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The Fixed Point Assignment Problem in Neural Networks and Its Application to Associative Memory*

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Abstract - A problem of assigning a prescribed set of vectors to asymptotically stable fixed points of a system arises from constructing associative memory using a neural network. This paper deals with this problem and discusses a method for constructing a neural network which satisfies the properties that not only a prescribed set of vectors is assigned to its fixed points but also each fixed point achieves a maximum convergence margin to improve the capability as associative memory. Finally to illustrate the result a simple numerical example is worked out.

Keywords: neural network, associative memory, stability, convergence margin

I. INTRODUCTION

In associative memory implemented by a neural network, information is memorized as a locally asymptotically stable fixed point of the network, and the content of information is recalled by only giving an incomplete content or a portion of the memorized information which is taken as an initial state of the network, so that its state converges asymptotically to the fixed point to recall the desired correct information. On the other hand, for memory in digital computers, information is stored in a memory device with a pre-assigned address, and the content of information is recalled by giving the exact address. Thus the associative memory much more resembles human's brain memory than the one used in digital computers.

A fundamental problem for implementing associative memory by a neural network is how to assign a prescribed set of points to locally asymptotically stable fixed points of the network, and this "fixed point assignment problem" has been extensively studied and a variety of methods have been proposed [2]-[9]. Among them, the so-called "orthogonal projection method" is a reasonably powerful method widely used to construct such a network for associative memory [2] - [5].

However, there is yet another important problem to be investigated, that is, a problem of how to enlarge and /or adjust the domain of attraction of asymptotically stable fixed points of such a neural network because the capability of associative memory for recalling correct information is dependent on the domains of attraction [7]-[9]. For this problem Shoji and Inaba [8] proposed a very powerful method which ensures a maximum domain of attraction for each fixed point. More precisely, the method enables us to construct a neural network in such a way that not only each prescribed point is assigned to its locally asymptotically stable fixed point by ensuring a maximum domain of attraction but also on the outside of this domain its behavior is exactly the same as that of the neural network constructed by the orthogonal projection method.

This paper first reexamines by means of some systems and control theoretical techniques the method proposed in [8] which assigns a prescribed set of points to asymptotically stable fixed points of a neural network so as to maximize the convergence margins. In fact, we introduce a state feedback structure into a neural network in such a way that the fixed points in the original network are unchanged but only their convergence margins are maximized. Further some numerical example is presented to illustrate the theoretical result obtained.

II. PRELIMINARIES AND THE ORTHOGONAL PROJECTION METHOD

First, let $\mathbb{B} := \{-1, 1\}$ and consider a neural network of the *McCulloch-Pitts Model* [1] defined over the state space $\mathbb{B}^n$ as

$$\text{MP:} \quad \begin{cases} x(k+1) = \text{Sgn}\{Wx(k) - h\}, \\ x(0) = x_0 \in \mathbb{B}^n \end{cases} \tag{1}$$

where $x(k) \in \mathbb{B}^n$ is the state, $W \in \mathbb{R}^{n\times n}$ the connection matrix, $h \in \mathbb{R}^n$ the threshold vector and $\text{Sgn}(\cdot)$ designates the vector-valued sign function, i.e.,

$$\text{Sgn}(\xi) := [\text{sgn}\,\xi_1 \quad \cdots \quad \text{sgn}\,\xi_n]^T$$

where $\xi = [\xi_1 \quad \cdots \quad \xi_n]^T \in \mathbb{R}^n$ and

$$\text{sgn}\,\eta := \begin{cases} 1, & \eta > 0 \\ -1, & \eta \le 0. \end{cases}$$

Further denote the solution of MP by $x_{MP}(k; x_0)$. Then it is clear that all the properties of solution $x_{MP}(k; x_0)$, or equivalently, of dynamical neural network MP are determined by the *parameter set* (W, h), and hence any design problem of such a network can be described as a problem of choosing an appro-

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priate parameter set (W,h).

