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Control of Travelling Pulses in Mems Arrays: Numerical Evidence of Practical Asymptotic Stabilization

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***Abstract*—** Control of large arrays of microactuators and sensors, are anticipated to be of much interest to the technological advances of tomorrow. Prominent among the requisite control tasks will be that of producing stable dynamic patterns. Here we address the problem of practical asymptotic stabilization of travelling pulses in a one dimensional array of microactuators. Solitons are used as models of travelling pulses. A method is described to embed descretized KdV equation in a microactuator array, and an essentially local feedback control scheme is developed, for the purpose of practical asymptotic stabilization of solitons. While no proofs of asymptotic stability are given, numerical evidence is presented to support the conjecture that the proposed control scheme is practically asymptotically stabilizing.

***Index Terms*—** mems arrays, transient response, soliton, KdV equation, practical asymptotic stabilization

I. INTRODUCTION

It is anticipated that mems technology and naotechnology will usher in an era of unprecedented technological capabilities. Anticipated devices such as micromechanisms that can swim, crawl, fly, autonomous micromanufacturing plants, microrobotic arrays, and next generation data storage, retrieval and search devices will all constitute of large arrays (perhaps millions or more) of mems elements operating in dynamic environments. In essence these arrays, due to their autonomous features, adaptability to varying environmental conditions, and their own complex dynamics, will qualify to be called artificial organisms. Empowering them with autonomous control features will be a necessary task which will be quite different from any of the control design problems encountered today, and perhaps the closest analogy will be the autonomous control features of living organisms of today.

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If we were to pursue this analogy, one of the key features of interest is that organisms seem to control their bodies by generating patterns. For example, eels, fish and snakes produce travelling waves. Human vocal cords produce a large number of patterns corresponding to various sound elements. Digestive tracts produce travelling pulses. In contrast to simple electrical engineering circuitry, all organisms seem to be capable of producing a large number of different patterns, and in addition capable of fast and graceful transition from one pattern to another. In human speech, transients between one sound and another are almost imperceptible. A snake may produce three waves along its body in normal movement, but several more waves while it is in fast retreat from an enemy, but the transition is fast and smooth.

It has been argued that futuristic artificial organisms, made of mems or nanotechnology, should also be empowered to produce a wide variety of patterns, and empowered to switch from one pattern to another in a quick and graceful manner. First aspect of this design challenge, i.e., what are appropriate patterns and how one may produce one in a dynamic array, has for the most part been already addressed by biologists, physicists and mathematicians. There is a large literature on pattern forming dynamic systems (see e.g. [7]). Embedding such an equation in a mems array with prescribed dynamics is a relatively easy task (see section II for an example). Second aspect of the problem, i.e., how to design a control system to ensure quick and graceful transition from one pattern to another, is almost completely unexplored. This paper is an attempt to bring attention to this problem.

As an illustrative example, let us consider the problem of producing a traveling pulse in a linear mems array. A futuristic application of such an array could be in a micromanufacturing plant as a mechanism to move parts along a conveyer mechanism. Let us say, in accordance

with the foregoing discussion that the controller should have following general features:
a) mems array elements are "essentially locally coupled", i.e. any feedback signal received by the controller of a particular mems element only take into account states of nearby elements. (This is in view of the fact that biological control systems have this feature).
b) mems array does not receive exogeneous timing signals. (This is required for autonomous operation).
c) mems array is capable of producing a family of traveling pulses, and when desired, dynamics can be switched from one traveling pulse pattern to another in a quick and smooth manner.

