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# Construction and sample path properties of Brownian house-moving between two curves

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**Abstract**

This study aims to construct a stochastic process called "Brownian house-moving," which is a Brownian bridge conditioned to stay between two curves. To construct this process, statements are prepared on the weak convergence of conditioned Brownian motions, conditioned Brownian bridges, and conditioned three-dimensional Bessel bridges. Moreover, the sample path properties of Brownian house-moving are studied as well.

## 1 Introduction

Recently, [5] developed a chain rule for Wiener path integrals between two curves that arise in the computation of first-order Greeks for barrier options, and demonstrated the effectiveness of this chain rule through numerical examples. In this chain rule, Brownian meander and BES(3)-bridge conditioned to stay between two curves played an important role. Furthermore, we are currently investigating higher-order chain rules for computing higher-order Greeks for barrier options, and we expect a stochastic process called "Brownian house-moving" to play an important role in their computation. A Brownian house-moving is defined as a Brownian bridge conditioned to stays between two curves. The purpose of this study is to construct these stochastic processes.

The remainder of this paper is organized as follows. In Section 2, we present the notation used in this study. Section 3 states the main results of this study. In Subsection 3.1, we construct the Brownian house-moving (Theorem 1). In addition, the sample path properties of Brownian house-moving (Corollaries 2, 3 and Theorem 2) are provided in this subsection. In Subsection 3.2, we construct the Brownian meander between two curves. In Subsection 3.3, we construct the BES(3)-bridge between two curves. To construct the Brownian house-moving, we state in Section 4 that a one-dimensional Brownian bridge conditioned to stay in $[-\varepsilon, \infty)$ converges weakly to a BES(3)-bridge as $\varepsilon \downarrow 0$. In Section 5, we prove the results for the distribution of the maximal value of the BES(3)-bridge used in this study. Sections 6, 7, 8, 9, and 10 are devoted to proving the main results in Section 3.

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## 2 Notation

For $0 \le s < t \le 1$, let $C([s,t],\mathbb{R})$ be the class of $\mathbb{R}$-valued continuous functions defined on $[s,t]$, and let

$$d_\infty(w,w') = \sup_{u\in[s,t]} |w(u) - w'(u)| \quad (w,w' \in C([s,t],\mathbb{R})).$$

$\mathcal{B}(C([s,t],\mathbb{R}))$ denotes the Borel $\sigma$-algebra with respect to the topology generated by the metric $d_\infty$. In addition, for $0 \le s < t \le 1$, $\pi_{[s,t]} : C([0,1],\mathbb{R}) \to C([s,t],\mathbb{R})$ denotes the restriction map.

Assume that $Y : (\Omega,\mathcal{F},P) \to (C([0,1],\mathbb{R}),\mathcal{B}(C([0,1],\mathbb{R})))$ is a random variable and $\Lambda \in \mathcal{B}(C([0,1],\mathbb{R}))$ satisfies $P(Y \in \Lambda) > 0$. Then, we define the probability measure $P_{Y^{-1}(\Lambda)}$ on $(Y^{-1}(\Lambda), Y^{-1}(\Lambda)\cap\mathcal{F})$ as

$$P_{Y^{-1}(\Lambda)}(A) := \frac{P(A)}{P(Y\in\Lambda)}, \qquad A \in Y^{-1}(\Lambda)\cap\mathcal{F} := \left\{Y^{-1}(\Lambda)\cap F \mid F \in \mathcal{F}\right\}.$$

Let $Y|_\Lambda$ denote the restriction $Y$ to $(Y^{-1}(\Lambda), Y^{-1}(\Lambda)\cap\mathcal{F}, P_{Y^{-1}(\Lambda)})$. Then,

$$Y|_\Lambda : (Y^{-1}(\Lambda), Y^{-1}(\Lambda)\cap\mathcal{F}, P_{Y^{-1}(\Lambda)}) \to (\Lambda,\mathcal{B}(\Lambda))$$

is a random variable. Throughout this study, $P_{Y^{-1}(\Lambda)}(Y|_\Lambda \in \Gamma)$ is often written as $P(Y|_\Lambda \in \Gamma)$, and $E^{P_{Y^{-1}(\Lambda)}}[f(Y|_\Lambda)]$ is often written as $E[f(Y|_\Lambda)]$.

For $s > 0$, we define

$$n_s(x) := \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{x^2}{2s}\right) \qquad (x \in \mathbb{R}).$$

$X_n \stackrel{\mathcal{D}}{\to} X$ denotes the convergence in distribution of the sequence of random variables $\{X_n\}_{n=1}^\infty$ to the random variable $X$. In addition, we write $X \stackrel{\mathcal{D}}{=} Y$ for random variables $X, Y$ that follow the same distribution.

Let $0 \le t_1 < t_2 \le 1$. Throughout this study, we use the following notation.

For $f, g \in C([0,1],\mathbb{R})$, we define

$$K_{[t_1,t_2]}(f,g) := \{w = \{w(t)\}_{t\in[t_1,t_2]} \in C([t_1,t_2],\mathbb{R}) \mid f(t) \le w(t) \le g(t),\ t_1 \le t \le t_2\},$$
$$K^+_{[t_1,t_2]}(f) := \bigcup_{n=1}^\infty K_{[t_1,t_2]}(f,n), \qquad K^-_{[t_1,t_2]}(g) := \bigcup_{n=1}^\infty K_{[t_1,t_2]}(-n,g),$$

and

$$K(f,g) := K_{[0,1]}(f,g), \qquad K^+(f) := K^+_{[0,1]}(f), \qquad K^-(g) := K^-_{[0,1]}(g),$$
$$K_t(f,g) := K_{[0,t]}(f,g), \qquad K^+_t(f) := K^+_{[0,t]}(f), \qquad K^-_t(g) := K^-_{[0,t]}(g).$$

For an $\mathbb{R}$-valued continuous process $X = \{X(t)\}_{t\in[0,1]}$, we write its maximal and minimal values as

$$M_{[t_1,t_2]}(X) = \max_{t_1\le u\le t_2} X(u), \qquad M_t(X) = M_{[0,t]}(X), \qquad M(X) = M_{[0,1]}(X),$$
$$m_{[t_1,t_2]}(X) = \min_{t_1\le u\le t_2} X(u), \qquad m_t(X) = m_{[0,t]}(X), \qquad m(X) = m_{[0,1]}(X).$$

Moreover, the natural filtration $\sigma(X(s) \mid 0 \le s \le t)$ of $X$ is denoted by $\mathcal{F}_t^X$.

$W = \{W(t)\}_{t\ge 0}$, $B^{a\to b} = \{B^{a\to b}(t)\}_{t\in[0,1]}$ $(a, b \in \mathbb{R})$, $W^+ = \{W^+(t)\}_{t\in[0,1]}$, and $r^{c\to d} = \{r^{c\to d}(t)\}_{t\in[0,1]}$ $(c, d \ge 0)$ denote standard one-dimensional Brownian motion, one-dimensional Brownian bridge from $a$ to $b$ on the time interval $[0, 1]$, Brownian meander on the time interval $[0, 1]$, and BES(3)-bridge from $c$ to $d$ on the time interval $[0, 1]$ defined on some probability space, respectively. For $a, b \in \mathbb{R}$ and $c, d \ge 0$, $W_{[t_1,t_2]}$, $B^{a\to b}_{[t_1,t_2]}$, $W^+_{[t_1,t_2]}$ and $r^{c\to d}_{[t_1,t_2]}$ denote one-dimensional Brownian motion, one-dimensional Brownian bridge from $a$ to $b$, Brownian meander, and BES(3)-bridge from $c$ to $d$ defined on $[t_1, t_2]$, respectively. Laws of $W_{[t_1,t_2]}$, $B^{a\to b}_{[t_1,t_2]}$, $W^+_{[t_1,t_2]}$, and $r^{c\to d}_{[t_1,t_2]}$ are given by

$$\begin{aligned}
\{W_{[t_1,t_2]}(u)\}_{u\in[t_1,t_2]} &\overset{\mathcal{D}}{=} \{W(u-t_1)\}_{u\in[t_1,t_2]}, \\
\left\{B^{a\to b}_{[t_1,t_2]}(u)\right\}_{u\in[t_1,t_2]} &\overset{\mathcal{D}}{=} \left\{ \sqrt{t_2-t_1}\, B^{\frac{a}{\sqrt{t_2-t_1}}\to\frac{b}{\sqrt{t_2-t_1}}}\left(\frac{u-t_1}{t_2-t_1}\right)\right\}_{u\in[t_1,t_2]}, \\
\left\{W^+_{[t_1,t_2]}(u)\right\}_{u\in[t_1,t_2]} &\overset{\mathcal{D}}{=} \left\{ \sqrt{t_2-t_1}\, W^+\left(\frac{u-t_1}{t_2-t_1}\right)\right\}_{u\in[t_1,t_2]}, \\
\left\{r^{c\to d}_{[t_1,t_2]}(u)\right\}_{u\in[t_1,t_2]} &\overset{\mathcal{D}}{=} \left\{ \sqrt{t_2-t_1}\, r^{\frac{c}{\sqrt{t_2-t_1}}\to\frac{d}{\sqrt{t_2-t_1}}}\left(\frac{u-t_1}{t_2-t_1}\right)\right\}_{u\in[t_1,t_2]}.
\end{aligned}$$

# 3 Main results

Let $g^-$ and $g^+$ be $\mathbb{R}$-valued $C^2$-functions defined on $[0, 1]$ that satisfy

$$\min_{0\le t\le 1}(g^+(t) - g^-(t)) > 0.$$

We assume that $\{\eta(\varepsilon)\}_{\varepsilon>0}$ satisfies

$$\eta(\varepsilon) \ge 0 \quad (\varepsilon > 0) \quad \text{and} \quad \eta(\varepsilon) \downarrow 0 \quad (\varepsilon \downarrow 0).$$

Let $0 \le t_1 < t_2 \le 1$. According to the values $g^-(t_1) \le a \le g^+(t_1)$ and $g^-(t_2) \le b \le g^+(t_2)$, the continuous process $X^{a,b,(g^-,g^+)}_{[t_1,t_2]}$ on $[t_1, t_2]$ is defined as follows (see also Lemma 3.1 below):

- in the case $a = g^-(t_1), b < g^+(t_2)$, the weak limit of $B^{a\to b}_{[t_1,t_2]}|_{K_{[t_1,t_2]}(g^- -\varepsilon, g^+ +\eta(\varepsilon))}$ as $\varepsilon \downarrow 0$;
- in the case $a > g^-(t_1), b = g^+(t_2)$, the weak limit of $B^{a\to b}_{[t_1,t_2]}|_{K_{[t_1,t_2]}(g^- -\eta(\varepsilon), g^+ +\varepsilon)}$ as $\varepsilon \downarrow 0$;
- in the case $g^-(t_1) < a < g^+(t_1), g^-(t_2) < b < g^+(t_2)$, the conditioned process $B^{a\to b}_{[t_1,t_2]}|_{K_{[t_1,t_2]}(g^-,g^+)}$.

In addition, according to the value $g^-(t_1) \le a < g^+(t_1)$, the continuous process $X^{a,(g^-,g^+)}_{[t_1,t_2]}$ on $[t_1, t_2]$ is defined as follows (see also Lemma 3.2 below):

- in the case $g^-(t_1) = a$, the weak limit of $(a + W_{[t_1,t_2]})\,|_{K_{[t_1,t_2]}(g^- -\varepsilon, g^+ +\eta(\varepsilon))}$ as $\varepsilon \downarrow 0$;
- in the case $g^-(t_1) < a$, the conditioned process $(a + W_{[t_1,t_2]})\,|_{K_{[t_1,t_2]}(g^-,g^+)}$.

For an $\mathbb{R}$-valued continuous process $X$ on $[t_1, t_2]$ and $\mathbb{R}$-valued $C^2$-function $g$ defined on $[t_1, t_2]$, we define

$$Z^g_{[t_1,t_2]}(X) := \exp\left\{g'(t_2)X(t_2) - g'(t_1)X(t_1) - \int_{t_1}^{t_2} X(u)g''(u)du - \frac{1}{2}\int_{t_1}^{t_2} g'(u)^2 du\right\}.$$

Therefore, if $X$ is $W_{[t_1,t_2]}$, then it follows from Itô's formula that

$$Z^g_{[t_1,t_2]}(W_{[t_1,t_2]}) = \exp\left\{\int_{t_1}^{t_2} g'(u)dW_{[t_1,t_2]}(u) - \frac{1}{2}\int_{t_1}^{t_2} g'(u)^2 du\right\}.$$

For ease of later computations, we define $\widetilde{Z}^g_{[t_1,t_2]}(X) := Z^g_{[t_1,t_2]}(X + g)$.

For $f \in C([t_1, t_2], \mathbb{R})$, we define $\overleftarrow{f} \in C([t_1, t_2], \mathbb{R})$ as

$$\overleftarrow{f}(t) := f(t_1 + t_2 - t), \quad t_1 \le t \le t_2.$$

**Lemma 3.1.** *Let $0 \le t_1 < t_2 \le 1$. $X^{a,b,(g^-,g^+)}_{[t_1,t_2]}$ exists and its distribution is given as follows. For every $\mathbb{R}$-valued bounded continuous function $F$ on $C([t_1, t_2], \mathbb{R})$,*

(1) *if $a = g^-(t_1)$, $g^-(t_2) \le b < g^+(t_2)$, then*

$$E\left[F(X^{a,b,(g^-,g^+)}_{[t_1,t_2]})\right] = \frac{E\left[F\left(r^{0\to b-g^-(t_2)}_{[t_1,t_2]}\Big|_{K^-_{[t_1,t_2]}(g^+-g^-)} + g^-\right)\widetilde{Z}^{g^- - a}_{[t_1,t_2]}\left(r^{0\to b-g^-(t_2)}_{[t_1,t_2]}\Big|_{K^-_{[t_1,t_2]}(g^+-g^-)}\right)^{-1}\right]}{E\left[\widetilde{Z}^{g^- - a}_{[t_1,t_2]}\left(r^{0\to b-g^-(t_2)}_{[t_1,t_2]}\Big|_{K^-_{[t_1,t_2]}(g^+-g^-)}\right)^{-1}\right]}, \tag{1}$$

(2) *if $g^-(t_1) < a \le g^+(t_1)$, $b = g^+(t_2)$, then*

$$E\left[F(X^{a,b,(g^-,g^+)}_{[t_1,t_2]})\right] = \frac{E\left[F\left(g^+ - \overleftarrow{r}^{0\to g^+(t_1)-a}_{[t_1,t_2]}\Big|_{K^-_{[t_1,t_2]}(g^+-g^-)}\right)\widetilde{Z}^{b-\overleftarrow{g}^+}_{[t_1,t_2]}\left(r^{0\to g^+(t_1)-a}_{[t_1,t_2]}\Big|_{K^-_{[t_1,t_2]}(\overleftarrow{g}^+-\overleftarrow{g}^-)}\right)^{-1}\right]}{E\left[\widetilde{Z}^{b-\overleftarrow{g}^+}_{[t_1,t_2]}\left(r^{0\to g^+(t_1)-a}_{[t_1,t_2]}\Big|_{K^-_{[t_1,t_2]}(\overleftarrow{g}^+-\overleftarrow{g}^-)}\right)^{-1}\right]}, \tag{2}$$

*where $\overleftarrow{r}^{0\to g^+(t_1)-a}_{[t_1,t_2]}$ denotes the continuous process $\{r^{0\to g^+(t_1)-a}_{[t_1,t_2]}(t_1 + t_2 - t)\}_{t\in[t_1,t_2]}$.*

**Lemma 3.2.** *Let $0 \le t_1 < t_2 \le 1$ and $a = g^-(t_1)$. $X^{a,(g^-,g^+)}_{[t_1,t_2]}$ exists and its distribution is given as follows. For every $\mathbb{R}$-valued bounded continuous function $F$ on $C([t_1, t_2], \mathbb{R})$,*

$$E\left[F(X^{a,(g^-,g^+)}_{[t_1,t_2]})\right] = \frac{E\left[F(W^+_{[t_1,t_2]}|_{K^-_{[t_1,t_2]}(g^+-g^-)} + g^-)\widetilde{Z}^{g^- - a}_{[t_1,t_2]}(W^+_{[t_1,t_2]}|_{K^-_{[t_1,t_2]}(g^+-g^-)})^{-1}\right]}{E\left[\widetilde{Z}^{g^- - a}_{[t_1,t_2]}(W^+_{[t_1,t_2]}|_{K^-_{[t_1,t_2]}(g^+-g^-)})^{-1}\right]}. \tag{3}$$

REMARK 3.1. *Let $A$ be a closed subset of $C([t_1, t_2], \mathbb{R})$, and let*

$$d_\infty(w, A) := \inf\{d_\infty(w, v) \mid v \in A\} \quad (w \in C([t_1, t_2], \mathbb{R})),$$

$$\varphi(x) := 1 - \int_0^1 1_{(-\infty,x]}(u)du \quad (x \in \mathbb{R}), \qquad F_n(w) := \varphi(nd_\infty(w, A)) \quad (w \in C([t_1, t_2], \mathbb{R})).$$

*Then, $F_n$ is a bounded continuous function on $C([t_1, t_2], \mathbb{R})$ and satisfies*

$$F_n(w) \downarrow 1_A(w), \qquad n \to \infty$$

*for $w \in C([t_1, t_2], \mathbb{R})$. Thus, the dominated convergence theorem implies that Lemmas 3.1 and 3.2 hold true for $F = 1_A$. Let $B \in \mathcal{B}(C([t_1, t_2], \mathbb{R}))$. Then, it follows from Dynkin's $\pi$-$\lambda$ theorem that Lemmas 3.1 and 3.2 hold true for $F = 1_B$.*

Further, we present the notation used in Subsections 3.1, 3.2 and 3.3.

Let $t_0 \in (t_1, t_2)$. For $w_1 \in C([t_1, t_0], \mathbb{R})$ and $w_2 \in C([t_0, t_2], \mathbb{R})$ that satisfy $w_1(t_0) = w_2(t_0)$, we define $w_1 \oplus_{t_0} w_2 \in C([t_1, t_2], \mathbb{R})$ as

$$(w_1 \oplus_{t_0} w_2)(t) := \begin{cases} w_1(t), & t_1 \le t \le t_0, \\ w_2(t), & t_0 \le t \le t_2. \end{cases}$$

For $0 < t < 1$, $0 \le t_1 < t_2 \le 1$ and $y \in (g^-(t), g^+(t))$, $y_i \in (g^-(t_i), g^+(t_i))$ $(i = 1, 2)$, we define

$$q_{[0,t]}^{(g^-,g^+),(\uparrow)}(y) = E\Big[\widetilde{Z}_{[0,t]}^{g^- - g^-(0)}\big(r_{[0,t]}^{0\to y - g^-(t)}|_{K^-_{[0,t]}(g^+ - g^-)}\big)^{-1}\Big] P\Big(r_{[0,t]}^{0\to y-g^-(t)} \in K^-_{[0,t]}(g^+ - g^-)\Big)\frac{P(W^+_{[0,t]}(t) \in dy - g^-(t))}{dy},$$

$$q_{[t,1]}^{(g^-,g^+),(\downarrow)}(y) = E\Big[\widetilde{Z}_{[t,1]}^{g^+(1) - \overleftarrow{g}^+}\Big(r_{[t,1]}^{0\to g^+(t) - y}|_{K^-_{[t,1]}(\overleftarrow{g}^+ - \overleftarrow{g}^-)}\Big)^{-1}\Big] P\Big(r_{[t,1]}^{0\to g^+(t)-y} \in K^-_{[t,1]}(\overleftarrow{g}^+ - \overleftarrow{g}^-)\Big)\frac{P(W^+_{[t,1]}(1) \in g^+(t) - dy)}{dy},$$

$$p_{[t_1,t_2]}^{(g^-,g^+)}(y_1) = P(y_1 + W_{[t_1,t_2]} \in K_{[t_1,t_2]}(g^-, g^+)),$$

$$p_{[t_1,t_2]}^{(g^-,g^+)}(y_1, y_2) = P(y_1 + W_{[t_1,t_2]} \in K_{[t_1,t_2]}(g^-, g^+), y_1 + W_{[t_1,t_2]}(t_2) \in dy_2)/dy_2.$$

## 3.1 Construction and sample path properties of Brownian house-moving

In this subsection, we define $b := g^+(1)$ and assume that $g^-(0) = 0$.

Assume that $\{\eta^-(\varepsilon)\}_{\varepsilon>0}$ and $\{\eta^+(\varepsilon)\}_{\varepsilon>0}$ satisfy

$$\eta^\pm(\varepsilon) > 0 \quad (\varepsilon > 0) \quad \text{and} \quad \eta^\pm(\varepsilon) \downarrow 0 \quad (\varepsilon \downarrow 0).$$

For $0 < t < 1$, $0 < t_1 < t_2 < 1$ and $y \in (g^-(t), g^+(t))$, $y_i \in (g^-(t_i), g^+(t_i))$ $(i = 1, 2)$, we define

$$h(t, y) = (C_{g^-,g^+})^{-1}\frac{1}{\sqrt{t}}q_{[0,t]}^{(g^-,g^+),(\uparrow)}(y)\frac{1}{\sqrt{1-t}}q_{[t,1]}^{(g^-,g^+),(\downarrow)}(y), \quad h(t_1, y_1, t_2, y_2) = \frac{p_{[t_1,t_2]}^{(g^-,g^+)}(y_1, y_2)\frac{1}{\sqrt{1-t_2}}q_{[t_2,1]}^{(g^-,g^+),(\downarrow)}(y_2)}{\frac{1}{\sqrt{1-t_1}}q_{[t_1,1]}^{(g^-,g^+),(\downarrow)}(y_1)},$$

where

$$C_{g^-,g^+} := \frac{\pi n_1(b)}{2}\lim_{\varepsilon\downarrow 0}\frac{P(B_{[0,1]}^{0\to b} \in K_{[0,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon)))}{\eta^-(\varepsilon)\eta^+(\varepsilon)}. \tag{4}$$

Our aim in this subsection is to prove the existence of the weak limit of $B_{[0,1]}^{0\to b}|_{K_{[0,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon))}$ as $\varepsilon \downarrow 0$. $H^{g^-\to g^+}$ denotes this weak limit. In this study, we call $H^{g^-\to g^+}$ "Brownian house-moving."

**Theorem 1.** *There exists an $\mathbb{R}$-valued continuous Markov process $H^{g^-\to g^+} = \{H^{g^-\to g^+}(t)\}_{t\in[0,1]}$ that satisfies*

$$\begin{aligned} E\left[F(H^{g^-\to g^+})\right] & \\ &= \lim_{\varepsilon\downarrow 0} E[F(B^{0\to b}_{[0,1]}|_{K_{[0,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon))})] \quad (5)\\ &= \int_{g^-(t)}^{g^+(t)} E\left[F(X^{0,y,(g^-,g^+)}_{[0,t]} \oplus_t X^{y,b,(g^-,g^+)}_{[t,1]})\right] h(t,y)dy \quad (6)\\ &= \int_{g^-(t_1)}^{g^+(t_1)}\int_{g^-(t_2)}^{g^+(t_2)} E[F(X^{0,y_1,(g^-,g^+)}_{[0,t_1]} \oplus_{t_1} X^{y_1,y_2,(g^-,g^+)}_{[t_1,t_2]} \oplus_{t_2} X^{y_2,b,(g^-,g^+)}_{[t_2,1]})]h(t_1,y_1)h(t_1,y_1,t_2,y_2)dy_1dy_2 \quad (7) \end{aligned}$$

*for every $\mathbb{R}$-valued bounded continuous function $F$ on $C([0,1],\mathbb{R})$, $0 < t < 1$ and $0 < t_1 < t_2 < 1$, where the respective processes that appear in* (6) *and* (7) *are independent of each other. Moreover, for $0 < t < 1$, $0 < t_1 < t_2 < 1$ and $y \in (g^-(t), g^+(t))$, $y_i \in (g^-(t_i), g^+(t_i))$ $(i = 1,2)$, the transition densities for $H^{g^-\to g^+}$ are given by*

$$P(H^{g^-\to g^+}(t) \in dy) = h(t,y)dy, \quad P(H^{g^-\to g^+}(t_2) \in dy_2 \mid H^{g^-\to g^+}(t_1) = y_1) = h(t_1,y_1,t_2,y_2)dy_2.$$

For $0 < t < 1$, $0 < t_1 < t_2 < 1$, $y, y_1, y_2 \in \mathbb{R}$ and $\eta > 0$, we define

$$\begin{aligned} J^{(\eta)}(t,y) &:= \sum_{k=-\infty}^{\infty} \frac{2(y+2k\eta)}{t} n_t(y+2k\eta),\\ \overline{J}^{(\eta)}(t,y) &:= \frac{\partial}{\partial\eta} J^{(\eta)}(t,y) = 4\sum_{k=-\infty}^{\infty} k\left(\frac{1}{t} - \frac{(y+2k\eta)^2}{t^2}\right) n_t(y+2k\eta),\\ J^{(\eta)}(t_1,y_1,t_2,y_2) &:= \sum_{k=-\infty}^{\infty} (n_{t_2-t_1}(y_2-y_1+2k\eta) - n_{t_2-t_1}(y_2+y_1+2k\eta)). \end{aligned}$$

Applying Theorem 1 (6) for $g^- \equiv 0$ and $g^+ \equiv b$, we obtain the next corollary.

