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Global Stabilization of a Class of Uncertain Nonlinear Systems Using Output Feedback*

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Abstract— **This paper studies the global stabilization by output feedback for uncertain nonlinear systems whose dynamic may not exactly known but satisfies some relaxed triangular-type conditions. Using a feedback domination design method, we explicitly construct a dynamic output compensator which globally stabilizes such a uncertain nonlinear system. The usefulness of our result is illustrated as an example.**

I. Introduction

The problem of controlling nonlinear systems by output feedback is one of most important problems in the field of nonlinear control. Unlike in the case of linear systems, the separation principle generally does not hold for nonlinear systems [7]. Due to this reason, the problem is more difficult and challenging. In recent years, many important results on the problem have been obtained. However, as investigated in [7], some extra growth conditions on the immeasurable states of the system are usually necessary for the global stabilization of nonlinear systems via output feedback. Since then, a great deal of subsequent research work has focused on the output feedback stabilization of nonlinear systems under various structural or growth conditions. For example, it is assumed that nonlinear terms of a given system satisfy triangular conditions in [2], [8] or some global Lipschitz-like condition in [1], etc.

In this paper, we consider essentially the same class of nonlinear systems as treated in [1,2], [5,6,8]. By far, it seems that one of most relaxed conditions imposed on the nonlinear terms of a given system is a triangular-type condition as far as the output feedback control is concerned as shown in [2]-[5,6]. Most recently, introducing a new way of understanding observers, a backstepping-like design procedure for observers was introduced in [2], [8], in which the global stabilization is achieved by a linear output feedback controller under the triangular condition.

The main purpose of this paper is to develop a global stabilizer for a class of uncertain nonlinear systems by linear output feedback under a furthermore relaxed condition on the nonlinear terms of a given system than a triangular-type condition.. In fact, we consider the following class of uncertain nonlinear systems:

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \delta_1(t,x,u)\\
\dot{x}_2 &= x_3 + \delta_2(t,x,u)\\
&\vdots\\
\dot{x}_n &= u \ + \delta_n(t,x,u)\\
y &= x_1
\end{aligned}
\tag{1}
$$

where $x=[x_1,x_2,\cdots,x_n]^T\in \boldsymbol{R}^n$ is the state, $u\in\boldsymbol{R}$ and $y\in\boldsymbol{R}$ are the input and the output of the system, respetively. A feature of this paper is that our design method of global stabilizing controllers does not require a detailed structure of the nonlinear terms $\delta_i:\boldsymbol{R}\times\boldsymbol{R}^n\times\boldsymbol{R}\to\boldsymbol{R}$ for $i=1,\ \cdots,n$, including a triangular-type condition (see (3) below), except that they are Lipschitz continuous and satisfy the following condition.

Assumption 1. For System (1), there exist some constants $c>0$ and $0<\alpha\le 1$ such that for any $s\in(0,\alpha)$ the inequality

$$
\sum_{i=1}^{n} s^{i-1}\left|\delta_i(t,x,u)\right| \le c\sum_{i=1}^{n} s^{i-1}\left|x_i\right|. \tag{2}
$$

is satisfied. □

It is not difficult to see that if the triangular condition imposed on $\delta_i(t,x,u)$ as in [2], [4-8], i.e.,

$$
\left|\delta_i(t,x,u)\right| \le c\sum_{j=1}^{i}\left|x_j\right| \tag{3}
$$

is satisfied, then Assumption 1 is always satisfied, but not vice versa. In fact, suppose that condition (3) is satisfied. Then, for any $s\in(0,\alpha)$

$$
\begin{aligned}
&\sum_{i=1}^{n} s^{i-1}\left|\delta_i\right|\\
&\quad\le c\left|x_1\right| + cs\left(\left|x_1\right|+\left|x_2\right|\right)+\cdots+cs^{n-1}\left(\left|x_1\right|+\cdots+\left|x_n\right|\right)\\
&\quad\le c\left(1+s+\cdots+s^{n-1}\right)\left|x_1\right|\\
&\qquad\quad + cs\left(1+s+\cdots+s^{n-2}\right)\left|x_2\right|+\cdots+cs^{n-1}\left|x_n\right|\\
&\quad\le c\left(\sum_{i=1}^{n} s^{i-1}\right)\sum_{i=1}^{n} s^{i-1}\left|x_i\right|
\end{aligned}
$$

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and hence Assumption 1 is satisfied, but it is clear that the converse may not always hold true.

