NONLINEAR OBSERVERS FOR TIME-VARYING SYSTEMS APPEARING IN DYNAMICAL MACHINE VISION[#]

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Abstract: Perspective dynamical systems arise in dynamic machine vision. This paper studies a Luenberger-type nonlinear observer for a perspective time-varying linear system in which the number of observing points is more than one. More precisely, assuming a given perspective time-varying linear system to be Lyapunov stable and to satisfy some sort of observability condition, it is shown that the estimation error converges exponentially to zero. *Copyright © 2005 IFAC*

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1. INTRODUCTION

The observation obtained by observing a moving rigid body by a camera is essentially the direction vector of a point observed, called *a perspective observation*, because all the points on a line passing through the center of the camera are projected to a single point of the image plane. And the basic problem in dynamical machine vision is how to determine the position and velocity of a moving body and/or any unknown parameters characterizing the motion and shape of the body from such perspective observation. A perspective dynamical system arises in mathematically describing such a dynamic machine vision problem and has been studied in a number of approaches in the framework of systems theory (See, e.g., Abdursul, *et al.,* 2004, Matveev, *et al.,* 2000, S. Soatto, *et al.,* 1996).

Some interesting works recently reported are concerned with nonlinear observers for estimating the unknown state of perspective time-varying linear systems (Abdursul, *et al.,* 2004, Matveev, *et al.,* 2000, S. Soatto, *et al.,* 1996). In particular, the paper (Abdursul, *et al.,* 2004) has proposed a nonlinear observer of the Luenberger-type for such a time-invariant system without transforming it into an implicit system as proposed and discussed in the papers (Matveev, *et al.,* 2000, S. Soatto, *et al.,* 1996) and shown that under some reasonable assumptions on the given system the estimation error of the proposed nonlinear observer converges exponentially to zero.

In these previous works it was assumed that only one point on a moving body is observed. However in more realistic situations the number of observing points, even the number of moving bodies, may be more than one. A similar situation occurs in computer vision problems. In fact the work (Chiusoa*, et a.,* 1995) discussed a problem of multiple observing points from the viewpoint of dynamical computer

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vision and a nonlinear filter was proposed in order to causally estimate the 3-dimentional shape by integrating noisy visual information over time. Further the nonlinear observer of the Luenberger-type developed in (Abdursul, *et al.,* 2004) has been extended to the case of multiple observing points by the recent work (Inaba, *et al.,* 2004).

The objective of the present paper is to further extend the work (Inaba and Abdursul, 2004) for time-invariant systems with multiple observing points to the case where the system considered is *time-varying* (see also Abdursul and Inaba, 2003), that is, the case of perspective dynamical *time-varying* linear systems with multiple observing points.

First, in Section 2, a perspective time-varying linear system considered in this investigation is described. In Section 3, a nonlinear observer of the Luenberger-type for perspective time-varying linear systems is discussed and then after obtaining some important properties of such systems a convergence theorem of the nonlinear observer is presented. More precisely, it is shown that, under suitable assumptions on a given perspective time-varying linear system, including that it is Lyapunov stable and satisfies some sort of observability condition, the estimation error of the nonlinear observer converges exponentially to zero. Finally, Section 4 gives some concluding remarks.

2. PERSPECTIVE TIME-VARYING LINEAR SYSTEMS

This section defines a mathematical model for a perspective dynamical time-varying system with more than one observing point. Throughout this investigation, let us denote the fields of real and complex numbers by $\mathbb{R}$ and $\mathbb{C}$, respectively.

First let us denote the number of observing points by $p \geq 1$. Then, as in the work (Inaba, *et al.*, 2004), *a perspective time-varying linear system with multiple observing points* considered in this investigation is given as

$$\begin{cases} \dot{x}(t) = A(t)x(t) + v(t), \;\; x(t_0) = x_0 \in \mathbb{R}^n \\ y(t) = H(Cx(t)) \in \mathbb{R}^{2p} \end{cases} \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ represents the entire state of the moving bodies, $v(t) \in \mathbb{R}^n$ the external input, $A(\cdot) : \mathbb{R} \to \mathbb{R}^{n \times n}$ a continuous and bounded matrix-valued function, $C \in \mathbb{R}^{(3p)\times n}$ and $t_0 \in \mathbb{R}$ the initial time. Finally, $y(t) \in \mathbb{R}^{2p}$ represents the *perspective observation vector* generated on the image plane by observing the p points, and as in the case of multiple observing points discussed in (Inaba, *et al.*, 2004) the function $H : \mathbb{R}^n \to \mathbb{R}^{2p}$ has the following form:

$$H := \left.\begin{bmatrix} h \\ \vdots \\ h \end{bmatrix}\right\} p, \; h(\xi) := \begin{bmatrix} \dfrac{\xi_1}{\xi_3} \\ \dfrac{\xi_2}{\xi_3} \end{bmatrix}, \; \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}, \; \xi_3 \neq 0 \,. \qquad (2)$$

For later convenience, the observation vector $y(t) \in \mathbb{R}^{2p}$ is expressed in the following form:

$$\begin{cases} y(t) = H(Cx(t)) = \begin{bmatrix} h(C^{(1)}x(t)) \\ \vdots \\ h(C^{(p)}x(t)) \end{bmatrix} = \begin{bmatrix} y^{(1)}(t) \\ \vdots \\ y^{(p)}(t) \end{bmatrix} \\ y^{(k)}(t) = h(C^{(k)}x(t)) \in \mathbb{R}^2 \end{cases} \qquad (3)$$

where $y^{(k)}(t) = h(C^{(k)}x(t))$ represents the perspective observation generated from the k-th observing point and the matrix $C \in \mathbb{R}^{(3p)\times n}$ is decomposed as

$$C = [C^{(1)T} \quad \cdots \quad C^{(p)T}]^T, \quad C^{(k)} \in \mathbb{R}^{3\times n}\,.$$

Now we investigate a Luenberger-type nonlinear observer for System (1) without converting it to an implicit system as discussed in (Matveev, *et al.*, 2000). As mentioned in the previous work (Abdursul, *et al.*, 2004), the advantage of this observer over the one in (Matveev, *et al.*, 2000) is that the former does not require a transformation that involves an integration of the input $v(s)$ $(0 \leq s \leq t)$, which may cause difficulty in implementation, in particular, for the case that the input $v(t)$ is generated in a state feedback form.

3. LUENBERGER-TYPE NONLINEAR OBSERVERS

In this section, we propose a Luenberger-type observer for a perspective time-varying linear system of the form (1), and show that under some suitable assumptions on System (1) the estimation error converges to zero exponentially.

First, notice that, denoting an estimate of the state $x(t)$ by $\hat{x}(t)$, a full-order state observer for System (1) is expressed generally in the form

$$\frac{d}{dt}\hat{x}(t) = \varphi(\hat{x}(t), v(t), y(t), t), \;\; \hat{x}(t_0) = \hat{x}_0 \in \mathbb{R}^n \quad (4)$$

and satisfies the condition that whenever $\hat{x}(t_0) = x(t_0)$ the solution $\hat{x}(t)$ of (4) coincides completely with the solution $x(t)$ of System (1) for any choice of $v(\cdot)$.

Therefore, under the condition that (4) has a unique solution, it is possible to assume that the function $\varphi(\hat{x}, v, y, t)$ has the form

$$\varphi(\hat{x}, v, y, t) = A(t)\hat{x} + v + r(\hat{x}, y, t)$$

where $r(\hat{x}, y, t)$ is any sufficiently smooth function satisfying the condition $r(x, h(Cx), t) = 0$ for all $x \in \mathbb{R}^n$ and all $t \geq t_0$. Among many functions $r(\hat{x}, y, t)$ satisfying this condition, it is desirable to choose a function, which is reasonably simple, but has sufficient freedom to adjust its characteristics. As such a function, we choose

$$r(\hat{x}, y, t) = K(y, \hat{x}, t)[y - h(C\hat{x})]$$

where $K(y,\hat{x},t)$ is any sufficiently smooth function. Then, (4) becomes *a nonlinear observer of the Luenberger-type*:

$$\frac{d}{dt}\hat{x}(t) = A(t)\hat{x}(t) + v(t) + K(y(t),\hat{x}(t),t) \times [y(t) - H(C\hat{x}(t))], \ \ \hat{x}(t_0) = \hat{x}_0 \in \mathbb{R}^n \quad (5)$$

where $K(y,\hat{x},t)$ is a suitable matrix-valued function of the form $K : \mathbb{R}^{2p} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times(2p)}$, called an *observer gain matrix*.