Now, we introduce basic notations used in the sequel. First, consider a general dynamical system over $\mathbb{B}^n$ described in the following form:

$$x(k+1) = f(x(k)), \quad x(0) = x_0 \in \mathbb{B}^n, \tag{2}$$

and denote the solution of (2) by $x(k;x_0)$ or simply $x(k)$. Then, a vector $\xi \in \mathbb{B}^n$ is said to be a *fixed point* or an *equilibrium solution* of (2) if $f(\xi) = \xi$ or equivalently $x(k;\xi) = \xi$ for all $k \geq 0$. The *domain of attraction* of a fixed point $\xi \in \mathbb{B}^n$ is defined as

$$\mathcal{D}(\xi) := \{x_0 \in \mathbb{B}^n \mid \exists k \geq 0 \text{ such that } x(k;x_0) = \xi\}. \tag{3}$$

Let the Hamming distance between ξ and ζ in $\mathbb{B}^n$ be denoted by $d_H(\zeta,\xi)$, and the δ-ball centered at $\xi \in \mathbb{B}^n$ by $\mathcal{N}_\delta(\xi)$, i.e.,

$$\mathcal{N}_\delta(\xi) := \{\zeta \in \mathbb{B}^n \mid d_H(\zeta,\xi) \leq \delta\}, \ \delta \geq 0.$$

Then, a fixed point $\xi \in \mathbb{B}^n$ is said to be *locally asymptotically stable* (or simply *stable*) if $\mathcal{D}(\xi) \supset \mathcal{N}_1(\xi)$. Finally, for a fixed point $\xi \in \mathbb{B}^n$, define

$$r(\xi) := \max\{\delta \geq 0 \mid \mathcal{N}_\delta(\xi) \subset \mathcal{D}(\xi)\}, \tag{4}$$

which will be used as a measure of *convergence margin* of the fixed point.

First we cite the following theorem [2], [5], [6], which provides a method for assigning a given set of vectors to its fixed points of a neural network.

THEOREM 1 (The Orthogonal Projection Method). Let $r < n$ and $\mathcal{P} := \{\xi^{(1)}, \cdots, \xi^{(r)}\} \subset \mathbb{B}^n$ be a set of distinct vectors, called *a set of prototype vectors* or simply *a prototype set*, and construct a dynamical neural network of MP with the parameter set (W,h) given by

$$\begin{cases} W := \Xi\Xi^\dagger \in \mathbb{R}^{n\times n}, \\ \Xi := [\xi^{(1)} \ \cdots \ \xi^{(r)}] \in \mathbb{B}^{n\times r} \\ h := [h_1 \ \cdots \ h_n]^T, \quad |h_k| \leq 1 \end{cases} \tag{5}$$

where $\Xi^\dagger \in \mathbb{B}^{r\times n}$ denotes the Moore-Penrose generalized inverse.

Then the following statements hold:

(i) Each $\xi^{(i)} \in \mathcal{P}$ is a fixed point of MP.

(ii) MP has no limit cycles. □

The name *Orthogonal Projection Method* comes from the fact that the matrix $W := \Xi\Xi^\dagger$ represents the orthogonal projection operator from $\mathbb{R}^n$ onto the subspace spanned by the column vectors in Ξ.

III. MODIFIED McCulloch-Pitts MODEL

Theorem 1 (the Orthogonal Projection Method) provides a reasonably powerful tool for constructing a parameter set (W,h) for a neural network such that a prototype set $\mathcal{P} := \{\xi^{(1)}, \cdots, \xi^{(r)}\} \subset \mathbb{B}^n$ is assigned to its fixed points. However it has been pointed out [8], [9] that a neural network constructed by this method may result in a very small convergence margin. To avoid this situation, Shoji and Inaba [8] studied a modified McCulloch-Pitts model and showed that this model provides not only the maximum convergence margin for each fixed point but also the exactly same behavior as that of MP model on the outside of the convergence margin. In this section, the modified model is reexamined from the viewpoint of systems and control theory and its basic properties are briefly discussed.

First, we consider the MP model of (1) with an arbitrary parameter set (W,h), and introduce to this system an input vector $u(k) \in \mathbb{R}^n$ and a state feedback to define the *Modified McCulloch-Pitts Model* as follows:

$$\text{MMP:} \quad \begin{cases} x(k+1) = \mathrm{Sgn}\{Wx(k) + u(k) - h\}, \\ u(k) = F\mathrm{Sgn}\{Vx(k) - g\} + \theta \\ x(0) = x_0 \in \mathbb{B}^n \end{cases} \tag{6}$$

