Calculating the infinity without continuity: proofs and diagrams in *De Quadratura Arithmetica*

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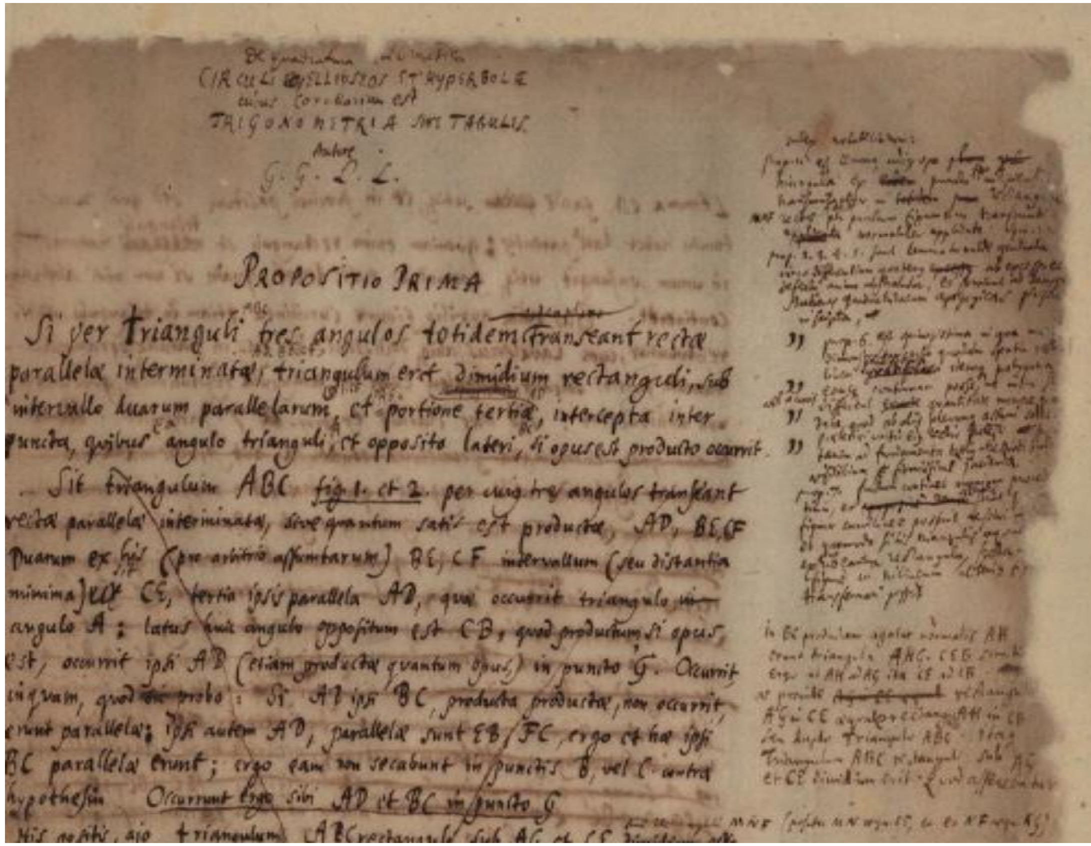
1: Introduction

- In this talk, we would analyze how mathematical diagrams works in proofs of *De quadratura arithmetica circuli ellipseos et hyperbolae cujus corollarium est trigonometria sine tabulis* (DQA)
- DQA is thought as the masterpiece of Leibniz’s mathematical study in Paris.
- In DQA, Leibniz succeeded a quadrature of the areas of curvilinear figures.



- DQA contains 51 propositions and 16 pictures. Almost half of all propositions utilize diagrams. So, DQA is a good material for understanding Leibniz's method of using diagrams.
- So far, DQA has been regarded as a philosophical source of foundation of infinitesimals.
- In fact, many scholars concentrate proposition 6 and 7, which do not utilize explicitly infinitesimals to determinate the area and claims that we could find the fictionalism of infinitesimals in DQA.
- Some scholars of history of mathematics criticize this tendency. (Blåsjö 2017)
- The importance of DQA is not limited to the foundational issue. For example, Grosholz maintains that DQA is an example of Leibniz's use of various representations, including geometric diagrams, algebraic equations, and tables (Grosholz 2007 pp.214-5).

- By discussing DQA in viewpoint of how diagrams are used, it is possible to support to getting an idea of what Leibniz’s mathematical practice was like.
- Such research will help to clarify the use of diagrams by European mathematicians in the 17th century and characterize the mathematical practice of this period.
- In this talk, we would make use of some results of philosophy of mathematical practice.

2: 'Practical' turn in the philosophy of mathematics

- Recently, some philosophers of mathematics focus on and analyze practices of mathematicians. (see Mancosu 2008)
- Various topics are discussed in the philosophy of mathematical practice. For example: cognitive feature of mathematical symbols, beauty of mathematical proof, social aspect of mathematics, and so on.
- In this talk, we would focus on roles of diagrams in proof.
- For a long time, philosophers and mathematicians have been wary of using diagrams in mathematical proof. One of the reasons is that using diagrams leads to inaccurate inferences. (Leibniz was one of those critics!)