Here we describe an approach to solving this problem via the use of a stabilized version of the Korteweg-de Vries equation (KdV). KdV is one of the most studied equations of the mathematical physics (see e.g. [1]). A family of solutions, called solitons, consists of a single (or a nonlinear superposition of several) travelling pulses. Thus, KdV solitons meet requirements (a) and (b) mentioned above. However, solitons are only orbitally stable, but not orbitally asymptotically stable. Thus, is order to meet (c) one has to device a local feedback control scheme, i.e., one in which feedback signals given to any individual mems element only use the state of nearby mems elements, that will asymptotically stabilize solitons. At this stage it isn't clear to us whether such control laws exist. Here we present a family of control laws that are essentially local, i.e. use very little global information, and conjecture that for any given bounded open neighborhood W in $C^{\infty}[0,1]$, and any given $L^2[0,1]$ neighborhood V of the soliton solution, a feedback control law from the family exists that will control any initial condition in W to V asymptotically. We present numerical evidence to show that elements of the family, even occurring very early in its hierarchy, can control "reasonably nice" initial conditions, i.e. sine waves, zero initial state, two solitons, other one solitions etc., to the vicinity of a desirable a priori choosen one solition.

II. Embedding a Pattern Forming Equation is a Mems Array

Let us consider a linear mems array consisting of N actuators. For the sake of simplicity here we assume that the element dynamics are uncoupled from each other. (How one may account for the presence of any local coupling, may be extrapolated rather easily). Let us suppose that the equations of motion of the array are:

$$\dot{\xi}_i \;=\; f(\xi_i) \;+\; g(\xi_i)u_i, \quad i = 1, \cdots, N \tag{1}$$

where, $\xi(i)$ is the state and u_i is the control input to the i^{th} actuator, and f and g are smooth vector fields. Here we assume that all mems elements are identical. Vector fields f and g will depend on the particulars of the mems technology. For example, for piston actuated capacitive mems devices, in a nondimensional form, a lumped parameter model may be written as,

$$\begin{aligned} f(\xi) = \quad &= \begin{bmatrix} \xi_2 \\ -2\tau\omega\xi_2 - \omega^2(\xi_1 - \hat{l}_0) - \xi_3^2/2 \\ -\xi_1\xi_3 \end{bmatrix}, \\ g(\xi) = \quad &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \end{aligned} \tag{2}$$

where the state x= [position, velocity, charge], ω and τ are constants representing the natural frequency and the damping constant respectively, and $\hat{l}_0$ is the free length of the gap between electrodes. Another popular model is the case in which each mems element is written as a second order linear system, in which case $f(\xi) = [\xi_2, -2\tau\omega\xi_2 - \omega^2\xi_1]^T$ and $g(\xi) = [0,1]^T$.

Now suppose that we have chosen a pattern forming ODE to embed,

$$\dot{z}_i \quad = \quad \phi_i(z_1, z_2, \cdots, z_N), \; i = 1, \cdots, N \tag{3}$$

where $z_i \in \Re$. Our problem now is to design a feedback control law such that $|\xi_{i,1}(t) - z_i(t)| \to 0$ asymptotically. In principal one must solve a nonlinear model matching problem, (see e.g. [8]) to embed (6) in (1). However, due to the nature of the equations one can get by with a standard back stepping computation. We will not discuss details of this aspect further. However, we wish to point out that if ϕ_i is local, i.e., depends only in z_i and a few other z_j where j is close to i, then the model matching control law will also be local. If in addition, $\{\phi_i\}_{i=1}^N$ utilize only a few global functions, then the model matching law can be designed to only need a few global functions. Therefore, we are only interested in embedding pattern forming equations which have local only coupling or failing that, predominantly local coupling.

Remark: One may observe that the equations are uncoupled, hence one may produce any desirable pattern by commanding individual actuators to follow requisite paths. However, such a scheme would require external timing signals for the array, hence impeding on autonomous operation. In addition, due to the absence of coupling between elements, one should not anticipate robustness from an uncoupled controller.

III. KdV solitons and lessons from the Fermi-Pasta-Ulam experiment

Pattern forming equation of choice in this is a discretization of a feedback controlled Kortweg de-Vries equation (KdV). First we discuss some relevant features of the KdV equation. We relegate details of the feedback control law to the next section.