where $F \in \mathbb{R}^{n\times m}$ with $m \leq n$, $V \in \mathbb{R}^{m\times n}$, $g \in \mathbb{R}^m$ and $\theta \in \mathbb{R}^n$. (m,F,V,g,θ) forms a *feedback parameter set* to be chosen so as to improve the convergence margin without changing its fixed points. Let the solution of MMP be denoted by $x_{MMP}(k;x_0)$ or simply $x_{MMP}(k)$ and consider a prototype set $\mathcal{P} := \{\xi^{(1)}, \cdots, \xi^{(r)}\} \subset \mathbb{B}^n$, which represent all the information to be memorized in the associative memory. Now, for each $i = 1, \cdots, r$, denote by d_i the minimum distance from $\xi^{(i)} \in \mathcal{P}$ to the others $\xi^{(j)}$ for $j \neq i$, i.e.,

$$d_i := \min\{d_H(\xi^{(i)}, \xi^{(j)}) \mid j = 1, \cdots, r \text{ and } j \neq i\}. \tag{7}$$

Then, the following theorem can be proved.

THEOREM 2. Let $\mathcal{P} := \{\xi^{(1)}, \cdots, \xi^{(r)}\} \subset \mathbb{B}^n$ be a prototype set of distinct vectors, and define

$$\Xi := [\xi^{(1)} \ \cdots \ \xi^{(r)}] \in \mathbb{B}^{n\times r}.$$

Consider the MP model (1) with an arbitrary parameter set (W,h) and the MMP model (6) with the feedback parameter set given by

$$\begin{cases} m := r \\ F := a\Xi, \quad V := \Xi^T \\ g := [g_1 \ \cdots \ g_r]^T, \quad g_i := n - d_i \\ \theta := a\sum_{j=1}^r \xi^{(j)} = a\Xi[1 \ \cdots \ 1]^T \end{cases} \tag{8}$$

where $W = (w_{ij})$, $h = (h_i)$ and $a > 0$ is any constant satisfying

$$a > \frac{1}{2}\max_{1\leq i\leq n}\left\{\sum_{j=1}^n |w_{ij}| + h_i\right\}. \tag{9}$$

Then, letting $x_0 \in \mathbb{B}^n$ be an initial state and $k \geq 0$ be an integer, the trajectories $x_{MP}(k)$ of MP model and $x_{MMP}(k)$ of MMP model satisfy the following properties:

(i) $x_{MMP}(k) \notin \bigcup_{j=1}^r \mathcal{N}_{(d_i-1)/2}(\xi^{(j)})$

$$\Rightarrow x_{MMP}(k+1) = x_{MP}(k+1)$$

(ii) $x_{MPF}(k) \in \mathcal{N}_{(d_i-1)/2}(\xi^{(i)})$ for some i

$$\Rightarrow x_{MMP}(k+1) = \xi^{(i)}$$

(iii) Every $\xi^{(i)} \in \mathcal{P}$ is a fixed point for MMP model.

PROOF: Only a sketch of proof is given here.

To prove (i), first notice from (6) and (8) that the closed-loop system is explicitly described as

$$x_{MMP}(k+1) = \mathrm{Sgn}[Wx_{MMP}(k) - h + a\{\Xi\,\mathrm{Sgn}(\Xi^T x_{MMP}(k) - g) + \sum_{j=1}^{r} \xi^{(j)}\}] \tag{10}$$

Therefore it suffices to show that if $y \notin \bigcup_{j=1}^{r} \mathcal{N}_{(d_i-1)/2}(\xi^{(j)})$ is arbitrary then

$$\Xi\,\mathrm{Sgn}(\Xi^T y - g) + \sum_{j=1}^{r} \xi^{(j)} = 0. \tag{11}$$

This is shown as follows. First, using the relation $x^T y = n - 2d_H(x,y)$ for any $x, y \in \mathbb{B}^n$, one obtains

$$\Xi^T y - g = \begin{bmatrix} \xi^{(1)T} y - n + d_1 \\ \vdots \\ \xi^{(r)T} y - n + d_r \end{bmatrix} = \begin{bmatrix} d_1 - 2d_H(\xi^{(1)}, y) \\ \vdots \\ d_r - 2d_H(\xi^{(r)}, y) \end{bmatrix}. \tag{12}$$

Then,

$$y \notin \bigcup_{j=1}^{r} \mathcal{N}_{(d_i-1)/2}(\xi^{(j)}) \Rightarrow d_j - 2d_H(\xi^{(j)}, y) \le 0, \quad \forall j$$

and hence (12) gives

$$\mathrm{Sgn}(\Xi^T y - g) = [-1 \ \cdots \ -1]^T,$$

which easily leads to the desired result (11).