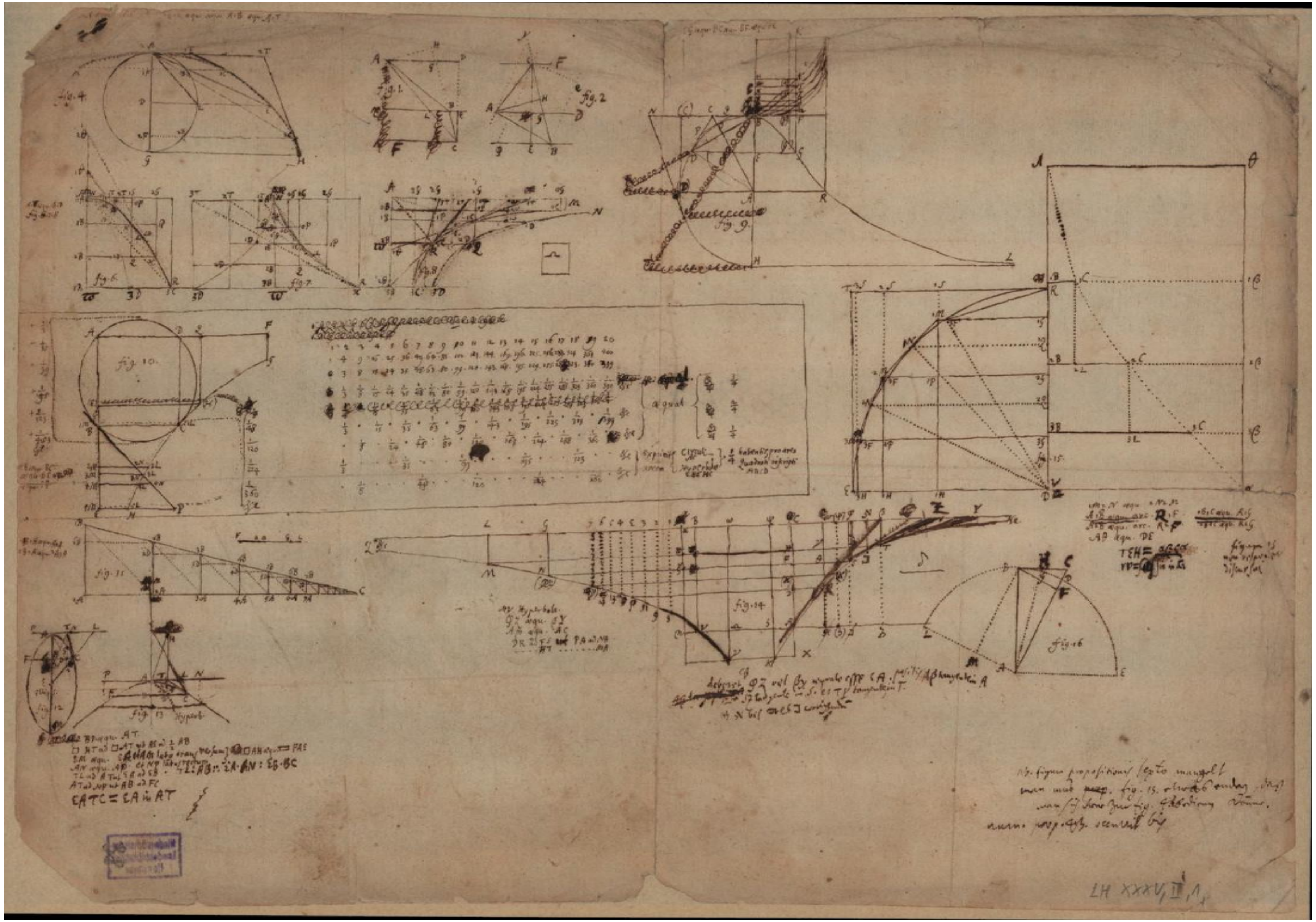
Exactness/Co-exactness distinction

- However, recently, some philosophers begin to acknowledge of the positive roles of diagrams in proofs.
- cf. Formalization of proof of Euclid' Elements. (Avigad et al 2010)
- Manders's distinction of Exactness/Co-exactness.(Manders 2008)
- Exactness: Quantitative properties such as the length of a line segment or the bisector of an angle. It is not possible, in a strict sense, to represent these properties using diagrams that always contain minute distortions.
- Co-exactness: Positional or topological relationships between diagram such as intersections and contacts. These can be adequately described by diagrams.

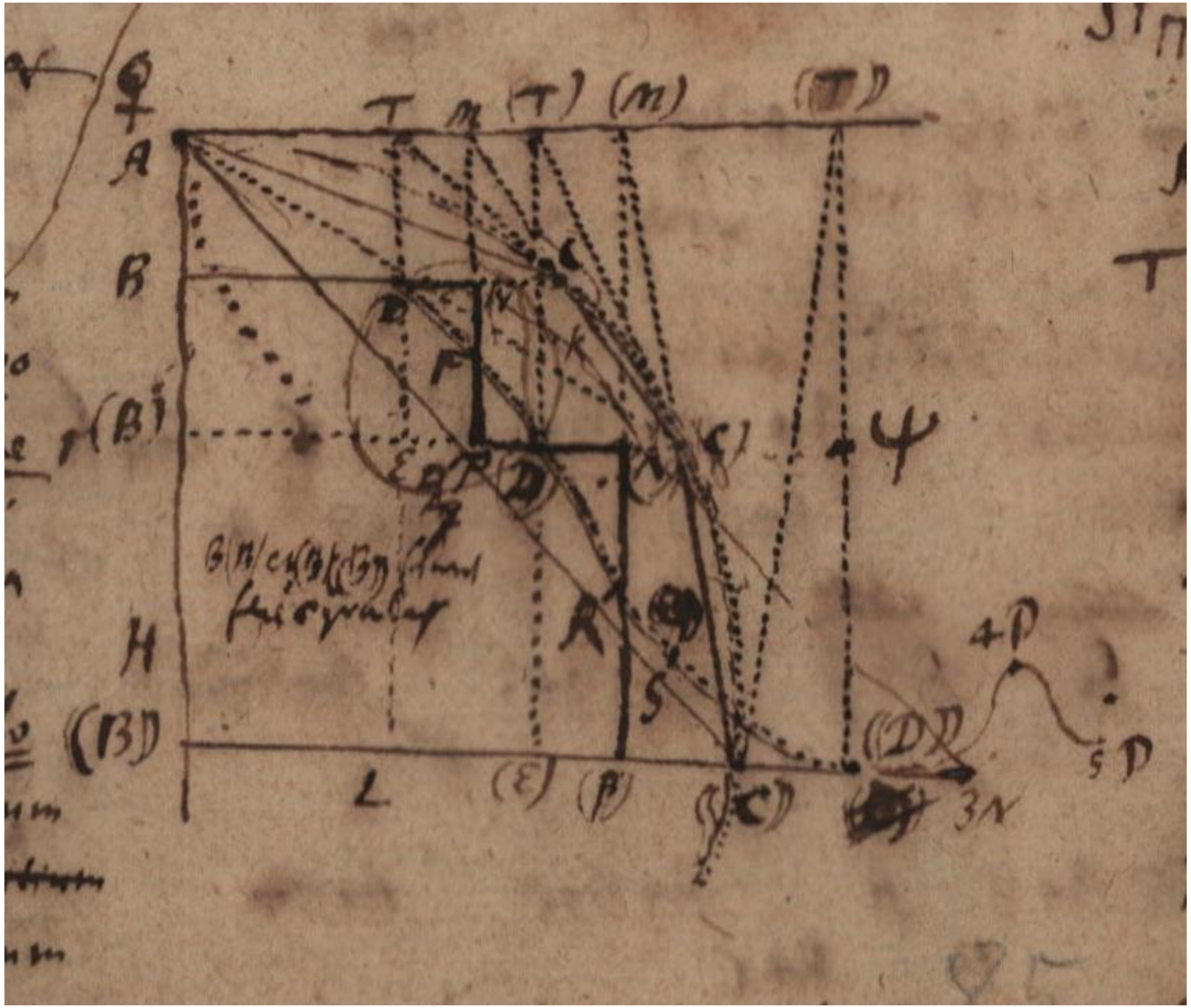
- Diagrams do not need to be drawn strictly as long as the positional relationship between diagrams can be expressed.
- The quantitative properties of diagrams (length of lines, size of angles) are specified by the text.
- When we do mathematics, we see texts and diagrams by associating them.
- Mathematics is “a workable cross-reference system between discursive text and diagram”(Manders 2008 p.108)