**Corollary 1.** *Let $b > 0$. It holds for every $\mathbb{R}$-valued bounded continuous function $F$ on $C([0,1],\mathbb{R})$ that*

$$\begin{aligned} E\left[F(H^{0\to b})\right] &= \lim_{\varepsilon\downarrow 0} E[F(B^{0\to b}_{[0,1]}|_{K_{[0,1]}(-\eta^-(\varepsilon), b+\eta^+(\varepsilon))})]\\ &= \int_0^b E\left[F\left(r^{0\to y}_{[0,t]}|_{K^-_{[0,t]}(b)} \oplus_t \left(b - \overleftarrow{r}^{0\to b-y}_{[t,1]}|_{K^-_{[t,1]}(b)}\right)\right)\right] P\left(H^{0\to b}(t) \in dy\right), \qquad 0 < t < 1, \end{aligned}$$

*where $r^{0\to y}_{[0,t]}|_{K^-_{[0,t]}(b)}$ and $\overleftarrow{r}^{0\to b-y}_{[t,1]}|_{K^-_{[t,1]}(b)}$ are chosen to be independent. Moreover, for $0 < s < t < 1$ and $x, y \in (0,b)$, the transition densities for $H^{0\to b}$ are given by*

$$\begin{aligned} P\left(H^{0\to b}(t) \in dy\right) &= \frac{J^{(b)}(t,y)\, J^{(b)}(1-t,b-y)}{\overline{J}^{(b)}(1,b)} dy,\\ P\left(H^{0\to b}(t) \in dy \mid H^{0\to b}(s) = x\right) &= \frac{J^{(b)}(s,x,t,y)\, J^{(b)}(1-t,b-y)}{J^{(b)}(1-s,b-x)} dy. \end{aligned}$$

REMARK 3.2. *Let $B \in \mathcal{B}(C([0,1],\mathbb{R}))$ be a measurable subset of $C([0,1],\mathbb{R})$. Then, it follows from the same argument in Remark 3.1 that Theorem 1 and Corollary 1 hold true for $F = 1_B$.*

**Corollary 2.** *Let $g$ be an $\mathbb{R}$-valued $C^1$-function defined on $[0,1]$ that satisfies*

$$g^-(t) < g(t) \le g^+(t), \quad 0 \le t \le 1.$$

*Then, for $t \in (0,1)$ and $g^-(t) \le z \le g(t)$, we have*

$$P\left(\min_{u\in[0,t]}\left\{g(u) - H^{g^-\to g^+}(u)\right\} = 0\right) = 0,$$
$$P\left(\min_{u\in[0,t]}\left\{g(u) - H^{g^-\to g^+}(u)\right\} \ge 0, H^{g^-\to g^+}(t) \le z\right) = \int_{g^-(t)}^{z} (C_{g^-,g^+})^{-1} \frac{1}{\sqrt{t}} q_{[0,t]}^{(g^-,g),(\uparrow)}(y) \frac{1}{\sqrt{1-t}} q_{[t,1]}^{(g^-,g^+),(\downarrow)}(y)dy.$$

**Corollary 3.** *Let $g$ be an $\mathbb{R}$-valued $C^1$-function defined on $[0,1]$ that satisfies*

$$g^-(t) \le g(t) < g^+(t), \quad 0 \le t \le 1.$$

*Then, for $t \in (0,1)$ and $g(t) \le z \le g^+(t)$, we have*

$$P\left(\min_{u\in[t,1]}\left\{H^{g^-\to g^+}(u) - g(u)\right\} = 0\right) = 0,$$
$$P\left(\min_{u\in[t,1]}\left\{H^{g^-\to g^+}(u) - g(u)\right\} \ge 0, H^{g^-\to g^+}(t) \le z\right) = \int_{g(t)}^{z} (C_{g^-,g^+})^{-1} \frac{1}{\sqrt{t}} q_{[0,t]}^{(g^-,g^+),(\uparrow)}(y) \frac{1}{\sqrt{1-t}} q_{[t,1]}^{(g,g^+),(\downarrow)}(y)dy.$$

REMARK 3.3. *Let $t \in (0,1)$. Applying Corollary 2 for $g = g^+$, we obtain*

$$P\left(m_{[0,t]}(g^+ - H^{g^-\to g^+}) = 0\right) = 0,$$
$$P\left(m_{[0,t]}(g^+ - H^{g^-\to g^+}) \ge 0\right) = P\left(m_{[0,t]}(g^+ - H^{g^-\to g^+}) \ge 0, H^{g^-\to g^+}(t) \le g^+(t)\right) = \int_{g^-(t)}^{g^+(t)} h(t,y)dy = 1,$$
$$P\left(m_{[0,t]}(g^+ - H^{g^-\to g^+}) > 0\right) = P\left(m_{[0,t]}(g^+ - H^{g^-\to g^+}) \ge 0\right) - P\left(m_{[0,t]}(g^+ - H^{g^-\to g^+}) = 0\right) = 1.$$

*On the other hand, applying Corollary 3 for $g = g^-$, we obtain*

$$P\left(m_{[t,1]}(H^{g^-\to g^+} - g^-) = 0\right) = 0,$$
$$P\left(m_{[t,1]}(H^{g^-\to g^+} - g^-) \ge 0\right) = P\left(m_{[t,1]}(H^{g^-\to g^+} - g^-) \ge 0, H^{g^-\to g^+}(t) \le g^+(t)\right) = \int_{g^-(t)}^{g^+(t)} h(t,y)dy = 1,$$
$$P\left(m_{[t,1]}(H^{g^-\to g^+} - g^-) > 0\right) = P\left(m_{[t,1]}(H^{g^-\to g^+} - g^-) \ge 0\right) - P\left(m_{[t,1]}(H^{g^-\to g^+} - g^-) = 0\right) = 1.$$

*Therefore, Brownian house-moving $H^{g^-\to g^+}$ satisfies*

$$P\left(\bigcap_{n\ge 2}\left\{\min_{0\le u\le 1-1/n}(g^+(u) - H^{g^-\to g^+}(u)) > 0, \ \min_{1/n\le u\le 1}(H^{g^-\to g^+}(u) - g^-(u)) > 0\right\}\right) = 1.$$

Let $t \in (0,1)$. Applying Theorem 1 (5) and a change of measure formula between Brownian meander and BES(3)-process ([4]), we obtain the Radon-Nikodym derivative of $\pi_{[0,t]} \circ H^{g^- \to g^+}$ with respect to $R_{[0,t]} + g^-$.

**Theorem 2.** *Let $t \in (0,1)$, and let $R_{[0,t]} = \{R_{[0,t]}(u)\}_{u \in [0,t]}$ be the BES(3)-process starting at 0 on $[0,t]$. Then, it holds that*

$$\frac{d\left(P \circ (\pi_{[0,t]} \circ H^{g^- \to g^+})^{-1}\right)}{d\left(P \circ (R_{[0,t]} + g^-)^{-1}\right)}(w)$$
$$= \sqrt{\frac{\pi}{2}} \cdot \frac{q_{[t,1]}^{(g^-,g^+),(\downarrow)}(w(t))}{C_{g^-,g^+}\sqrt{1-t} \cdot (w(t) - g^-(t)) \cdot Z_{[0,t]}^{g^-}(w)} \cdot 1_{K_{[0,t]}^{-}(g^+)}(w), \quad w \in C([0,t],\mathbb{R}).$$

REMARK 3.4. *In [6], using Corollary 1 and a Monte Carlo sampling technique for BES(3)-bridges, we numerically generated Brownian house-moving $H^{0 \to b}$ at discrete times. On the other hand, this sampling method does not work effectively for general Brownian house-moving $H^{g^- \to g^+}$. However, combining Theorem 2 and a Monte Carlo sampling technique for the BES(3)-process, we can approximate the expected values of the functional of $H^{g^- \to g^+}$.*

## 3.2 Construction of Brownian meander between two curves

In this subsection, we assume that $g^-(0) = 0$.

For $0 < t < 1$, $0 < t_1 < t_2 < 1$ and $y \in (g^-(t), g^+(t))$, $y_i \in (g^-(t_i), g^+(t_i))$ $(i = 1,2)$, we define

$$k(t,y) = (\widetilde{C}_{g^-,g^+})^{-1} \frac{1}{\sqrt{t}} q_{[0,t]}^{(g^-,g^+),(\uparrow)}(y) p_{[t,1]}^{(g^-,g^+)}(y), \quad k(t_1,y_1,t_2,y_2) = \frac{p_{[t_1,t_2]}^{(g^-,g^+)}(y_1,y_2) p_{[t_2,1]}^{(g^-,g^+)}(y_2)}{p_{[t_1,1]}^{(g^-,g^+)}(y_1)},$$

where

$$\widetilde{C}_{g^-,g^+} := \sqrt{\frac{\pi}{2}} \lim_{\varepsilon \downarrow 0} \frac{P(W_{[0,1]} \in K_{[0,1]}(g^- - \varepsilon, g^+))}{\varepsilon}.$$

$W^{+,(g^-,g^+)}$ denotes $X_{[0,1]}^{0,(g^-,g^+)}$, which is the weak limit of $W_{[0,1]}|_{K_{[0,1]}(g^- - \varepsilon, g^+)}$ as $\varepsilon \downarrow 0$. In this study, we call $W^{+,(g^-,g^+)}$ "Brownian meander between two curves." $W^{+,(g^-,g^+)}$ played an important role in [5]. Our aim in this subsection is to prove that $W^{+,(g^-,g^+)}$ is an $\mathbb{R}$-valued continuous Markov process on $[0,1]$.

**Theorem 3.** *There exists an $\mathbb{R}$-valued continuous Markov process $W^{+,(g^-,g^+)} = \{W^{+,(g^-,g^+)}(t)\}_{t \in [0,1]}$ that satisfies*

$$E\left[F(W^{+,(g^-,g^+)})\right]$$
$$= \lim_{\varepsilon \downarrow 0} E[F(W_{[0,1]}|_{K_{[0,1]}(g^- - \varepsilon, g^+)})] \tag{8}$$
$$= \int_{g^-(t)}^{g^+(t)} E\left[F(X_{[0,t]}^{0,y,(g^-,g^+)} \oplus_t X_{[t,1]}^{y,(g^-,g^+)})\right] k(t,y)dy \tag{9}$$
$$= \int_{g^-(t_1)}^{g^+(t_1)} \int_{g^-(t_2)}^{g^+(t_2)} E[F(X_{[0,t_1]}^{0,y_1,(g^-,g^+)} \oplus_{t_1} X_{[t_1,t_2]}^{y_1,y_2,(g^-,g^+)} \oplus_{t_2} X_{[t_2,1]}^{y_2,(g^-,g^+)})] k(t_1,y_1) k(t_1,y_1,t_2,y_2) dy_1 dy_2 \tag{10}$$

*for every* $\mathbb{R}$*-valued bounded continuous function* $F$ *on* $C([0,1],\mathbb{R})$*,* $0 < t < 1$ *and* $0 < t_1 < t_2 < 1$*, where the respective processes that appear in* (9) *and* (10) *are independent of each other. Moreover, for* $0 < t < 1$*,* $0 < t_1 < t_2 < 1$ *and* $y \in (g^-(t), g^+(t))$*,* $y_i \in (g^-(t_i), g^+(t_i))$ $(i = 1, 2)$*, the transition densities for* $W^{+,(g^-,g^+)}$ *are given by*

$$P(W^{+,(g^-,g^+)}(t) \in dy) = k(t,y)dy, \quad P(W^{+,(g^-,g^+)}(t_2) \in dy_2 \mid W^{+,(g^-,g^+)}(t_1) = y_1) = k(t_1, y_1, t_2, y_2)dy_2.$$

REMARK 3.5. *Let* $B \in \mathcal{B}(C([0,1],\mathbb{R}))$ *be a measurable subset of* $C([0,1],\mathbb{R})$*. Then, it follows from the same argument in Remark 3.1 that Theorem 3 holds true for* $F = 1_B$*.*

## 3.3 Construction of BES(3)-bridge between two curves

In this subsection, we assume that $g^-(0) = 0$ and $g^-(1) < c < b := g^+(1)$.

For $0 < t < 1$, $0 < t_1 < t_2 < 1$ and $y \in (g^-(t), g^+(t))$, $y_i \in (g^-(t_i), g^+(t_i))$ $(i = 1, 2)$, we define

$$l(t,y) = (\widehat{C}_{g^-,g^+})^{-1} \frac{1}{\sqrt{t}} q^{(g^-,g^+),(\uparrow)}_{[0,t]}(y) p^{(g^-,g^+)}_{[t,1]}(y,c), \quad l(t_1,y_1,t_2,y_2) = \frac{p^{(g^-,g^+)}_{[t_1,t_2]}(y_1,y_2) p^{(g^-,g^+)}_{[t_2,1]}(y_2,c)}{p^{(g^-,g^+)}_{[t_1,1]}(y_1,c)},$$

where

$$\widehat{C}_{g^-,g^+} := \sqrt{\frac{\pi}{2}} n_1(c) \lim_{\varepsilon \downarrow 0} \frac{P(B^{0\to c}_{[0,1]} \in K_{[0,1]}(g^- - \varepsilon, g^+))}{\varepsilon}.$$

$r^{0\to c,(g^-,g^+)}$ denotes $X^{0,c,(g^-,g^+)}_{[0,1]}$, which is the weak limit of $B^{0\to c}_{[0,1]}|_{K_{[0,1]}(g^- - \varepsilon, g^+)}$ as $\varepsilon \downarrow 0$. In this study, we call $r^{0\to c,(g^-,g^+)}$ "BES(3)-bridge between two curves." $r^{0\to c,(g^-,g^+)}$ played an important role in [5]. Our aim in this subsection is to prove that $r^{0\to c,(g^-,g^+)}$ is an $\mathbb{R}$-valued continuous Markov process on $[0,1]$.

**Theorem 4.** *There exists an* $\mathbb{R}$*-valued continuous Markov process* $r^{0\to c,(g^-,g^+)} = \{r^{0\to c,(g^-,g^+)}(t)\}_{t\in[0,1]}$ *that satisfies*

$$\begin{aligned} E&\left[F(r^{0\to c,(g^-,g^+)})\right] \\ &= \lim_{\varepsilon\downarrow 0} E[F(B^{0\to c}_{[0,1]}|_{K_{[0,1]}(g^- - \varepsilon, g^+)})] \qquad (11) \\ &= \int_{g^-(t)}^{g^+(t)} E\left[F(X^{0,y,(g^-,g^+)}_{[0,t]} \oplus_t X^{y,c,(g^-,g^+)}_{[t,1]})\right] l(t,y)dy \qquad (12) \\ &= \int_{g^-(t_1)}^{g^+(t_1)} \int_{g^-(t_2)}^{g^+(t_2)} E[F(X^{0,y_1,(g^-,g^+)}_{[0,t_1]} \oplus_{t_1} X^{y_1,y_2,(g^-,g^+)}_{[t_1,t_2]} \oplus_{t_2} X^{y_2,c,(g^-,g^+)}_{[t_2,1]})] l(t_1,y_1) l(t_1,y_1,t_2,y_2) dy_1 dy_2 \qquad (13) \end{aligned}$$

*for every* $\mathbb{R}$*-valued bounded continuous function* $F$ *on* $C([0,1],\mathbb{R})$*,* $0 < t < 1$ *and* $0 < t_1 < t_2 < 1$*, where the respective processes that appear in* (12) *and* (13) *are independent of each other. Moreover, for* $0 < t < 1$*,* $0 < t_1 < t_2 < 1$ *and* $y \in (g^-(t), g^+(t))$*,* $y_i \in (g^-(t_i), g^+(t_i))$ $(i = 1, 2)$*, the transition densities for* $r^{0\to c,(g^-,g^+)}$ *are given by*

$$P(r^{0\to c,(g^-,g^+)}(t) \in dy) = l(t,y)dy, \quad P(r^{0\to c,(g^-,g^+)}(t_2) \in dy_2 \mid r^{0\to c,(g^-,g^+)}(t_1) = y_1) = l(t_1,y_1,t_2,y_2)dy_2.$$

REMARK 3.6. *Let $B \in \mathcal{B}(C([0,1],\mathbb{R}))$ be a measurable subset of $C([0,1],\mathbb{R})$. Then, it follows from the same argument in Remark 3.1 that Theorem 4 holds true for $F = 1_B$.*

We also prove that $r^{0\to b,(g^-,g^++\eta)}$ converges weakly to $H^{g^-\to g^+}$ as $\eta \downarrow 0$.

**Theorem 5.** *For every $\mathbb{R}$-valued bounded continuous function $F$ on $C([0,1],\mathbb{R})$, we have*

$$E\left[F(H^{g^-\to g^+})\right] = \lim_{\eta\downarrow 0} E[F(r^{0\to b,(g^-,g^++\eta)})].$$

REMARK 3.7. *Let $R = \{R(t)\}_{t\ge 0}$ be 3-dimensional Bessel process (BES(3) process for short) starting from 0, and let $\tau_b$ ($b > 0$) denotes the first hitting time of the point $b$ by $R$:*

$$\tau_b := \inf\{r \ge 0 \mid R(r) = b\}.$$

*It has been shown in [7] that Brownian house-moving $H^{0\to b} = \{H^{0\to b}(t)\}_{t\in[0,1]}$ satisfies*

$$P\left(H^{0\to b}(t) \in dy\right) = P\left(R(t) \in dy \mid \tau_b = 1\right),$$
$$P\left(H^{0\to b}(t) \in dy \mid H^{0\to b}(s) = x\right) = P\left(R(t) \in dy \mid R(s) = x, \tau_b = 1\right)$$

*for $0 < s < t < 1$ and $x, y \in (0, b)$.*

# 4 Weak convergence to BES(3)-bridge

It has been shown in [2] that the one-dimensional Brownian bridge from 0 to 0 conditioned to stay in $[-\varepsilon,\infty)$ converges weakly to the Brownian excursion (i.e., the BES(3)-bridge from 0 to 0). Motivated by this research, we prove the following weak convergence that is used to construct the Brownian house-moving.

**Theorem 6.** *Let $b \ge 0$ and $B^{0\to b} = \{B^{0\to b}(t)\}_{t\in[0,1]}$ be the one-dimensional Brownian bridge from 0 to $b$ on $[0,1]$, and let $r^{0\to b} = \{r^{0\to b}(t)\}_{t\in[0,1]}$ be the BES(3)-bridge from 0 to $b$ on $[0,1]$. Then, we have*

$$B^{0\to b}|_{K^+(-\varepsilon)} \xrightarrow{\mathcal{D}} r^{0\to b}, \qquad \varepsilon \downarrow 0,$$

*where $K^+(-\varepsilon) := \{w = \{w(t)\}_{t\in[0,1]} \in C([0,1],\mathbb{R}) \mid -\varepsilon \le w(t),\ 0 \le t \le 1\}$.*

In [2], we can find the proof of Theorem 6 for $b = 0$. Thus, in this section, we seek a proof of Theorem 6 for $b > 0$. To this end, using a well-known fact about weak convergence (Theorem 7), it suffices to show the following conditions:

[T6] the family $\{B^{0\to b}|_{K^+(-\varepsilon)}\}_{0<\varepsilon<\varepsilon_0}$ is tight for some $\varepsilon_0 > 0$;

[F6] the finite-dimensional distribution of $B^{0\to b}|_{K^+(-\varepsilon)}$ converges to that of $r^{0\to b}$ as $\varepsilon \downarrow 0$ .

Now, Proposition A.3 yields the Markov property of $B^{0\to b}|_{K^+(-\varepsilon)}$ and $r^{0\to b}$. Therefore, according to Lemma A.10, [F6] follows from Corollary 4 expressed below.

**Lemma 4.1.** *For $0 < s < t < 1$ and $x, y > -\varepsilon$, we have*

$$P\left(B^{0\to b}|_{K^+(-\varepsilon)}(t) \in dy\right) = \frac{(n_{1-t}(b-y) - n_{1-t}(b+y+2\varepsilon))\,(n_t(y) - n_t(y+2\varepsilon))}{n_1(b) - n_1(b+2\varepsilon)} dy, \tag{14}$$

$$P\left(B^{0\to b}|_{K^+(-\varepsilon)}(t) \in dy \mid B^{0\to b}|_{K^+(-\varepsilon)}(s) = x\right)$$
$$= \frac{(n_{t-s}(y-x) - n_{t-s}(y+x+2\varepsilon))\,(n_{1-t}(b-y) - n_{1-t}(b+y+2\varepsilon))}{n_{1-s}(b-x) - n_{1-s}(b+x+2\varepsilon)} dy. \tag{15}$$

Proof. Using (87) and (89), we have

$$\begin{aligned} P\left(B^{0\to b}|_{K^+(-\varepsilon)}(t) \in dy\right) &= P\left(B^{0\to b}(t) \in dy \mid m(B^{0\to b}) \geq -\varepsilon\right) \\ &= \frac{P\left(W(t) \in dy, m(W) \geq -\varepsilon, W(1) \in db\right)}{P\left(m(W) \geq -\varepsilon, W(1) \in db\right)} \\ &= \frac{(n_{1-t}(b-y) - n_{1-t}(b+y+2\varepsilon))\,(n_t(y) - n_t(y+2\varepsilon))}{n_1(b) - n_1(b+2\varepsilon)} dy. \end{aligned}$$

Using (89) and (90), we have

$$\begin{aligned} &P\left(B^{0\to b}|_{K^+(-\varepsilon)}(t) \in dy \mid B^{0\to b}|_{K^+(-\varepsilon)}(s) = x\right) \\ &= P\left(B^{0\to b}(t) \in dy \mid B^{0\to b}(s) = x, m(B^{0\to b}) > -\varepsilon\right) \\ &= \frac{P\left(W(t) \in dy, W(s) \in dx, m(W) \geq -\varepsilon, W(1) \in db\right)}{P\left(W(s) \in dx, m(W) \geq -\varepsilon, W(1) \in db\right)} \\ &= \frac{(n_{t-s}(y-x) - n_{t-s}(y+x+2\varepsilon))\,(n_{1-t}(b-y) - n_{1-t}(b+y+2\varepsilon))}{n_{1-s}(b-x) - n_{1-s}(b+x+2\varepsilon)} dy. \end{aligned}$$

□

**Corollary 4.** *For $0 < s < t < 1$ and $x, y > 0$, we have*

$$\lim_{\varepsilon\downarrow 0} P\left(B^{0\to b}|_{K^+(-\varepsilon)}(t) \in dy\right) = P\left(r^{0\to b}(t) \in dy\right),$$
$$\lim_{\varepsilon\downarrow 0} P\left(B^{0\to b}|_{K^+(-\varepsilon)}(t) \in dy \mid B^{0\to b}|_{K^+(-\varepsilon)}(s) = x\right) = P\left(r^{0\to b}(t) \in dy \mid r^{0\to b}(s) = x\right).$$

Proof. Let us define

$$\psi_1(\varepsilon) := (n_{1-t}(b-y) - n_{1-t}(b+y+2\varepsilon))\,(n_t(y) - n_t(y+2\varepsilon)), \qquad \psi_2(\varepsilon) := n_1(b) - n_1(b+2\varepsilon).$$

Then, simple calculations imply that

$$\lim_{\varepsilon\downarrow 0} \psi_i(\varepsilon) = 0, \quad i = 1, 2, \qquad \lim_{\varepsilon\downarrow 0} \frac{d}{d\varepsilon}\psi_2(\varepsilon) = 2bn_1(b),$$
$$\lim_{\varepsilon\downarrow 0} \frac{d}{d\varepsilon}\psi_1(\varepsilon) = (n_{1-t}(b-y) - n_{1-t}(b+y))\,\frac{2y}{t} n_t(y). \tag{16}$$

Using (14), (15), (16), L'Hôpital's rule, and

$$P\left(r^{0\to b}(t) \in dy\right) = \frac{yn_t(y)\,(n_{1-t}(b-y) - n_{1-t}(b+y))}{tbn_1(b)}dy,$$
$$P\left(r^{0\to b}(t) \in dy \mid r^{0\to b}(s) = x\right) = \frac{(n_{t-s}(y-x) - n_{t-s}(y+x))\,(n_{1-t}(b-y) - n_{1-t}(b+y))}{n_{1-s}(b-x) - n_{1-s}(b+x)}dy,$$

we establish the assertion. □

Over the remainder of this section, we prove [T6]. Theorem 8 is known to be a sufficient condition for the tightness of the family of the laws of continuous processes. To use Theorem 8 for [T6], we prepare the following inequalities. Let

$$f(x) = (b+x)n_1(b+x).$$

Since $f(0) = bn_1(b) > 0$ and $f$ is continuous at $x = 0$, we can take $\delta > 0$ such that $f(x) > \frac{1}{2}f(0) = \frac{1}{2}bn_1(b)$ holds for $0 < x < \delta$. Throughout this section, we fix such a $\delta$ and define

$$\varepsilon_0 := \min\left\{1, \frac{\delta}{2}\right\}. \tag{17}$$

We establish the moment inequalities of $B^{0\to b}|_{K^+(-\varepsilon)}$.

**Lemma 4.2.** *For each $m \in \mathbb{N}$, we can find a constant $C_m > 0$ depending only on $m$ (and $b$) such that*

$$(1) \qquad \sup_{0<\varepsilon<\varepsilon_0} E\left[\left|B^{0\to b}|_{K^+(-\varepsilon)}(r)\right|^{2m}\right] \le C_m \frac{r^{m-1}}{\sqrt{1-r}}, \qquad r \in (0,1),$$

$$(2) \qquad \sup_{0<\varepsilon<\varepsilon_0} E\left[\left|B^{0\to b}|_{K^+(-\varepsilon)}(1-r) - b\right|^{2m}\right] \le C_m \frac{r^{m}}{\sqrt{(1-r)^3}}, \qquad r \in (0,1),$$

$$(3) \qquad \sup_{0<\varepsilon<\varepsilon_0} E\left[\left|B^{0\to b}|_{K^+(-\varepsilon)}(t) - B^{0\to b}|_{K^+(-\varepsilon)}(s)\right|^{2m}\right] \le \frac{C_m}{s\sqrt{1-t}}\,|t-s|^m, \qquad s,t \in (0,1).$$

Proof. Let $C = 4b^{-1}\exp(b^2/2)$ and $0 < \varepsilon < \varepsilon_0$. First, we prove inequality (1). We estimate the density $P\left(B^{0\to b}|_{K^+(-\varepsilon)}(r) \in dz\right)$ given by (14). According to Taylor's theorem, there exists $\theta \in (0,1)$ that satisfies

$$n_1(b) - n_1(b+2\varepsilon) = \int_b^{b+2\varepsilon} zn_1(z)dz = 2\varepsilon(b+2\varepsilon\theta)n_1(b+2\varepsilon\theta) = 2\varepsilon f(2\varepsilon\theta) \ge \varepsilon bn_1(b). \tag{18}$$

Note that we have

$$n_{1-r}(b-z) - n_{1-r}(b+z+2\varepsilon) \le \frac{1}{\sqrt{2\pi(1-r)}}. \tag{19}$$

Thus, it follows from (18), (19), and Lemma A.1 that

$$P\left(B^{0\to b}|_{K^+(-\varepsilon)}(r) \in dz\right) < \frac{1}{\varepsilon bn_1(b)}\frac{1}{\sqrt{2\pi(1-r)}}\frac{4\varepsilon}{r}n_r\left(z/\sqrt{2}\right)dz = \frac{C}{r\sqrt{1-r}}n_r\left(z/\sqrt{2}\right)dz.$$

Therefore,

$$E\left[\left|B^{0\to b}|_{K^+(-\varepsilon)}(r)\right|^{2m}\right] \le \frac{C2^m\sqrt{2}}{r\sqrt{1-r}}\int_{\mathbb{R}}|x|^{2m}n_r(x)dx = \frac{C2^m\sqrt{2}}{r\sqrt{1-r}}(2m-1)!!r^m$$

holds, and we obtain inequality (1).