The KdV has the form,

$$q_t = -(qq_x + q_{xxx}), \quad (4)$$

where $x \in [0, 1]$, $t \geq 0$ and q_t, q_x, q_{xxx} are respective partial derivatives of q with respect to variables t and x. There is a rich theory of the KdV equation (see e.g. [1], [2]. For the purposes of this paper, the most important fact is that the KdV equation admits travelling pulses as solutions. These, so called one solitons, are parameterized as,

$$q(t, x) = a^2 sech^2(a(x - a^2 t/3 - x_0)/\sqrt{12}), \quad (5)$$

where, $a > 0$ and x_0 are constants. Indeed, there is a soliton hierarchy, i.e, one solitons, two solitons, etc., and elements at the two and higher levels are nonlinear superpositions of one solitons, in the sense that they themselves consist of finitely many moving pulses, and when the pulses are sufficiently far apart, they are well approximated by linear superpositions of one solitons.

Developments of the theory of solitons originated from attempts to explain an apparent anomaly in a simulation experiment carried out by Fermi, Pasta and Ulam in the Los Alamos Maniac I supercomputer (see [9] for a description). Expectation at the time was that nonlinear spring forces in a high dimensional spring-mass array would settle at a thermodynamic equilibrium at which energy would be shared equally by all linear modes. However, simulations showed that energy was confined to a few nonlinear oscillatory modes. This was later explained in the celebrated papers [4], [5]. What Fermi-Pasta-Ulam observed was a solution in the close proximity of an N-soliton. (Simulations of the Fermi-Pasta-Ulam experiment are plenty on the world wide web.) Indeed, it was shown later that for a smooth initial profile most of the energy goes into relatively few soliton states. This is the key to our controller design. For smooth initial conditions, solutions of the KdV equation can be approximated with finitely many parameters, hence these solutions can be approximately controlled from one to another, and in particular to a desired one soliton, by using finite dimensional approximations. Such approximations will be developed in the section V.

IV. Integral Invariants of the KdV

KdV equation on $[0, 1]$ with periodic boundary conditions, admits infinitely many integral invariants [6]. A linearly independent set of them are usually written in a hierarchical form. The first few are (see e.g. [6], [3]):

$$\begin{aligned}
H_1(q) &= \int q dx, \\
H_2(q) &= \int q^2 dx, \\
H_3(q) &= \int (q_x^2 - \frac{1}{3}q^3) dx, \\
H_4(q) &= \int (\frac{9}{5}q_{xx}^2 - 3qq_x^2 + \frac{1}{4}q^4) dx, \\
H_5(q) &= \int (\frac{1}{5}q^5 - 6q^2 q_x^2 + \frac{36}{5}qq_{xx}^2 - \frac{108}{35}q_{xxx}^2) dx, \\
H_6(q) &= \int (\frac{1}{6}q^6 - 10q^3 q_x^2 + 18q^2 q_{xx}^2 - 5q_x^4 \\
&\quad - \frac{108}{7}qq_{xxx}^2 + \frac{120}{7}q_{xx}^3 + \frac{36}{7}q_{xxxx}^2) dx, \quad (6)
\end{aligned}$$

Details on how to compute other elements of the hierarchy may be found in [6], [2] etc.

V. Control Design

Let us write the KdV equation in the form,

$$q_t = F(q),$$

where, $F(q) = -(qq_x + q_{xxx})$. Here $x \in [0, 1]$ and boundary conditions are periodic. Since our objective is to control solutions, let us add a control term $v = v(q)$, and write the controlled equation as,

$$q_t(t, x) = F(q)(t, x) + v(q)(x). \quad (7)$$

We remark here that the aim is to design v as a predominantly local feedback control law, i.e. one which depends on q and a few of its partial derivatives, and a few global functions. In other words we seek $v(q)(x)$ in the form $\theta(q, q_x, \cdots, \partial^k q(x), w_1(q), w_2(q), \cdots, w_l(q))$, where w_i are global functions. The reason for this restriction is that upon spatial discretization partial derivatives of q becomes functions of the state at the element at x and a few of its nearest neighbors. The global functions w_i are common to all elements, hence can be computed by sharing resources.