3: Diagrams in DQA

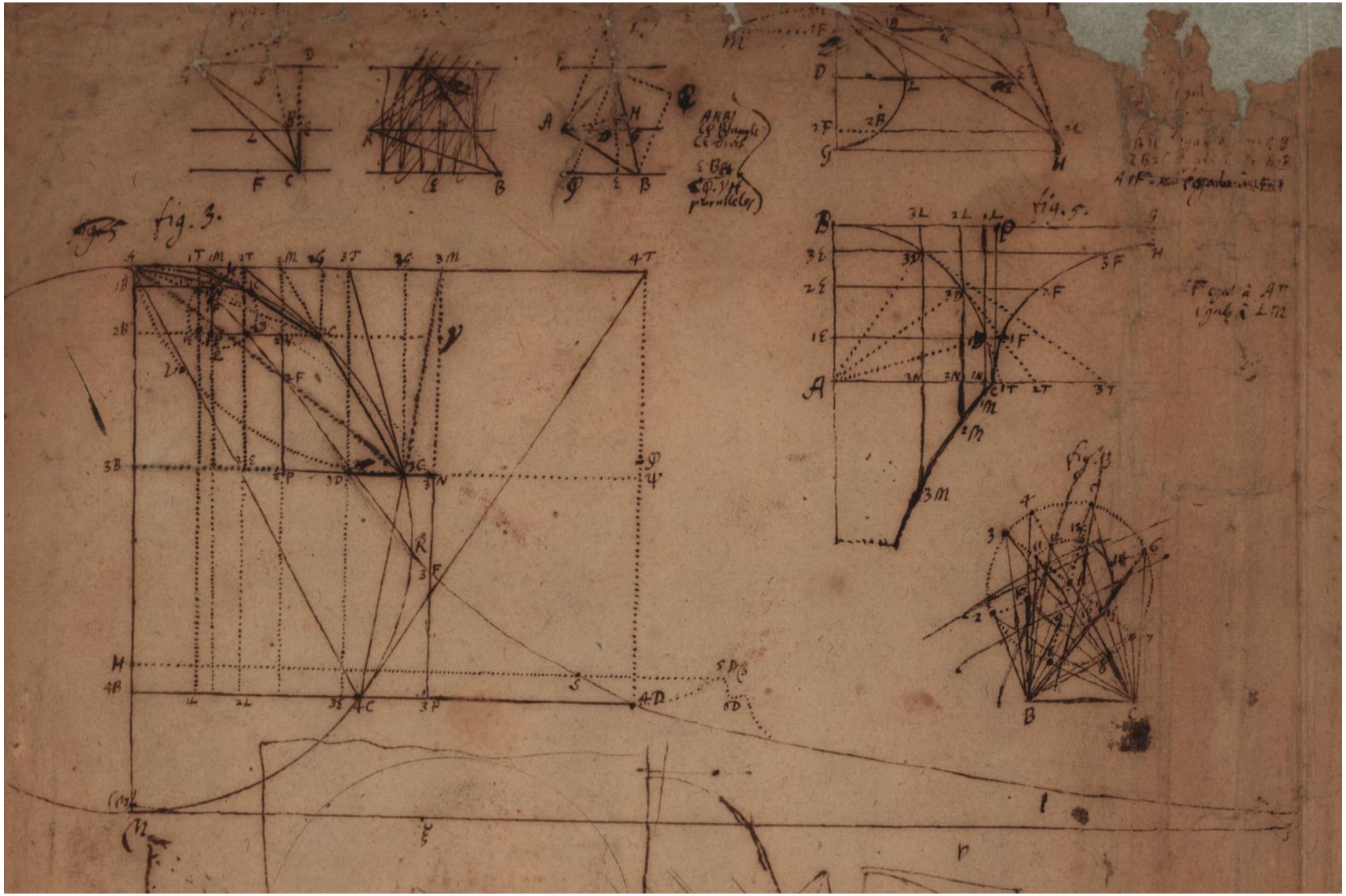
- DQA is one of the desirable text of Leibniz‘s mathematical study in that it contains much proofs and diagrams. Moreover, there are related texts of DQA, so focusing of DQA and its drafts, we could understand better the development of Leibniz’s practice of mathematics. (Crippa 2019 pp.99-100)



- DQA's diagrams
- LH 35 2 1 38r

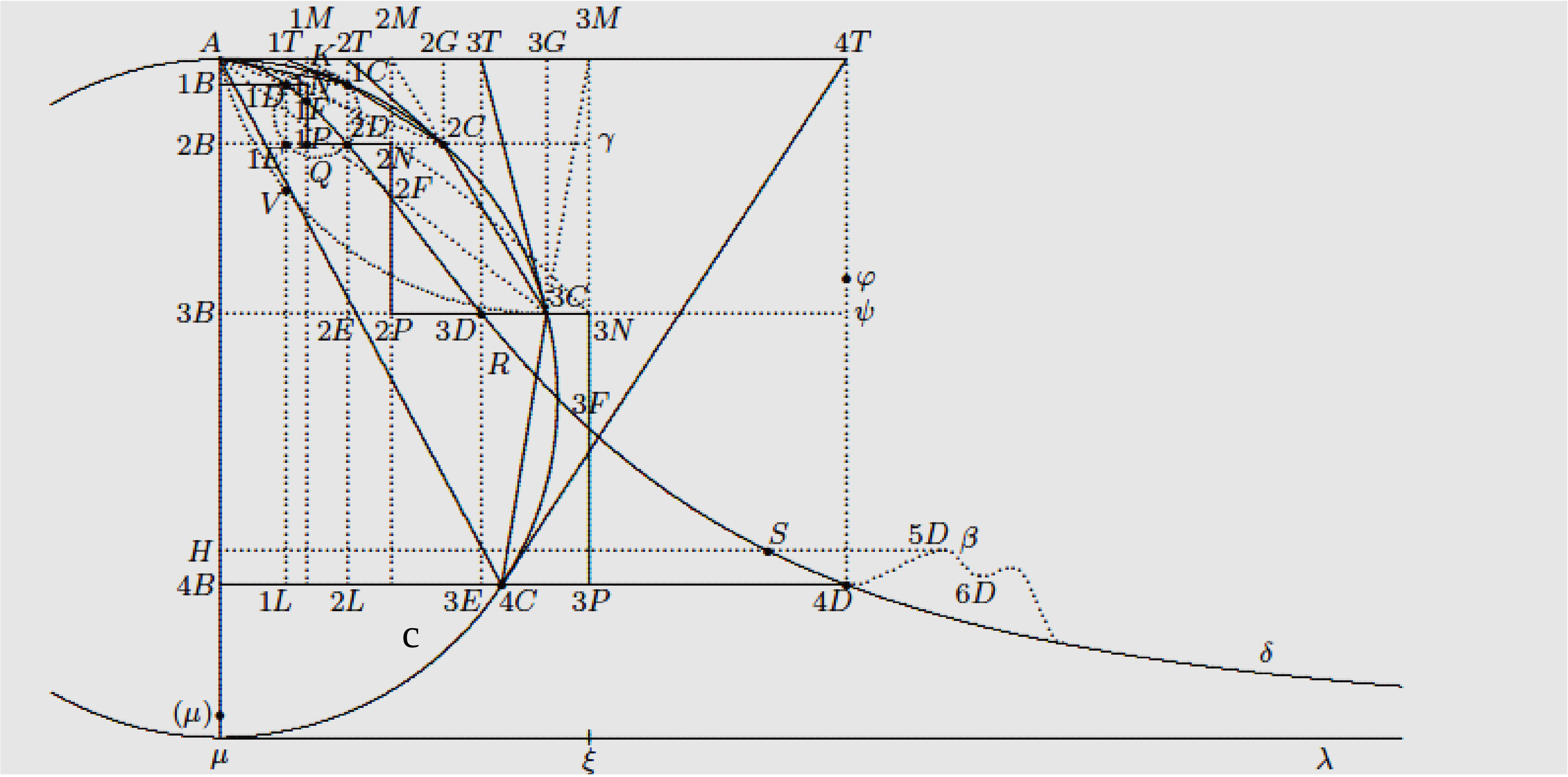


- DISSERTATIONIS DE ARITHMETICA CIRCULI QUADRATURA PROPOSITIONES SEPTEM
- Spring, 1676
- A VII 6 N.14
- LH, 35, 2, 1, 85r