The fundamental problem we must answer is how to choose a gain matrix $K(y,\hat{x},t)$ in (5) so as to satisfy the condition $\hat{x}(t) \to x(t)$ as $t \to \infty$. To do so, first introduce the following notation:

$$Cx = \begin{bmatrix} C^{(1)} \\ \vdots \\ C^{(p)} \end{bmatrix} x =: \begin{bmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \\ \xi_3^{(1)} \\ \vdots \\ \xi_1^{(p)} \\ \xi_2^{(p)} \\ \xi_3^{(p)} \end{bmatrix} = \xi \in \mathbb{R}^{3p},$$

and a similar notation for $C\hat{x} =: \hat{\xi} \in R^{3p} \in \mathbb{R}^{3p}$. Then using these notations, simplify the term $y - H(C\hat{x})$ as follows:

$$\begin{aligned} y - H(C\hat{x}) &= H(Cx) - H(C\hat{x}) \\ &= \begin{bmatrix} \frac{\xi_1^{(1)}}{\xi_3^{(1)}} - \frac{\hat{\xi}_1^{(1)}}{\hat{\xi}_3^{(1)}} \\ \frac{\xi_2^{(1)}}{\xi_3^{(1)}} - \frac{\hat{\xi}_2^{(1)}}{\hat{\xi}_3^{(1)}} \\ \vdots \\ \frac{\xi_1^{(p)}}{\xi_3^{(p)}} - \frac{\hat{\xi}_1^{(p)}}{\hat{\xi}_3^{(p)}} \\ \frac{\xi_2^{(p)}}{\xi_3^{(p)}} - \frac{\hat{\xi}_2^{(p)}}{\hat{\xi}_3^{(p)}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\hat{\xi}_3^{(1)}} \begin{bmatrix} 1 & 0 & -\frac{\xi_1^{(1)}}{\xi_3^{(1)}} \\ 0 & 1 & -\frac{\xi_2^{(1)}}{\xi_3^{(1)}} \end{bmatrix} \begin{bmatrix} \xi_1^{(1)} - \hat{\xi}_1^{(1)} \\ \xi_2^{(1)} - \hat{\xi}_2^{(1)} \\ \xi_3^{(1)} - \hat{\xi}_3^{(1)} \end{bmatrix} \\ \vdots \\ \frac{1}{\hat{\xi}_3^{(p)}} \begin{bmatrix} 1 & 0 & -\frac{\xi_1^{(p)}}{\xi_3^{(p)}} \\ 0 & 1 & -\frac{\xi_2^{(p)}}{\xi_3^{(p)}} \end{bmatrix} \begin{bmatrix} \xi_1^{(p)} - \hat{\xi}_1^{(p)} \\ \xi_2^{(p)} - \hat{\xi}_2^{(p)} \\ \xi_3^{(p)} - \hat{\xi}_3^{(p)} \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\hat{\xi}_3^{(1)}} \begin{bmatrix} I_2 & -y^{(1)} \end{bmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\hat{\xi}_3^{(p)}} \begin{bmatrix} I_2 & -y^{(p)} \end{bmatrix} \end{bmatrix} (\xi - \hat{\xi}) \\ &= E(\hat{x})B(y)C\rho \end{aligned} \quad (6)$$

where I_2 indicates the 2×2 identity matrix and

$$\begin{cases} E(\hat{x}) := \begin{bmatrix} \frac{1}{C_3^{(1)}\hat{x}} I_2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{C_3^{(p)}\hat{x}} I_2 \end{bmatrix} \in \mathbb{R}^{2p} \\ B(y) := \begin{bmatrix} \begin{bmatrix} I_2 & -y^{(1)} \end{bmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{bmatrix} I_2 & -y^{(p)} \end{bmatrix} \end{bmatrix} \in \mathbb{R}^{(2p)\times(3p)} \end{cases} \quad (7)$$

$$\rho := x - \hat{x}. \quad (8)$$

Using (6), one obtains

$$K(y,\hat{x},t)[y - H(C\hat{x})] = K(y,\hat{x},t)E(\hat{x})B(y)C\rho \quad (9)$$

and hence to eliminate from (9) all the denominators $C_3^{(k)}\hat{x}$ appearing in $E(\hat{x})$, one can choose a gain matrix $K(y,\hat{x})$ of the form:

$$K(y,\hat{x},t) = P(t)^{-1}C^*B^*(y)E^{-1}(\hat{x}) \quad (10)$$

where C^* indicates the complex conjugate transpose of C and $P(t) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is an appropriately chosen matrix-valued function, which is considered to be a *free parameter* for the gain matrix. And with this choice for $K(y,\hat{x},t)$, the Luenberger-type nonlinear observer (5) becomes

$$\frac{d}{dt}\hat{x}(t) = A(t)\hat{x}(t) + v(t) + K(y(t),\hat{x}(t),t) \times [y(t) - H(C\hat{x}(t))], \ \ \hat{x}(t_0) = \hat{x}_0 \in \mathbb{R}^n \quad (11)$$

and it follows from (1), (11) and (6) that the differential equation for the estimation error $\rho := x - \hat{x}$ is obtained as

$$\frac{d}{dt}\rho(t) = [A(t) - P(t)^{-1}C^*B^*(y(t))B(y(t))C]\rho(t), \quad \rho(t_0) = x(t_0) - \hat{x}(t_0) \in \mathbb{R}^n. \quad (12)$$

Next, we make suitable assumptions on System (1), which seem to be necessary and/or reasonable from the viewpoint of dynamical machine vision.

Assumption 3.1. System (1) is assumed to satisfy the following conditions.

(i) System (1) is Lyapunov stable. That is there exists $b > 0$ such that

$$\|\Phi(t,s)\| \le b < \infty, \qquad \forall t,s \in \mathbb{R}$$

where $\Phi(t,s)$ is the transition matrix for $A(t)$.

(ii) The observation vector $y(t)$ is a continuous and bounded function of t, that is,

$$y(\cdot) \in C^m[t_0,\infty) \cap L_\infty^m[t_0,\infty). \quad (13)$$

(iii) There exist $T > 0$ and $\varepsilon > 0$ such that

$$\int_0^T \Phi(t+\tau,t)^* C^* B^*(y(t+\tau)) \times B(y(t+\tau)) C\Phi(t+\tau,t) d\tau \ge \varepsilon I_r,\ \forall t \ge t_0. \tag{14}$$

□

Remark 3.2. All the conditions given in Assumption 3.1 are necessary and/or reasonable requirements from the viewpoint of machine vision.

(i) The condition (i) is imposed to ensure that if $v(t) \equiv 0$ then the motion of a moving body takes place within a bounded space.

(ii) The condition (ii) is imposed to ensure that the motion is smooth enough and takes place inside a conical region centered at the camera to produce a continuous and bounded measurement $y(t)$ on the image plane.

(iii) The condition (iii) is imposed to ensure some sort of detestability for the perspective system (1). In fact, the inequality (14) implies that $(C, A(t))$ is a uniformly observable pair. This fact is verified in Proposition 3.3. □

The following proposition gives some system theoretical implications of Assumption 3.1 (iii).

Proposition 3.3. Consider System (1), and assume that Assumption 3.1 (iii) is satisfied. Then the pair $(C, A(t))$ is uniformly observable, that is, there exist some $T > 0$ and $\hat{\varepsilon} > 0$ such that

$$\int_0^T \Phi(t+\tau,t)^* C^* C\Phi(t+\tau,t) d\tau \ge \hat{\varepsilon} I_r,\ \forall t \ge t_0.$$

Proof: Using (7), define a symmetric nonnegative matrix by

$$S(y) := B^T(y)B(y)$$

$$= \begin{bmatrix} \begin{bmatrix} I_2 & -y^{(1)} \\ -y^{(1)} & \left\|y^{(1)}\right\|^2 \end{bmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{bmatrix} I_2 & -y^{(p)} \\ -y^{(p)} & \left\|y^{(p)}\right\|^2 \end{bmatrix} \end{bmatrix}$$

Then it is not difficult to see that

$$\det(\lambda I_{3p} - S(y)) = \lambda^p (\lambda - 1)^p (\lambda - 1 - \left\|y^{(1)}\right\|^2) \cdots (\lambda - 1 - \left\|y^{(p)}\right\|^2)$$

and hence the eigenvalues of $S(y)$ are

$$\lambda(S(y)) = 0, 1, 1 + \left\|y^{(1)}\right\|^2, \cdots, 1 + \left\|y^{(p)}\right\|^2.$$