To prove (ii), assume that

$$x_{MMP}(k) \in \mathcal{N}_{(d_i-1)/2}(\xi^{(i)})$$

for some $k \ge 0$ and some $\xi^{(i)}$, and hence that

$$d_H(\xi^{(i)}, x_{MMP}(k)) \le (d_i - 1)/2. \tag{13}$$

Then, using the relation (12), one obtains

$$z := \mathrm{Sgn}(\Xi^T x_{MMP}(k) - g) = \mathrm{Sgn}\begin{bmatrix} d_1 - 2d_H(\xi^{(1)}, x_{MMP}(k)) \\ \vdots \\ d_r - 2d_H(\xi^{(r)}, x_{MMP}(k)) \end{bmatrix}. \tag{14}$$

Further one obtains the inequality

$$d_j - 2d_H(\xi^{(j)}, x_{MMP}(k)) < 0, \quad \forall j \ne i. \tag{15}$$

Thus, using (15) and (13), (14), (10) can be reduced to

$$\begin{aligned} x_{MMP}(k+1) &= \mathrm{Sgn}[Wx_{MMP}(k) - h + \{a\Xi z + a\Xi[1 \cdots 1]^T\}] \\ &= \mathrm{Sgn}\{Wx_{MMP}(k) - h + 2a\xi^{(i)}\}. \end{aligned} \tag{16}$$

Since a is chosen to satisfy (9), it is not difficult to see that (16) gives the desired conclusion

$$x_{MMP}(k+1) = \xi^{(i)}.$$

Finally the statement (iii) is obvious form (ii). □

COROLLARY 3. Let all the notations be the same as those in Theorem 2. Then, the following statements are satisfied:

(i) For every $\xi^{(i)} \in \mathcal{P}$,

$$\mathcal{D}(\xi^{(i)}) = \{\xi \in \mathbb{B}^n \mid \exists k \ge 0 \text{ such that } x_{MP}(k;\xi) \in \mathcal{N}_{(d_i-1)/2}(\xi^{(i)}) \text{ and } x_{MP}(l;\xi) \notin \bigcup_{j=1}^{r} \mathcal{N}_{(d_j-1)/2}(\xi^{(j)}),\ \forall\, l < k\}.$$

(ii) For every $\xi^{(i)} \in \mathcal{P}$,

$$\mathcal{N}_{(d_i-1)/2}(\xi^{(i)}) \subset \mathcal{D}(\xi^{(i)}).$$

(iii) If $d_i \ge 3$, then $\xi^{(i)}$ is a stable fixed point.

(iv) If the MP model has no limit cycles, then the MMP model has also no limit cycles. □

Finally the following theorem is easily verified using Theorem 2 and Corollary 3.

THEOREM 4. Let all the notations be the same as those in Theorem 2 except an arbitrary parameter set (W, h) is replaced with the parameter set constructed as in Theorem 1 (the Orthogonal Projection Method).

Then, if $d_i \ge 3$ for all $i = 1, \cdots r$,

(i) each assigned fixed point $\xi^{(i)}$ is locally asymptotically stable

(ii) there is no other fixed point in $\mathcal{N}_{(d_i-1)/2}(\xi^{(i)})$ and its convergence margin is given as $r(\xi^{(i)}) = (d_i - 1)/2$, that is, each fixed point $\xi^{(i)}$ has the maximum convergence margin. □

Figure 1 below explains the result of Theorem 4 together with Theorem 2, that is, on the outside of the convergence margin $(d_i - 1)/2 \ge 1$ the two trajectories $x_{MMP}(k)$ and $x_{MP}(k; x_0)$ are exactly the same until they reach the convergence margin and then at the next moment $x_{MMP}(k)$ immediately moves to the fixed point $\xi^{(i)}$ but $x_{MP}(k)$ may travel more and eventually reach a fictitious fixed point ξ^* created by MP model.