Second, we prove inequality (2). We make an estimation different from the one expressed above on the density $P\left(B^{0\to b}|_{K^+(-\varepsilon)}(1-r)\in dz\right)$ given by (14). Using Lemma A.1, we have

$$n_{1-r}(z) - n_{1-r}(z+2\varepsilon) \le \frac{4\varepsilon}{1-r}n_{1-r}(z/\sqrt{2}) \le \frac{4\varepsilon}{\sqrt{2\pi(1-r)^3}}, \qquad z \ge -\varepsilon.$$

Combining these inequalities and (18), we obtain

$$P\left(B^{0\to b}|_{K^+(-\varepsilon)}(1-r)\in dz\right) \le \frac{4\varepsilon}{\sqrt{2\pi(1-r)^3}}\frac{1}{\varepsilon b n_1(b)}n_r(b-z)dz = \frac{C}{\sqrt{(1-r)^3}}n_r(b-z)dz.$$

Therefore,

$$E\left[\left|B^{0\to b}|_{K^+(-\varepsilon)}(1-r)-b\right|^{2m}\right] \le \frac{C}{\sqrt{(1-r)^3}}\int_{\mathbb{R}}|z-b|^{2m}n_r(b-z)dz = \frac{C}{\sqrt{(1-r)^3}}(2m-1)!!r^m$$

holds, and we have established inequality (2).

Finally, we prove inequality (3). Let $t, s \in (0,1)$ satisfy $s < t$. By (14) and (15),

$$\begin{aligned}
&P\left(B^{0\to b}|_{K^+(-\varepsilon)}(t)\in dy, B^{0\to b}|_{K^+(-\varepsilon)}(s)\in dx\right)\\
&\quad= P\left(B^{0\to b}|_{K^+(-\varepsilon)}(t)\in dy \mid B^{0\to b}|_{K^+(-\varepsilon)}(s)=x\right)P\left(B^{0\to b}|_{K^+(-\varepsilon)}(s)\in dx\right)\\
&\quad= \frac{(n_{t-s}(y-x)-n_{t-s}(y+x+2\varepsilon))(n_{1-t}(b-y)-n_{1-t}(b+y+2\varepsilon))(n_s(x)-n_s(x+2\varepsilon))}{n_1(b)-n_1(b+2\varepsilon)}dxdy
\end{aligned}$$

holds. In addition, we have

$$n_{t-s}(y-x)-n_{t-s}(y+x+2\varepsilon) \le n_{t-s}(y-x), \qquad n_{1-t}(b-y)-n_{1-t}(b+y+2\varepsilon) \le \frac{1}{\sqrt{2\pi(1-t)}}. \tag{20}$$

Thus, it follows from Lemma A.1 and (20) that

$$\begin{aligned}
P\left(B^{0\to b}|_{K^+(-\varepsilon)}(t)\in dy, B^{0\to b}|_{K^+(-\varepsilon)}(s)\in dx\right) &\le \frac{1}{\varepsilon b n_1(b)}\frac{4\varepsilon}{s}n_s\left(x/\sqrt{2}\right)\frac{1}{\sqrt{2\pi(1-t)}}n_{t-s}(y-x)dxdy\\
&= \frac{C}{s\sqrt{1-t}}n_s\left(x/\sqrt{2}\right)n_{t-s}(y-x)dxdy.
\end{aligned}$$

Therefore,

$$\begin{aligned}
E\left[\left|B^{0\to b}|_{K^+(-\varepsilon)}(t)-B^{0\to b}|_{K^+(-\varepsilon)}(s)\right|^{2m}\right] &\le \int_{[-\varepsilon,\infty)^2}|y-x|^{2m}\frac{C}{s\sqrt{1-t}}n_s\left(x/\sqrt{2}\right)n_{t-s}(y-x)dxdy\\
&\le \frac{C}{s\sqrt{1-t}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|y-x|^{2m}n_{t-s}(y-x)dy\right)n_s\left(x/\sqrt{2}\right)dx\\
&= \frac{C\sqrt{2}}{s\sqrt{1-t}}(2m-1)!!|t-s|^m
\end{aligned}$$

holds, and inequality (3) is obtained. □

The following is obtained by applying Lemma 4.2 (1) for $m = 1$ and Lemma 4.2 (3) for $m = 2$.

**Corollary 5.** (1) *There exists $\nu > 0$ that satisfies*

$$\sup_{0<\varepsilon<\varepsilon_0} E\left[\left|B^{0\to b}|_{K^+(-\varepsilon)}(r)\right|^\nu\right] < \infty, \qquad r \in [0,1].$$

(2) *For each $u \in \left(0, \frac{1}{2}\right)$, there exist $\alpha, \beta, C > 0$ that satisfy*

$$\sup_{0<\varepsilon<\varepsilon_0} E\left[\left|B^{0\to b}|_{K^+(-\varepsilon)}(t) - B^{0\to b}|_{K^+(-\varepsilon)}(s)\right|^\alpha\right] \le C|t-s|^{1+\beta}, \qquad t, s \in [u, 1-u].$$

*Therefore, for each $u \in \left(0, \frac{1}{2}\right)$, the family $\{\pi_{[u,1-u]} \circ B^{0\to b}|_{K^+(-\varepsilon)}\}_{0<\varepsilon<\varepsilon_0}$ is tight.*

**Lemma 4.3.** *For each $\xi > 0$,*

$$\lim_{u\downarrow 0} \sup_{0<\varepsilon<\varepsilon_0} P\left(\sup_{0\le t\le u} \left|B^{0\to b}|_{K^+(-\varepsilon)}(t)\right| > \xi\right) = 0, \qquad \lim_{u\downarrow 0} \sup_{0<\varepsilon<\varepsilon_0} P\left(\sup_{1-u\le t\le 1} \left|B^{0\to b}|_{K^+(-\varepsilon)}(t) - b\right| > \xi\right) = 0$$

*hold, where $\varepsilon_0$ is the number defined in (17).*

Proof. Applying Lemma 4.2 (1)–(3) for $m = 4$, for $0 < r < 1$ and $0 < s < t < 1$, we have

$$\sup_{0<\varepsilon<\varepsilon_0} E\left[\left|B^{0\to b}|_{K^+(-\varepsilon)}(r)\right|^8\right] \le \frac{C_4}{\sqrt{1-r}} r^3, \tag{21}$$

$$\sup_{0<\varepsilon<\varepsilon_0} E\left[\left|B^{0\to b}|_{K^+(-\varepsilon)}(1-r) - b\right|^8\right] \le C_4 \frac{r^4}{\sqrt{(1-r)^3}}, \tag{22}$$

$$\sup_{0<\varepsilon<\varepsilon_0} E\left[\left|B^{0\to b}|_{K^+(-\varepsilon)}(t) - B^{0\to b}|_{K^+(-\varepsilon)}(s)\right|^8\right] \le \frac{C_4}{s\sqrt{1-t}} |t-s|^4 . \tag{23}$$

Let $\gamma = \frac{1}{16}$, $0 < \varepsilon < \varepsilon_0$ and $n \in \mathbb{N}$. We define

$$F_n^\varepsilon = \left\{\max_{1\le k\le 2^{n-1}} \left|B^{0\to b}|_{K^+(-\varepsilon)}\left(\frac{k-1}{2^n}\right) - B^{0\to b}|_{K^+(-\varepsilon)}\left(\frac{k}{2^n}\right)\right| \ge 2^{-n\gamma}\right\},$$

$$\widetilde{F}_n^\varepsilon = \left\{\max_{2^{n-1}\le k\le 2^n} \left|B^{0\to b}|_{K^+(-\varepsilon)}\left(\frac{k-1}{2^n}\right) - B^{0\to b}|_{K^+(-\varepsilon)}\left(\frac{k}{2^n}\right)\right| \ge 2^{-n\gamma}\right\},$$

$$a(n,k,\varepsilon) = P\left(\left|B^{0\to b}|_{K^+(-\varepsilon)}\left(\frac{k-1}{2^n}\right) - B^{0\to b}|_{K^+(-\varepsilon)}\left(\frac{k}{2^n}\right)\right| \ge 2^{-n\gamma}\right), \qquad 1 \le k \le 2^n.$$

Then, by Chebyshev's inequality, we have

$$a(n,k,\varepsilon) \le (2^{n\gamma})^8 E\left[\left|B^{0\to b}|_{K^+(-\varepsilon)}\left(\frac{k-1}{2^n}\right) - B^{0\to b}|_{K^+(-\varepsilon)}\left(\frac{k}{2^n}\right)\right|^8\right], \qquad 1 \le k \le 2^n. \tag{24}$$

Using (21), (22), (23), and (24), we have

$$a(n,1,\varepsilon) \le 2^{\frac{n}{2}} \frac{C_4}{\sqrt{1-\frac{1}{2^n}}} \left(\frac{1}{2^n}\right)^3 \le 2^{\frac{n}{2}} C_4 2^{\frac{1}{2}} 2^{-3n} < C_4 2^{-n} 2^{-\frac{n}{2}},$$

$$a(n,2^n,\varepsilon) \le 2^{\frac{n}{2}} \frac{C_4}{\sqrt{\left(1-\frac{1}{2^n}\right)^3}} \left(\frac{1}{2^n}\right)^4 \le C_4 2^{-3n} < C_4 2^{-n} 2^{-\frac{n}{2}},$$

$$a(n,k,\varepsilon) \le 2^{\frac{n}{2}} \left(\frac{2^n}{k-1}\right) \sqrt{\frac{2^n}{2^n-k}} C_4 \left(\frac{1}{2^n}\right)^4 \le C_4 2^{\frac{n}{2}} 2^n 2^{\frac{n}{2}} 2^{-4n} < C_4 2^{-n} 2^{-\frac{n}{2}}, \qquad 2 \le k \le 2^n - 1.$$

Thus, it follows that

$$P\left(F_n^{\varepsilon}\right) \le \sum_{k=1}^{2^{n-1}} a(n,k,\varepsilon) \le C_4 2^{-\frac{n}{2}}, \qquad P\left(\widetilde{F}_n^{\varepsilon}\right) \le \sum_{k=2^{n-1}}^{2^n} a(n,k,\varepsilon) \le C_4 2^{-\frac{n}{2}}.$$

Therefore, Lemmas A.12 and A.13 prove the desired results. □

By Corollary 5 and Lemma 4.3, we can apply Theorem 8 for $\{B^{0\to b}|_{K^+(-\varepsilon)}\}_{0<\varepsilon<\varepsilon_0}$ and obtain [T6].

# 5 Distribution of the maximal value of the BES(3)-bridge

As an application of Theorem 6, we derive the distribution of the maximal value of the BES(3)-bridge $r^{0\to b}$ $(b > 0)$.

**Proposition 5.1.** *For each $x > b > 0$, we have*

$$P\left(M(r^{0\to b}) \le x\right) = \frac{J^{(x)}(1,b)}{2bn_1(b)} > 0.$$

Proof. Using (89) and (90), we have

$$\begin{aligned} P\left(M(B^{0\to b}|_{K^+(-\varepsilon)}) \le x\right) &= P\left(M(B^{0\to b}) \le x \mid -\varepsilon \le m(B^{0\to b})\right) \\ &= \frac{P\left(-\varepsilon \le m(W) < M(W) \le x, W(1) \in db\right)}{P\left(-\varepsilon \le m(W), W(1) \in db\right)} = \frac{\psi_1(\varepsilon)}{\psi_2(\varepsilon)} \end{aligned}$$

for $\varepsilon > 0$, where

$$\psi_1(\varepsilon) := \sum_{k=-\infty}^{\infty} \left(n_1(b+2k(x+\varepsilon)) - n_1(2\varepsilon + b + 2k(x+\varepsilon))\right), \qquad \psi_2(\varepsilon) := n_1(b) - n_1(b+2\varepsilon).$$

By simple calculations, we obtain

$$\lim_{\varepsilon\downarrow 0} \psi_i(\varepsilon) = 0, \quad i = 1,2, \quad \lim_{\varepsilon\downarrow 0} \frac{\partial}{\partial\varepsilon}\psi_2(\varepsilon) = 2bn_1(b), \quad \lim_{\varepsilon\downarrow 0} \frac{\partial}{\partial\varepsilon}\psi_1(\varepsilon) = J^{(x)}(1,b).$$

By combining Theorem 6, the Portmanteau theorem, Lemma A.18, and L'Hôpital's rule, we obtain

$$P\left(M(r^{0\to b}) \le x\right) = \lim_{\varepsilon\downarrow 0} P\left(M(B^{0\to b}|_{K^+(-\varepsilon)}) \le x\right) = \frac{J^{(x)}(1,b)}{2bn_1(b)}. \tag{25}$$

Now, we define the domain $D$, and the function $f$ on $D$ as

$$D = \{z = x + \mathrm{i}y \mid x \in (0,\infty), y \in (-b/2, b/2)\},$$
$$f(z) = \sum_{k=-\infty}^{\infty} (b + 2k(b+z)) \exp\left(-\frac{(b+2k(b+z))^2}{2}\right), \qquad z \in D,$$

where i is the imaginary unit. Then, we have

$$P\left(M(r^{0\to b}) \le b + \eta\right) = \frac{J^{(b+\eta)}(1,b)}{2bn_1(b)} = \frac{f(\eta)}{b\exp\left(-\frac{b^2}{2}\right)}, \qquad \eta > 0,$$

by (25). Furthermore, we define

$$D_R = \{z = x + \mathrm{i}y \mid x \in (0,R), y \in (-b/2, b/2)\}, \qquad R > 0.$$

For $R > 0$, $z \in D_R$, and $k \in \mathbb{Z}$, we have

$$|b + 2k(b+z)| \le b + 2|k|(b + |x| + |y|) \le b + 2|k|(2b + R)$$

and

$$\begin{aligned}\exp\left(-\frac{(b+2k(b+z))^2}{2}\right) &= \exp\left(-\frac{1}{2}b^2 - 2k(b+x)b - 2k^2(b+x)^2 + 2k^2y^2\right)\\ &\le \exp\left(2|k|(b+R)b - 2k^2b^2 + \frac{1}{2}k^2b^2\right)\\ &= \exp\left(-\frac{3}{2}k^2b^2 + 2|k|(b+R)b\right).\end{aligned}$$

Thus, we see that $f$ is a holomorphic function on $D$.

For the sake of contradiction, assume that $f(\eta_0) = 0$ holds for some $\eta_0 > 0$. Then, because $f$ is a non-decreasing and non-negative function on $(0,\infty)$, $f(z) = 0$, $z \in D$ holds by the identity theorem. However, this contradicts

$$\lim_{\eta\to\infty} f(\eta) = b\exp\left(-\frac{b^2}{2}\right)\lim_{\eta\to\infty} P(M(r^{0\to b}) \le b + \eta) = b\exp\left(-\frac{b^2}{2}\right) > 0.$$

□

REMARK 5.1. *More generally, in [9] p. 8 (28), Proposition 5.1 has been shown by the expanded Gikhman–Kiefer formula for BES(δ)-bridges.*

**Corollary 6.** *For $0 \le s < t < \infty$, it holds that*

$$P\left(r^{0\to y}_{[s,t]} \in K^-_{[s,t]}(c)\right) = \frac{(t-s)J^{(c)}(t-s,y)}{2yn_{t-s}(y)} > 0, \qquad 0 < y < c.$$

Proof. Using Proposition 5.1, we obtain

$$P\left(r^{0\to y}_{[s,t]} \in K^-_{[s,t]}(c)\right) = P\left(M(r^{0\to y/\sqrt{t-s}}) \le c/\sqrt{t-s}\right) = \frac{\sqrt{t-s}\cdot J^{(c/\sqrt{t-s})}(1, y/\sqrt{t-s})}{2yn_1(y/\sqrt{t-s})} = \frac{(t-s)J^{(c)}(t-s,y)}{2yn_{t-s}(y)}.$$

□

**Corollary 7.** *Assume that $g \in C([0,1],\mathbb{R})$ satisfies $\min_{0\le t\le 1} g(t) > 0$. Then, we have*

$$(A)\quad P(W^+ \in K^-(g)) > 0 \qquad \text{and} \qquad (B)\quad P(r^{0\to b} \in K^-(g)) > 0, \quad 0 < b < g(1).$$

Proof. Let $b \in (0, g(1))$. Take $\delta \in (0,1)$ such that

$$c_1 := \min_{t\in[1-\delta,1]} g(t) \ge \frac{1}{2}(g(1)+b)$$

holds. Let $t_0 = 1-\delta$ and $c_0 = b \wedge \min_{0\le u\le 1} g(u) > 0$. Then, Lemmas A.4 and A.5 imply

$$P(r^{0\to b} \in K^-(g)) \ge P\left(r^{0\to b} \in \pi^{-1}_{[0,t_0]}(K^-_{[0,t_0]}(c_0)) \cap \pi^{-1}_{[t_0,1]}(K^-_{[t_0,1]}(c_1))\right) = \int_0^{c_0} \kappa_{t_0}(y) P\left(r^{0\to b}(t_0) \in dy\right),$$

where

$$\kappa_{t_0}(y) = P\left(r^{0\to y}_{[0,t_0]} \in K^-_{[0,t_0]}(c_0)\right) \frac{P(B^{y\to b}_{[t_0,1]} \in K_{[t_0,1]}(0,c_1))}{P(B^{y\to b}_{[t_0,1]} \in K^+_{[t_0,1]}(0))}.$$

Using Proposition 5.1 and Lemma A.7, we have $\kappa_{t_0}(y) > 0$ on $y \in (0, c_0)$, and obtain $(B)$.

Because $W|_{K^+(-\varepsilon)} \xrightarrow{\mathcal{D}} W^+$ $(\varepsilon \downarrow 0)$ holds, the Markov property of $W$ and Theorem 6 imply that

$$P(W^+ \in K^-(g)) \ge P\left(W^+ \in \pi^{-1}_{[0,t_1]}(K^-_{[0,t_1]}(c_2)) \cap \pi^{-1}_{[t_1,1]}(K^-_{[t_1,1]}(c_2))\right) = \int_0^{c_2} \widetilde{\kappa}_{t_1}(y) P\left(W^+(t_1) \in dy\right),$$

with $0 < t_1 < 1$ and $c_2 = \min_{0\le u\le 1} g(u) > 0$, where

$$\widetilde{\kappa}_{t_1}(y) = P\left(r^{0\to y}_{[0,t_1]} \in K^-_{[0,t_1]}(c_2)\right) \frac{P\left(y + W_{[t_1,1]} \in K_{[t_1,1]}(0,c_2)\right)}{P\left(y + W_{[t_1,1]} \in K^+_{[t_1,1]}(0)\right)}.$$

Using Proposition 5.1, we have $\widetilde{\kappa}_{t_1}(y) > 0$ on $y \in (0, c_2)$, and obtain $(A)$. □

# 6 Proofs of Lemma 3.1 and Lemma 3.2

In this section, we prove Lemma 3.1 and Lemma 3.2.

## 6.1 Proof of Lemma 3.1

To prove (1), it suffices to show that the limit

$$\lim_{\varepsilon\downarrow 0}\frac{E[F(a+W_{[t_1,t_2]})\ ;\ a+W_{[t_1,t_2]}(t_2)\in db, a+W_{[t_1,t_2]}\in K_{[t_1,t_2]}(g^- -\varepsilon, g^+ +\eta(\varepsilon))]}{P(a+W_{[t_1,t_2]}(t_2)\in db, a+W_{[t_1,t_2]}\in K_{[t_1,t_2]}(g^- -\varepsilon, g^+ +\eta(\varepsilon)))} \tag{26}$$

exists and coincides with the right-hand side of (1). For each $F$ and $\varepsilon > 0$, Girsanov's theorem yields

$$\begin{aligned}
&E[F(a+W_{[t_1,t_2]})\ ;\ a+W_{[t_1,t_2]}(t_2)\in db, a+W_{[t_1,t_2]}\in K_{[t_1,t_2]}(g^- -\varepsilon, g^+ +\eta(\varepsilon))]\\
&\quad = E[F(W_{[t_1,t_2]}+g^-)\widetilde{Z}^{g^- -a}_{[t_1,t_2]}(W_{[t_1,t_2]})^{-1}\ ;\ W_{[t_1,t_2]}(t_2)\in db-g^-(t_2), W_{[t_1,t_2]}\in K_{[t_1,t_2]}(-\varepsilon, g^+ - g^- +\eta(\varepsilon))]\\
&\quad = E[F(B^{0\to b-g^-(t_2)}_{[t_1,t_2]}+g^-)\widetilde{Z}^{g^- -a}_{[t_1,t_2]}(B^{0\to b-g^-(t_2)}_{[t_1,t_2]})^{-1}\ ;\ B^{0\to b-g^-(t_2)}_{[t_1,t_2]}\in K_{[t_1,t_2]}(-\varepsilon, g^+ - g^- +\eta(\varepsilon))]\\
&\qquad \times P(W_{[t_1,t_2]}(t_2)\in db-g^-(t_2))\\
&\quad = E\Big[F(B^{0\to b-g^-(t_2)}_{[t_1,t_2]}|_{K_{[t_1,t_2]}(-\varepsilon,g^+-g^-+\eta(\varepsilon))}+g^-)\widetilde{Z}^{g^- -a}_{[t_1,t_2]}(B^{0\to b-g^-(t_2)}_{[t_1,t_2]}|_{K_{[t_1,t_2]}(-\varepsilon,g^+-g^-+\eta(\varepsilon))})^{-1}\Big]\\
&\qquad \times P(B^{0\to b-g^-(t_2)}_{[t_1,t_2]}\in K_{[t_1,t_2]}(-\varepsilon, g^+ - g^- +\eta(\varepsilon)))P(W_{[t_1,t_2]}(t_2)\in db-g^-(t_2)).
\end{aligned}$$

Therefore, taking the limit $\varepsilon\downarrow 0$ in (26), we obtain (1) by Proposition A.2.

To prove (2), it suffices to show that the limit

$$\lim_{\varepsilon\downarrow 0}\frac{E[F(a+W_{[t_1,t_2]})\ ;\ a+W_{[t_1,t_2]}(t_2)\in db, a+W_{[t_1,t_2]}\in K_{[t_1,t_2]}(g^- -\eta(\varepsilon), g^+ +\varepsilon)]}{P(a+W_{[t_1,t_2]}(t_2)\in db, a+W_{[t_1,t_2]}\in K_{[t_1,t_2]}(g^- -\eta(\varepsilon), g^+ +\varepsilon))} \tag{27}$$

exists and coincides with the right-hand side of (2). Because $W_{[t_1,t_2]}(\cdot)\overset{\mathcal{D}}{=}W_{[t_1,t_2]}(t_2)-W_{[t_1,t_2]}(t_1+t_2-\cdot)$ holds, Girsanov's theorem yields

$$\begin{aligned}
&E[F(a+W_{[t_1,t_2]})\ ;\ a+W_{[t_1,t_2]}(t_2)\in db, a+W_{[t_1,t_2]}\in K_{[t_1,t_2]}(g^- -\eta(\varepsilon), g^+ +\varepsilon)]\\
&\quad = E[F(b-\overleftarrow{W}_{[t_1,t_2]})\ ;\ a+W_{[t_1,t_2]}(t_2)\in db, W_{[t_1,t_2]}\in K_{[t_1,t_2]}(b-\overleftarrow{g}^+ -\varepsilon, b-\overleftarrow{g}^- +\eta(\varepsilon))]\\
&\quad = E\Big[F(g^+ -\overleftarrow{W}_{[t_1,t_2]})\widetilde{Z}^{b-\overleftarrow{g}^+}_{[t_1,t_2]}(W_{[t_1,t_2]})^{-1}\ ;\\
&\qquad\qquad a+W_{[t_1,t_2]}(t_2)\in db-(b-g^+(t_1)), W_{[t_1,t_2]}\in K_{[t_1,t_2]}(-\varepsilon, \overleftarrow{g}^+ - \overleftarrow{g}^- +\eta(\varepsilon))\Big]\\
&\quad = E\Big[F(g^+ -\overleftarrow{B}^{0\to g^+(t_1)-a}_{[t_1,t_2]})\widetilde{Z}^{b-\overleftarrow{g}^+}_{[t_1,t_2]}(B^{0\to g^+(t_1)-a}_{[t_1,t_2]})^{-1}\ ;\ B^{0\to g^+(t_1)-a}_{[t_1,t_2]}\in K_{[t_1,t_2]}(-\varepsilon, \overleftarrow{g}^+ - \overleftarrow{g}^- +\eta(\varepsilon))\Big]\\
&\qquad \times P(a+W_{[t_1,t_2]}(t_2)\in db-(b-g^+(t_1)))\\
&\quad = E\Big[F\Big(g^+ -\overleftarrow{B}^{0\to g^+(t_1)-a}_{[t_1,t_2]}|_{K_{[t_1,t_2]}(-\varepsilon,g^+-g^-+\eta(\varepsilon))}\Big)\widetilde{Z}^{b-\overleftarrow{g}^+}_{[t_1,t_2]}\Big(B^{0\to g^+(t_1)-a}_{[t_1,t_2]}|_{K_{[t_1,t_2]}(-\varepsilon,\overleftarrow{g}^+-\overleftarrow{g}^-+\eta(\varepsilon))}\Big)^{-1}\Big]\\
&\qquad \times P\Big(B^{0\to g^+(t_1)-a}_{[t_1,t_2]}\in K_{[t_1,t_2]}(-\varepsilon, \overleftarrow{g}^+ - \overleftarrow{g}^- +\eta(\varepsilon))\Big)P(a+W_{[t_1,t_2]}(t_2)\in db-(b-g^+(t_1))),
\end{aligned}$$

where $\overleftarrow{B}^{0\to g^+(t_1)-a}_{[t_1,t_2]}$ denotes the continuous process $\{B^{0\to g^+(t_1)-a}_{[t_1,t_2]}(t_1+t_2-t)\}_{t\in[t_1,t_2]}$. Therefore, taking the limit $\varepsilon\downarrow 0$ in (27), we can obtain (2) by Proposition A.2.