Thus one obtains

$$\max \lambda(S(y(t)) \le (1 + \|y(\cdot)\|_\infty) < \infty, \forall t > t_0,$$

and accordingly the inequalities

$$(1 + \|y(\cdot)\|_\infty) I_{3p} \ge S(y(t)) \ge 0, \forall t \ge t_0.$$

Now it follows from the above and (14) that for any $t \ge t_0$

$$\int_0^T \Phi(t+\tau,t)^* C^* C\Phi(t+\tau,t) d\tau \ge \frac{1}{1 + \|y(\cdot)\|_\infty} \int_0^T \Phi(t+\tau,t)^* C^* S(t+\tau,t) \times C\Phi(t+\tau,t) d\tau \ge \frac{1}{1 + \|y(\cdot)\|_\infty} \varepsilon I = \hat{\varepsilon} I$$

where

$$\hat{\varepsilon} := \frac{1}{1 + \|y(\cdot)\|_\infty} \varepsilon > 0.$$

This verifies the statement. □

For a general time-varying linear system with system matrix $A(t)$ and output matrix $C(t)$, i.e., for a system of the form

$$\begin{cases} \dot{x}(t) = A(t)x(t) + v(t), & x(t_0) = x_0 \\ y(t) = C(t)x(t) \end{cases}$$

its observability and detectability are completely determined by the pair $(C(t), A(t))$, and completely independent of the input and initial state. However for a general nonlinear system they are generally dependent on the input and initial state. It should be noted that the inequality (14) implies that the pair $(C, A(t))$ is uniformly observable, but this uniform observability does not necessarily imply the observability and detectability of System (1). However it is seen from the next theorem that the condition (14) is sufficient for existence of a nonlinear observer and hence implies the observability or at least detectability along the trajectory $x(t)$ that is generated by a given input $v(t)$ and initial state $x(t_0)$.

Now, it is ready to state our main theorem. However, since its proof requires a number of lengthy and cumbersome technical arguments, only a sketch of the proof is given here.

Theorem 3.4 (Nonlinear Observers). Assume that System (1) satisfies Assumption 3.1 and the matrix differential inequality

$$A^T(t)P(t) + P(t)A(t) + \dot{P}(t) \le 0, \quad \forall\, t \ge t_0 \tag{15}$$

has symmetric positive-definite matrix-valued solution $P(\cdot) : [t_0, \infty) \to \mathbb{R}^{n \times n}$ such that for some $\delta > 0$

$$P(t) \ge \delta I > 0, \quad \forall\, t \ge t_0.$$

Consider a nonlinear observer of the Luenberger-type (5), that is,

$$\frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + v(t) + P^{-1}C^* B^*(y(t)) E^{-1}(\hat{x}(t)) \times [y(t) - H(C\hat{x}(t))],\ \hat{x}(t_0) = \hat{x}_0 \in \mathbb{R}^n \tag{16}$$

with a gain matrix of the form (10), i.e.,

$$K(y, \hat{x}, t) := P(t)^{-1} C^* B^*(y) E^{-1}(\hat{x}) \tag{17}$$

where $B(y)$ is given by (7).

Then, the following statements hold.

(i) The estimation error $\rho(t) := x(t) - \hat{x}(t)$ satisfies the differential equation

$$\frac{d}{dt}\rho(t) = [A(t) - P(t)^{-1}C^* B^*(y(t))B(y(t))C]\rho(t), \quad \rho(t_0) = x(t_0) - \hat{x}(t_0) \in \mathbb{R}^n.$$

(ii) $\rho(t)$ converges exponentially to zero, that is, there exist $\alpha > 0, \beta > 0$ such that

$$\|\rho(t)\| := \|x(t) - \hat{x}(t)\| \le \beta e^{-\alpha t}\|\rho(t_0)\|, \quad \forall t \ge t_0.$$

Proof: The statement (i) can be easily verified by differentiating $\rho(t) = x(t) - \hat{x}(t)$ and substituting (1), (16) and (17) to the resultant. Therefore, only the statement (ii) will be proved in detail. To prove this, it suffices to show by virtue of Lemma 5.1.2 (Curtain and Zwart, 1995) that the estimation error $\rho(t)$ satisfies the inequality

$$\int_{t_0}^{\infty} \|\rho(t)\|^2 dt \le \gamma \|\rho(t_0)\|^2 \tag{18}$$

where $\gamma > 0$ is some constant, independent on $\rho(t_0)$.