II. Global Stabilization by Output Feedback

In this section, we prove that there exists a dynamic output compensator of the form

$$\dot{\xi} = f(\xi, y), \quad u = h(\xi, y) . \tag{4}$$

such that the closed-loop system (1) with the dynamic output compensator (4) satisfies

$$\lim_{t\to\infty}(x(t), \xi(t)) = (0,0)$$

That is to say that system (1) is stabilized by the dynamic output compensator (4). The dynamic output compensator we propose is made of a linear high gain observer and a linear high gain controller as follows.

Theorem1. *Under Assumption1, there is a dynamic output compensator of the form (4) that solves the global stabilization problem for a uncertain nonlinear system of the form (1).*

Proof: We begin by introducing the following dynamic system:

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + ra_1(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_2 &= \hat{x}_3 + r^2 a_2(x_1 - \hat{x}_1) \\ &\vdots \\ \dot{\hat{x}}_n &= u + r^n a_n(x_1 - \hat{x}_1) \end{aligned} \tag{5}$$

where $r \geq 1$ is a gain parameter to be determined later, and a_i $(i = 1, \cdots, n)$ are the coefficients of any Hurwitz polynomial $\rho^n + a_1\rho^{n-1} + \cdots + a_{n-1}\rho + a_n$.

Next, treating that (5) is an observer for system (1), consider the estimation error

$$e_i = x_i - \hat{x}_i, \quad 1 \leq i \leq n \tag{6}$$

then it follows from (1) and (5) that

$$\begin{aligned} \dot{e}_1 &= e_2 - ra_1e_1 + \delta_1(t,x,u) \\ \dot{e}_2 &= e_3 - r^2 a_2 e_1 + \delta_2(t,x,u) \\ &\vdots \\ \dot{e}_n &= -r^n a_n e_1 + \delta_n(t,x,u). \end{aligned} \tag{7}$$

Further, introduce the scaled estimation error ε by

$$\varepsilon_i = \frac{1}{r^{i-1}} e_i, \quad 1 \leq i \leq n \tag{8}$$

and

$$\varepsilon = [\varepsilon_1 \quad \varepsilon_2 \quad \cdots \quad \varepsilon_n]^T \in \boldsymbol{R}^n . \tag{9}$$

Then one obtains

$$\begin{aligned} \dot{\varepsilon}_1 &= r(\varepsilon_2 - a_1\varepsilon_1) + \delta_1(t,x,u) \\ \dot{\varepsilon}_2 &= r(\varepsilon_3 - a_2\varepsilon_1) + \frac{1}{r}\delta_2(t,x,u) \\ &\vdots \\ \dot{\varepsilon}_n &= \qquad -ra_n\varepsilon_1 + \frac{1}{r^{n-1}}\delta_n(t,x,u) \end{aligned} \tag{10}$$

or equivalently

$$\dot{\varepsilon} = rA\varepsilon + \Phi_1 \tag{11}$$

where

$$\Phi_1 = \left[\delta_1(t,x,u), \frac{1}{r}\delta_2(t,x,u), \quad \ldots \quad \frac{1}{r^{n-1}}\delta_n(t,x,u)\right]^T \tag{12}$$

$$A = \begin{pmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{pmatrix} .$$