## 6.2 Proof of Lemma 3.2

It suffices to show that the limit

$$\lim_{\varepsilon\downarrow 0} \frac{E\left[F\left(a+W_{[t_1,t_2]}\right)\ ;\ a+W_{[t_1,t_2]}\in K_{[t_1,t_2]}(g^- -\varepsilon, g^+ +\eta(\varepsilon))\right]}{P\left(a+W_{[t_1,t_2]}\in K_{[t_1,t_2]}(g^- -\varepsilon, g^+ +\eta(\varepsilon))\right)} \tag{28}$$

exists and coincides with the right-hand side of the desired result. Girsanov's theorem yields

$$\begin{aligned}
&E\left[F(a+W_{[t_1,t_2]})\ ;\ a+W_{[t_1,t_2]}\in K_{[t_1,t_2]}(g^- -\varepsilon, g^+ +\eta(\varepsilon))\right] \\
&\quad = E\left[F(W_{[t_1,t_2]}+g^-)\widetilde{Z}^{g^- -a}_{[t_1,t_2]}(W_{[t_1,t_2]})^{-1}\ ;\ W_{[t_1,t_2]}\in K_{[t_1,t_2]}(-\varepsilon, g^+ - g^- +\eta(\varepsilon))\right] \\
&\quad = E\left[F(W_{[t_1,t_2]}|_{K_{[t_1,t_2]}(-\varepsilon,g^+-g^-+\eta(\varepsilon))}+g^-)\widetilde{Z}^{g^- -a}_{[t_1,t_2]}(W_{[t_1,t_2]}|_{K_{[t_1,t_2]}(-\varepsilon,g^+-g^-+\eta(\varepsilon))})^{-1}\right] \\
&\qquad \times P\left(W_{[t_1,t_2]}\in K_{[t_1,t_2]}(-\varepsilon, g^+ - g^- +\eta(\varepsilon))\right),
\end{aligned}$$

and taking the limit $\varepsilon\downarrow 0$ in (28), we obtain (3) by Proposition A.2.

# 7 Preparation for proofs of the main results

In this section, we prove some lemmas in preparation for proofs of the main results.

**Lemma 7.1.** *Assume that $h^-$ and $h^+$ are $\mathbb{R}$-valued $C^2$-functions defined on $[0,1]$ satisfying*

$$h^-(0)<0<h^+(0) \quad \text{and} \quad \min_{0\le t\le 1}(h^+(t)-h^-(t))>0.$$

*Then, for every $\mathbb{R}$-valued bounded continuous function $F$ on $C([0,1],\mathbb{R})$, $0<s<t<1$ and $h^-(1)<b<h^+(1)$, we have*

$$\begin{aligned}
&E[F(W_{[0,1]})\ ;\ W_{[0,1]}(1)\in db, W_{[0,1]}\in K_{[0,1]}(h^-,h^+)] \\
&= \int_{h^-(t)}^{h^+(t)} E[F(X^{0,y,(h^-,h^+)}_{[0,t]}\oplus_t X^{y,b,(h^-,h^+)}_{[t,1]})] \qquad (29)\\
&\qquad\qquad \times P(W_{[0,t]}\in K_{[0,t]}(h^-,h^+), W_{[0,t]}(t)\in dy) \\
&\qquad\qquad \times P(y+W_{[t,1]}\in K_{[t,1]}(h^-,h^+), y+W_{[t,1]}(1)\in db) \\
&= \int_{h^-(s)}^{h^+(s)} dx \int_{h^-(t)}^{h^+(t)} dy\, E[F(X^{0,x,(h^-,h^+)}_{[0,s]}\oplus_s X^{x,y,(h^-,h^+)}_{[s,t]}\oplus_t X^{y,b,(h^-,h^+)}_{[t,1]})] \qquad (30)\\
&\qquad\qquad\qquad \times P(W_{[0,s]}\in K_{[0,s]}(h^-,h^+), W_{[0,s]}(s)\in dx)/dx \\
&\qquad\qquad\qquad \times P(x+W_{[s,t]}\in K_{[s,t]}(h^-,h^+), x+W_{[s,t]}(t)\in dy)/dy \\
&\qquad\qquad\qquad \times P(y+W_{[t,1]}\in K_{[t,1]}(h^-,h^+), y+W_{[t,1]}(1)\in db),
\end{aligned}$$

*where the respective processes that appear in* (29) *and* (30) *are independent of each other.*

Proof. The Markov property of $W_{[0,1]}$ yields

$$
\begin{aligned}
&E[F(W_{[0,1]})\ ;\ W_{[0,1]}(1) \in db, W_{[0,1]} \in K_{[0,1]}(h^-, h^+)] \\
&= \int_{h^-(t)}^{h^+(t)} E[F(W_{[0,1]})\ ;\ W_{[0,1]}(1) \in db, W_{[0,1]} \in K_{[0,1]}(h^-, h^+), W_{[0,1]}(t) \in dy] \\
&= \int_{h^-(t)}^{h^+(t)} E[F(W_{[0,t]} \oplus_t (y + W_{[t,1]}))\ ;\ W_{[0,t]} \in K_{[0,t]}(h^-, h^+), W_{[0,t]}(t) \in dy, \\
&\qquad\qquad y + W_{[t,1]} \in K_{[t,1]}(h^-, h^+), y + W_{[t,1]}(1) \in db] \qquad (31) \\
&= \int_{h^-(s)}^{h^+(s)} \int_{h^-(t)}^{h^+(t)} E[F(W_{[0,s]} \oplus_s (x + W_{[s,t]}) \oplus_t (y + W_{[t,1]}))\ ; \qquad (32) \\
&\qquad\qquad W_{[0,s]} \in K_{[0,s]}(h^-, h^+), W_{[0,s]}(s) \in dx, \\
&\qquad\qquad x + W_{[s,t]} \in K_{[s,t]}(h^-, h^+), x + W_{[s,t]}(t) \in dy, \\
&\qquad\qquad y + W_{[t,1]} \in K_{[t,1]}(h^-, h^+), y + W_{[t,1]}(1) \in db],
\end{aligned}
$$

where the respective processes that appear in (31) and (32) are independent of each other. Using (31) and (32), we obtain (29) and (30), respectively. □

In a similar manner to the above lemma, we can obtain the following.

**Lemma 7.2.** *Under the same assumption as that of Lemma 7.1, we have*

$$
\begin{aligned}
&E[F(W_{[0,1]})\ ;\ W_{[0,1]} \in K_{[0,1]}(h^-, h^+)] \\
&= \int_{h^-(t)}^{h^+(t)} E[F(X_{[0,t]}^{0,y,(h^-,h^+)} \oplus_t X_{[t,1]}^{y,(h^-,h^+)})] \qquad (33) \\
&\qquad \times P(W_{[0,t]} \in K_{[0,t]}(h^-, h^+), W_{[0,t]}(t) \in dy) P(y + W_{[t,1]} \in K_{[t,1]}(h^-, h^+)) \\
&= \int_{h^-(s)}^{h^+(s)} dx \int_{h^-(t)}^{h^+(t)} dy\ E[F(X_{[0,s]}^{0,x,(h^-,h^+)} \oplus_s X_{[s,t]}^{x,y,(h^-,h^+)} \oplus_t X_{[t,1]}^{y,(h^-,h^+)})] \qquad (34) \\
&\qquad \times P(W_{[0,s]} \in K_{[0,s]}(h^-, h^+), W_{[0,s]}(s) \in dx)/dx \\
&\qquad \times P(x + W_{[s,t]} \in K_{[s,t]}(h^-, h^+), x + W_{[s,t]}(t) \in dy)/dy \\
&\qquad \times P(y + W_{[t,1]} \in K_{[t,1]}(h^-, h^+)),
\end{aligned}
$$

*for every $\mathbb{R}$-valued bounded continuous function $F$ on $C([0,1],\mathbb{R})$ and $0 < s < t < 1$, where the respective processes that appear in* (33) *and* (34) *are independent of each other.*

Applying Girsanov's theorem, we obtain Lemmas 7.3 and 7.4.

**Lemma 7.3.** *Assume that $h^-$ and $h^+$ are $\mathbb{R}$-valued $C^2$-functions defined on $[0,1]$ satisfying $h^-(0) = 0$ and*

$$\min_{0 \le t \le 1} (h^+(t) - h^-(t)) > 0. \qquad (35)$$

*Then, for* $0 < t < 1$, $\varepsilon > 0$ *and* $y \in (h^-(t) - \varepsilon, h^+(t))$, *we have*

$$\begin{aligned}
&\frac{P(W_{[0,t]}(t) \in dy, W_{[0,t]} \in K_{[0,t]}(h^- - \varepsilon, h^+))}{P(W_{[0,t]} \in K^+_{[0,t]}(-\varepsilon))} \\
&= E\left[\widetilde{Z}^{h^-}_{[0,t]}(B^{0\to y-h^-(t)}_{[0,t]}|_{K_{[0,t]}(-\varepsilon, h^+-h^-)})^{-1}\right] P\left(B^{0\to y-h^-(t)}_{[0,t]}|_{K^+_{[0,t]}(-\varepsilon)} \in K^-_{[0,t]}(h^+ - h^-)\right) \\
&\quad \times P\left(W_{[0,t]}|_{K^+_{[0,t]}(-\varepsilon)}(t) \in dy - h^-(t)\right).
\end{aligned}$$

Proof. Girsanov's theorem implies

$$\begin{aligned}
&\frac{P(W_{[0,t]}(t) \in dy, W_{[0,t]} \in K_{[0,t]}(h^- - \varepsilon, h^+))}{P(W_{[0,t]} \in K^+_{[0,t]}(-\varepsilon))} \\
&= \frac{E[\widetilde{Z}^{h^-}_{[0,t]}(W_{[0,t]})^{-1}\ ;\ W_{[0,t]}(t) \in dy - h^-(t), W_{[0,t]} \in K_{[0,t]}(-\varepsilon, h^+ - h^-)]}{P(W_{[0,t]} \in K^+_{[0,t]}(-\varepsilon))} \\
&= E[\widetilde{Z}^{h^-}_{[0,t]}(B^{0\to y-h^-(t)}_{[0,t]}|_{K^+_{[0,t]}(-\varepsilon)})^{-1}\ ;\ B^{0\to y-h^-(t)}_{[0,t]}|_{K^+_{[0,t]}(-\varepsilon)} \in K^-_{[0,t]}(h^+ - h^-)] \\
&\quad \times P\left(W_{[0,t]}|_{K^+_{[0,t]}(-\varepsilon)}(t) \in dy - h^-(t)\right) \\
&= E\left[\widetilde{Z}^{h^-}_{[0,t]}(B^{0\to y-h^-(t)}_{[0,t]}|_{K_{[0,t]}(-\varepsilon, h^+-h^-)})^{-1}\right] P\left(B^{0\to y-h^-(t)}_{[0,t]}|_{K^+_{[0,t]}(-\varepsilon)} \in K^-_{[0,t]}(h^+ - h^-)\right) \\
&\quad \times P\left(W_{[0,t]}|_{K^+_{[0,t]}(-\varepsilon)}(t) \in dy - h^-(t)\right).
\end{aligned}$$

□

**Lemma 7.4.** *Assume that* $h^-$ *and* $h^+$ *are* $\mathbb{R}$*-valued* $C^2$*-functions defined on* $[0, 1]$ *satisfying* (35). *Let* $b = h^+(1)$. *Then, for* $0 < t < 1$, $\varepsilon > 0$ *and* $y \in (h^-(t), h^+(t) + \varepsilon)$, *we have*

$$\begin{aligned}
&\frac{P(y + W_{[t,1]}(1) \in db, y + W_{[t,1]} \in K_{[t,1]}(h^-, h^+ + \varepsilon))}{P(W_{[t,1]} \in K^+_{[t,1]}(-\varepsilon))db} \\
&= E\Big[\widetilde{Z}^{b-\overleftarrow{h}^+}_{[t,1]}\Big(B^{0\to h^+(t)-y}_{[t,1]}|_{K_{[t,1]}(-\varepsilon, \overleftarrow{h}^+ - \overleftarrow{h}^-)}\Big)^{-1}\Big] P\Big(B^{0\to h^+(t)-y}_{[t,1]}|_{K^+_{[t,1]}(-\varepsilon)} \in K^-_{[t,1]}(\overleftarrow{h}^+ - \overleftarrow{h}^-)\Big) \\
&\quad \times P(W_{[t,1]}|_{K^+_{[t,1]}(-\varepsilon)}(1) \in h^+(t) - dy)/dy.
\end{aligned}$$

Proof. Combining $W_{[t,1]}(\cdot) \stackrel{\mathcal{D}}{=} W_{[t,1]}(1) - W_{[t,1]}(t+1-\cdot)$ and Girsanov's theorem, we obtain

$$\begin{aligned}
&\frac{P(y + W_{[t,1]}(1) \in db, y + W_{[t,1]} \in K_{[t,1]}(h^-, h^+ + \varepsilon))}{P(W_{[t,1]} \in K^+_{[t,1]}(-\varepsilon))} \\
&= \frac{P(y + W_{[t,1]}(1) \in db, W_{[t,1]} \in K_{[t,1]}(b - \overleftarrow{h}^+ - \varepsilon, b - \overleftarrow{h}^-))}{P(W_{[t,1]} \in K^+_{[t,1]}(-\varepsilon))} \\
&= \frac{E[\widetilde{Z}^{b-\overleftarrow{h}^+}_{[t,1]}(W_{[t,1]})^{-1} \ ; \ W_{[t,1]}(1) \in db - y - (b - h^+(t)), W_{[t,1]} \in K_{[t,1]}(-\varepsilon, \overleftarrow{h}^+ - \overleftarrow{h}^-)]}{P(W_{[t,1]} \in K^+_{[t,1]}(-\varepsilon))} \\
&= E[\widetilde{Z}^{b-\overleftarrow{h}^+}_{[t,1]}(B^{0\to h^+(t)-y}_{[t,1]}|_{K^+_{[t,1]}(-\varepsilon)})^{-1} \ ; \ B^{0\to h^+(t)-y}_{[t,1]}|_{K^+_{[t,1]}(-\varepsilon)} \in K^-_{[t,1]}(\overleftarrow{h}^+ - \overleftarrow{h}^-)] \\
&\quad \times P\left(W_{[t,1]}|_{K^+_{[t,1]}(-\varepsilon)}(1) \in db - y - (b - h^+(t))\right) \\
&= E[\widetilde{Z}^{b-\overleftarrow{h}^+}_{[t,1]}(B^{0\to h^+(t)-y}_{[t,1]}|_{K_{[t,1]}(-\varepsilon, \overleftarrow{h}^+ - \overleftarrow{h}^-)})^{-1}] P(B^{0\to h^+(t)-y}_{[t,1]}|_{K^+_{[t,1]}(-\varepsilon)} \in K^-_{[t,1]}(\overleftarrow{h}^+ - \overleftarrow{h}^-)) \\
&\quad \times (P(W_{[t,1]}|_{K^+_{[t,1]}(-\varepsilon)}(1) \in h^+(t) - dy)/dy) db.
\end{aligned}$$

□

Applying Lemma A.8 and the fact that $W_{[s,t]}|_{K^+_{[s,t]}(-\varepsilon)}$ converges weakly to $W^+_{[s,t]}$ ([2]), we obtain the following Lemma.

**Lemma 7.5.** *For* $0 \le s < t \le 1$*, we have*

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} \left| P\left(W_{[s,t]}|_{K^+_{[s,t]}(-\varepsilon)}(u) \in dy\right)/dy - P\left(W^+_{[s,t]}(u) \in dy\right)/dy \right| dy = 0 \qquad (u \in [s,t]).$$

# 8 Proofs of the main results in Subsection 3.1

In this section, we prove the main results in Subsection 3.1.

## 8.1 Proof of Theorem 1

In this subsection, we assume that all $X^{x,y,(g^-,g^+)}_{[s,t]}$ are independent. For each $\mathbb{R}$-valued bounded continuous function $G$ on $C([0,1],\mathbb{R})$ and $\varepsilon > 0$, we define

$$I(\varepsilon, G) := E[G(W_{[0,1]}) \ ; \ W_{[0,1]}(1) \in db, W_{[0,1]} \in K_{[0,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon))].$$

Then, we have

$$E[F(B^{0\to b}_{[0,1]}|_{K_{[0,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon))})] = \frac{I(\varepsilon, F)}{I(\varepsilon, 1)}. \tag{36}$$

Further, by Lemma 7.1, we obtain

$$I(\varepsilon, F) = \int_{g^-(t)-\eta^-(\varepsilon)}^{g^+(t)+\eta^+(\varepsilon)} E\left[F\left(X_{[0,t]}^{0,y,(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon))} \oplus_t X_{[t,1]}^{y,b,(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon))}\right)\right] \tag{37}$$
$$\times P(W_{[0,t]} \in K_{[0,t]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon)), W_{[0,t]}(t) \in dy)$$
$$\times P(y + W_{[t,1]} \in K_{[t,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon)), y + W_{[t,1]}(1) \in db).$$

It follows from (37), Lemmas 7.3, 7.4, Proposition A.2, Lemma 7.5, Lemma 3.1 and Lemma A.14 that

$$\begin{aligned} I(F) :=& \lim_{\varepsilon \downarrow 0} \frac{I(\varepsilon, F)}{\eta^-(\varepsilon)\eta^+(\varepsilon)} \\ =& \lim_{\varepsilon \downarrow 0} \frac{I(\varepsilon, F)}{P(W_{[0,t]} \in K_{[0,t]}^+(-\eta^-(\varepsilon)))P(W_{[t,1]} \in K_{[t,1]}^+(-\eta^+(\varepsilon)))} \\ & \times \lim_{\varepsilon \downarrow 0} \frac{P(W_{[0,t]} \in K_{[0,t]}^+(-\eta^-(\varepsilon)))P(W_{[t,1]} \in K_{[t,1]}^+(-\eta^+(\varepsilon)))}{\eta^-(\varepsilon)\eta^+(\varepsilon)} \\ =& \frac{2}{\pi} \int_{g^-(t)}^{g^+(t)} E\left[F\left(X_{[0,t]}^{0,y,(g^-,g^+)} \oplus_t X_{[t,1]}^{y,b,(g^-,g^+)}\right)\right] \frac{1}{\sqrt{t}} q_{[0,t]}^{(g^-,g^+),(\uparrow)}(y) \frac{1}{\sqrt{1-t}} q_{[t,1]}^{(g^-,g^+),(\downarrow)}(y) dy db. \end{aligned} \tag{38}$$

Applying the above argument also for $F = 1$, we have

$$\begin{aligned} I(1) :=& \frac{2}{\pi} \int_{g^-(t)}^{g^+(t)} \frac{1}{\sqrt{t}} q_{[0,t]}^{(g^-,g^+),(\uparrow)}(y) \frac{1}{\sqrt{1-t}} q_{[t,1]}^{(g^-,g^+),(\downarrow)}(y) dy db \\ =& \lim_{\varepsilon \downarrow 0} \frac{P(W(1) \in db, W \in K_{[0,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon)))}{\eta^-(\varepsilon)\eta^+(\varepsilon)} \\ =& n_1(b) \lim_{\varepsilon \downarrow 0} \frac{P(B_{[0,1]}^{0\to b} \in K_{[0,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon)))}{\eta^-(\varepsilon)\eta^+(\varepsilon)} db = \frac{2}{\pi} C_{g^-,g^+} db. \end{aligned} \tag{39}$$

Combining (36), (38) and (39), we obtain

$$C_{g^-,g^+} = \int_{g^-(t)}^{g^+(t)} \frac{1}{\sqrt{t}} q_{[0,t]}^{(g^-,g^+),(\uparrow)}(y) \frac{1}{\sqrt{1-t}} q_{[t,1]}^{(g^-,g^+),(\downarrow)}(y) dy \in (0, \infty)$$

and

$$\lim_{\varepsilon \downarrow 0} E[F(B_{[0,1]}^{0\to b}|_{K_{[0,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon))})] = \frac{I(F)}{I(1)} = \int_{g^-(t)}^{g^+(t)} E\left[F\left(X_{[0,t]}^{0,y,(g^-,g^+)} \oplus_t X_{[t,1]}^{y,b,(g^-,g^+)}\right)\right] h(t, y) dy.$$

Therefore, we can define the probability measure $\widetilde{P}_H$ on $(C([0,1], \mathbb{R}), \mathcal{B}(C([0,1], \mathbb{R})))$ as

$$\widetilde{P}_H(A) := \int_{g^-(t)}^{g^+(t)} P\left(X_{[0,t]}^{0,y,(g^-,g^+)} \oplus_t X_{[t,1]}^{y,b,(g^-,g^+)} \in A\right) h(t, y) dy \quad (A \in \mathcal{B}(C([0,1], \mathbb{R}))),$$

and there exists an $\mathbb{R}$-valued continuous stochastic process $H^{g^- \to g^+} = \{H^{g^- \to g^+}(t)\}_{t \in [0,1]}$ that satisfies (5) and (6). Thus, a limit argument on $F$ yields

$$P(H^{g^- \to g^+}(t) \in dy) = h(t, y) dy \quad (y \in (g^-(t), g^+(t))).$$

On the other hand, by Lemma 7.1, we obtain

$$
\begin{aligned}
I(\varepsilon, F) = &\int_{g^-(t_2)-\eta^-(\varepsilon)}^{g^+(t_2)+\eta^+(\varepsilon)} dy_2 \int_{g^-(t_1)-\eta^-(\varepsilon)}^{g^+(t_1)+\eta^+(\varepsilon)} dy_1 \\
&\times E\left[F\left(X_{[0,t_1]}^{0,y_1,(g^- -\eta^-(\varepsilon),g^+ +\eta^+(\varepsilon))} \oplus_{t_1} X_{[t_1,t_2]}^{y_1,y_2,(g^- -\eta^-(\varepsilon),g^+ +\eta^+(\varepsilon))} \oplus_{t_2} X_{[t_2,1]}^{y_2,b,(g^- -\eta^-(\varepsilon),g^+ +\eta^+(\varepsilon))}\right)\right] \\
&\times P(y_2 + W_{[t_2,1]} \in K_{[t_2,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon)), y_2 + W_{[t_2,1]}(1) \in db) \\
&\times P(y_1 + W_{[t_1,t_2]} \in K_{[t_1,t_2]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon)), y_1 + W_{[t_1,t_2]}(t_2) \in dy_2)/dy_2 \\
&\times P(W_{[0,t_1]} \in K_{[0,t_1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon)), W_{[0,t_1]}(t_1) \in dy_1)/dy_1.
\end{aligned} \tag{40}
$$

By (40), Lemmas 7.3, 7.4, Proposition A.2, Lemma 7.5, Lemma 3.1 and Lemma A.14, $I(F)$ satisfies

$$
\begin{aligned}
I(F) = &\lim_{\varepsilon\downarrow 0} \frac{I(\varepsilon, F)}{P(W_{[0,t_1]} \in K^+_{[0,t_1]}(-\eta^-(\varepsilon)))P(W_{[t_2,1]} \in K^+_{[t_2,1]}(-\eta^+(\varepsilon)))} \\
&\times \lim_{\varepsilon\downarrow 0} \frac{P(W_{[0,t_1]} \in K^+_{[0,t_1]}(-\eta^-(\varepsilon)))P(W_{[t_2,1]} \in K^+_{[t_2,1]}(-\eta^+(\varepsilon)))}{\eta^-(\varepsilon)\eta^+(\varepsilon)} \\
= &\frac{2}{\pi} \int_{g^-(t_1)}^{g^+(t_1)} \int_{g^-(t_2)}^{g^+(t_2)} E\left[F\left(X_{[0,t_1]}^{0,y_1,(g^-,g^+)} \oplus_{t_1} X_{[t_1,t_2]}^{y_1,y_2,(g^-,g^+)} \oplus_{t_2} X_{[t_2,1]}^{y_2,b,(g^-,g^+)}\right)\right] \\
&\qquad \times \frac{1}{\sqrt{t_1}} q_{[0,t_1]}^{(g^-,g^+),(\uparrow)}(y_1) p_{[t_1,t_2]}^{(g^-,g^+)}(y_1,y_2) \frac{1}{\sqrt{1-t_2}} q_{[t_2,1]}^{(g^-,g^+),(\downarrow)}(y_2) dy_1 dy_2 db \\
= &\frac{2}{\pi} C_{g^-,g^+} \int_{g^-(t_1)}^{g^+(t_1)} \int_{g^-(t_2)}^{g^+(t_2)} E\left[F\left(X_{[0,t_1]}^{0,y_1,(g^-,g^+)} \oplus_{t_1} X_{[t_1,t_2]}^{y_1,y_2,(g^-,g^+)} \oplus_{t_2} X_{[t_2,1]}^{y_2,b,(g^-,g^+)}\right)\right] \\
&\qquad \times h(t_1,y_1)h(t_1,y_1,t_2,y_2) dy_1 dy_2 db.
\end{aligned} \tag{41}
$$

It follows from (36), (39) and (41) that

$$
\begin{aligned}
&E[F(H^{g^-\to g^+})] \\
&\quad = \lim_{\varepsilon\downarrow 0} E[F(B_{[0,1]}^{0\to b}|_{K_{[0,1]}(g^- -\eta^-(\varepsilon),g^+ +\eta^+(\varepsilon))})] \\
&\quad = \int_{g^-(t_1)}^{g^+(t_1)} \int_{g^-(t_2)}^{g^+(t_2)} E\left[F\left(X_{[0,t_1]}^{0,y_1,(g^-,g^+)} \oplus_{t_1} X_{[t_1,t_2]}^{y_1,y_2,(g^-,g^+)} \oplus_{t_2} X_{[t_2,1]}^{y_2,b,(g^-,g^+)}\right)\right] h(t_1,y_1)h(t_1,y_1,t_2,y_2) dy_1 dy_2.
\end{aligned}
$$

Hence, (7) holds. Similarly, using a limit argument on $F$, we can deduce for $y_1 \in (g^-(t_1), g^+(t_1))$ and $y_2 \in (g^-(t_2), g^+(t_2))$ that

$$
\begin{aligned}
&P(H^{g^-\to g^+}(t_1) \in dy_1, H^{g^-\to g^+}(t_2) \in dy_2) = h(t_1,y_1)h(t_1,y_1,t_2,y_2)dy_1dy_2, \\
&P(H^{g^-\to g^+}(t_2) \in dy_2 \mid H^{g^-\to g^+}(t_1) = y_1) = h(t_1,y_1,t_2,y_2)dy_2.
\end{aligned} \tag{42}
$$

If we define $I_{t_1}(\varepsilon, y_1)$ ($y_1 \in (g^-(t_1), g^+(t_1))$) to be

$$
I_{t_1}(\varepsilon, y_1) := P(y_1 + W_{[t_1,1]} \in K_{[t_1,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon)), y_1 + W_{[t_1,1]}(1) \in db),
$$

then

$$\begin{aligned} I_{t_1}(y_1) &:= \lim_{\varepsilon\downarrow 0} \frac{I_{t_1}(\varepsilon, y_1)}{\eta^+(\varepsilon)} \\ &= \lim_{\varepsilon\downarrow 0} \frac{I_{t_1}(\varepsilon, y_1)}{P(W_{[t_1,1]} \in K^+_{[t_1,1]}(-\eta^+(\varepsilon)))} \lim_{\varepsilon\downarrow 0} \frac{P(W_{[t_1,1]} \in K^+_{[t_1,1]}(-\eta^+(\varepsilon)))}{\eta^+(\varepsilon)} \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{1-t_1}} q^{(g^-,g^+),(\downarrow)}_{[t_1,1]}(y_1) db \qquad (y_1 \in (g^-(t_1), g^+(t_1))) \end{aligned} \tag{43}$$

holds by Lemma 7.4 and Proposition A.2. On the other hand, because we have

$$\begin{aligned} I_{t_1}(\varepsilon, y_1) = \int_{g^-(t_2)-\eta^-(\varepsilon)}^{g^+(t_2)+\eta^+(\varepsilon)} & P(y_1 + W_{[t_1,t_2]} \in K_{[t_1,t_2]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon)), y_1 + W_{[t_1,t_2]}(t_2) \in dy_2) \\ & \times P(y_2 + W_{[t_2,1]} \in K_{[t_2,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon)), y_2 + W_{[t_2,1]}(1) \in db), \end{aligned}$$

for $y_1 \in (g^-(t_1), g^+(t_1))$, it follows from Lemma 7.4, Proposition A.2 and Lemma 7.5 that

$$\begin{aligned} I_{t_1}(y_1) &= \lim_{\varepsilon\downarrow 0} \frac{I_{t_1}(\varepsilon, y_1)}{P(W_{[t_2,1]} \in K^+_{[t_2,1]}(-\eta^+(\varepsilon)))} \lim_{\varepsilon\downarrow 0} \frac{P(W_{[t_2,1]} \in K^+_{[t_2,1]}(-\eta^+(\varepsilon)))}{\eta^+(\varepsilon)} \\ &= \sqrt{\frac{2}{\pi}} \int_{g^-(t_2)}^{g^+(t_2)} p^{(g^-,g^+)}_{[t_1,t_2]}(y_1, y_2) \frac{1}{\sqrt{1-t_2}} q^{(g^-,g^+),(\downarrow)}_{[t_2,1]}(y_2) dy_2 db \qquad (y_1 \in (g^-(t_1), g^+(t_1))). \end{aligned} \tag{44}$$

Combining (43) and (44), we obtain

$$\frac{1}{\sqrt{1-t_1}} q^{(g^-,g^+),(\downarrow)}_{[t_1,1]}(y_1) = \int_{g^-(t_2)}^{g^+(t_2)} p^{(g^-,g^+)}_{[t_1,t_2]}(y_1, y_2) \frac{1}{\sqrt{1-t_2}} q^{(g^-,g^+),(\downarrow)}_{[t_2,1]}(y_2) dy_2 \qquad (y_1 \in (g^-(t_1), g^+(t_1)))$$

and

$$\int_{g^-(t_2)}^{g^+(t_2)} h(t_1, y_1, t_2, y_2) dy_2 = 1 \qquad (y_1 \in (g^-(t_1), g^+(t_1))). \tag{45}$$

Assume that $t_3$ satisfies $0 < t_1 < t_2 < t_3 < 1$. Because we have

$$p^{(g^-,g^+)}_{[t_1,t_3]}(y_1, y_3) = \int_{g^-(t_2)}^{g^+(t_2)} p^{(g^-,g^+)}_{[t_1,t_2]}(y_1, y_2) p^{(g^-,g^+)}_{[t_2,t_3]}(y_2, y_3) dy_2$$

for $y_1 \in (g^-(t_1), g^+(t_1))$ and $y_3 \in (g^-(t_3), g^+(t_3))$, we can deduce for $y_1 \in (g^-(t_1), g^+(t_1))$ and $y_3 \in (g^-(t_3), g^+(t_3))$ that the following Chapman–Kolmogorov identity holds:

$$h(t_1, y_1, t_3, y_3) = \int_{g^-(t_2)}^{g^+(t_2)} h(t_1, y_1, t_2, y_2) h(t_2, y_2, t_3, y_3) dy_2. \tag{46}$$

Therefore, (42), (45) and (46) imply that $H^{g^-\to g^+} = \{H^{g^-\to g^+}(t)\}_{t\in[0,1]}$ is a Markov process.