First, using (12) and (15), one can easily obtain

$$\begin{aligned}\frac{d}{dt}\rho(t)^T P(t)\rho(t) &= \rho(t)^T[A^T(t)P(t) + P(t)A(t) + \dot{P}(t)]\rho(t) \\ &\quad - 2\rho(t)^T C^T B^T(y(t))B(y(t))C\rho(t) \\ &\le -2\|B(y(t))C\rho(t)\|^2.\end{aligned}$$

Hence, integrating the above from 0 to t gives

$$0 \le \rho(t)^T P(t)\rho(t) \le \rho(t_0)^T P(t_0)\rho(t_0) - 2\int_0^t \|B(y(s))C\rho(s)\|^2 ds, \quad \forall t > t_0$$

which immediately leads to

$$\begin{cases} \rho(t)^T P(t)\rho(t) \le \rho(t_0)^T P(t_0)\rho(t_0) \le \frac{1}{2}\|P(t_0)\|\|\rho(t_0)\|^2, \quad t \ge t_0 \\ \int_{t_0}^t \|B(y(s))C\rho(s)\|^2 ds \le \frac{1}{2}\rho(t_0)^T P(t_0)\rho(t_0) \le \frac{1}{2}\|P(t_0)\|\|\rho(t_0)\|^2, \forall t \ge t_0. \end{cases} \tag{19}$$

First note that it easily follows from (7) and Assumption 3.1 (ii) that

$$\|B(y(\cdot))\|_\infty \le 1 + \|y(\cdot)\|_\infty < \infty.$$

Now expressing (12) in the form

$$\dot{\rho}(t) = A(t)\rho(t) - \theta(t) \tag{20}$$

where

$$\theta(t) := P^{-1}(t)C^T B^T(y(t))B(y(t))C\rho(t), \tag{21}$$

one can write the solution of (20) in the form

$$\rho(t+s) = \Phi(t+s,t)\rho(t) - \int_t^{t+s} \Phi(t+s,\tau)\theta(\tau)d\tau, \quad \forall s > 0. \tag{22}$$

Next using (14) and (22) one can evaluate

$$\begin{aligned}\int_{t_0}^{\infty}\|\rho(t)\|^2 dt &= \frac{1}{\varepsilon}\int_{t_0}^{\infty}\rho^*(t)(\varepsilon I)\rho(t)dt \\ &\le \frac{1}{\varepsilon}\int_{t_0}^{\infty}\Big[\rho^*(t)\Big\{\int_0^T \Phi^*(t+s,t)C^*B^*(y(t+s)) \\ &\qquad \times B(y(t+s))C\Phi(t+s,t)ds\Big\}\rho(t)\Big]dt \\ &= \frac{1}{\varepsilon}\int_{t_0}^{\infty}\Big[\int_0^T \|B(y(t+s))C\Phi(t+s,t)\rho(t)\|^2 ds\Big]dt \\ &= \frac{1}{\varepsilon}\int_{t_0}^{\infty}\Big[\int_0^T \Big\|B(y(t+s))C\Big\{\rho(t+s) \\ &\qquad + \int_t^{t+s}\Phi(t+s,\tau)\theta(\tau)d\tau\Big\}\Big\|^2 ds\Big]dt \\ &\le \frac{2}{\varepsilon}\Bigg[\underbrace{\int_{t_0}^{\infty}\Big\{\int_0^T \|B(y(t+s))C\rho(t+s)\|^2 ds\Big\}dt}_{=:Q_1} \\ &\qquad + \underbrace{\int_{t_0}^{\infty}\Big\{\int_0^T \Big\|B(y(t+s))C\int_t^{t+s}\Phi(t+s,\tau)\theta(\tau)d\tau\Big\|^2 ds\Big\}dt}_{=:Q_2}\Bigg].\end{aligned}$$