Now consider the quadratic function

$$V_1 := \varepsilon^T P \varepsilon , \tag{13}$$

where P is a positive definite symmetric matrix satisfying

$$A^T P + PA = -I . \tag{14}$$

Then it follows from (10) and (14) that the time derivative of V_1 along the solution of (11) satisfies

$$\begin{aligned} \dot{V}_1 &= r\varepsilon^T(A^T P + PA)\varepsilon + 2\varepsilon^T P\Phi_1 \\ &\leq -r\|\varepsilon\|^2 + 2\varepsilon^T P\Phi_1 \\ &\leq -r\|\varepsilon\|^2 + 2\varepsilon^T P\Phi_1. \end{aligned} \tag{15}$$

From Assumption1 and the fact that $r \geq 1$, one gets

$$\begin{aligned} \|\Phi_1\| &\leq \left(|\delta_1| + \frac{1}{r}|\delta_2| + \cdots + \frac{1}{r^{n-1}}|\delta_n|\right) \\ &\leq c\sum_{j=1}^{n}\left(\frac{1}{r}\right)^{j-1}\sum_{i=1}^{n}\left(\frac{1}{r}\right)^{i-1}|x_i| \\ &\leq nc\sum_{i=1}^{n}\frac{1}{r^{i-1}}|x_i|. \end{aligned}$$

Further a simple computation with (6) and (8) gives

$$\begin{aligned} 2\varepsilon^T P\Phi_1 &\leq 2\|\varepsilon\|\|P\|nc\sum_{i=1}^{n}\frac{1}{r^{i-1}}|x_i| \\ &\leq 2\|\varepsilon\|\|P\|nc\sum_{i=1}^{n}\left(\frac{1}{r^{i-1}}|\hat{x}_i| + |\varepsilon_i|\right) \\ &= 2\|\varepsilon\|\|P\|nc\left(\sum_{i=1}^{n}\frac{1}{r^{i-1}}|\hat{x}_i| + \sum_{i=1}^{n}|\varepsilon_i|\right) \end{aligned}$$

$$
\begin{aligned}
&\le 2\|\varepsilon\|\|P\|nc\left(\sum_{i=1}^{n}\frac{1}{r^{i-1}}|\hat{x}_i|+\sqrt{n}\|\varepsilon\|\right)\\
&\le 2\|P\|nc\left(\sum_{i=1}^{n}\frac{1}{r^{i-1}}|\hat{x}_i|\|\varepsilon\|+\sqrt{n}\|\varepsilon\|^2\right)\\
&\le 2\|P\|nc\left(\sum_{i=1}^{n}\frac{1}{2}\left(\frac{1}{r^{2(i-1)}}|\hat{x}_i|^2+\|\varepsilon\|^2\right)+\sqrt{n}\|\varepsilon\|^2\right)\\
&\le n\left(2\sqrt{n}+n\right)c\|P\|\sum_{i=1}^{n}\frac{1}{r^{2(i-1)}}|\hat{x}_i|^2\\
&\quad + n\left(2\sqrt{n}+n\right)c\|P\|\|\varepsilon\|^2\\
&\le k_1\|\varepsilon\|^2+k_1\sum_{i=1}^{n}\frac{1}{r^{2(i-1)}}\hat{x}_i^2
\end{aligned}
$$

where $k_1 = cn\left(2\sqrt{n}+n\right)\|P\|$. Then from (15) one obtains

$$
\dot{V}_1 \le -\left(r-k_1\right)\|\varepsilon\|^2 + k_1\sum_{i=1}^{n}\frac{1}{r^{2(i-1)}}\hat{x}_i^2 . \tag{16}
$$

Next introduce $\xi = [\xi_1 \quad \xi_2 \quad \cdots \quad \xi_n]^T \in \boldsymbol{R}^n$ by

$$
\xi_i = \frac{\hat{x}_i}{r^{i-1}}, \quad 1 \le i \le n .
$$

Then

$$
\begin{aligned}
\dot{\xi}_1 &= r\xi_2 + ra_1\varepsilon_1\\
\dot{\xi}_2 &= r\xi_3 + ra_2\varepsilon_1\\
&\vdots\\
\dot{\xi}_n &= r\left(\frac{1}{r^n}u\right)+ra_n\varepsilon_1
\end{aligned} \tag{17}
$$

and hence the inequality (17) can be written as

$$
\dot{V}_1 \le -\left(r-k_1\right)\|\varepsilon\|^2 + k_1\|\xi\|^2 . \tag{18}
$$