- Quadraturae Circuli Arithmeticae pars prima April -July 1676
- A VII 6 N.20
- LH 35, 2, 1, 159v

4: Proof of proposition 6



- Proposition 6 and 7 achieve the quadrature of certain curve by using the method of exhaust and reductio proof.
- As mentioned earlier, these two propositions are interpreted as a philosophical basis of infinitesimals. (Arthur, Levey, Rabouin,…)
- However, rather, from these two, we could know more about Leibniz's practice of mathematics.

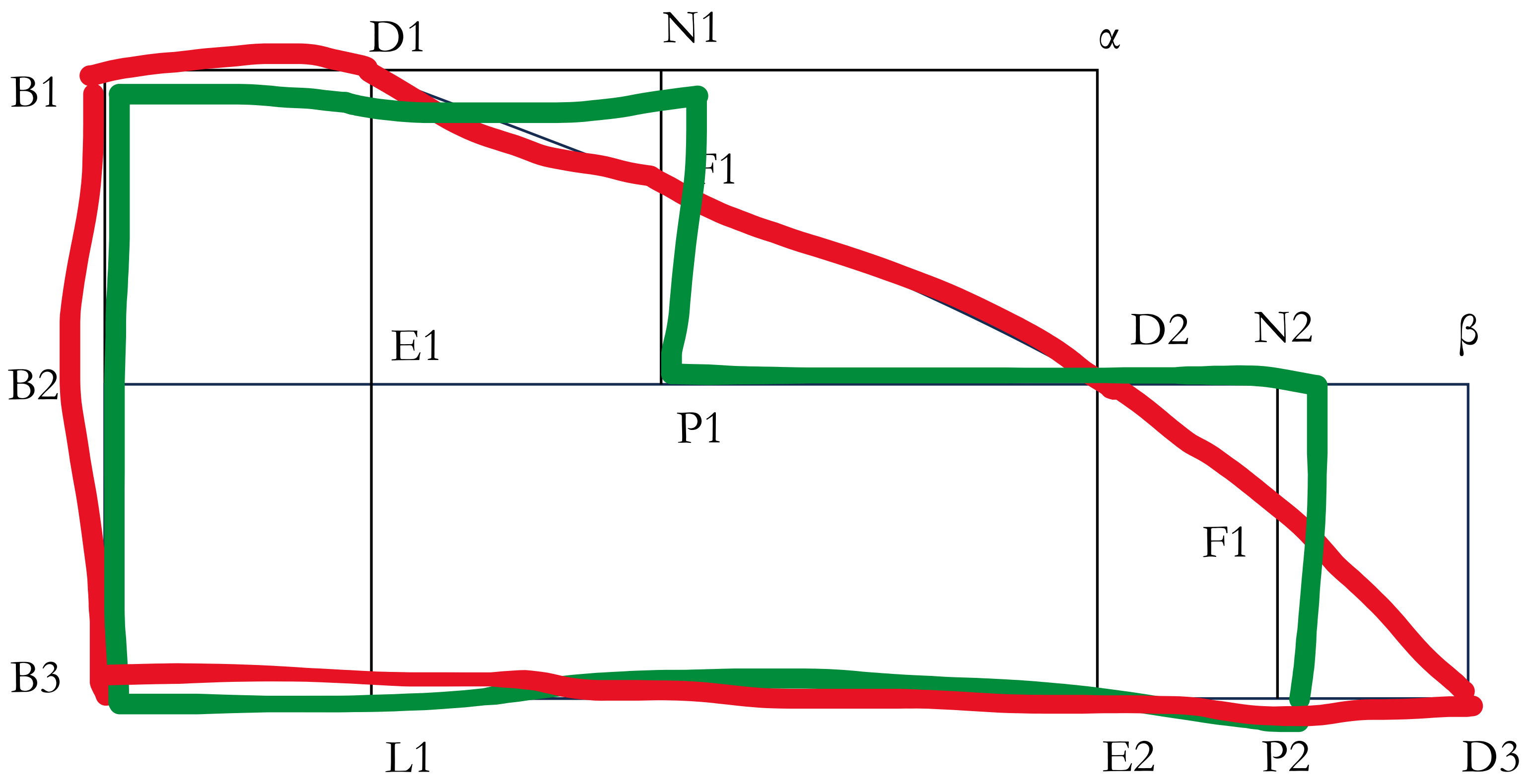
- Proposition 6:“Continue to construct quadrilaterals, staircase figures, and polygons, and rigorously prove that the difference between them and between a curve and those diagrams can be smaller than a given quantity (which is often assumed by other authors)” (A.VII,6,521)
- Leibniz thinks that this proof could be skipped for the beginner.
- ‘Reading this proposition can be omitted if one does not wish <u>a rigorous proof</u> of Proposition 7. And it would be better to ignore it at first and read it after one has understood the whole, lest <u>its rigor exhaust the immature mind and keep it away from the other attractive parts</u>. ’(A VII, 6, 527)
- Leibniz seems to think that the proof of proposition 6 is less attractive?

Reorganize and analyze proof of proposition 6

- Proof of proposition 6 consists of 8 steps. But now, we would reorganize this proof as 4 steps:

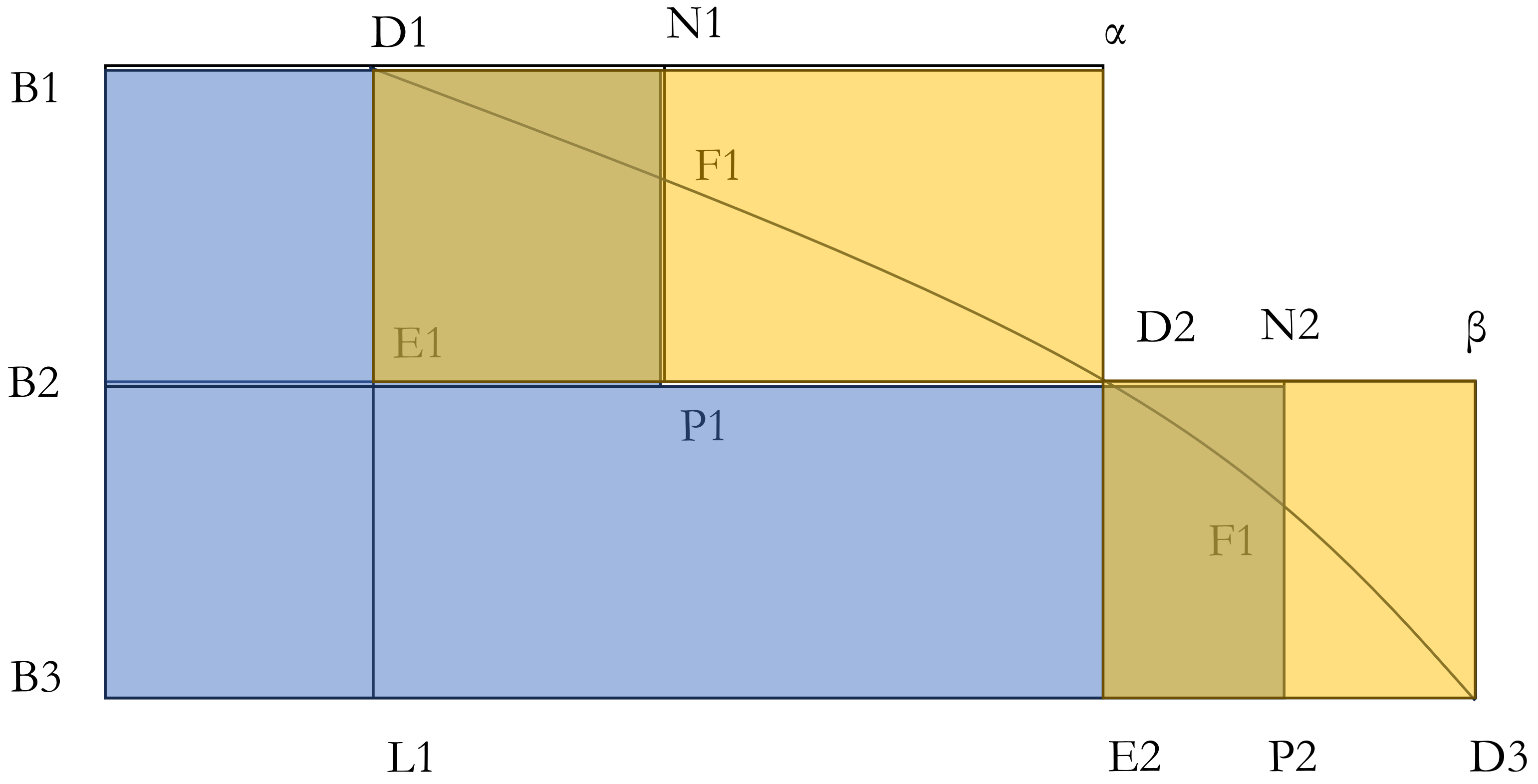
- 1st step: Drawing (1)
- 2nd step: Diagrammatic reasoning (2-4)
- 3rd step: Generalization (5-6)
- 4th step: Drawing (in imagination) (7-8)

1st step: Drawing



- Proposition 6 demonstrates that the difference between B1D1D2D3B3B1 and B1N1P1N2P2B3B1 can be smaller than any given quantity.