## 8.2 Proof of Corollary 1

Let $0 < s < t < 1$ and $x, y \in (0, b)$. Then, by Corollary 6, we obtain

$$q_{[0,t]}^{(0,b),(\uparrow)}(y) = \frac{tJ^{(b)}(t,y)}{2yn_t(y)}P(\sqrt{t}W^+(1) \in dy)/dy = \sqrt{\frac{\pi t}{2}}J^{(b)}(t,y)$$

and

$$q_{[t,1]}^{(0,b),(\downarrow)}(y) = \frac{(1-t)J^{(b)}(1-t,b-y)}{2(b-y)n_{1-t}(b-y)}P(\sqrt{1-t}W^+(1) \in b-dy)/dy = \sqrt{\frac{\pi(1-t)}{2}}J^{(b)}(1-t,b-y).$$

Further, by Lemma A.3 and L'Hôpital's rule, it holds that

$$\begin{aligned} C_{0,b} &= \frac{\pi n_1(b)}{2}\lim_{\varepsilon\downarrow 0}\frac{P(B_{[0,1]}^{0\to b} \in K_{[0,1]}(-\varepsilon, b+\varepsilon))}{\varepsilon^2} \\ &= \frac{\pi}{2}\lim_{\varepsilon\downarrow 0}\frac{P(W(1)\in db,\ -\varepsilon \le m(W) < M(W) \le b+\varepsilon)}{\varepsilon^2} \\ &= \frac{\pi}{2}\overline{J}^{(b)}(1,b). \end{aligned}$$

On the other hand, $p_{[s,t]}^{(0,b)}(x,y)$ is written as

$$\begin{aligned} p_{[s,t]}^{(0,b)}(x,y) &= P(x + W_{[s,t]} \in K_{[s,t]}(0,b), x + W_{[s,t]}(t) \in dy)/dy \\ &= P\left(W(t-s) \in dy - x, -x \le m_{t-s}(W) < M_{t-s}(W) \le b - x\right)/dy \\ &= \sum_{k=-\infty}^{\infty}(n_{t-s}(y - x + 2kb) - n_{t-s}(2(k+1)b - y - x)) \\ &= J^{(b)}(s,x,t,y) \end{aligned}$$

by (88). Therefore, by Theorem 1, we obtain

$$P\left(H^{0\to b}(t) \in dy\right) = \frac{\frac{1}{\sqrt{t}}q_{[0,t]}^{(0,b),(\uparrow)}(y)\frac{1}{\sqrt{1-t}}q_{[t,1]}^{(0,b),(\downarrow)}(y)}{C_{0,b}} = \frac{J^{(b)}(t,y)J^{(b)}(1-t,b-y)}{\overline{J}^{(b)}(1,b)},$$

$$P\left(H^{0\to b}(t) \in dy \mid H^{0\to b}(s) = x\right) = \frac{p_{[s,t]}^{(0,b)}(x,y)\frac{1}{\sqrt{1-t}}q_{[t,1]}^{(0,b),(\downarrow)}(y)}{\frac{1}{\sqrt{1-s}}q_{[s,1]}^{(0,b),(\downarrow)}(x)} = \frac{J^{(b)}(s,x,t,y)J^{(b)}(1-t,b-y)}{J^{(b)}(1-s,b-x)}.$$

## 8.3 Proof of Corollary 2

Let $A_i$ $(i = 1, 2)$ be closed subsets of $C([0,1],\mathbb{R})$ given by

$$\begin{aligned} A_1 &:= \left\{w \in C([0,1],\mathbb{R}) \;\middle|\; \min_{u\in[0,t]}\{g(u) - w(u)\} = 0\right\}, \\ A_2 &:= \left\{w \in C([0,1],\mathbb{R}) \;\middle|\; \min_{u\in[0,t]}\{g(u) - w(u)\} \ge 0,\ w(t) \le z\right\}. \end{aligned}$$

Remark 3.2 implies that Theorem 1 can be applied for $F = 1_{A_i}$ $(i = 1, 2)$. Thus, we obtain

$$P\left(\min_{u\in[0,t]}\left\{g(u) - H^{g^-\to g^+}(u)\right\} = 0\right) = \int_{g^-(t)}^{g(t)} P\left(X^{0,y,(g^-,g^+)}_{[0,t]} \in \partial K^-_{[0,t]}(g)\right) h(t,y)dy, \tag{47}$$

$$P\left(\min_{u\in[0,t]}\left\{g(u) - H^{g^-\to g^+}(u)\right\} \geq 0, H^{g^-\to g^+}(t) \leq z\right)$$
$$= \int_{g^-(t)}^{z} P\left(X^{0,y,(g^-,g^+)}_{[0,t]} \in K^-_{[0,t]}(g),\ X^{0,y,(g^-,g^+)}_{[0,t]}(t) \leq z\right) h(t,y)dy. \tag{48}$$

It follows from Lemma A.18 that

$$P\left(r^{0\to y-g^-(t)}_{[0,t]} \in \partial K^-_{[0,t]}(g-g^-)\right) = 0 \qquad (g^-(t) < y < g(t)). \tag{49}$$

Combining Remark 3.1 and (49), we obtain

$$P\left(X^{0,y,(g^-,g^+)}_{[0,t]} \in \partial K^-_{[0,t]}(g)\right) = 0 \qquad (g^-(t) < y < g(t)). \tag{50}$$

Thus, by (47) and (50), it holds that

$$P\left(\min_{u\in[0,t]}\left\{g(u) - H^{g^-\to g^+}(u)\right\} = 0\right) = \int_{g^-(t)}^{g(t)} P\left(X^{0,y,(g^-,g^+)}_{[0,t]} \in \partial K^-_{[0,t]}(g)\right) h(t,y)dy = 0.$$

On the other hand, we obtain

$$P\left(X^{0,y,(g^-,g^+)}_{[0,t]} \in K^-_{[0,t]}(g)\right)$$
$$= \frac{E\left[\widetilde{Z}^{g^-}_{[0,t]}\left(r^{0\to y-g^-(t)}_{[0,t]}\big|_{K^-_{[0,t]}(g-g^-)}\right)^{-1}\right]}{E\left[\widetilde{Z}^{g^-}_{[0,t]}\left(r^{0\to y-g^-(t)}_{[0,t]}\big|_{K^-_{[0,t]}(g^+-g^-)}\right)^{-1}\right]} \cdot \frac{P\left(r^{0\to y-g^-(t)}_{[0,t]} \in K^-_{[0,t]}(g-g^-)\right)}{P\left(r^{0\to y-g^-(t)}_{[0,t]} \in K^-_{[0,t]}(g^+-g^-)\right)} \qquad (g^-(t) < y < g(t)) \tag{51}$$

by Remark 3.1. Combining (48) and (51), we obtain

$$P\left(\min_{u\in[0,t]}\left\{g(u) - H^{g^-\to g^+}(u)\right\} \geq 0, H^{g^-\to g^+}(t) \leq z\right)$$
$$= \int_{g^-(t)}^{z} P\left(X^{0,y,(g^-,g^+)}_{[0,t]} \in K^-_{[0,t]}(g)\right) h(t,y)dy$$
$$= \int_{g^-(t)}^{z} (C_{g^-,g^+})^{-1} \frac{1}{\sqrt{t}} q^{(g^-,g),(\uparrow)}_{[0,t]}(y) \frac{1}{\sqrt{1-t}} q^{(g^-,g^+),(\downarrow)}_{[t,1]}(y)dy.$$

## 8.4 Proof of Corollary 3

Let $b = g^+(1)$, and let $A_i$ $(i = 1, 2)$ be closed subsets of $C([0,1],\mathbb{R})$ given by

$$A_1 := \left\{w \in C([0,1],\mathbb{R}) \;\middle|\; \min_{u\in[t,1]}\{w(u) - g(u)\} = 0\right\},$$
$$A_2 := \left\{w \in C([0,1],\mathbb{R}) \;\middle|\; \min_{u\in[t,1]}\{w(u) - g(u)\} \geq 0,\ w(t) \leq z\right\}.$$

Remark 3.2 implies that Theorem 1 can be applied for $F = 1_{A_i}$ $(i = 1, 2)$. Thus, we obtain

$$P\left(\min_{u\in[t,1]}\left\{H^{g^-\to g^+}(u) - g(u)\right\} = 0\right) = \int_{g(t)}^{g^+(t)} P\left(X_{[t,1]}^{y,b,(g^-,g^+)} \in \partial K_{[t,1]}^+(g)\right) h(t,y)dy, \tag{52}$$

$$\begin{aligned} &P\left(\min_{u\in[t,1]}\left\{H^{g^-\to g^+}(u) - g(u)\right\} \geq 0, H^{g^-\to g^+}(t) \leq z\right) \\ &\quad = \int_{g(t)}^{z} P\left(X_{[t,1]}^{y,b,(g^-,g^+)} \in K_{[t,1]}^+(g),\ X_{[t,1]}^{y,b,(g^-,g^+)}(t) \leq z\right) h(t,y)dy. \end{aligned} \tag{53}$$

It follows from Lemma A.18 that

$$P\left(r_{[t,1]}^{0\to g^+(t)-y} \in \partial K_{[t,1]}^-(\overleftarrow{g}^+ - \overleftarrow{g})\right) = 0 \qquad (g(t) < y < g^+(t)). \tag{54}$$

Combining Remark 3.1 and (54), we obtain

$$P\left(X_{[t,1]}^{y,b,(g^-,g^+)} \in \partial K_{[t,1]}^+(g)\right) = 0 \qquad (g(t) < y < g^+(t)). \tag{55}$$

Thus, by (52) and (55), it holds that

$$P\left(\min_{u\in[t,1]}\left\{H^{g^-\to g^+}(u) - g(u)\right\} = 0\right) = \int_{g(t)}^{g^+(t)} P\left(X_{[t,1]}^{y,b,(g^-,g^+)} \in \partial K_{[t,1]}^+(g)\right) h(t,y)dy = 0.$$

On the other hand, we obtain

$$\begin{aligned} &P\left(X_{[t,1]}^{y,b,(g^-,g^+)} \in K_{[t,1]}^+(g)\right) \\ &\quad = \frac{E\left[\widetilde{Z}_{[t,1]}^{b-\overleftarrow{g}^+}\left(r_{[t,1]}^{0\to g^+(t)-y}\Big|_{K_{[t,1]}^-(\overleftarrow{g}^+ - \overleftarrow{g})}\right)^{-1}\right]}{E\left[\widetilde{Z}_{[t,1]}^{b-\overleftarrow{g}^+}\left(r_{[t,1]}^{0\to g^+(t)-y}\Big|_{K_{[t,1]}^-(\overleftarrow{g}^+ - \overleftarrow{g}^-)}\right)^{-1}\right]} \cdot \frac{P\left(r_{[t,1]}^{0\to g^+(t)-y} \in K_{[t,1]}^-(\overleftarrow{g}^+ - \overleftarrow{g})\right)}{P\left(r_{[t,1]}^{0\to g^+(t)-y} \in K_{[t,1]}^-(\overleftarrow{g}^+ - \overleftarrow{g}^-)\right)} \qquad (g(t) < y < g^+(t)) \end{aligned} \tag{56}$$

by Remark 3.1. Combining (53) and (56), we obtain

$$\begin{aligned} &P\left(\min_{u\in[t,1]}\left\{H^{g^-\to g^+}(u) - g(u)\right\} \geq 0, H^{g^-\to g^+}(t) \leq z\right) \\ &\quad = \int_{g(t)}^{z} P\left(X_{[t,1]}^{y,b,(g^-,g^+)} \in K_{[t,1]}^+(g)\right) h(t,y)dy \\ &\quad = \int_{g(t)}^{z} (C_{g^-,g^+})^{-1} \frac{1}{\sqrt{t}} q_{[0,t]}^{(g^-,g^+),(\uparrow)}(y) \frac{1}{\sqrt{1-t}} q_{[t,1]}^{(g,g^+),(\downarrow)}(y)dy. \end{aligned}$$

## 8.5 Proof of Theorem 2

Let $P^X$ denote the measure induced by a continuous process $X = \{X(t)\}_{t\in[0,1]}$. In addition, for a continuous process $X = \{X(t)\}_{t\in[0,1]}$, we write the expectation with respect to the probability $P^X$ as $E^X$, and we define

$$P_t^X := P \circ (\pi_{[0,t]} \circ X)^{-1}, \quad 0 < t < 1.$$

First, we prepare two lemmas.

**Lemma 8.1.** *Let $a, c \in \mathbb{R}$. For $t \in (0,1)$, we have*

$$\frac{dP_t^{B^{a\to c}}}{dP_t^{a+W}}(w) = \frac{n_{1-t}(w(t)-c)}{n_1(a-c)}, \quad w \in C([0,t],\mathbb{R}).$$

Proof. Let $A \in \mathcal{B}(C([0,t],\mathbb{R}))$ be fixed. By the Markov property of $a + W$, we obtain the assertion as follows:

$$\begin{aligned} P_t^{B^{a\to c}}(A) &= \frac{P^{a+W}\left(\pi_{[0,t]}^{-1}(A), w(1) \in dc\right)}{P^{a+W}(w(1) \in dc)} \\ &= \frac{E^{a+W}\left[1_{\pi_{[0,t]}^{-1}(A)}(w) \cdot P^{a+W}(w(1) \in dc \mid w(t))\right]}{P^{a+W}(w(1) \in dc)} \\ &= \int_{\pi_{[0,t]}^{-1}(A)} \frac{P^{a+W}(w(1) \in dc \mid w(t))}{P^{a+W}(w(1) \in dc)} P^{a+W}(dw) \\ &= \int_A \frac{n_{1-t}(w(t)-c)}{n_1(a-c)} P_t^{a+W}(dw). \end{aligned}$$

□

**Lemma 8.2.** *Let $t \in (0,1)$, and let $R_{[0,t]} = \{R_{[0,t]}(u)\}_{u\in[0,t]}$ be the BES(3)-process starting at 0 on $[0,t]$. Then, we have*

$$\begin{aligned} &E\left[F(\pi_{[0,t]}(H^{g^-\to g^+}))\right] \\ &= \sqrt{\frac{\pi}{2}} E\left[F\left(R_{[0,t]} + g^-\right) \frac{q_{[t,1]}^{(g^-,g^+),(\downarrow)}\left(R_{[0,t]}(t) + g^-(t)\right)}{C_{g^-,g^+}\sqrt{1-t} \cdot R_{[0,t]}(t) \cdot \widetilde{Z}_{[0,t]}^{g^-}(R_{[0,t]})} 1_{K^-_{[0,t]}(g^+-g^-)}\left(R_{[0,t]}\right)\right] \\ &= \sqrt{\frac{\pi}{2}} \int_{C([0,t],\mathbb{R})} F(w) \frac{q_{[t,1]}^{(g^-,g^+),(\downarrow)}(w(t))}{C_{g^-,g^+}\sqrt{1-t} \cdot (w(t) - g^-(t)) \cdot Z_{[0,t]}^{g^-}(w)} 1_{K^-_{[0,t]}(g^+)}(w)\, P\left(R_{[0,t]} + g^- \in dw\right) \end{aligned}$$

*for every $\mathbb{R}$-valued bounded continuous function $F$ on $C([0,t],\mathbb{R})$.*

Proof. By the Markov property of $B_{[0,1]}^{0\to b}$ and Lemma 8.1, we obtain

$$\begin{aligned} &E[F(\pi_{[0,t]}(B_{[0,1]}^{0\to b}))1_{K_{[0,1]}(g^- -\eta^-(\varepsilon), g^+ +\eta^+(\varepsilon))}(B_{[0,1]}^{0\to b})] \\ &= \int_{C([0,1],\mathbb{R})} F(\pi_{[0,t]}(w))1_{K_{[0,t]}(g^- -\eta^-(\varepsilon), g^+ +\eta^+(\varepsilon))}(\pi_{[0,t]}(w))1_{K_{[t,1]}(g^- -\eta^-(\varepsilon), g^+ +\eta^+(\varepsilon))}(\pi_{[t,1]}(w))P(B_{[0,1]}^{0\to b} \in dw) \\ &= \int_{C([0,1],\mathbb{R})} F(\pi_{[0,t]}(w))1_{K_{[0,t]}(g^- -\eta^-(\varepsilon), g^+ +\eta^+(\varepsilon))}(\pi_{[0,t]}(w)) \\ &\qquad \times P(B_{[t,1]}^{w(t)\to b} \in K_{[t,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon)))P(B_{[0,1]}^{0\to b} \in dw) \\ &= \int_{C([0,t],\mathbb{R})} F(w)1_{K_{[0,t]}(g^- -\eta^-(\varepsilon), g^+ +\eta^+(\varepsilon))}(w)P(B_{[t,1]}^{w(t)\to b} \in K_{[t,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon)))P_t^{B^{0\to b}}(dw) \\ &= \int_{C([0,t],\mathbb{R})} F(w)P(B_{[t,1]}^{w(t)\to b} \in K_{[t,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon))) \qquad (57) \\ &\qquad \times \frac{n_{1-t}(w(t)-b)}{n_1(b)} 1_{K_{[0,t]}(g^- -\eta^-(\varepsilon), g^+ +\eta^+(\varepsilon))}(w)P_t^W(dw). \end{aligned}$$

Then, by (57), it holds that

$$
\begin{aligned}
&E[F(\pi_{[0,t]}(B^{0\to b}_{[0,1]}|_{K_{[0,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon))}))] \\
&\quad = \frac{E[F(\pi_{[0,t]}(B^{0\to b}_{[0,1]}))\ ;\ B^{0\to b}_{[0,1]} \in K_{[0,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon))]}{P(B^{0\to b}_{[0,1]} \in K_{[0,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon)))} \\
&\quad = \frac{\pi}{2} \cdot \frac{2\eta^-(\varepsilon)\eta^+(\varepsilon)}{\pi n_1(b) P(B^{0\to b}_{[0,1]} \in K_{[0,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon)))} \cdot \frac{P\big(W_{[0,t]} \in K_{[0,t]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon))\big)}{\eta^-(\varepsilon)} \\
&\quad\quad \times \int_{C([0,t],\mathbb{R})} F(w) \frac{P\big(B^{w(t)\to b}_{[t,1]} \in K_{[t,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon))\big)}{\eta^+(\varepsilon)} n_{1-t}(w(t) - b) \qquad (58) \\
&\quad\quad\quad\quad \times P\big(W_{[0,t]}|_{K_{[0,t]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon))} \in dw\big).
\end{aligned}
$$

On the other hand, using Lemma 7.4 and Proposition A.2, we obtain

$$
\begin{aligned}
&\frac{P\big(B^{a\to b}_{[t,1]} \in K_{[t,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon))\big)}{\eta^+(\varepsilon)} n_{1-t}(a - b) \\
&= \frac{P(a + W_{[t,1]}(1) \in db,\ a + W_{[t,1]} \in K_{[t,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon)))}{P(a + W_{[t,1]}(1) \in db)\eta^+(\varepsilon)} n_{1-t}(a - b) \\
&= \frac{P(a + W_{[t,1]}(1) \in db,\ a + W_{[t,1]} \in K_{[t,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon)))}{P(W_{[t,1]} \in K^+_{[t,1]}(-\eta^+(\varepsilon)))db} \cdot \frac{P(W_{[t,1]} \in K^+_{[t,1]}(-\eta^+(\varepsilon)))}{\eta^+(\varepsilon)} \\
&\to q^{(g^-,g^+),(\downarrow)}_{[t,1]}(a) \cdot \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{1-t}}, \qquad \varepsilon \downarrow 0, \qquad (59)
\end{aligned}
$$

for $g^-(t) < a < g^+(t)$. In addition, by Girsanov's theorem and Proposition A.2, we have

$$
\begin{aligned}
&\frac{P(W_{[0,t]} \in K_{[0,t]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon)))}{\eta^-(\varepsilon)} \\
&\quad = \frac{P(W_{[0,t]} \in K_{[0,t]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon)))}{P(W_{[0,t]} \in K^+_{[0,t]}(-\eta^-(\varepsilon)))} \cdot \frac{P(W_{[0,t]} \in K^+_{[0,t]}(-\eta^-(\varepsilon)))}{\eta^-(\varepsilon)} \\
&\quad = \frac{E[\widetilde{Z}^{g^-}_{[0,t]}(W_{[0,t]})^{-1}\ ;\ W_{[0,t]} \in K_{[0,t]}(-\eta^-(\varepsilon), g^+ - g^- + \eta^+(\varepsilon))]}{P(W_{[0,t]} \in K^+_{[0,t]}(-\eta^-(\varepsilon)))} \cdot \frac{P(W_{[0,t]} \in K^+_{[0,t]}(-\eta^-(\varepsilon)))}{\eta^-(\varepsilon)} \\
&\quad = E\left[\widetilde{Z}^{g^-}_{[0,t]}(W_{[0,t]}|_{K_{[0,t]}(-\eta^-(\varepsilon), g^+ - g^- + \eta^+(\varepsilon))})^{-1}\right] P\left(W_{[0,t]}|_{K^+_{[0,t]}(-\eta^-(\varepsilon))} \in K^-_{[0,t]}(g^+ - g^- + \eta^+(\varepsilon))\right) \\
&\quad\quad \times \frac{P(W_{[0,t]} \in K^+_{[0,t]}(-\eta^-(\varepsilon)))}{\eta^-(\varepsilon)} \\
&\quad \to E\left[\widetilde{Z}^{g^-}_{[0,t]}(W^+_{[0,t]}|_{K^-_{[0,t]}(g^+ - g^-)})^{-1}\right] P\left(W^+_{[0,t]} \in K^-_{[0,t]}(g^+ - g^-)\right) \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{t}}, \qquad \varepsilon \downarrow 0. \qquad (60)
\end{aligned}
$$

Therefore, it follows from Theorem 1, (58), (4), (59), (60) and Lemma 3.2 that

$$E\left[F(\pi_{[0,t]}(H^{g^-\to g^+}))\right] = \lim_{\varepsilon\downarrow 0} E[F(\pi_{[0,t]}(B^{0\to b}_{[0,1]}|_{K_{[0,1]}(g^- - \eta^-(\varepsilon), g^+ + \eta^+(\varepsilon))}))]$$

$$= E\left[F\left(W^+_{[0,t]} + g^-\right) \frac{q^{(g^-,g^+),(\downarrow)}_{[t,1]}\left(W^+_{[0,t]}(t) + g^-(t)\right)}{C_{g^-,g^+}\sqrt{t(1-t)}\cdot \widetilde{Z}^{g^-}_{[0,t]}(W^+_{[0,t]})} 1_{K^-_{[0,t]}(g^+ - g^-)}\left(W^+_{[0,t]}\right)\right]. \tag{61}$$