Let $H_i,\ i = 1, \cdots$ be the integral invariants of the KdV described in section IV. Let us write them as $H_i(q) = \int_0^1 \varphi_i(q, \cdots, \partial^{i-2}q) dx$. By definition, directional derivatives of H_i along F are all equal to zero, hence,

$$\begin{aligned}
\frac{d}{dt}H_i(q) &= \int_0^1 \partial_{u_j}\varphi_i(q, \cdots)\partial^j v(x) dx, \\
&= \int_0^1 \delta\varphi_i(q, \cdots, \partial^{i-2}q) v(x) dx \quad (8)
\end{aligned}$$

where $\delta\varphi_i = \sum_j (-1)^j \partial^j \frac{\partial}{\partial q_j}\varphi_i$, and where q_j denoted the j^{th} partial derivative of q with respect to x. Observe that this sum is finite since φ_i depends only on the first $i-2$ derivatives of q.

Now we may write,

$$\frac{d}{dt}\begin{bmatrix} H_1(q) \\ \vdots \\ H_m(q) \end{bmatrix} = \int_0^1 A(q, q_1, \cdots, q_{2m-2}) v(x) dx, \quad (9)$$

for arbitrary m where, $A(q, q_1, \cdots, q_{2m-2}) = [\delta\varphi_1, \cdots, \varphi_m)]^T$.

Let us suppose that it is desired to stabilize the solution at a desired one soliton q_0. Since any initial condition may be approximated closely by a multi-soliton, it follows that information contained in the initial data can be essentially captured by the values of the first few integral invariants. Thus, the problem of controlling the state from a given initial condition to the vicinity of the desired soliton may be rephrased as a problem of controlling the first few integral invariants from their initial values to their respective values at the desired soliton. This is the heuristic basis of the proposed control law.

Thus, the proposed control law is,

$$v(x) = -A^T(q, \cdots, q_{2m-2})\Lambda \begin{bmatrix} H_1(q) - H_1(q_0) \\ \vdots \\ H_m(q) - H_m(q_0) \end{bmatrix} \quad (10)$$

where, Λ is a positive diagonal matrix, and m is an integer such that $H_1, \cdots, H_m$ captures the initial state reasonably well.

This control law is inspired by the theory of nonlinear output regulation developed by Byrnes and Isidori [8]. We wish to remark that, while it is obvious that the control law stabilizes $[H_1, \cdots, H_m]$ at $[H_1(q_0), \cdots, H_m(q_0)]$, it does not prove that the solutions will converge for the reason that the complementary dynamics, (on may think of this as zero dynamics of outputs $H_1, \cdots, H_m$) are only stable, and not necessarily asymptotically stable. Indeed it is possible that the control law may destabilize the complimentary dynamics. Proving that this isn't the case is a challenge which we haven't succeeded in resolving as of yet. In the section VI we provide numerical evidence to support the conjecture that for sufficiently large m the control law will control the state to an arbitrarily small neighborhood of the desired soliton.

VI. Simulation Evidence

In this section we describe simulation results that seem to support the conjecture that the proposed control law indeed controls a reasonable initial state to the vicinity of a desired soliton. In the simulations, we have chosen following parameters in the controller (10).

$$\begin{aligned} m &= 5 \\ \Lambda &= \mathrm{diag}[1, 1, 0.1, 10^{-5}, 10^{-6}] \\ q_0(t,x) &= a^2 sech^2(a(x - a^2t/3)/\sqrt{12}) \\ a &= 3.5. \end{aligned}$$

Instead of $x \in [0,1]$ we hav taken $x \in [0,15]$. This larger spatial domain allows to work with a fewer number of integral invariants in computing control laws. Numerical simulations of the controlled system are carried out by first discretizing the system spatially using symmetric difference method with a step size of 0.2 and integrating the resulting system of ordinary differential equations using the 4th order Runge-Kutta routine using Matlab. Initial states chosen were (a) a sum of two gaussians, (b) zero initial state, and (c) a half sine wave (as in the Fermi-Pasta-Ulam experiment).

VII. Concluding Remarks

Whereas the control methodology is explained for embedded KdV solitons, the same procedure works for the Toda lattice, and other soliton equations of mathematical physics. Soliton control for the Toda lattice would be essentially the same, however embedding in a mems array would be slightly more complex since Toda lattice dynamics are second order.