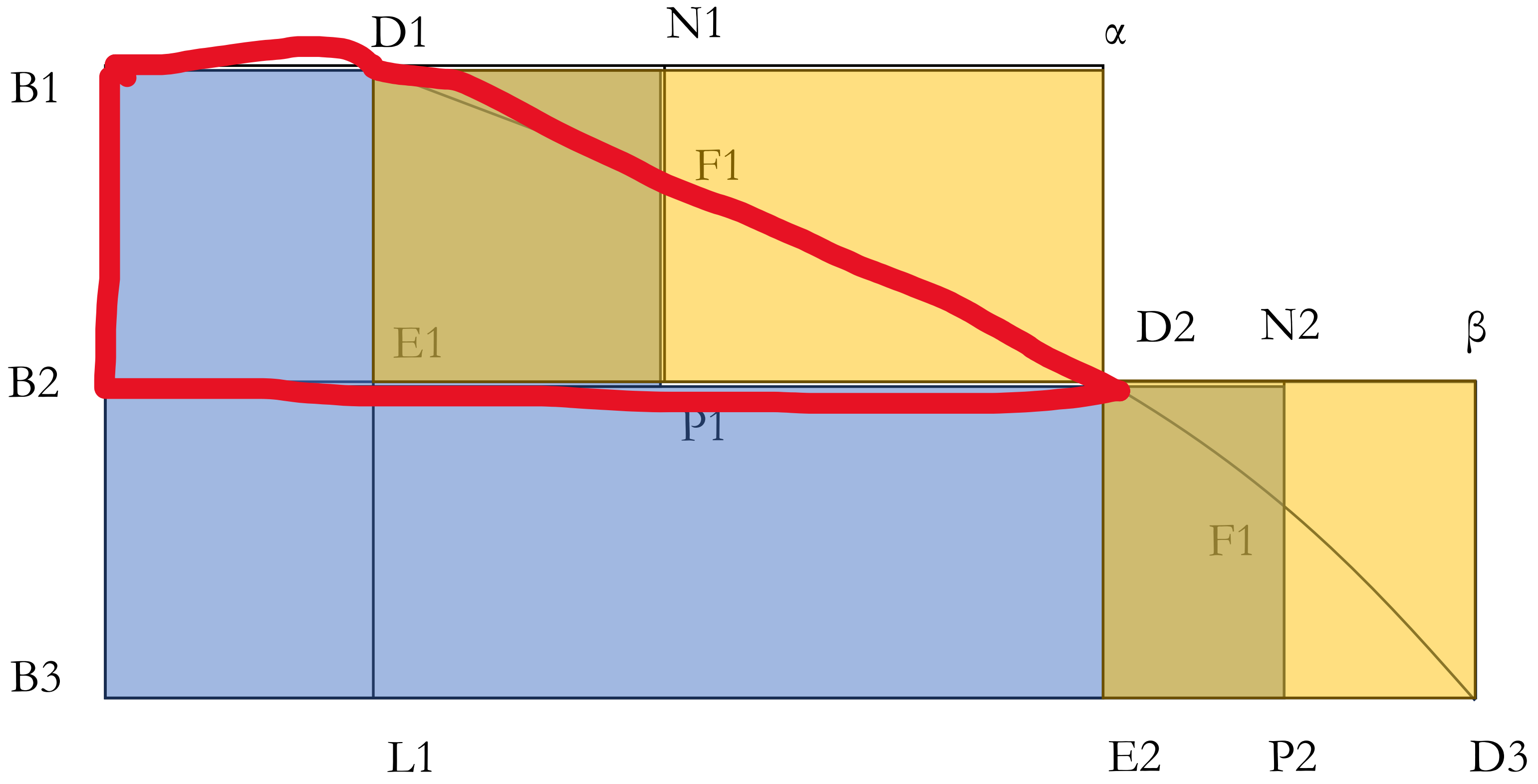
1st step: Drawing



elementary rectangle

complementary rectangle

2nd step: Diagrammatic reasoning

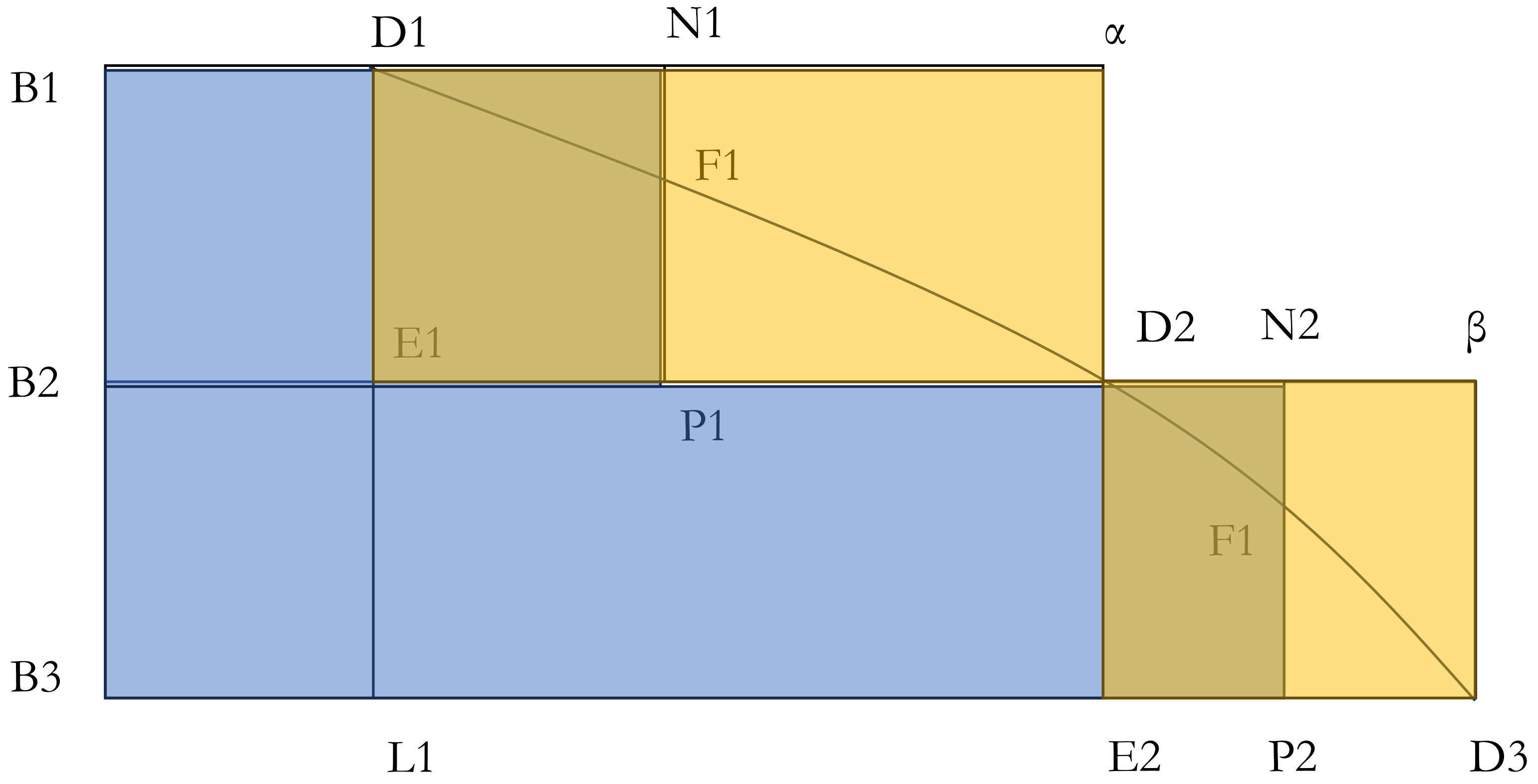


B1B2D2D1 - B1B2P1N1

⇔F1P1D2 – D1F1N1

|B1B2D2D1 - B1B2P1N1| < |D1αD2E1|

2nd step: Diagrammatic reasoning

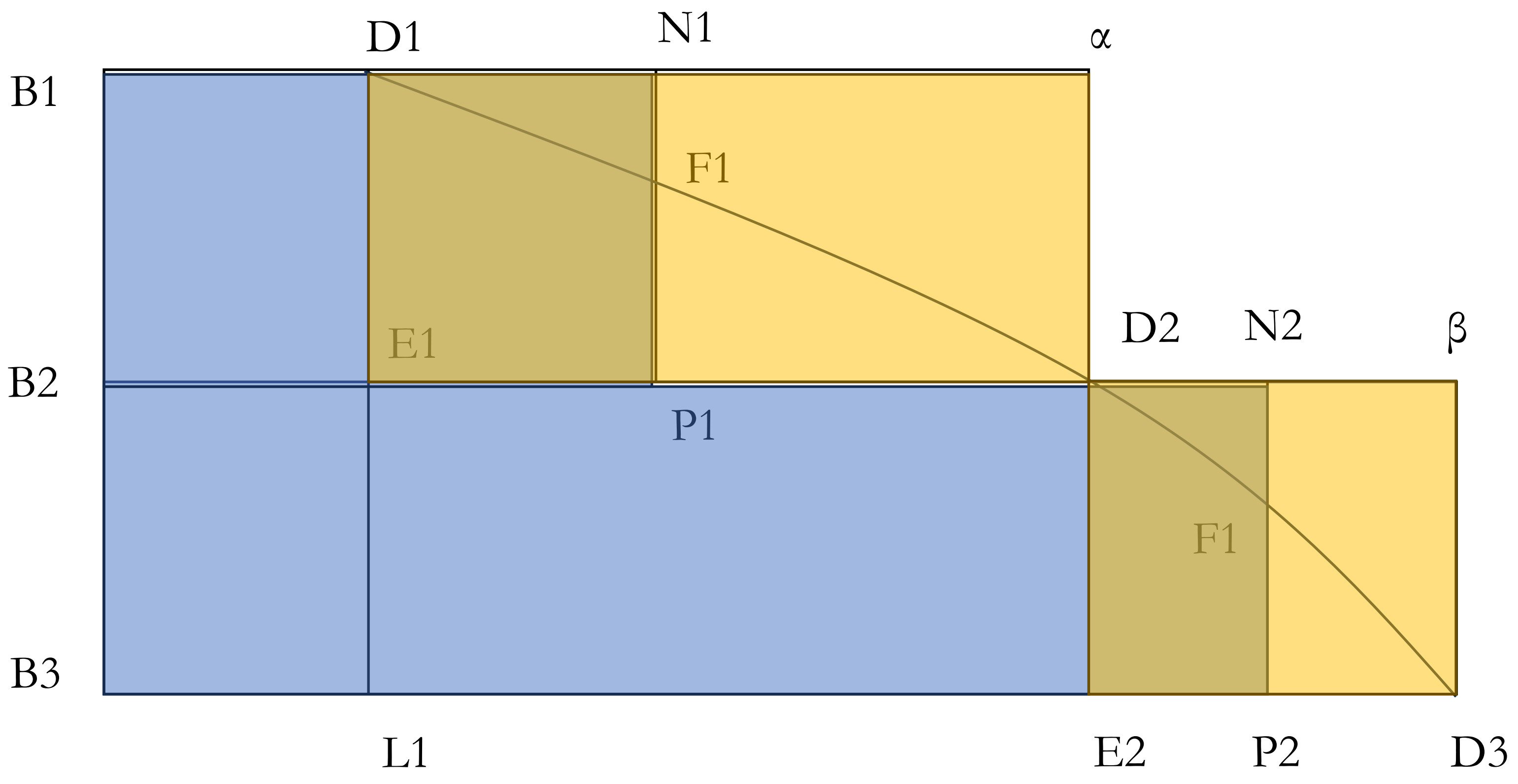


|B1B2D2D1 - B1B2P1N1| < | D1αD2E1|

2^{nd} step: Diagrammatic reasoning

- |B1B2D2D1 - B1B2P1N1| < | D1αD2E1|
- This relation is demonstrated by relying on co-exactness properties.

