P}(\omega)/\mathcal{I}$ adds an ultrafilter $\mathcal{U}$ which is a P-point without rapid RK-predecessors, but $\mathcal{U}^*$ is not a P^+-ideal [17]. Thus this answers Canjar's question in the negative. Moreover, A. Blass, M. Hrušák and J. Verner [4] (also using our theorem) showed that $\mathbb{M}_{\mathcal{U}}$ doesn't add any dominating reals if and only if $\mathcal{U}$ is strong P-point.

4. Concluding remarks and open problems.

It is still interesting to try to better understand ideals $\mathcal{I}$ for which $\mathbb{M}_{\mathcal{I}^*}$ doesn't add any dominating reals. An interesting class of ideals in this respect are those generated by maximal almost disjoint (mad) families.

Theorem 4.1. [6] *Assume $\mathfrak{b} = \mathfrak{c}$. Then there exists a mad family $\mathcal{A}$ such that $\mathbb{M}_{\mathcal{I}(\mathcal{A})}$ adds a dominating real.*

Question 4.2. [6] *Is it consistent that there is no mad family $\mathcal{A}$ such that $\mathbb{M}_{\mathcal{I}(\mathcal{A})}$ adds a dominating real?*

As far as definable ideals are concern, J. Brendle has in [6] that an $\mathbb{M}_{\mathcal{I}^*}$ doesn't add any dominating reals for any F_σ-ideal $\mathcal{I}$. This follows directly from our characterization. However, it is not clear whether this characterizes F_σ-ideals among Borel ones:

Question 4.3. *Is there a Borel ideal $\mathcal{I}$ which is not F_σ, yet $\mathbb{M}_{\mathcal{I}^*}$ doesn't add any dominating reals?*

However, we have the following useful approximation:

Theorem 4.4. *Suppose $\mathcal{I}$ is a Borel ideal. Then the following are equivalent.*

(1) $\mathcal{I}$ can be extended to an ideal $\mathcal{J}$ such that $\mathbb{M}_{\mathcal{J}^*}$ doesn't add any dominating reals.
(2) $\mathcal{I}$ can be extended to a P^+-ideal.
(3) $\mathcal{I}$ can be extended to an F_σ-ideal.

Proof. (3) implies (1).

This is proved in Brendle's paper [6], but it also follows from our theorem using the following two simple observations:

(i) If $\mathcal{I}$ is F_σ then $\mathcal{I}^{<\omega}$ is F_σ, and
(ii) every F_σ-ideal is P^+.

(1) implies (2).

Suppose (2) doesn't hold. Then every $\mathcal{J}$ extending $\mathcal{I}$ is not P^+.

Claim 4.5. *If $\mathcal{J}^{<\omega}$ is P^+, then $\mathcal{J}$ is P^+.*

Proof of Claim. Let $\{Y_n : n \in \omega\}$ be a decreasing sequence of $\mathcal{J}^+$. Put $Y_n^* = \{\{k\} : k \in Y_n\}$ for $n < \omega$. Then $Y_n^* \in (\mathcal{J}^{<\omega})^+$. By assumption, there exists $Y^* \in (\mathcal{J}^{<\omega})^+$ such that $Y^* \subset^* Y_n^*$ for $n < \omega$. Put $Y = \bigcup Y^*$. Then $Y \in \mathcal{J}^+$ and $Y \subset^* Y_n$ for $n < \omega$. So $\mathcal{J}$ is P^+. □

By this claim, $\mathcal{J}^{<\omega}$ is not P^+. So $\mathbb{M}_{\mathcal{J}^*}$ adds a dominating real by our theorem.

(2) implies (3).

This follows from a theorem of D. Meza and M. Hrušák (see [16]).

□

This, in particular, shows that the ideal $\mathcal{Z}$ of sets Banach density zero can not be extended to an ideal $\mathcal{I}$ such that $\mathbb{M}_{\mathcal{I}^*}$ doesn't add any dominating reals, as it can not be extended to an F_σ-ideal.

Corollary 4.6. *Let $\mathcal{Z}$ be the density zero ideal. If $\mathcal{Z}^* \subset \mathcal{F}$, then $\mathbb{M}_{\mathcal{F}}$ adds a dominating real.*

Proof. Recall that $\mathcal{Z} = \{A \subset \omega : \lim_{n\to\infty} \frac{|A \cap n|}{n} = 0\}$. Suppose $\mathcal{Z} \subset \mathcal{I}$. Let $X^j_n = \{k \cdot n! + j : k \in \omega\}$ for $n \geq 1$ and $k < n!$. Then $\{X^i_{n+1} : i < (n+1)!$ and $X^i_{n+1} \subset X^j_n\}$ is a partition of X^j_n into finitely many pieces. So if $X^j_n \in \mathcal{I}^+$, then there exists $X^i_{n+1} \in \mathcal{I}^+$ such that $X^i_{n+1} \subset X^j_n$ for some $i < (n+1)!$. By induction on n, we can construct a decreasing sequence $\{X^{j_n}_n : n \in \omega\}$ of $\mathcal{I}$-positive sets. Since $X \subset^* X^{j_n}_n$ implies

$$\lim_{k\to\infty} \frac{|X \cap k|}{k} \leq \lim_{k\to\infty} \frac{|X^{j_n}_n \cap k|}{k} \leq \frac{1}{n!},$$

every pseudointersection of $\{X^{j_n}_n : n \in \omega\}$ is in $\mathcal{Z} \subset \mathcal{I}$. Hence $\mathcal{Z}$ cannot be extended to a P^+-ideal. So $\mathcal{Z}$ cannot be extended to an ideal $\mathcal{J}$ such that $\mathbb{M}_{\mathcal{J}^*}$ adds a dominating real.

□

Question 4.7. *Is there forcing notion $\mathbb{P}$ which destroys $\mathcal{Z}$ and doesn't add a dominating real?*

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