EXAMPLE. Figure 2 depicts a simple numerical example of associative memories for English alphabets $A, B, \cdots, Z$ and Blank. Each letter is divided into 10x10 pixels as in Figure 2 and each pixel is represented by 1 or -1 according to black or white, forming prototype vectors $\xi^{(A)}, \xi^{(B)}, \cdots, \xi^{(Z)}, \xi^{(\square)}$ in $\mathbb{B}^{100}$. The parameter set (W, h) of the MP model is constructed by the Orthogonal Projection Method given as in Theorem 1 with $h = 0$, and the feedback parameter set (m, F, V, g, θ) of the MMP model is computed according to Theorem 2. In fact, for these prototype vectors, all the convergence margins satisfy $d_A, d_B, \cdots, d_Z, d_\square \ge 4$, and hence all the prototype vectors $\xi^{(A)}, \xi^{(B)}, \cdots, \xi^{(Z)}, \xi^{(\square)}$ are assigned to fixed points of the MP model and to asymptotically stable fixed points of the MMP model and further all the properties stated in Corollary 3 and Theorem 4 are satisfied.

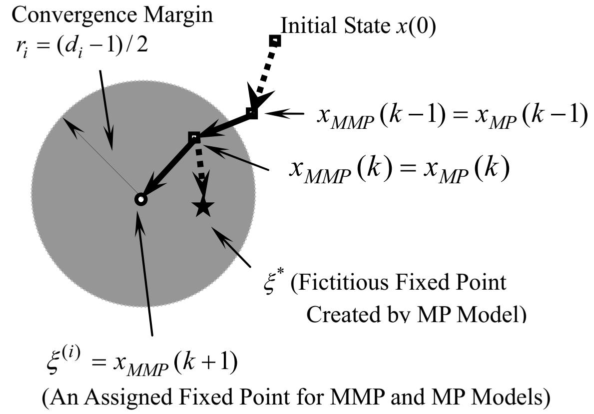

Figure 1. The Trajectories of MMP and MP Models

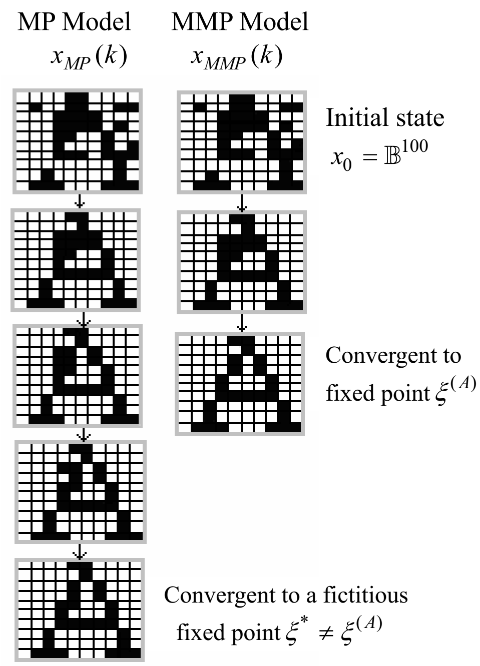

Figure 2. Trajectories of MP and MMP Models

In fact, it is seen from Figure 2 that starting both the MP model and the MMP model from the same initial state $x_0 \in \mathbb{B}^{100}$ obtained from $\xi^{(A)}$ by adding noises, $x_{MMP}(k)$ converges to the prototype vector $\xi^{(A)}$, but surprising enough $x_{MP}(k; x_0)$ converges to a very nearby point, i.e., a fictitious fixed point ξ^*, which differs only at one pixel, that is, $d_H(\xi^{(A)}, \xi^*) = 1$. That is to say that the Orthogonal Projection Method ensures to assign all the prototype vectors to fixed points but simultaneously may produce a fictitious fixed point just next to a prototype vector.

IV. CONCLUDING REMARKS

Implementing an associative memory by a neural network, it is important to improve the convergence margin of each assigned stable fixed point, which corresponds to the capability of associative memory. This paper studied a method to obtain the maximum convergence margin for each fixed point without affecting the assigned stable fixed points. To illustrate the result a simple numerical is presented.

A more sophisticated associated memory has been considered by introducing limit cycles to memorize information [6]. For this type of associative memories, it would also be interesting to study a similar convergence margin as a future problem. Finally it should be mentioned that the result obtained seems to have various control applications, in particular to intelligent control area, and possible applications to Human Adaptive Mechatronics [10] have been considered.

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