3rd step: generalization



ER:elementary rectangle

CR:complementary rectangle

|B1B2D2D1 - B1B2P1N1| < | D1αD2E1|

⇕

|ER1 – W1| < | CR1|

- $|W1 + W2| - |ER1 + ER2| < |W1 + W2 - (ER1 + ER2)| < |W1 - ER1| + |W2 - ER2|$
- Leibniz relies on $|A| - |B| < |A - B|$, which is equivalent to the result of proposition 3.
- This generalization can be restated as follows:

1: Case B1 and C1 is demonstrated.

2: The same procedure which conducts the result for case B1 and C1 can be utilized for case B2 and C2.

3: This transit of procedure can be repeated infinitely to conclude the intended result.

- $|W1 + W2| - |ER1 + ER2| < |W1 + W2 - (ER1 + ER2)| < |W1 - ER1| + |W2 - ER2|$
- Leibniz relies on $|A| - |B| < |A - B|$, which is equivalent to the result of …on 3.
- …eralization can be restated as follows:

via diagram

1: Case B1 and C1 is demonstrated.

via diagram and parameter

…rocedure which conducts the result for case B1 and …1 can be utilized for case B2 and C2.

3: This transit of procedure can be repeated infinitely to conclude the intended result.

4^{th} step: drawing (in imagination)

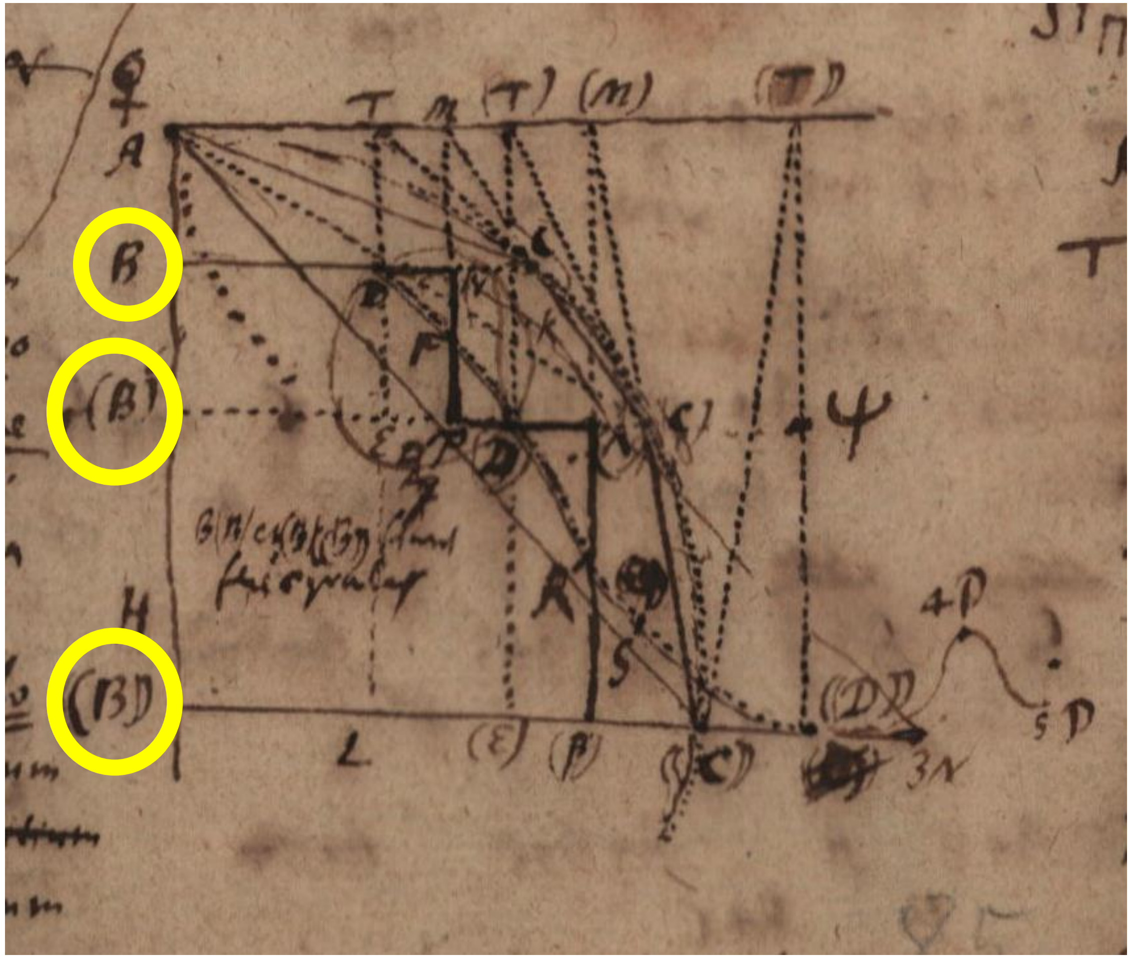
- The sum of the areas of CRn is evaluated.
- The heights BiBj of each CR are taken not to be equal, but if the heights of all CR are equal, then the sum of all the base EiD_{i+1} multiplied by the height is equal to CR.
- If the heights are different, then the greatest height multiplied by the sum of the base is equal to or greater than CR.
- So, if the value obtained by multiplying the maximum height by the sum of the base is H, CR = CR1 + CR2 + CR3 … < H is obtained.

4th step: drawing (in imagination)