We have provided no proofs of practical asymptotic stability of the soliton orbit here. We are vigorously pursuing this and other related theoretical aspects.

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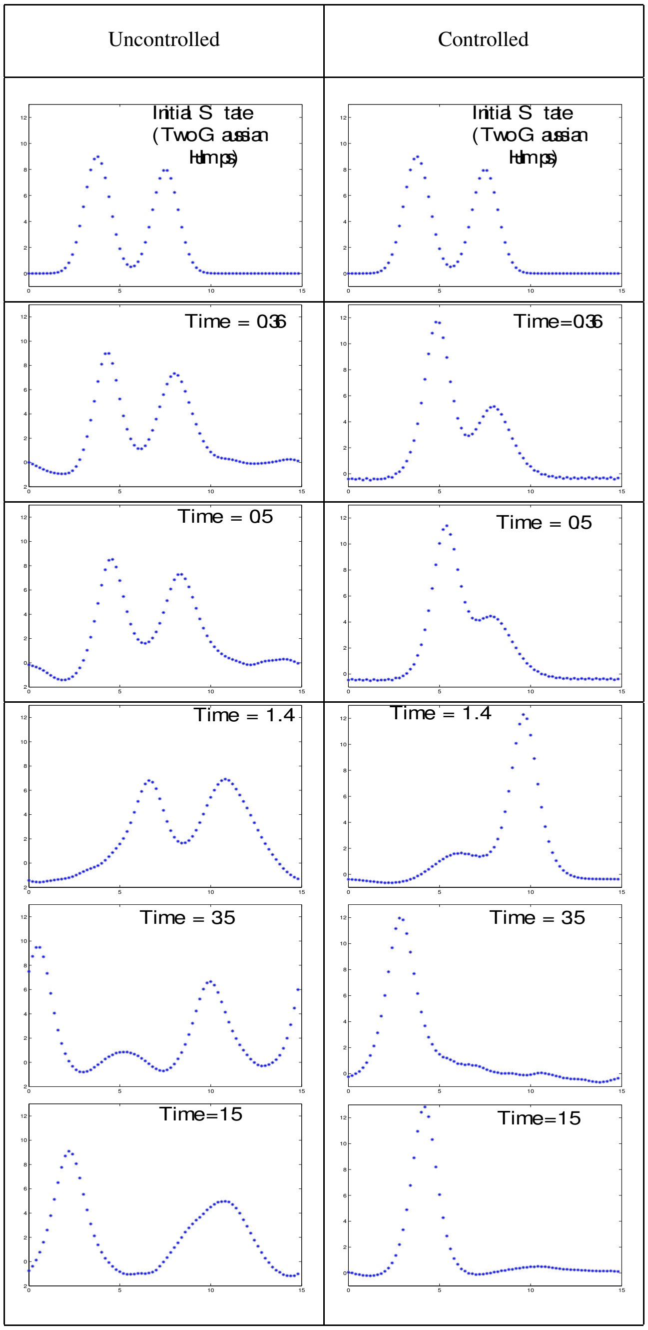


TABLE I

SOLITON STABILIZATION LAW WITH TWO GAUSSIAN HUMPS AS THE INITIAL STATE. HORIZONTAL AXIS REPRESENT MEMS ELEMENT LOCATIONS. VERTICAL AXIS REPRESENT DISPLACEMENT OF EACH ELEMENT. TIME EVOLUTION IS INDICATED THE THE FIGURES.