Further Q_1 can be evaluated as follows:

$$\begin{aligned}Q_1 &= \int_{t_0}^{\infty}dt\int_0^T \|B(y(t+s))C\rho(t+s)\|^2 ds \\ &= \int_{t_0}^{\infty}\Big\{\int_t^{t+T}\|B(y(\tau))C\rho(\tau)\|^2 d\tau\Big\}dt \\ &= \int_{t_0}^{\infty}\Big\{\|B(y(\tau))C\rho(\tau)\|^2\int_{\max\{t_0,\tau-T\}}^{\tau} 1dt\Big\}d\tau \\ &\le T\int_{t_0}^{\infty}\|B(y(t))C\rho(t)\|^2 dt \le \frac{T\delta^2 M}{2}\|\rho(t_0)\|^2 \\ &=: \gamma_1\|\rho(t_0)\|^2\end{aligned} \tag{23}$$

where $\gamma_1 > 0$ is a constant, independent of $\rho(\cdot)$ and $t_0 \ge 0$.

To evaluate Q_2, first note that, since by Assumption 3.1 (i) $A(t)$ is Lyapunov stable, there is a constant $b > 0$ such that

$$\|\Phi(t+s,t)\| \le b < \infty \text{ for all } s,t \ge 0.$$

Using this fact, one obtains

$$\begin{aligned}Q_2 &= \int_{t_0}^{\infty}\Big\{\int_0^T \Big\|B(y(t+s))C\int_t^{t+s}\Phi(t+s,\tau)\theta(\tau)d\tau\Big\|^2 ds\Big\}dt \\ &\le (1+\|y(\cdot)\|_\infty)^2\|C\|^2\int_{t_0}^{\infty}\Big\{\int_0^T\Big\|\int_t^{t+s}\Phi(t+s,\tau)\theta(\tau)d\tau\Big\|^2 ds\Big\}dt \\ &\le (1+\|y(\cdot)\|_\infty)^2\|C\|^2 b^2\int_{t_0}^{\infty}\Big\{\Big\|\int_t^{t+T}\theta(\tau)d\tau\Big\|^2\int_0^T 1ds\Big\}dt \\ &= (1+\|y(\cdot)\|_\infty)^2\|C\|^2 b^2 T\int_{t_0}^{\infty}\Big\|\int_t^{t+T}\theta(\tau)d\tau\Big\|^2 dt.\end{aligned}$$

and further, using (21) and (19), the above expression for Q_2 can be further reduced to

$$
\begin{aligned}
Q_2 &\le (1+\|y(\cdot)\|_\infty)^2 \|C\|^2 b^2 T^2 \int_{t_0}^{\infty} \|\theta(\tau)\|^2 \, dt \\
&\le (1+\|y(\cdot)\|_\infty)^2 \|C\|^2 b^2 T^2 \\
&\quad \times \int_{t_0}^{\infty} \left\| P^{-1}(t) C^* B^*(y(t),\hat{x}(t)) B(y(t),\hat{x}(t)) C \rho(t) \right\|^2 dt \\
&\le \frac{1}{m^2} (1+\|y(\cdot)\|_\infty)^2 \|C\|^2 b^2 T^2 \|C\|^2 (1+\|y(\cdot)\|_\infty)^2 \\
&\quad \times \frac{1}{\delta^2} \int_{t_0}^{\infty} \|B(y) C \rho(t)\|^2 \, dt \\
&\le \frac{1}{\delta^2 m^2} (1+\|y(\cdot)\|_\infty)^4 \|C\|^4 b^2 T^2 \frac{\delta^2 M}{2} \|\rho(t_0)\|^2 \\
&=: \gamma_2 \|\rho(t_0)\|^2
\end{aligned}
\tag{24}
$$

where $\gamma_2 > 0$ is a constant, independent of $\rho(\cdot)$ and $t_0 \ge 0$.

Finally, it follows from (23) and (24) the desired inequality

$$
\begin{aligned}
\int_{t_0}^{\infty} \|\rho(t)\|^2 dt &\le \frac{2}{\varepsilon} (\gamma_1 + \gamma_2) \|\rho(t_0)\|^2 \\
&=: \gamma \|\rho(t_0)\|^2 .
\end{aligned}
$$

This completes the proof of Theorem 3.4. □

4. CONCLUDING REMARKS

This paper studied a nonlinear observer for perspective time-varying linear systems arising in dynamical machine vision. First a Luenberger-type nonlinear observer was proposed, and then under some reasonable assumptions on a given perspective system, it was shown that it is possible to construct such a Luenberger-type nonlinear observer whose estimation error converges exponentially to zero.

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