Now, we design a compensator of the form

$$
u = -r^n\left(b_n\xi_1 + b_{n-1}\xi_2 + \cdots + b_1\xi_n\right) \tag{19}
$$

where b_i are the coefficients of any Hurwitz polynomial $\rho^n + b_1\rho^{n-1} + \cdots + b_{n-1}\rho + b_n$. Then it is easy to verify that ξ -subsystem (17) with the controller (19) can be expressed as

$$
\dot{\xi} = rB\xi + r\varepsilon_1 col\left[a_1, a_2, \cdots, a_n\right], \tag{20}
$$

where

$$
B = \begin{pmatrix} 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\\ -b_n & -b_{n-1} & \cdots & -b_1 \end{pmatrix}.
$$

Further choose a quadratic function of the form

$$
V_2 := \xi^T Q\xi , \tag{21}
$$

where Q is a positive definite symmetric matrix satisfying

$$
B^T Q + QB = -2I . \tag{22}
$$

Then one can easily obtain the inequality

$$
\begin{aligned}
\dot{V}_2 &= \dot{\xi}^T Q\xi + \xi^T Q\dot{\xi}\\
&= r\xi^T(B^T Q + QB)\xi + 2r\xi^T Qa\varepsilon_1\\
&\le -2r\|\xi\|^2 + 2r\xi^T Qa\varepsilon_1\\
&\le -2r\|\xi\|^2 + 2r\xi^T Qa\varepsilon_1
\end{aligned} \tag{23}
$$

and similarly

$$
\begin{aligned}
2r\xi^T Qa\varepsilon_1 &\le 2r\|\xi\|\|Q\|\|col\left[a_1,\cdots,a_n\right]\|\|\varepsilon\|\\
&\le 2r\left(\frac{1}{2}\|\xi\|^2 + \frac{1}{2}\left(\|Q\|\|col\left[a_1,\cdots,a_n\right]\|\|\varepsilon\|\right)^2\right)\\
&\le r\|\xi\|^2 + r\left(\|Q\|\|col\left[a_1,\cdots,a_n\right]\|\right)^2\|\varepsilon\|^2\\
&\le r\|\xi\|^2 + rk_2\|\varepsilon\|^2
\end{aligned}
$$

where $k_2 = \left(\|Q\|\|col\left[a_1,\cdots,a_n\right]\|\right)^2$ is a constant, independent of r. Thus the inequality (23) can be written as

$$
\dot{V}_2 \le -r\|\xi\|^2 + rk_2\|\varepsilon\|^2 . \tag{24}
$$

Next, we observe that the closed-loop system (1) with (5) and (19) can be treated as an interconnection of ε -subsystem and ξ -subsystem. Now, consider the function

$$
V := (k_2+1)V_1 + V_2 = (k_2+1)\varepsilon^T P\varepsilon + \xi^T Q\xi . \tag{25}
$$

It easily follows from (18), (24) that

$$
\begin{aligned}
\dot{V} &= \left(k_2+1\right)\dot{V}_1 + \dot{V}_2\\
&\le -\left(r-k_1\right)\left(k_2+1\right)\|\varepsilon\|^2 + k_1\left(k_2+1\right)\|\xi\|^2\\
&\qquad\qquad - r\|\xi\|^2 + rk_2\|\varepsilon\|^2\\
&\le -\left(r-k_1(k_2+1)\right)\|\varepsilon\|^2 - \left(r-k_1(k_2+1)\right)\|\xi\|^2 .
\end{aligned} \tag{26}
$$

Clearly, if we choose the gain parameter r to be

$$
r \ge 1 + k_1(k_2+1)
$$

then

$$
\dot{V}_2 \le -\left(\|\varepsilon\|^2 + \|\xi\|^2\right).
$$

This implies

$$
\varepsilon(t) \to 0, \quad \xi(t) \to 0 \quad \text{as} \quad t \to \infty ,
$$

and hence that the closed-loop system (1) with (5) and (19) is globally asymptotically stable. This completes the proof. □

The new approach proposed not need to go through the recursive design procedure as in [8]. It can determine all the observer and controller parameters in one step, rather than n-steps [2], [8].