Further, combining (61) and a change of measure formula between Brownian meander and BES(3)-process ([4]), we obtain

$$\begin{aligned} &E\left[F(\pi_{[0,t]}(H^{g^-\to g^+}))\right] \\ &= \sqrt{\frac{\pi}{2}} E\left[F\left(R_{[0,t]} + g^-\right) \frac{q^{(g^-,g^+),(\downarrow)}_{[t,1]}\left(R_{[0,t]}(t) + g^-(t)\right)}{C_{g^-,g^+}\sqrt{1-t}\cdot R_{[0,t]}(t)\cdot \widetilde{Z}^{g^-}_{[0,t]}(R_{[0,t]})} 1_{K^-_{[0,t]}(g^+ - g^-)}\left(R_{[0,t]}\right)\right] \\ &= \sqrt{\frac{\pi}{2}} \int_{C([0,t],\mathbb{R})} F(w) \frac{q^{(g^-,g^+),(\downarrow)}_{[t,1]}\left(w(t)\right)}{C_{g^-,g^+}\sqrt{1-t}\cdot (w(t) - g^-(t))\cdot Z^{g^-}_{[0,t]}(w)} 1_{K^-_{[0,t]}(g^+)}(w)\, P\left(R_{[0,t]} + g^- \in dw\right). \end{aligned}$$

□

Now, we prove Theorem 2. Let $A$ be a closed subset of $C([0,t],\mathbb{R})$. By Remark 3.1, there exists a sequence $\{F_n\}$ of bounded continuous functions on $C([0,t],\mathbb{R})$ that satisfies

$$F_n(w) \downarrow 1_A(w), \qquad n\to\infty$$

for $w\in C([0,t],\mathbb{R})$. Thus, by Lemma 8.2 and Lebesgue's dominated convergence theorem, it holds that

$$\begin{aligned} &P\left(\pi_{[0,t]}\circ H^{g^-\to g^+} \in A\right) \\ &= \lim_{n\to\infty} E\left[F_n\left(\pi_{[0,t]}\circ H^{g^-\to g^+}\right)\right] \\ &= \lim_{n\to\infty} \sqrt{\frac{\pi}{2}} \int_{C([0,t],\mathbb{R})} F_n(w) \frac{q^{(g^-,g^+),(\downarrow)}_{[t,1]}\left(w(t)\right)}{C_{g^-,g^+}\sqrt{1-t}\cdot (w(t) - g^-(t))\cdot Z^{g^-}_{[0,t]}(w)} 1_{K^-_{[0,t]}(g^+)}(w)\, P\left(R_{[0,t]} + g^- \in dw\right) \\ &= \sqrt{\frac{\pi}{2}} \int_{C([0,t],\mathbb{R})} 1_A(w) \frac{q^{(g^-,g^+),(\downarrow)}_{[t,1]}\left(w(t)\right)}{C_{g^-,g^+}\sqrt{1-t}\cdot (w(t) - g^-(t))\cdot Z^{g^-}_{[0,t]}(w)} 1_{K^-_{[0,t]}(g^+)}(w)\, P\left(R_{[0,t]} + g^- \in dw\right). \end{aligned} \tag{62}$$

Using (62) and Dynkin's lemma, we can completely prove the assertion. □

# 9 Proofs of Theorem 3

In this subsection, we assume that all $X^{x,y,(g^-,g^+)}_{[s,t]}$ and $X^{z,(g^-,g^+)}_{[s,t]}$ are independent. For each $\mathbb{R}$-valued bounded continuous function $G$ on $C([0,1],\mathbb{R})$ and $\varepsilon>0$, we define

$$I(\varepsilon, G) := E[G(W_{[0,1]})\ ;\ W_{[0,1]} \in K_{[0,1]}(g^- - \varepsilon, g^+)].$$

Then, we have

$$E[F(W_{[0,1]}|_{K_{[0,1]}(g^- -\varepsilon, g^+)})] = \frac{E[F(W_{[0,1]})\ ;\ W_{[0,1]} \in K_{[0,1]}(g^- - \varepsilon, g^+)]}{P(W_{[0,1]} \in K_{[0,1]}(g^- - \varepsilon, g^+))} = \frac{I(\varepsilon, F)}{I(\varepsilon, 1)}. \tag{63}$$

Further, by Lemma 7.2, we obtain

$$I(\varepsilon, F) = \int_{g^-(t)-\varepsilon}^{g^+(t)} E\left[F\left(X_{[0,t]}^{0,y,(g^- -\varepsilon, g^+)} \oplus_t X_{[t,1]}^{y,(g^- -\varepsilon, g^+)}\right)\right] P(W_{[0,t]} \in K_{[0,t]}(g^- - \varepsilon, g^+), W_{[0,t]}(t) \in dy) \times P(y + W_{[t,1]} \in K_{[t,1]}(g^- - \varepsilon, g^+)). \tag{64}$$

It follows from (64), Lemma 7.3, Proposition A.2, Lemma 7.5, Lemma 3.1 and Lemma A.14 that

$$\begin{aligned} I(F) &:= \lim_{\varepsilon \downarrow 0} \frac{I(\varepsilon, F)}{\varepsilon} \\ &= \lim_{\varepsilon \downarrow 0} \frac{I(\varepsilon, F)}{P(W_{[0,t]} \in K_{[0,t]}^+(-\varepsilon))} \times \lim_{\varepsilon \downarrow 0} \frac{P(W_{[0,t]} \in K_{[0,t]}^+(-\varepsilon))}{\varepsilon} \\ &= \sqrt{\frac{2}{\pi}} \int_{g^-(t)}^{g^+(t)} E\left[F\left(X_{[0,t]}^{0,y,(g^-,g^+)} \oplus_t X_{[t,1]}^{y,(g^-,g^+)}\right)\right] \frac{1}{\sqrt{t}} q_{[0,t]}^{(g^-,g^+),(\uparrow)}(y) p_{[t,1]}^{(g^-,g^+)}(y) dy. \end{aligned} \tag{65}$$

Applying the above argument also for $F = 1$, we have

$$I(1) := \sqrt{\frac{2}{\pi}} \int_{g^-(t)}^{g^+(t)} \frac{1}{\sqrt{t}} q_{[0,t]}^{(g^-,g^+),(\uparrow)}(y) p_{[t,1]}^{(g^-,g^+)}(y) dy = \lim_{\varepsilon \downarrow 0} \frac{P(W \in K_{[0,1]}(g^- - \varepsilon, g^+))}{\varepsilon} = \sqrt{\frac{2}{\pi}} \widetilde{C}_{g^-,g^+}. \tag{66}$$

Combining (63), (65) and (66), we obtain

$$\widetilde{C}_{g^-,g^+} = \int_{g^-(t)}^{g^+(t)} \frac{1}{\sqrt{t}} q_{[0,t]}^{(g^-,g^+),(\uparrow)}(y) p_{[t,1]}^{(g^-,g^+)}(y) dy \in (0, \infty)$$

and

$$\lim_{\varepsilon \downarrow 0} E[F(W_{[0,1]}|_{K_{[0,1]}(g^- -\varepsilon, g^+)})] = \frac{I(F)}{I(1)} = \int_{g^-(t)}^{g^+(t)} E\left[F(X_{[0,t]}^{0,y,(g^-,g^+)} \oplus_t X_{[t,1]}^{y,(g^-,g^+)})\right] k(t, y) dy.$$

Therefore, we can define the probability measure $\widetilde{P}_+$ on $(C([0,1], \mathbb{R}), \mathcal{B}(C([0,1], \mathbb{R})))$ as

$$\widetilde{P}_+(A) := \int_{g^-(t)}^{g^+(t)} P\left(X_{[0,t]}^{0,y,(g^-,g^+)} \oplus_t X_{[t,1]}^{y,(g^-,g^+)} \in A\right) k(t, y) dy \quad (A \in \mathcal{B}(C([0,1], \mathbb{R}))),$$

and there exists an $\mathbb{R}$-valued continuous stochastic process $W^{+,(g^-,g^+)} = \{W^{+,(g^-,g^+)}(t)\}_{t \in [0,1]}$ that satisfies (8) and (9). Thus, a limit argument on $F$ yields

$$P(W^{+,(g^-,g^+)}(t) \in dy) = k(t, y) dy \quad (y \in (g^-(t), g^+(t))).$$

On the other hand, by Lemma 7.2, we obtain

$$I(\varepsilon,F)=\int_{g^-(t_2)-\varepsilon}^{g^+(t_2)}dy_2\int_{g^-(t_1)-\varepsilon}^{g^+(t_1)}dy_1E\left[F\left(X_{[0,t_1]}^{0,y_1,(g^--\varepsilon,g^+)}\oplus_{t_1}X_{[t_1,t_2]}^{y_1,y_2,(g^--\varepsilon,g^+)}\oplus_{t_2}X_{[t_2,1]}^{y_2,(g^--\varepsilon,g^+)}\right)\right] \tag{67}$$
$$\times P(y_2+W_{[t_2,1]}\in K_{[t_2,1]}(g^--\varepsilon,g^+))$$
$$\times P(y_1+W_{[t_1,t_2]}\in K_{[t_1,t_2]}(g^--\varepsilon,g^+),y_1+W_{[t_1,t_2]}(t_2)\in dy_2)/dy_2$$
$$\times P(W_{[0,t_1]}\in K_{[0,t_1]}(g^--\varepsilon,g^+),W_{[0,t_1]}(t_1)\in dy_1)/dy_1.$$

By (67), Lemma 7.3, Proposition A.2, Lemma 7.5, Lemma 3.1 and Lemma A.14, $I(F)$ satisfies

$$\begin{aligned}I(F)&=\lim_{\varepsilon\downarrow 0}\frac{I(\varepsilon,F)}{P(W_{[0,t_1]}\in K^+_{[0,t_1]}(-\varepsilon))}\times\lim_{\varepsilon\downarrow 0}\frac{P(W_{[0,t_1]}\in K^+_{[0,t_1]}(-\varepsilon))}{\varepsilon}\\
&=\sqrt{\frac{2}{\pi}}\int_{g^-(t_1)}^{g^+(t_1)}\int_{g^-(t_2)}^{g^+(t_2)}E\left[F\left(X_{[0,t_1]}^{0,y_1,(g^-,g^+)}\oplus_{t_1}X_{[t_1,t_2]}^{y_1,y_2,(g^-,g^+)}\oplus_{t_2}X_{[t_2,1]}^{y_2,(g^-,g^+)}\right)\right]\\
&\qquad\times\frac{1}{\sqrt{t_1}}q_{[0,t_1]}^{(g^-,g^+),(\uparrow)}(y_1)p_{[t_1,t_2]}^{(g^-,g^+)}(y_1,y_2)p_{[t_2,1]}^{(g^-,g^+)}(y_2)dy_1dy_2\\
&=\sqrt{\frac{2}{\pi}}\widetilde{C}_{g^-,g^+}\int_{g^-(t_1)}^{g^+(t_1)}\int_{g^-(t_2)}^{g^+(t_2)}E\left[F\left(X_{[0,t_1]}^{0,y_1,(g^-,g^+)}\oplus_{t_1}X_{[t_1,t_2]}^{y_1,y_2,(g^-,g^+)}\oplus_{t_2}X_{[t_2,1]}^{y_2,(g^-,g^+)}\right)\right]\\
&\qquad\times k(t_1,y_1)k(t_1,y_1,t_2,y_2)dy_1dy_2.\end{aligned}\tag{68}$$

It follows from (63), (66) and (68) that

$$\begin{aligned}&E[F(W^{+,(g^-,g^+)})]\\
&\quad=\lim_{\varepsilon\downarrow 0}E[F(W_{[0,1]}|_{K_{[0,1]}(g^--\varepsilon,g^+)})]\\
&\quad=\int_{g^-(t_1)}^{g^+(t_1)}\int_{g^-(t_2)}^{g^+(t_2)}E\left[F\left(X_{[0,t_1]}^{0,y_1,(g^-,g^+)}\oplus_{t_1}X_{[t_1,t_2]}^{y_1,y_2,(g^-,g^+)}\oplus_{t_2}X_{[t_2,1]}^{y_2,(g^-,g^+)}\right)\right]k(t_1,y_1)k(t_1,y_1,t_2,y_2)dy_1dy_2.\end{aligned}$$

Hence, (10) holds. Similarly, using a limit argument on $F$, we can deduce for $y_1\in(g^-(t_1),g^+(t_1))$ and $y_2\in(g^-(t_2),g^+(t_2))$ that

$$\begin{aligned}&P(W^{+,(g^-,g^+)}(t_1)\in dy_1,W^{+,(g^-,g^+)}(t_2)\in dy_2)=k(t_1,y_1)k(t_1,y_1,t_2,y_2)dy_1dy_2,\\
&P(W^{+,(g^-,g^+)}(t_2)\in dy_2\mid W^{+,(g^-,g^+)}(t_1)=y_1)=k(t_1,y_1,t_2,y_2)dy_2.\end{aligned}\tag{69}$$

Because we have

$$p_{[t_1,1]}^{(g^-,g^+)}(y_1)=\int_{g^-(t_2)}^{g^+(t_2)}p_{[t_1,t_2]}^{(g^-,g^+)}(y_1,y_2)p_{[t_2,1]}^{(g^-,g^+)}(y_2)dy_2\qquad(y_1\in(g^-(t_1),g^+(t_1))),$$

we can deduce that

$$\int_{g^-(t_2)}^{g^+(t_2)}k(t_1,y_1,t_2,y_2)dy_2=1\qquad(y_1\in(g^-(t_1),g^+(t_1))).\tag{70}$$

Assume that $t_3$ satisfies $0 < t_1 < t_2 < t_3 < 1$. Because we have

$$p^{(g^-,g^+)}_{[t_1,t_3]}(y_1, y_3) = \int_{g^-(t_2)}^{g^+(t_2)} p^{(g^-,g^+)}_{[t_1,t_2]}(y_1, y_2)p^{(g^-,g^+)}_{[t_2,t_3]}(y_2, y_3)dy_2$$

for $y_1 \in (g^-(t_1), g^+(t_1))$ and $y_3 \in (g^-(t_3), g^+(t_3))$, we can deduce for $y_1 \in (g^-(t_1), g^+(t_1))$ and $y_3 \in (g^-(t_3), g^+(t_3))$ that the following Chapman–Kolmogorov identity holds:

$$k(t_1, y_1, t_3, y_3) = \int_{g^-(t_2)}^{g^+(t_2)} k(t_1, y_1, t_2, y_2)k(t_2, y_2, t_3, y_3)dy_2. \tag{71}$$

Therefore, (69), (70) and (71) imply that $W^{+,(g^-,g^+)} = \{W^{+,(g^-,g^+)}(t)\}_{t\in[0,1]}$ is a Markov process.

# 10 Proofs of the main results in Subsection 3.3

In this section, we prove Theorems 4 and 5.

## 10.1 Proof of Theorem 4

In this subsection, we assume that all $X^{x,y,(g^-,g^+)}_{[s,t]}$ are independent. For each $\mathbb{R}$-valued bounded continuous function $G$ on $C([0, 1], \mathbb{R})$ and $\varepsilon > 0$, we define

$$I(\varepsilon, G) := E[G(W_{[0,1]})\ ;\ W_{[0,1]}(1) \in dc, W_{[0,1]} \in K_{[0,1]}(g^- - \varepsilon, g^+)].$$

Then, we have

$$E[F(B^{0\to c}_{[0,1]}|_{K_{[0,1]}(g^- -\varepsilon,g^+)})] = \frac{E[F(W_{[0,1]})\ ;\ W_{[0,1]}(1) \in dc, W_{[0,1]} \in K_{[0,1]}(g^- - \varepsilon, g^+)]}{P(W_{[0,1]}(1) \in dc, W_{[0,1]} \in K_{[0,1]}(g^- - \varepsilon, g^+))} = \frac{I(\varepsilon, F)}{I(\varepsilon, 1)}. \tag{72}$$

Further, by Lemma 7.1, we obtain

$$\begin{aligned} I(\varepsilon, F) = \int_{g^-(t)-\varepsilon}^{g^+(t)} E\left[F\left(X^{0,y,(g^- -\varepsilon,g^+)}_{[0,t]} \oplus_t X^{y,c,(g^- -\varepsilon,g^+)}_{[t,1]}\right)\right] P(W_{[0,t]} \in K_{[0,t]}(g^- - \varepsilon, g^+), W_{[0,t]}(t) \in dy) \\ \times P(y + W_{[t,1]} \in K_{[t,1]}(g^- - \varepsilon, g^+), y + W_{[t,1]}(1) \in dc). \end{aligned} \tag{73}$$

It follows from (73), Lemma 7.3, Proposition A.2, Lemma 7.5, Lemma 3.1 and Lemma A.14 that

$$\begin{aligned} I(F) &:= \lim_{\varepsilon\downarrow 0} \frac{I(\varepsilon, F)}{\varepsilon} \\ &= \lim_{\varepsilon\downarrow 0} \frac{I(\varepsilon, F)}{P(W_{[0,t]} \in K^+_{[0,t]}(-\varepsilon))} \times \lim_{\varepsilon\downarrow 0} \frac{P(W_{[0,t]} \in K^+_{[0,t]}(-\varepsilon))}{\varepsilon} \\ &= \sqrt{\frac{2}{\pi}} \int_{g^-(t)}^{g^+(t)} E\left[F\left(X^{0,y,(g^-,g^+)}_{[0,t]} \oplus_t X^{y,c,(g^-,g^+)}_{[t,1]}\right)\right] \frac{1}{\sqrt{t}} q^{(g^-,g^+),(\uparrow)}_{[0,t]}(y) p^{(g^-,g^+)}_{[t,1]}(y, c)dydc. \end{aligned} \tag{74}$$

Applying the above argument also for $F = 1$, we have

$$
\begin{aligned}
I(1) :=& \sqrt{\frac{2}{\pi}} \int_{g^-(t)}^{g^+(t)} \frac{1}{\sqrt{t}} q_{[0,t]}^{(g^-,g^+),(\uparrow)}(y) p_{[t,1]}^{(g^-,g^+)}(y,c) dy dc \\
=& \lim_{\varepsilon \downarrow 0} \frac{P(W(1) \in dc, W \in K_{[0,1]}(g^- - \varepsilon, g^+))}{\varepsilon} \\
=& n_1(c) dc \lim_{\varepsilon \downarrow 0} \frac{P(B_{[0,1]}^{0 \to c} \in K_{[0,1]}(g^- - \varepsilon, g^+))}{\varepsilon} = \sqrt{\frac{2}{\pi}} \widehat{C}_{g^-,g^+} dc.
\end{aligned} \tag{75}
$$

Combining (72), (74) and (75), we obtain

$$
\widehat{C}_{g^-,g^+} = \int_{g^-(t)}^{g^+(t)} \frac{1}{\sqrt{t}} q_{[0,t]}^{(g^-,g^+),(\uparrow)}(y) p_{[t,1]}^{(g^-,g^+)}(y,c) dy \in (0,\infty)
$$

and

$$
\lim_{\varepsilon \downarrow 0} E[F(B_{[0,1]}^{0 \to c}|_{K_{[0,1]}(g^- - \varepsilon, g^+)})] = \frac{I(F)}{I(1)} = \int_{g^-(t)}^{g^+(t)} E\left[F\left(X_{[0,t]}^{0,y,(g^-,g^+)} \oplus_t X_{[t,1]}^{y,c,(g^-,g^+)}\right)\right] l(t,y) dy.
$$

Therefore, we can define the probability measure $\widehat{P}^{0 \to c}$ on $(C([0,1],\mathbb{R}), \mathcal{B}(C([0,1],\mathbb{R})))$ as

$$
\widehat{P}^{0 \to c}(A) := \int_{g^-(t)}^{g^+(t)} P\left(X_{[0,t]}^{0,y,(g^-,g^+)} \oplus_t X_{[t,1]}^{y,c,(g^-,g^+)} \in A\right) l(t,y) dy \quad (A \in \mathcal{B}(C([0,1],\mathbb{R}))),
$$

and there exists an $\mathbb{R}$-valued continuous stochastic process $r^{0 \to c,(g^-,g^+)} = \{r^{0 \to c,(g^-,g^+)}(t)\}_{t \in [0,1]}$ that satisfies (11) and (12). Thus, a limit argument on $F$ yields

$$
P(r^{0 \to c,(g^-,g^+)}(t) \in dy) = l(t,y) dy \quad (y \in (g^-(t), g^+(t))).
$$

On the other hand, by Lemma 7.1, we obtain

$$
\begin{aligned}
I(\varepsilon, F) = \int_{g^-(t_2)-\varepsilon}^{g^+(t_2)} dy_2 \int_{g^-(t_1)-\varepsilon}^{g^+(t_1)} dy_1 E\left[F\left(X_{[0,t_1]}^{0,y_1,(g^- -\varepsilon,g^+)} \oplus_{t_1} X_{[t_1,t_2]}^{y_1,y_2,(g^- -\varepsilon,g^+)} \oplus_{t_2} X_{[t_2,1]}^{y_2,c,(g^- -\varepsilon,g^+)}\right)\right] \\
\times P(y_2 + W_{[t_2,1]} \in K_{[t_2,1]}(g^- - \varepsilon, g^+), y_2 + W_{[t_2,1]}(1) \in dc) \\
\times P(y_1 + W_{[t_1,t_2]} \in K_{[t_1,t_2]}(g^- - \varepsilon, g^+), y_1 + W_{[t_1,t_2]}(t_2) \in dy_2)/dy_2 \\
\times P(W_{[0,t_1]} \in K_{[0,t_1]}(g^- - \varepsilon, g^+), W_{[0,t_1]}(t_1) \in dy_1)/dy_1.
\end{aligned} \tag{76}
$$

By (76), Lemma 7.3, Proposition A.2, Lemma 7.5, Lemma 3.1 and Lemma A.14, $I(F)$ satisfies

$$
\begin{aligned}
I(F) &= \lim_{\varepsilon\downarrow 0} \frac{I(\varepsilon,F)}{P(W_{[0,t_1]}\in K^+_{[0,t_1]}(-\varepsilon))} \times \lim_{\varepsilon\downarrow 0} \frac{P(W_{[0,t_1]}\in K^+_{[0,t_1]}(-\varepsilon))}{\varepsilon} \\
&= \sqrt{\frac{2}{\pi}} \int_{g^-(t_1)}^{g^+(t_1)} \int_{g^-(t_2)}^{g^+(t_2)} E\left[F\left(X^{0,y_1,(g^-,g^+)}_{[0,t_1]} \oplus_{t_1} X^{y_1,y_2,(g^-,g^+)}_{[t_1,t_2]} \oplus_{t_2} X^{y_2,c,(g^-,g^+)}_{[t_2,1]}\right)\right] \\
&\qquad\qquad \times \frac{1}{\sqrt{t_1}} q^{(g^-,g^+),(\uparrow)}_{[0,t_1]}(y_1) p^{(g^-,g^+)}_{[t_1,t_2]}(y_1,y_2) p^{(g^-,g^+)}_{[t_2,1]}(y_2,c) dy_1 dy_2 dc \\
&= \sqrt{\frac{2}{\pi}} \widehat{C}_{g^-,g^+} \int_{g^-(t_1)}^{g^+(t_1)} \int_{g^-(t_2)}^{g^+(t_2)} E\left[F\left(X^{0,y_1,(g^-,g^+)}_{[0,t_1]} \oplus_{t_1} X^{y_1,y_2,(g^-,g^+)}_{[t_1,t_2]} \oplus_{t_2} X^{y_2,c,(g^-,g^+)}_{[t_2,1]}\right)\right] \\
&\qquad\qquad \times l(t_1,y_1) l(t_1,y_1,t_2,y_2) dy_1 dy_2 dc.
\end{aligned} \tag{77}
$$

It follows from (72), (75) and (77) that

$$
\begin{aligned}
&E[F(r^{0\to c,(g^-,g^+)})] \\
&\quad = \lim_{\varepsilon\downarrow 0} E[F(B^{0\to c}_{[0,1]}|_{K_{[0,1]}(g^- -\varepsilon, g^+)})] \\
&\quad = \int_{g^-(t_1)}^{g^+(t_1)} \int_{g^-(t_2)}^{g^+(t_2)} E\left[F\left(X^{0,y_1,(g^-,g^+)}_{[0,t_1]} \oplus_{t_1} X^{y_1,y_2,(g^-,g^+)}_{[t_1,t_2]} \oplus_{t_2} X^{y_2,c,(g^-,g^+)}_{[t_2,1]}\right)\right] l(t_1,y_1) l(t_1,y_1,t_2,y_2) dy_1 dy_2.
\end{aligned}
$$

Hence, (13) holds. Similarly, using a limit argument on $F$, we can deduce for $y_1 \in (g^-(t_1), g^+(t_1))$ and $y_2 \in (g^-(t_2), g^+(t_2))$ that

$$
\begin{aligned}
&P(r^{0\to c,(g^-,g^+)}(t_1)\in dy_1, r^{0\to c,(g^-,g^+)}(t_2)\in dy_2) = l(t_1,y_1) l(t_1,y_1,t_2,y_2) dy_1 dy_2, \\
&P(r^{0\to c,(g^-,g^+)}(t_2)\in dy_2 \mid r^{0\to c,(g^-,g^+)}(t_1) = y_1) = l(t_1,y_1,t_2,y_2) dy_2.
\end{aligned} \tag{78}
$$

Because we have

$$
p^{(g^-,g^+)}_{[t_1,1]}(y_1,c) = \int_{g^-(t_2)}^{g^+(t_2)} p^{(g^-,g^+)}_{[t_1,t_2]}(y_1,y_2) p^{(g^-,g^+)}_{[t_2,1]}(y_2,c) dy_2 \qquad (y_1 \in (g^-(t_1), g^+(t_1))),
$$

we can deduce that

$$
\int_{g^-(t_2)}^{g^+(t_2)} l(t_1,y_1,t_2,y_2) dy_2 = 1 \qquad (y_1 \in (g^-(t_1), g^+(t_1))). \tag{79}
$$

Assume that $t_3$ satisfies $0 < t_1 < t_2 < t_3 < 1$. Because we have

$$
p^{(g^-,g^+)}_{[t_1,t_3]}(y_1,y_3) = \int_{g^-(t_2)}^{g^+(t_2)} p^{(g^-,g^+)}_{[t_1,t_2]}(y_1,y_2) p^{(g^-,g^+)}_{[t_2,t_3]}(y_2,y_3) dy_2
$$

for $y_1 \in (g^-(t_1), g^+(t_1))$ and $y_3 \in (g^-(t_3), g^+(t_3))$, we can deduce for $y_1 \in (g^-(t_1), g^+(t_1))$ and $y_3 \in (g^-(t_3), g^+(t_3))$ that the following Chapman–Kolmogorov identity holds:

$$
l(t_1,y_1,t_3,y_3) = \int_{g^-(t_2)}^{g^+(t_2)} l(t_1,y_1,t_2,y_2) l(t_2,y_2,t_3,y_3) dy_2. \tag{80}
$$

Therefore, (78), (79) and (80) imply that $r^{0\to c,(g^-,g^+)} = \{r^{0\to c,(g^-,g^+)}(t)\}_{t\in[0,1]}$ is a Markov process.