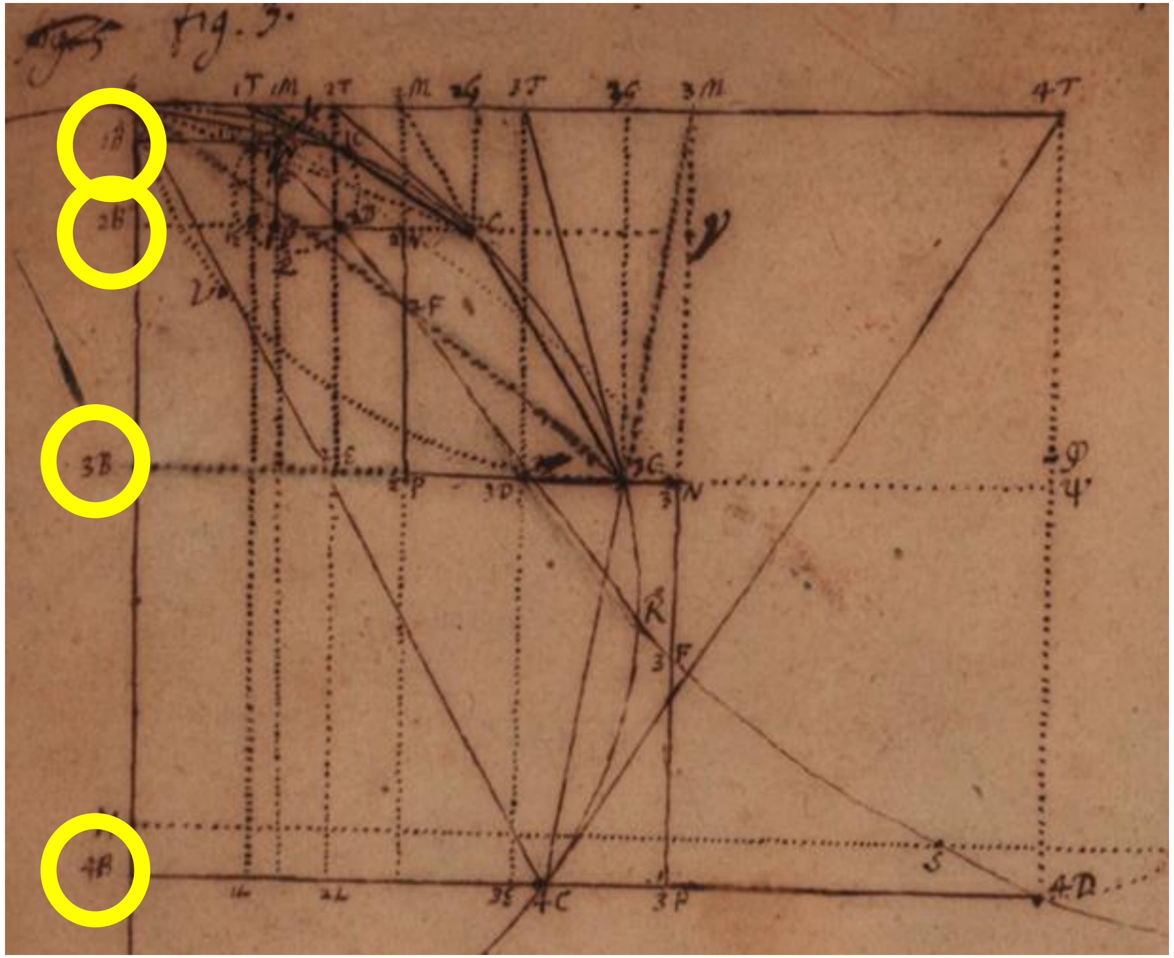
- From above, we can claim that |W1 +W2| - |ER1 + ER2| < |W1 +W2 - (ER1 + ER2)| < |W1 - ER1| + |W2 - ER2|+ …< CR1 + CR2 +…<H .
- H is the sum of the base multiplied by the maximum height of CR, but by making the curve segment finer, the 'maximum height' can also be made smaller.
- Therefore, the value of H can also be made smaller. This results that the difference between the sum of the ER and the area of linear and curve can be reduced to any extent.
- By continuing the division of the curve indefinitely, the difference between the sum of ER and the area of W can be reduced to zero.
- In other words, it can be concluded that the sum of ER and the area of W are equal.

- What makes this generalization possible?

Which is more productive?



B, (B), ((B)), (((B))),…..



B1, B2, B3, B4….

- ‘Semper ergo summa horum rectangulorum adeoque et differentia spatii gradiformis et mixtilinei novi,quae ipsa minor ostensa est, erit quantitas aliqua quantumlibet parva, adeoque qualibet assignata minor reddi potest. Quod ostendendum erat./Si curva aliqua transire intelligeretur, per omnia puncta N (N) ((N)) adhuc facilius id demonstrabitur, quia et summa exiguorum Triangulorum ut NP(N) etiam quavis data quantitate minor eodem modo reddi potest.
- [If it were understood that a certain curve should pass through all the points N (N) ((N)) it would still be more easily demonstrated, because the sum of small triangles such as NP(N) can also be rendered smaller by any given quantity in the same way]’ (A VII 6 140)

- This draft says that it it the role of the text that guarantees that recursive procedure is possible.
- In DQA, this guarantee made by diagrams and parameter.

5: Visualization of proof procedure

- We claim that diagram in proof 6 has a function that visualizing a recursive procedure.
- Leibniz utilizes not only diagrams, but also index of symbols to determinate the area bounded by curve by finite procedure.
- More specifically, firstly the proof is made based on diagrammatic reasoning, then goes to generalization by letting the reader draw imaginatively the same procedure.

- It is visually shown that if the same operation is continued, the difference will be zero.
- “The earlier objects used in the method are interpreted as special cases of new objects”. (Knobloch 2016 p.103)
- Notation of B1, B2, B3, B4…., not B, (B), ((B)), (((B))),….. could be thought as an assist to such generalization.

With or without loss of generality?

- It is true that Leibniz utilized diagrams effectively in mathematical proof, however, the question of whether this really worked remains.
- For example: the problem of generality. Relying diagrams in a proof may lose generality.
- As for the proof of proposition 6, Leibniz keep generality of procedure, that is, he made possible to transit from an earlier step to a newer step based on diagrams and parameter.
- Of course, this method itself does not always work. In the letter to Bodenhausen in 5. Nov. 1690 , Leibniz confesses that his method in DQA is overcome by his new differencial calculus. (A III 4 637, cf. Knobloch 2016)

6: Conclusion

- We analyze the proof of proposition 6 in DQA from viewpoint of philosophy of mathematical practice.
- Leibniz uses diagram and parameter to prove that the procedure which used for a particular case can be repeated infinitely to determinate the area.
- Some change of symbolic notation from the draft of DQA to DQA shows that we could see some development of ideas of how diagrams are used in Leibniz's mathematics.
- By conducting such analysis more broadly, we could reveal more detailed of the mathematical practice of Leibniz.

Reference

- Viktor Blåsjö, "On what has been called Leibniz's rigorous foundation of infinitesimal geometry by means of Riemannian sums", *Historia Mathematica* 44 (2017) 134–149
- Davide Crippa, *The Impossibility of Squaring the Circle in the 17th Century:A Debate Among Gregory, Huygens and Leibniz*, Birkhauser, 2019.
- Emily Grosholz, *Representation and Productive Ambiguity in Mathematics and the Sciences*, Oxford University Press, 2007.
- Eberhard Knobloch, "Generality in Leibniz's Mathematics", *The Oxford Handbook of Generality in Mathematics and the Sciences*, Karine Chemla, Renaud Chorlay, David Rabouin eds., Oxford University Press, 2016, pp.90-109.
- Paolo Mancosu, *The Philosophy of Mathematical Practice*, Oxford University Press, 2008.
- Kenneth Manders, "Diagram-Based Geometric Practice", Paolo Mancosu, ed, *The Philosophy of Mathematical Practice*, Oxford University Press, 2008, pp.65-79.
- Kenneth Manders, "The Euclidean diagram(1995)", Paolo Mancosu, ed, *The Philosophy of Mathematical Practice*, Oxford University Press, 2008, pp.80-133.

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