Example: Consider the following systems:

$$\dot{x}_1 = x_2 + \frac{x_1}{(1-c_1x_2)^2 + x_2^2}$$

$$\dot{x}_2 = u + \ln\left(1+\left(x_2^2\right)^{c_2}\right) \tag{27}$$

$$y = x_1$$

where c_1 and $c_2 \geq 1$ are constants. It is easy to check that the system (27) satisfies Assumption 1. Thus, by Theorem1, a globally stabilizing output dynamic compensator can be constructed. To construct such a compensator by following the proof of Theorem 1, choose the coefficients of the two Hurwitz polynomials to be $a_1 = a_2 = 1$ and $b_1 = 11/4,\ b_2 = 20$. Then the compensator given by (19) is now described as

$$\dot{\hat{x}}_1 = \hat{x}_2 + r\left(y - \hat{x}_1\right)$$

$$\dot{\hat{x}}_2 = u + r^2\left(y - \hat{x}_1\right) \quad (28).$$

$$u = -r(b_2 r\hat{x}_1 + b_1\hat{x}_2).$$

For our numerical simulation, we chose $r \geq 8339$ and the initial states to be $\left(x_1(0), x_2(0), \hat{x}_1(0), \hat{x}_2(0)\right) = (1,5,3,5)$. Then the simulation results shown in Fig.1 demonstrates the effectiveness of the output dynamic compensator (28).

III. Conclusion

We have presented the new result on global stabilization of a class of uncertain nonlinear systems by a dynamic output compensator. By integrating the idea of the use the output feedback domination design method [2], we gave an explicit method for constructing a globally stabilizing output dynamic compensator for a family of uncertain nonlinear systems.

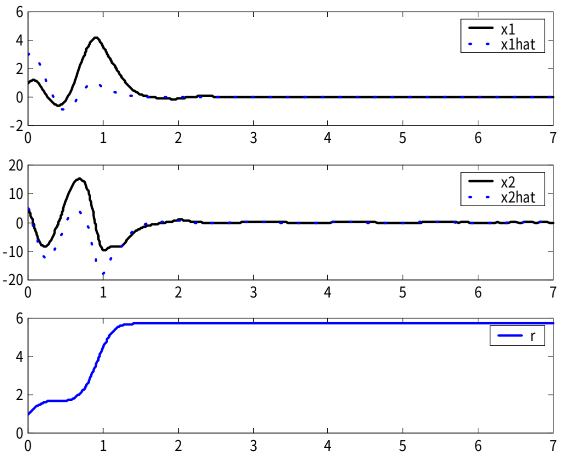

Fig. 1. By the proposed method ($c_1 = c_2 = 5$).

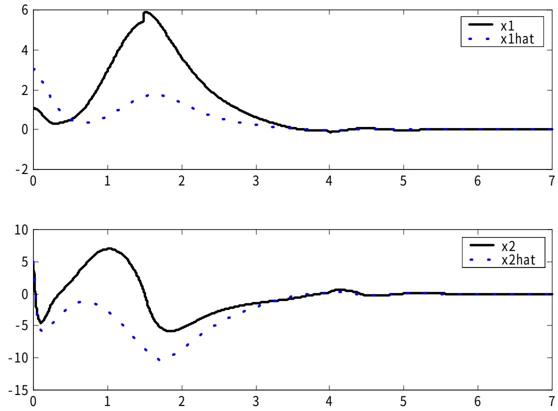

Fig. 2. By the method in [8] ($c_1 = c_2 = 5$).

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