## 10.2 Proof of Theorem 5

In this subsection, we assume that all $X_{[s,t]}^{x,y,(g^-,g^+)}$ are independent. For each $F$, $\varepsilon > 0$ and $\eta > 0$, we have

$$E[F(B_{[0,1]}^{0\to b}|_{K_{[0,1]}(g^- -\varepsilon, g^+ +\eta)})] = \frac{I(\varepsilon,\eta,F)}{I(\varepsilon,\eta,1)}, \tag{81}$$

where

$$I(\varepsilon,\eta,F) := E[F(W_{[0,1]})\ ;\ W_{[0,1]}(1) \in db, W_{[0,1]} \in K_{[0,1]}(g^- - \varepsilon, g^+ + \eta)].$$

Then, by Lemma 7.1, we obtain

$$\begin{aligned} I(\varepsilon,\eta,F) = \int_{g^-(t)-\varepsilon}^{g^+(t)+\eta} & E\left[F\left(X_{[0,t]}^{0,y,(g^- -\varepsilon,g^+ +\eta)} \oplus_t X_{[t,1]}^{y,b,(g^- -\varepsilon,g^+ +\eta)}\right)\right] \\ & \times P(W_{[0,t]} \in K_{[0,t]}(g^- - \varepsilon, g^+ + \eta), W_{[0,t]}(t) \in dy) \\ & \times P(y + W_{[t,1]} \in K_{[t,1]}(g^- - \varepsilon, g^+ + \eta), y + W_{[t,1]}(1) \in db). \end{aligned} \tag{82}$$

By (82), Lemma 7.3, Proposition A.2, Lemma 7.5, Lemma 3.1 and Lemma A.14, it holds that

$$\begin{aligned} I(\eta,F) &:= \lim_{\varepsilon\downarrow 0} \frac{I(\varepsilon,\eta,F)}{\varepsilon} \\ &= \sqrt{\frac{2}{\pi}} \int_{g^-(t)}^{g^+(t)+\eta} E\left[F\left(X_{[0,t]}^{0,y,(g^-,g^+ +\eta)} \oplus_t X_{[t,1]}^{y,b,(g^-,g^+ +\eta)}\right)\right] \frac{1}{\sqrt{t}} q_{[0,t]}^{(g^-,g^+ +\eta),(\uparrow)}(y)dy \\ &\qquad \times P(y + W_{[t,1]} \in K_{[t,1]}(g^-, g^+ + \eta), y + W_{[t,1]}(1) \in db). \end{aligned} \tag{83}$$

On the other hand, using (83), Lemma 7.4, Proposition A.2, Lemma 7.5, Lemma 3.1 and Lemma A.14, we obtain

$$\begin{aligned} I(F) &:= \lim_{\eta\downarrow 0} \frac{I(\eta,F)}{\eta} \\ &= \frac{2}{\pi} \int_{g^-(t)}^{g^+(t)} E\left[F\left(X_{[0,t]}^{0,y,(g^-,g^+)} \oplus_t X_{[t,1]}^{y,b,(g^-,g^+)}\right)\right] \frac{1}{\sqrt{t}} q_{[0,t]}^{(g^-,g^+),(\uparrow)}(y) \frac{1}{\sqrt{1-t}} q_{[t,1]}^{(g^-,g^+),(\downarrow)}(y)dydb. \end{aligned} \tag{84}$$

By (39) and (84), it holds that

$$I(1) := \frac{2}{\pi} \int_{g^-(t)}^{g^+(t)} \frac{1}{\sqrt{t}} q_{[0,t]}^{(g^-,g^+),(\uparrow)}(y) \frac{1}{\sqrt{1-t}} q_{[t,1]}^{(g^-,g^+),(\downarrow)}(y)dydb = \frac{2}{\pi} C_{g^-,g^+}db. \tag{85}$$

Combining Theorem 4 and (81), we obtain

$$\lim_{\eta\downarrow 0} E[F(r^{0\to b,(g^-,g^+ +\eta)})] = \lim_{\eta\downarrow 0}\lim_{\varepsilon\downarrow 0} E[F(B_{[0,1]}^{0\to b}|_{K_{[0,1]}(g^- -\varepsilon,g^+ +\eta)})] = \lim_{\eta\downarrow 0}\lim_{\varepsilon\downarrow 0} \frac{I(\varepsilon,\eta,F)}{I(\varepsilon,\eta,1)} = \frac{I(F)}{I(1)}. \tag{86}$$

Therefore, it follows from (84), (85), (86) and Theorem 1 that

$$\lim_{\eta\downarrow 0} E[F(r^{0\to b,(g^-,g^+ +\eta)})] = \int_{g^-(t)}^{g^+(t)} E\left[F\left(X_{[0,t]}^{0,y,(g^-,g^+)} \oplus_t X_{[t,1]}^{y,b,(g^-,g^+)}\right)\right] h(t,y)dy = E\left[F(H^{g^-\to g^+})\right].$$

# A Appendix

In this appendix, we prepare several lemmas. Although some of the results in this appendix are either well known or easy to obtain, we prove them for completeness.

**Lemma A.1.** *Let* $r, \varepsilon \in (0, 1]$. *It holds that*

$$n_r(z) - n_r(z + 2\varepsilon) \leq \frac{4\varepsilon}{r} n_r\left(\frac{z}{\sqrt{2}}\right), \qquad z \geq -\varepsilon.$$

Proof. We define

$$g^{(\varepsilon,r)}(z) := 2\sqrt{\frac{2\pi}{r}}(z+\varepsilon)n_r\left(\frac{z}{\sqrt{2}}\right) = \frac{2(z+\varepsilon)}{r}\exp\left(-\frac{z^2}{4r}\right), \qquad z \in \mathbb{R}.$$

Then, by a simple calculation, we have

$$\frac{d}{dz}g^{(\varepsilon,r)}(z) = 2\sqrt{\frac{2\pi}{r}}\left\{1 - \frac{z(z+\varepsilon)}{2r}\right\} n_r\left(\frac{z}{\sqrt{2}}\right).$$

Thus we obtain

$$\max_{z \geq z_0^-} g^{(\varepsilon,r)}(z) = g^{(\varepsilon,r)}(z_0^+), \quad \text{where} \quad z_0^{\pm} = -\frac{1}{2}\varepsilon \pm \sqrt{2r + \frac{1}{4}\varepsilon^2} \quad \text{(the plus-minus signs correspond)},$$

and

$$\max_{z \geq z_0^-} g^{(\varepsilon,r)}(z) = g^{(\varepsilon,r)}\left(z_0^+\right) < \frac{\varepsilon + 2\sqrt{2 + \frac{1}{4}\varepsilon^2}}{r} < \frac{4}{r}.$$

Therefore, combining the inequality $1 - \exp(-x) \leq x$ $(x \geq 0)$, we can deduce

$$n_r(z) - n_r(z + 2\varepsilon) \leq n_r(z)\frac{2\varepsilon(z+\varepsilon)}{r} = \varepsilon g^{(\varepsilon,r)}(z) n_r\left(z/\sqrt{2}\right) < \frac{4\varepsilon}{r} n_r\left(z/\sqrt{2}\right), \qquad z \geq -\varepsilon.$$

□

**Lemma A.2.** *Let* $W = \{W(t)\}_{t \geq 0}$ *be the standard one-dimensional Brownian motion defined on* $(\Omega, \mathcal{F}, P)$. *For* $t > 0$, *we have*

$$P\left(W(t) \in dz, m_t(W) \geq -\varepsilon\right) = \left(n_t(z) - n_t(z + 2\varepsilon)\right)dz, \qquad (z > -\varepsilon), \tag{87}$$

$$P\left(W(t) \in dz, -\varepsilon \leq m_t(W) < M_t(W) \leq \eta\right) \tag{88}$$

$$= \sum_{k=-\infty}^{\infty}\left(n_t(z + 2k(\eta+\varepsilon)) - n_t(2\eta - z + 2k(\eta+\varepsilon))\right)dz, \qquad (-\varepsilon < z < \eta).$$

*For* $0 < t < u$, *we have*

$$P\left(W(t) \in dy, W(u) \in dz, m_u(W) \geq -\varepsilon\right) \tag{89}$$
$$= \left(n_{u-t}(z-y) - n_{u-t}(z+y+2\varepsilon)\right)\left(n_t(y) - n_t(y+2\varepsilon)\right) dydz, \qquad (y, z > -\varepsilon).$$

*For* $0 < s < t < u$, *we have*

$$P\left(W(s) \in dx, W(t) \in dy, W(u) \in dz, m_u(W) \geq -\varepsilon\right) \qquad (x, y, z > -\varepsilon) \tag{90}$$
$$= \left(n_{u-t}(z-y) - n_{u-t}(z+y+2\varepsilon)\right)\left(n_{t-s}(y-x) - n_{t-s}(y+x+2\varepsilon)\right)\left(n_s(x) - n_s(x+2\varepsilon)\right) dxdydz.$$

Proof. In this proof, $(\Omega, \mathcal{F})$, $W = \{W(t)\}_{t \geq 0}$, $(P^a)_{a \in \mathbb{R}}$ denotes the one-dimensional Brownian family, and $P^0$ is written simply as $P$. We can find (87) and (88) in [1]. Using the Markov property of $W$, (87), and (88), we have

$$\begin{aligned}
(89) \quad & P\left(W(t) \in dy, W(u) \in dz, m_u(W) \geq -\varepsilon\right) \\
&= E\left[P\left(W(t) \in dy, m_u(W) \geq -\varepsilon, W(u) \in dz \mid \mathcal{F}_t^W\right)\right] \\
&= P^y\left(W(u-t) \in dz, m_{u-t}(W) \geq -\varepsilon\right) P\left(W(t) \in dy, m_t(W) \geq -\varepsilon\right) \\
&= P\left(y + W(u-t) \in dz, y + m_{u-t}(W) \geq -\varepsilon\right) P\left(W(t) \in dy, m_t(W) \geq -\varepsilon\right) \\
&= \left(n_{u-t}(z-y) - n_{u-t}(z+y+2\varepsilon)\right)\left(n_t(y) - n_t(y+2\varepsilon)\right) dydz.
\end{aligned}$$

Using the Markov property of $W$, (87), and (89), we also have

$$\begin{aligned}
(90) \quad & P\left(W(s) \in dx, W(t) \in dy, W(u) \in dz, m_u(W) \geq -\varepsilon\right) \\
&= E\left[P\left(W(s) \in dx, W(t) \in dy, W(u) \in dz, m_u(W) \geq -\varepsilon \mid \mathcal{F}_t^W\right)\right] \\
&= P^y\left(W(u-t) \in dz, m_{u-t}(W) \geq -\varepsilon\right) P\left(W(s) \in dx, W(t) \in dy, m_t(W) \geq -\varepsilon\right) \\
&= P\left(y + W(u-t) \in dz, y + m_{u-t}(W) \geq -\varepsilon\right) P\left(W(s) \in dx, W(t) \in dy, m_t(W) \geq -\varepsilon\right) \\
&= \left(n_{u-t}(z-y) - n_{u-t}(z+y+2\varepsilon)\right)\left(n_{t-s}(y-x) - n_{t-s}(y+x+2\varepsilon)\right)\left(n_s(x) - n_s(x+2\varepsilon)\right) dxdydz.
\end{aligned}$$

□

**Lemma A.3.** *Let* $W = \{W(t)\}_{t \geq 0}$ *be the standard one-dimensional Brownian motion defined on* $(\Omega, \mathcal{F}, P)$. *For* $t, \varepsilon, \eta > 0$ *and* $-\varepsilon \leq z \leq \eta$, *we define*

$$\psi_t(\varepsilon, \eta, z) := P\left(W(t) \in dz, -\varepsilon \leq m_t(W) < M_t(W) \leq \eta\right) / dz.$$

*Then, we have*

$$\lim_{\varepsilon \downarrow 0} \frac{\partial}{\partial \varepsilon} \psi_t(\varepsilon, \eta + \varepsilon, \eta) = 0, \quad \lim_{\varepsilon \downarrow 0} \frac{\partial^2}{\partial \varepsilon^2} \psi_t(\varepsilon, \eta + \varepsilon, \eta) = 2\overline{J}^{(\eta)}(t, \eta), \qquad \eta > 0. \tag{91}$$

Proof. By (88), the derivative of $\psi_t(\varepsilon, \eta + \varepsilon, \eta)$ satisfies

$$\begin{aligned}
\frac{\partial}{\partial \varepsilon} \psi_t(\varepsilon, \eta + \varepsilon, \eta) &= \frac{\partial}{\partial \varepsilon} \sum_{k=-\infty}^{\infty} \left(n_t(\eta + 2k(\eta + 2\varepsilon)) - n_t((2k+1)(\eta + 2\varepsilon))\right) \\
&= \sum_{k=-\infty}^{\infty} \left(4k n_t'(\eta + 2k(\eta + 2\varepsilon)) - 2(2k+1) n_t'((2k+1)(\eta + 2\varepsilon))\right) \\
&\to -\sum_{k=-\infty}^{\infty} n_t'((2k+1)\eta) = 0, \qquad \varepsilon \downarrow 0,
\end{aligned}$$

and

$$\frac{\partial^2}{\partial\varepsilon^2}\psi_t(\varepsilon,\eta+\varepsilon,\eta) = \frac{\partial}{\partial\varepsilon}\sum_{k=-\infty}^{\infty}(4kn_t'(\eta+2k(\eta+2\varepsilon)) - 2(2k+1)n_t'((2k+1)(\eta+2\varepsilon)))$$
$$= \sum_{k=-\infty}^{\infty}\Big((4k)^2 n_t''(\eta+2k(\eta+2\varepsilon)) - 2^2(2k+1)^2 n_t''((2k+1)(\eta+2\varepsilon))\Big)$$
$$\to \sum_{k=-\infty}^{\infty}\Big((4k)^2 n_t''((2k+1)\eta) - 2^2(2k+1)^2 n_t''((2k+1)\eta)\Big) =: \widehat{\Phi}, \qquad \varepsilon\downarrow 0.$$

Here, by $n_t''(z) = -n_t(z)/t + (z/t)^2 n_t(z)$, it holds that

$$\widehat{\Phi} = -4\sum_{k=-\infty}^{\infty}(4k+1)n_t''((2k+1)\eta) = 4\sum_{k=-\infty}^{\infty}(4k+1)\left\{\frac{1}{t} - \frac{(\eta+2k\eta)^2}{t^2}\right\}n_t((2k+1)\eta).$$

Because we have

$$\sum_{k=-\infty}^{\infty}(2k+1)\left\{\frac{1}{t} - \frac{((2k+1)\eta)^2}{t^2}\right\}n_t((2k+1)\eta) = 0,$$

it follows that

$$\widehat{\Phi} = 4\sum_{k=-\infty}^{\infty}2k\left\{\frac{1}{t} - \frac{(\eta+2k\eta)^2}{t^2}\right\}n_t((2k+1)\eta) = 2\overline{J}^{(\eta)}(t,\eta).$$

□

**Lemma A.4.** *Let $T>0$ and $a,b>0$. Then, for $0<s<t<T$ and $x,y>0$, we have*

$$P\left(B^{a\to b}_{[0,T]}|_{K^+(0)}(t)\in dy\right) = P\left(r^{a\to b}_{[0,T]}(t)\in dy\right) = \left(\frac{T}{2\pi t(T-t)}\right)^{\frac{1}{2}}\frac{\left(e^{-\frac{(y-a)^2}{2t}} - e^{-\frac{(y+a)^2}{2t}}\right)\left(e^{-\frac{(b-y)^2}{2(T-t)}} - e^{-\frac{(b+y)^2}{2(T-t)}}\right)}{e^{-\frac{(b-a)^2}{2T}} - e^{-\frac{(b+a)^2}{2T}}},$$
$$P\left(B^{a\to b}_{[0,T]}|_{K^+(0)}(t)\in dy \mid B^{a\to b}_{[0,T]}|_{K^+(0)}(s)=x\right)$$
$$= P\left(r^{a\to b}_{[0,T]}(t)\in dy \mid r^{a\to b}_{[0,T]}(s)=x\right) = \left(\frac{T-s}{2\pi(t-s)(T-t)}\right)^{\frac{1}{2}}\frac{\left(e^{-\frac{(y-x)^2}{2(t-s)}} - e^{-\frac{(y+x)^2}{2(t-s)}}\right)\left(e^{-\frac{(b-y)^2}{2(T-t)}} - e^{-\frac{(b+y)^2}{2(T-t)}}\right)}{e^{-\frac{(b-x)^2}{2(T-s)}} - e^{-\frac{(b+x)^2}{2(T-s)}}}.$$

*Therefore, the Markov processes $B^{a\to b}_{[0,T]}|_{K^+(0)}$ and $r^{a\to b}_{[0,T]}$ follow the same distribution.*

Proof. $P\left(r^{a\to b}_{[0,T]}(t)\in dy\right)$ and $P\left(r^{a\to b}_{[0,T]}(t)\in dy \mid r^{a\to b}_{[0,T]}(s)=x\right)$ are given in [10] p. 463. On the other hand, we can calculate $P\left(B^{a\to b}_{[0,T]}|_{K^+(0)}(t)\in dy\right)$ and $P\left(B^{a\to b}_{[0,T]}|_{K^+(0)}(t)\in dy \mid B^{a\to b}_{[0,T]}|_{K^+(0)}(s)=x\right)$ based on Chapter 5, Problem 6.11 in [8]. Therefore, the desired result is obtained by direct calculation. □

**Lemma A.5.** *Let $\delta > 0$, $a \geq 0$, and $b > 0$. Then, for the BES($\delta$)-bridge $r^{a\to b} = \{r^{a\to b}(t)\}_{t\in[0,1]}$ from $a$ to $b$ on $[0,1]$, we have*

$$P\left(r^{a\to b}(t) \in dy, M(r^{a\to b}) \leq x\right) = P\left(r^{a\to y}_{[0,t]} \in K^-_{[0,t]}(x)\right) P\left(r^{y\to b}_{[t,1]} \in K^-_{[t,1]}(x)\right) P\left(r^{a\to b}(t) \in dy\right)$$

*for $0 < t < 1$ and $b \leq x$, $0 \leq y \leq x$. Here, $r^{c\to d}_{[t_1,t_2]} = \{r^{c\to d}_{[t_1,t_2]}(t)\}_{t\in[t_1,t_2]}$ denotes the BES($\delta$)-bridge from $c$ to $d$ on $[t_1, t_2]$.*

Proof. In this proof, the pair $(R = \{R(t)\}_{t\geq 0}, P^{(\delta)}_a)$ denotes a BES($\delta$)-process starting from $a \geq 0$: $P^{(\delta)}_a(R(0) = a) = 1$. Then, by the Markov property of $R$, we have

$$\begin{aligned} P\left(r^{a\to b}(t) \in dy, M(r^{a\to b}) \leq x\right) &= \frac{P^{(\delta)}_a\left(R(t) \in dy, M(R) \leq x, R(1) \in db\right)}{P^{(\delta)}_a\left(R(1) \in db\right)} \\ &= \frac{P^{(\delta)}_y\left(R(1-t) \in db, M_{1-t}(R) \leq x\right) P^{(\delta)}_a\left(R(t) \in dy, M_t(R) \leq x\right)}{P^{(\delta)}_a\left(R(1) \in db\right)} \end{aligned}$$

and

$$P\left(r^{a\to b}(t) \in dy\right) = \frac{P^{(\delta)}_a\left(R(t) \in dy, R(1) \in db\right)}{P^{(\delta)}_a\left(R(1) \in db\right)} = \frac{P^{(\delta)}_y\left(R(1-t) \in db\right) P^{(\delta)}_a\left(R(t) \in dy\right)}{P^{(\delta)}_a\left(R(1) \in db\right)}.$$

Therefore, because we have

$$\begin{aligned} P^{(\delta)}_y\left(R(1-t) \in db, M_{1-t}(R) \leq x\right) &= P\left(M_{1-t}(r^{y\to b}) \leq x\right) P^{(\delta)}_y\left(R(1-t) \in db\right) \\ &= P\left(r^{y\to b}_{[t,1]} \in K^-_{[t,1]}(x)\right) P^{(\delta)}_y\left(R(1-t) \in db\right), \\ P^{(\delta)}_a\left(R(t) \in dy, M_t(R) \leq x\right) &= P\left(M_t(r^{a\to y}) \leq x\right) P^{(\delta)}_a\left(R(t) \in dy\right) \\ &= P\left(r^{a\to y}_{[0,t]} \in K^-_{[0,t]}(x)\right) P^{(\delta)}_a\left(R(t) \in dy\right), \end{aligned}$$

it follows that

$$\begin{aligned} &P\left(r^{a\to b}(t) \in dy, M(r^{a\to b}) \leq x\right) \\ &\quad = P\left(r^{a\to y}_{[0,t]} \in K^-_{[0,t]}(x)\right) P\left(r^{y\to b}_{[t,1]} \in K^-_{[t,1]}(x)\right) \frac{P^{(\delta)}_a\left(R(t) \in dy\right) P^{(\delta)}_y\left(R(1-t) \in db\right)}{P^{(\delta)}_a\left(R(1) \in db\right)} \\ &\quad = P\left(r^{a\to y}_{[0,t]} \in K^-_{[0,t]}(x)\right) P\left(r^{y\to b}_{[t,1]} \in K^-_{[t,1]}(x)\right) P\left(r^{a\to b}(t) \in dy\right). \end{aligned}$$

□

**Lemma A.6.** *For $c > 0$ and $a, b \in (0, c)$, it holds that $P(B^{a\to b} \in K(0, c)) > 0$.*

Proof. It holds that

$$f(z) = \sum_{k=-\infty}^{\infty} \left(n_1(b - a + 2k(a \vee b + z)) - n_1(b + a + 2k(a \vee b + z))\right)$$

defines a holomorphic function on

$$D = \{z = x + \mathrm{i}y \mid x \in (0, \infty), y \in (-(b + a)/2, (b + a)/2)\}.$$

Lemma A.2 implies that

$$f(\eta) = P(W \in K(-a, a \vee b + \eta - a), W(1) \in db - a)/db, \qquad \eta > 0.$$

Observe that $f$ is non-decreasing and non-negative on $(0, \infty)$. Assume, for the sake of contradiction, that $f(\eta_0) = 0$ holds for some $\eta_0 > 0$. Then, it follows from the identity theorem that $f(z) = 0$ holds for every $z \in D$. This contradicts

$$\lim_{\eta\to\infty} f(\eta) = P(W \in K^{+}(-a), W(1) \in db - a)/db > 0.$$

Therefore, $f(\eta) > 0$ holds for $\eta > 0$, and hence we obtain

$$P(B^{a\to b} \in K(0, c)) = \frac{f(c - a \vee b)db}{P(W(1) \in db - a)} > 0.$$

□

**Lemma A.7.** *Let $a, b \in \mathbb{R}$. Assume that $\mathbb{R}$-valued $C^1$-functions $g^-, g^+$ defined on $[0, 1]$ satisfy the following conditions:*

$$\min_{0\le t\le 1}(g^+(t) - g^-(t)) > 0, \quad g^-(0) < a < g^+(0), \quad g^-(1) < b < g^+(1).$$

*Then, we have $P(B^{a\to b} \in K(g^-, g^+)) > 0$.*

Proof. Girsanov's theorem implies

$$\begin{aligned}
&P(a + W(1) \in db, a + W \in K(g^-, g^+)) \\
&\quad = P\left(a + W(1) \in db, a + W - (g^- - g^-(0)) \in K(g^-(0), g^+ - g^- + g^-(0))\right) \\
&\quad = E\left[\widetilde{Z}_{[0,1]}^{g^- - g^-(0)}(W)^{-1}\ ;\ a - g^-(0) + W(1) \in db - g^-(1),\ a - g^-(0) + W \in K(0, g^+ - g^-)\right] \\
&\quad = E\left[\widetilde{Z}_{[0,1]}^{g^- - g^-(0)}(B^{a-g^-(0)\to b-g^-(1)} - a + g^-(0))^{-1}\ ;\ B^{a-g^-(0)\to b-g^-(1)} \in K(0, g^+ - g^-)\right] \\
&\qquad \times P(a - g^-(0) + W(1) \in db - g^-(1)).
\end{aligned}$$

Thus, it holds that

$$\begin{aligned}
&P(B^{a\to b} \in K(g^-, g^+)) \\
&\quad = \frac{P(a + W \in K(g^-, g^+),\ a + W(1) \in db)}{P(a + W(1) \in db)} \\
&\quad = E\left[\widetilde{Z}_{[0,1]}^{g^- - g^-(0)}(B^{a-g^-(0)\to b-g^-(1)} - a + g^-(0))^{-1}\ ;\ B^{a-g^-(0)\to b-g^-(1)} \in K(0, g^+ - g^-)\right] \\
&\qquad \times \frac{P(a - g^-(0) + W(1) \in db - g^-(1))}{P(a + W(1) \in db)} \\
&\quad \ge C_{g^-,a,b} \times P\left(B^{a-g^-(0)\to b-g^-(1)} \in K(0, g^+ - g^-)\right) \frac{P(a - g^-(0) + W(1) \in db - g^-(1))}{P(a + W(1) \in db)},
\end{aligned}$$

for some $C_{g^-,a,b} > 0$ depending only on $g^-$, $a$, and $b$. Therefore, we may assume that $g^- = 0$, $0 < a < g^+(0)$, and $0 < b < g^+(1)$. Take $\delta \in (0, 1/2)$ such that

$$\min_{t\in[0,\delta]} g^+(t) \geq \frac{1}{2}(a + g^+(0)), \qquad \min_{t\in[1-\delta,1]} g^+(t) \geq \frac{1}{2}(b + g^+(1)),$$

and let $t_1 = \delta$, $t_2 = 1 - \delta$ and

$$c_1 = \min_{t\in[0,\delta]} g^+(t), \quad c_2 = \min_{t\in[\delta,1-\delta]} g^+(t), \quad c_3 = \min_{t\in[1-\delta,1]} g^+(t), \quad c_4 = a \wedge b \wedge c_1 \wedge c_2 \wedge c_3 > 0.$$

Then, we have

$$P(B^{a\to b} \in K(0, g^+)) \geq \int_0^{c_4} \int_0^{c_4} h(y_1, y_2) \frac{n_{t_1}(y_1 - a) n_{t_2-t_1}(y_2 - y_1) n_{1-t_2}(b - y_2)}{n_1(a, b)} dy_1 dy_2,$$

where

$$h(y_1, y_2) = P\left(B^{a\to y_1}_{[0,t_1]} \in K_{[0,t_1]}(0, c_1)\right) P\left(B^{y_1\to y_2}_{[t_1,t_2]} \in K_{[t_1,t_2]}(0, c_4)\right) P\left(B^{y_2\to b}_{[t_2,1]} \in K_{[t_2,1]}(0, c_2)\right).$$