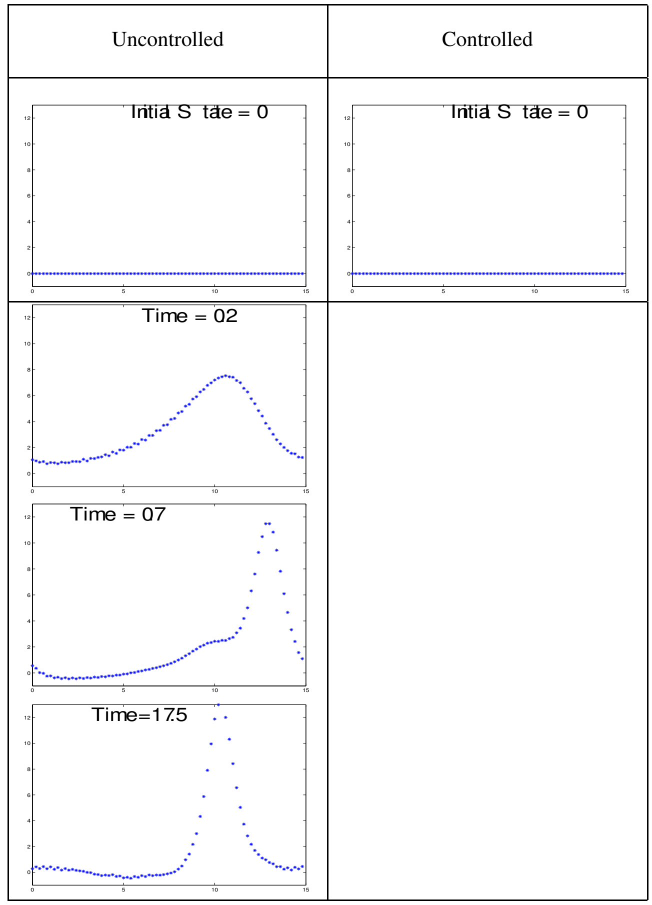


TABLE II

SOLITON STABILIZATION FROM ZERO INITIAL STATE. HORIZONTAL AXIS REPRESENT MEMS ELEMENT LOCATIONS. VERTICAL AXIS REPRESENT DISPLACEMENT OF EACH ELEMENT. TIME EVOLUTION IS INDICATED THE THE FIGURES.

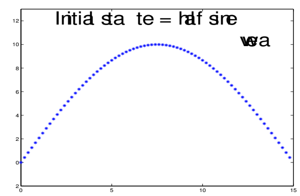
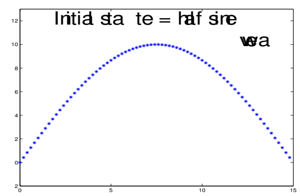
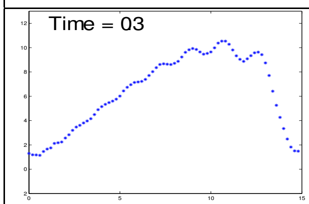
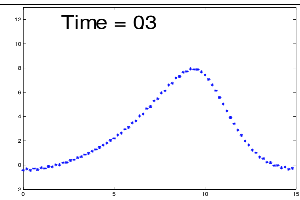
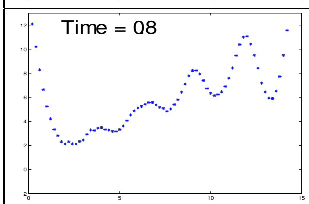
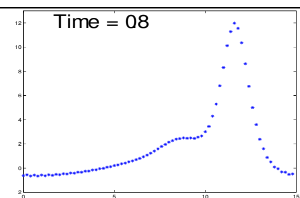
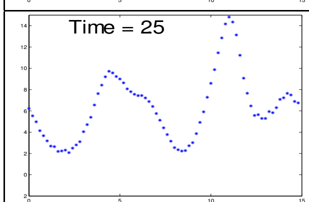
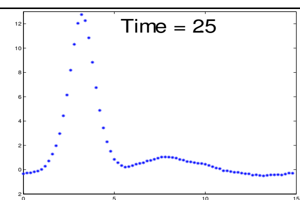
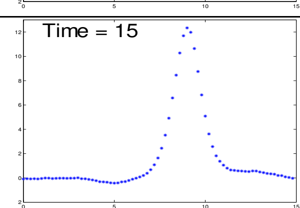
Uncontrolled	Controlled
	
	
	
	
	

TABLE III

SOLITON STABILIZATION FROM INITIAL STATE OF HALF SINE WAVE. HORIZONTAL AXIS REPRESENT MEMS ELEMENT LOCATIONS. VERTICAL AXIS REPRESENT DISPLACEMENT OF EACH ELEMENT. TIME EVOLUTION IS INDICATED THE THE FIGURES.