By Lemma A.6, it holds that $h(y_1, y_2) > 0$ on $(y_1, y_2) \in (0, c_4) \times (0, c_4)$. Therefore, we obtain our assertion. □

**Theorem 7.** *(Chapter 2, Theorem 4.15 in [8]) Let $\{X_n\}_{n=1}^\infty$ be the family of $C([0, 1], \mathbb{R}^d)$-valued random variables. If the family $\{X_n\}_{n=1}^\infty$ is tight and the finite-dimensional distribution of $X_n$ converges to that of some $X$, then $X_n \xrightarrow{\mathcal{D}} X$ holds.*

**Lemma A.8.** *(Scheffé's Theorem) Let $(X, \mathfrak{A}, \mu)$ be a measure space, and let $p, p_n : X \to [0, \infty)$ be $\mathfrak{A}/\mathcal{B}([0, \infty))$-measurable and satisfy*

$$\int_X p(x)\mu(dx) = 1, \qquad \int_X p_n(x)\mu(dx) = 1.$$

*If $p_n \to p$ holds $\mu$-a.e., then we have*

$$\sup_{E\in\mathfrak{A}} \left| \int_E p(x)\mu(dx) - \int_E p_n(x)\mu(dx) \right| \leq \frac{1}{2} \int_X |p(x) - p_n(x)|\mu(dx) \to 0, \qquad n \to \infty.$$

Proof. Because the function $q_n = p - p_n$ satisfies $\int_X q_n(x)\mu(dx) = 0$, we have

$$\int_{\{p_n>p\}} q_n(x)\mu(dx) = \int_{\{p_n\leq p\}} q_n(x)\mu(dx)$$

for $n$. Therefore, by the inequality

$$0 \leq q_n 1_{\{p_n<p\}} \leq q_n 1_{\{p_n\leq p\}} \leq p,$$

we can use the dominated convergence theorem and obtain

$$\int_X |q_n(x)|\mu(dx) = \int_{\{p_n<p\}} q_n(x)\mu(dx) - \int_{\{p_n>p\}} q_n(x)\mu(dx) = \int_{\{p_n<p\}} q_n(x)\mu(dx) + \int_{\{p_n\leq p\}} q_n(x)\mu(dx) \to 0$$

as $n \to \infty$. The inequality

$$\sup_{E\in\mathfrak{A}} \left| \int_E p(x)\mu(dx) - \int_E p_n(x)\mu(dx) \right| \le \frac{1}{2} \int_X |p(x) - p_n(x)|\mu(dx)$$

follows from the identity

$$\int_E q_n(x)\mu(dx) = -\int_{E^c} q_n(x)\mu(dx), \qquad E \in \mathfrak{A}.$$

□

**Lemma A.9.** *Let $(X, \mathfrak{A}, \mu)$ be a $\sigma$-finite measure space, and let $\mathfrak{A}$-measurable functions $p_n, q_n : X \to [0, \infty)$ satisfy*

$$0 \le p_n \le q_n, \qquad \int_X q_n(x)\mu(dx) < \infty$$

*for $n \in \mathbb{N}$. If $p := \lim_{n\to\infty} p_n$ exists $\mu$-a.e., and there exists a $\mathfrak{A}$-measurable function $q : X \to [0, \infty)$ that satisfies*

$$\lim_{n\to\infty} \int_X |q_n(x) - q(x)|\mu(dx) = 0,$$

*then we have*

$$\lim_{n\to\infty} \int_X |p_n(x) - p(x)|\mu(dx) = 0.$$

Proof. Let $\varepsilon > 0$ be fixed. Then, by the $\sigma$-finiteness of $\mu$, we can find $S \in \mathfrak{A}$ such that

$$\mu(S) < \infty, \qquad \int_{X\setminus S} q(x)\mu(dx) < \varepsilon$$

holds. Because $\mu(S) < \infty$ and

$$\lim_{n\to\infty} \int_S |q_n(x) - q(x)|\mu(dx) = 0$$

hold, we obtain the uniform integrability of $\{q_n 1_S\}_n$:

$$\lim_{K\to\infty} \sup_{n\in\mathbb{N}} \int_{S\cap\{q_n\ge K\}} q_n(x)\mu(dx) = 0.$$

Thus,

$$\lim_{K\to\infty} \sup_{n\in\mathbb{N}} \int_{S\cap\{p_n\ge K\}} p_n(x)\mu(dx) = 0 \tag{92}$$

holds. By combining (92) and $\mu$-*a.e.* convergence $p_n 1_S \to p1_S$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \int_S |p_n(x) - p(x)|\mu(dx) = 0.$$

Hence, because we have

$$\int_{X\setminus S} p(x)\mu(dx) \leq \underline{\lim}_{n\to\infty} \int_{X\setminus S} p_n(x)\mu(dx) \leq \overline{\lim}_{n\to\infty} \int_{X\setminus S} q_n(x)\mu(dx) = \int_{X\setminus S} q(x)\mu(dx) < \varepsilon,$$

it holds that

$$\overline{\lim}_{n\to\infty} \int_X |p_n(x) - p(x)|\mu(dx) \leq \overline{\lim}_{n\to\infty} \int_S |p_n(x) - p(x)|\mu(dx) + 2\varepsilon = 2\varepsilon,$$

and the proof is completed. □

**Lemma A.10.** *Let $\mathbb{R}^d$-valued Markov processes $X_n, X$ on $[0,1]$ have transition densities*

$$P(X_n(t) \in dy) = q_n(t,y)dy, \qquad P(X_n(t) \in dy \mid X_n(s) = x) = q_n(s,x,t,y)dy,$$
$$P(X(t) \in dy) = q(t,y)dy, \qquad P(X(t) \in dy \mid X(s) = x) = q(s,x,t,y)dy$$

*for $0 \leq s < t \leq 1$, $x, y \in \mathbb{R}^d$, and $n \in \mathbb{N}$. If we have*

$$\lim_{n\to\infty} q_n(t,y) = q(t,y), \qquad a.e.\ y \in \mathbb{R}^d,$$
$$\lim_{n\to\infty} q_n(s,x,t,y) = q(s,x,t,y), \qquad a.e.\ (x,y) \in \mathbb{R}^d \times \mathbb{R}^d,$$

*for $0 \leq s < t \leq 1$, then the finite-dimensional distribution of $X_n$ converges to that of $X$ as $n \to \infty$.*

Proof. Let $0 \leq t_1 < \cdots < t_l \leq 1$ be given. Then,

$$f_n(x_1,\ldots,x_l) = q_n(t_1,x_1)\prod_{i=1}^{l-1} q_n(t_i,x_i,t_{i+1},x_{i+1}), \qquad f(x_1,\ldots,x_l) = q(t_1,x_1)\prod_{i=1}^{l-1} q(t_i,x_i,t_{i+1},x_{i+1})$$

satisfy

$$\int_{\mathbb{R}^{dl}} f_n(x)dx = 1, \qquad \int_{\mathbb{R}^{dl}} f(x)dx = 1$$

and $f_n \to f$ holds by assumption. Therefore, for every $\mathbb{R}$-valued bounded continuous function $g$ defined on $\mathbb{R}^{dl}$, it follows from Lemma A.8 that

$$\left|E\left[g\left(X_n(t_1),\ldots,X_n(t_l)\right)\right] - E\left[g\left(X(t_1),\ldots,X(t_l)\right)\right]\right| \leq \sup_{z\in\mathbb{R}^{dl}} |g(z)| \int_{\mathbb{R}^{dl}} |f_n(x) - f(x)|dx \to 0, \qquad n \to \infty.$$

□

**Theorem 8.** *Let $\mathcal{E}$ be a nonempty set. For $\varepsilon \in \mathcal{E}$, $X^{(\varepsilon)}$ is a $(C([0,1],\mathbb{R}^d), \mathcal{B}(C([0,1],\mathbb{R}^d)))$-valued random variable defined on $(\Omega^{(\varepsilon)}, \mathcal{F}^{(\varepsilon)}, P^{(\varepsilon)})$. Assume that $\{X^{(\varepsilon)}(0)\}_{\varepsilon\in\mathcal{E}}$ is uniformly integrable, and the following conditions hold.*

(1) *For each $u \in \left(0, \frac{1}{2}\right)$, $\{\pi_{[u,1-u]} \circ X^{(\varepsilon)}\}_{\varepsilon\in\mathcal{E}}$ is tight.*

(2) *For each $\xi > 0$, it holds that*

$$\lim_{u\downarrow 0} \sup_{\varepsilon\in\mathcal{E}} P^{(\varepsilon)}\left(\sup_{0\le t\le u} |X^{(\varepsilon)}(t) - X^{(\varepsilon)}(0)| > \xi\right) = 0, \qquad \lim_{u\downarrow 0} \sup_{\varepsilon\in\mathcal{E}} P^{(\varepsilon)}\left(\sup_{1-u\le t\le 1} |X^{(\varepsilon)}(t) - X^{(\varepsilon)}(1)| > \xi\right) = 0.$$

*Then, the family $\{X^{(\varepsilon)}\}_{\varepsilon\in\mathcal{E}}$ is tight.*

Proof. For each $w \in C := C([0,1],\mathbb{R}^d)$, $0 \le a < b \le 1$, and $\delta > 0$, we define

$$m^{a,b}(w,\delta) = \max_{\substack{a\le t,s\le b\\ |t-s|\le\delta}} |w(t) - w(s)|.$$

If we write the law of $X^{(\varepsilon)}$ as $P^\varepsilon := P^{(\varepsilon)} \circ (X^{(\varepsilon)})^{-1}$, then based on Chapter 2, Theorem 4.10 in [8], what we must prove is

$$\lim_{\lambda\uparrow\infty} \sup_{\varepsilon\in\mathcal{E}} P^\varepsilon\left(\left\{w \in C \,\middle|\, |w(0)| > \lambda\right\}\right) = 0, \tag{93}$$

$$\lim_{\delta\downarrow 0} \sup_{\varepsilon\in\mathcal{E}} P^\varepsilon\left(\left\{w \in C \,\middle|\, m^{0,1}(w,\delta) > \eta\right\}\right) = 0, \qquad \eta > 0. \tag{94}$$

Since (93) follows from assumption (1), hereinafter we deal with (94). For each $\eta > 0$ and $u \in (0, \frac{1}{2})$, we have

$$\begin{aligned}
&\left\{w \in C \,\middle|\, m^{0,1}(w,\delta) \ge \eta\right\}\\
&\quad\subset \left\{w \in C \,\middle|\, m^{0,u}(w,\delta) \ge \frac{\eta}{3}\right\} \cup \left\{w \in C \,\middle|\, m^{u,1-u}(w,\delta) \ge \frac{\eta}{3}\right\} \cup \left\{w \in C \,\middle|\, m^{1-u,1}(w,\delta) \ge \frac{\eta}{3}\right\}.
\end{aligned}$$

Therefore, for any $\gamma > 0$, we only have to find $u_0 \in (0, \frac{1}{2})$ and $\delta > 0$ such that

$$\begin{cases}
\sup_{\varepsilon\in\mathcal{E}} P^\varepsilon\left(\left\{w \in C \,\middle|\, m^{0,u_0}(w,\delta) \ge \frac{\eta}{3}\right\}\right) < \frac{\gamma}{3},\\
\sup_{\varepsilon\in\mathcal{E}} P^\varepsilon\left(\left\{w \in C \,\middle|\, m^{u_0,1-u_0}(w,\delta) \ge \frac{\eta}{3}\right\}\right) < \frac{\gamma}{3},\\
\sup_{\varepsilon\in\mathcal{E}} P^\varepsilon\left(\left\{w \in C \,\middle|\, m^{1-u_0,1}(w,\delta) \ge \frac{\eta}{3}\right\}\right) < \frac{\gamma}{3}.
\end{cases}$$

Now, for $u > 0$ and $w \in \left\{w \in C \,\middle|\, m^{0,u}(w,u) \ge \frac{\eta}{3}\right\}$, we have

$$\frac{\eta}{3} \le m^{0,u}(w,u) = \max_{\substack{0\le s,t\le u\\ |t-s|\le u}} |w(t) - w(s)| \le 2 \sup_{0\le t\le u} |w(t) - w(0)|.$$

Therefore, by assumption (2),

$$\sup_{\varepsilon\in\mathcal{E}} P^\varepsilon\left(\left\{w \in C \,\middle|\, m^{0,u}(w,u) \ge \frac{\eta}{3}\right\}\right) \le \sup_{\varepsilon\in\mathcal{E}} P^\varepsilon\left(\left\{w \in C \,\middle|\, \sup_{0\le t\le u} |w(t) - w(0)| \ge \frac{\eta}{6}\right\}\right) < \frac{\gamma}{3}$$

holds for sufficiently small $u > 0$. By the same argument, $\sup_{\varepsilon\in\mathcal{E}} P^{\varepsilon}\left(\left\{w \in C \mid m^{1-u,1}(w,u) \geq \frac{\eta}{3}\right\}\right) < \frac{\gamma}{3}$ holds for sufficiently small $u > 0$. Hence, we can find $u_0 \in (0, \frac{1}{2})$ that satisfies

$$\sup_{\varepsilon\in\mathcal{E}} P^{\varepsilon}\left(\left\{w \in C \mid m^{0,u_0}(w,u_0) \geq \frac{\eta}{3}\right\}\right) < \frac{\gamma}{3}, \qquad \sup_{\varepsilon\in\mathcal{E}} P^{\varepsilon}\left(\left\{w \in C \mid m^{1-u_0,1}(w,u_0) \geq \frac{\eta}{3}\right\}\right) < \frac{\gamma}{3}.$$

On the other hand, because $\{\pi_{[u_0,1-u_0]} \circ X^{(\varepsilon)}\}_{\varepsilon\in\mathcal{E}}$ is tight by assumption (1), based on Chapter 2, Theorem 4.10 in [8], we can find $\delta \in (0, u_0)$ such that $\sup_{\varepsilon\in\mathcal{E}} P^{\varepsilon}\left(\left\{w \in C \mid m^{u_0,1-u_0}(w,\delta) \geq \frac{\eta}{3}\right\}\right) < \frac{\gamma}{3}$. □

**Lemma A.11.** *(Chapter 2, Problem 4.11 in [8]) Let $\mathcal{E}$ be a nonempty set. For $\varepsilon \in \mathcal{E}$, $X^{(\varepsilon)}$ is a $(C([0,1],\mathbb{R}^d), \mathcal{B}(C([0,1],\mathbb{R}^d)))$-valued random variable defined on $(\Omega^{(\varepsilon)}, \mathcal{F}^{(\varepsilon)}, P^{(\varepsilon)})$. Assume that $\{X^{(\varepsilon)}\}_{\varepsilon\in\mathcal{E}}$ satisfies the following conditions:*

(1) *There exists $\nu > 0$ that satisfies*

$$\sup_{\varepsilon\in\mathcal{E}} E^{(\varepsilon)}\left[\left|X^{(\varepsilon)}(0)\right|^{\nu}\right] < \infty.$$

(2) *There exist $\alpha, \beta, C > 0$ that satisfy*

$$\sup_{\varepsilon\in\mathcal{E}} E^{(\varepsilon)}\left[\left|X^{(\varepsilon)}(t) - X^{(\varepsilon)}(s)\right|^{\alpha}\right] \leq C\,|t-s|^{1+\beta}, \qquad t, s \in [0,1].$$

*Then, $\{X^{(\varepsilon)}\}_{\varepsilon\in\mathcal{E}}$ is tight.*

**Lemma A.12.** *Let $\mathcal{E}$ be a nonempty set and $\gamma > 0$. For $\varepsilon \in \mathcal{E}$, $X^{(\varepsilon)}$ is a $(C([0,1],\mathbb{R}^d), \mathcal{B}(C([0,1],\mathbb{R}^d)))$-valued random variable defined on $(\Omega^{(\varepsilon)}, \mathcal{F}^{(\varepsilon)}, P^{(\varepsilon)})$. Assume that*

$$F_l^{\varepsilon} := \left\{\max_{1\leq k\leq 2^{l-1}} \left|X^{(\varepsilon)}\left(\frac{k-1}{2^l}\right) - X^{(\varepsilon)}\left(\frac{k}{2^l}\right)\right| \geq 2^{-l\gamma}\right\} \in \mathcal{F}^{(\varepsilon)}, \qquad \varepsilon \in \mathcal{E}, \quad l = 1, 2, \ldots$$

*satisfy $\sum_{l=1}^{\infty} \sup_{\varepsilon\in\mathcal{E}} P^{(\varepsilon)}(F_l^{\varepsilon}) < \infty$, then we have*

$$\lim_{u\downarrow 0} \sup_{\varepsilon\in\mathcal{E}} P^{(\varepsilon)}\left(\sup_{0\leq t\leq u}\left|X^{(\varepsilon)}(t) - X^{(\varepsilon)}(0)\right| > \xi\right) = 0, \qquad \xi > 0.$$

Proof. We define

$$\Omega_m^{\varepsilon} = \bigcap_{l=m}^{\infty} (F_l^{\varepsilon})^c.$$

For any $\eta > 0$, we can find $m \in \mathbb{N}$ such that $\sum_{l=m}^{\infty} \sup_{\varepsilon\in\mathcal{E}} P^{(\varepsilon)}\left(F_l^{\varepsilon}\right) < \eta$. Thus,

$$P^{(\varepsilon)}\left((\Omega_m^{\varepsilon})^c\right) = P^{(\varepsilon)}\left(\bigcup_{l=m}^{\infty} F_l^{\varepsilon}\right) \leq \sum_{l=m}^{\infty} P^{(\varepsilon)}\left(F_l^{\varepsilon}\right) < \eta$$

holds for $\varepsilon \in \mathcal{E}$. Therefore, for $\varepsilon \in \mathcal{E}$, we have

$$P^{(\varepsilon)}\left(\sup_{0\le t\le u}\left|X^{(\varepsilon)}(t)-X^{(\varepsilon)}(0)\right|>\xi\right)<\eta+P^{(\varepsilon)}\left(\left\{\sup_{0\le t\le u}\left|X^{(\varepsilon)}(t)-X^{(\varepsilon)}(0)\right|>\xi\right\}\cap\Omega_m^{\varepsilon}\right).$$

Now, let $\omega \in \Omega_m^{\varepsilon}$ and $l \ge m$ be fixed. We can prove by induction on $n > l$ that

$$\left|X^{(\varepsilon)}(t)-X^{(\varepsilon)}(s)\right|\le 2\sum_{j=l+1}^{n}2^{-\gamma j},\qquad t,s\in D_n,\ 0<t-s<2^{-l} \tag{95}$$

holds. Here, $D_n$ denotes $\{k/2^n \mid 0 \le k \le 2^{n-1}\}$. In fact, for $n = l+1$, (95) holds since $\omega \in (F_{l+1}^{\varepsilon})^c$. Suppose that (95) is valid for $n = l+1, \ldots N-1$. For $t, s \in D_N$ that satisfy $0 < t-s < 2^{-l}$, we set $t^1 = \max\{u \in D_{N-1} \mid u \le t\}$ and $s^1 = \min\{u \ge s \mid u \in D_{N-1}\}$. Since $\omega \in (F_N^{\varepsilon})^c$ and $s^1 - s \le 2^{-N}$, $t - t^1 \le 2^{-N}$ hold, we have

$$\left|X^{(\varepsilon)}(t)-X^{(\varepsilon)}(t^1)\right|\le 2^{-\gamma N},\qquad \left|X^{(\varepsilon)}(s^1)-X^{(\varepsilon)}(s)\right|\le 2^{-\gamma N}.$$

Therefore, combining the assumption of the induction and the inequality $t^1 - s^1 \le t - s < 2^{-l}$, we obtain

$$\begin{aligned}\left|X^{(\varepsilon)}(t)-X^{(\varepsilon)}(s)\right| &\le \left|X^{(\varepsilon)}(t)-X^{(\varepsilon)}(t^1)\right|+\left|X^{(\varepsilon)}(t^1)-X^{(\varepsilon)}(s^1)\right|+\left|X^{(\varepsilon)}(s^1)-X^{(\varepsilon)}(s)\right|\\ &\le 2^{-\gamma N}+2\sum_{j=l+1}^{N-1}2^{-\gamma j}+2^{-\gamma N}\\ &= 2\sum_{j=l+1}^{N}2^{-\gamma j},\end{aligned}$$

and (95) is valid for $n = N$.

Again, let $\omega \in \Omega_m^{\varepsilon}$. For $t, s \in \bigcup_{n=1}^{\infty} D_n$ with $0 < t-s \le 2^{-(m+1)}$, we can find $l \ge m$ such that $2^{-(l+1)} \le t-s < 2^{-l}$. For this $l$, it follows from (95) that

$$\left|X^{(\varepsilon)}(t)-X^{(\varepsilon)}(s)\right|\le 2\sum_{j=l+1}^{\infty}2^{-\gamma j}=\frac{2}{1-2^{-\gamma}}2^{-\gamma(l+1)}\le\frac{2}{1-2^{-\gamma}}|t-s|^{\gamma}.$$

Hence, by the continuity of both sides,

$$\max_{\substack{0\le t,s\le\frac{1}{2}\\ 0<|t-s|\le 2^{-(m+1)}}}\frac{|X^{(\varepsilon)}(t)-X^{(\varepsilon)}(s)|}{|t-s|^{\gamma}}\le\frac{2}{1-2^{-\gamma}}$$

holds on $\Omega_m^{\varepsilon}$. Therefore, if $u$ satisfies $u^{\gamma} < \min\left\{\left(\frac{2}{1-2^{-\gamma}}\right)^{-1}\xi, 2^{-(m+1)}\right\}$, then

$$\left\{\sup_{0\le t\le u}\left|X^{(\varepsilon)}(t)-X^{(\varepsilon)}(0)\right|>\xi\right\}\cap\Omega_m^{\varepsilon}=\emptyset$$

holds for $\varepsilon \in \mathcal{E}$. □

**Lemma A.13.** *Under the same assumption of Lemma A.12, if*

$$\widetilde{F}_l^\varepsilon = \left\{ \max_{2^{l-1} \le k \le 2^l} \left| X^{(\varepsilon)}\left(\frac{k-1}{2^l}\right) - X^{(\varepsilon)}\left(\frac{k}{2^l}\right) \right| \ge 2^{-l\gamma} \right\} \in \mathcal{F}^{(\varepsilon)}, \qquad \varepsilon \in \mathcal{E}, \quad l = 1, 2, \dots$$

*satisfy* $\sum_{l=1}^\infty \sup_{\varepsilon \in \mathcal{E}} P^{(\varepsilon)}(\widetilde{F}_l^\varepsilon) < \infty$, *then we have*

$$\lim_{u \downarrow 0} \sup_{\varepsilon \in \mathcal{E}} P^{(\varepsilon)} \left( \sup_{0 \le t \le u} \left| X^{(\varepsilon)}(1-t) - X^{(\varepsilon)}(1) \right| > \xi \right) = 0, \qquad \xi > 0.$$

Proof. This lemma can be proven by employing the same argument as that used for Lemma A.12. □

**Proposition A.1.** *Let* $X_n$, $n = 1, 2, \dots$ *be a sequence of* $C([0,1], \mathbb{R}^d)$*-valued random variables with each component defined on* $(\Omega_n, \mathcal{F}_n, P_n)$*, and let* $X$ *be a* $C([0,1], \mathbb{R}^d)$*-valued random variable defined on* $(\Omega, \mathcal{F}, P)$*. Suppose that we have the following.*

*(1)* $(X_n(t_0), \dots, X_n(t_l)) \xrightarrow{\mathcal{D}} (X(t_0), \dots, X(t_l))$ *for* $l \in \mathbb{N}$ *and* $0 = t_0 < \cdots < t_l < 1$.

*(2) For each* $\delta > 0$*, it holds that*

$$\lim_{u \downarrow 0} \sup_{n \in \mathbb{N}} P_n \left( \sup_{0 \le t \le u} |X_n(1-t) - X_n(1)| > \delta \right) = 0.$$

*Then,* $X_n$ *converges to* $X$ *in the finite-dimensional distributional sense.*

Proof. Let $0 = t_0 < \cdots < t_l < t_{l+1} = 1$, $\xi_0, \dots, \xi_l, \xi_{l+1} \in \mathbb{R}^d$, and $\varepsilon > 0$ be given. Take $\delta > 0$ such that $|e^{\mathrm{i}\xi_{l+1} \cdot x} - 1| < \varepsilon$ holds for $x \in [-\delta, \delta]^d$. By assumption (2), we can find $u_0 \in (0, 1 - t_l)$ that satisfies

$$P\left( \sup_{0 \le s \le u_0} |X(1-s) - X(1)| > \delta \right) \le \varepsilon, \qquad \sup_{n \in \mathbb{N}} P_n \left( \sup_{0 \le s \le u_0} |X_n(1-s) - X_n(1)| > \delta \right) \le \varepsilon.$$

Then, we have

$$\begin{aligned}
&\left| E_n \left[ \textstyle\prod_{j=0}^{l+1} e^{\mathrm{i}\xi_j \cdot X_n(t_j)} - e^{\mathrm{i}\xi_{l+1} \cdot X_n(1-u_0)} \prod_{j=0}^{l} e^{\mathrm{i}\xi_j \cdot X_n(t_j)} \right] \right| \\
&\quad \le E_n[|e^{\mathrm{i}\xi_{l+1} \cdot (X_n(1-u_0) - X_n(1))} - 1|] \\
&\quad \le E_n[|e^{\mathrm{i}\xi_{l+1} \cdot (X_n(1-u_0) - X_n(1))} - 1| \,;\, |X_n(1-u_0) - X_n(1))| \le \delta] + 2P_n \left( \sup_{0 \le t \le u} |X_n(1-t) - X_n(1)| > \delta \right) \\
&\quad \le 3\varepsilon.
\end{aligned}$$

In the same manner, we obtain

$$\left| E \left[ \textstyle\prod_{j=0}^{l+1} e^{\mathrm{i}\xi_j \cdot X(t_j)} - e^{\mathrm{i}\xi_{l+1} \cdot X(1-u_0)} \prod_{j=0}^{l} e^{\mathrm{i}\xi_j \cdot X(t_j)} \right] \right| \le 3\varepsilon.$$

Hence, it follows from assumption (1) that

$$\begin{aligned}
&\overline{\lim_{n \to \infty}} \left| E_n \left[ \textstyle\prod_{j=0}^{l+1} e^{\mathrm{i}\xi_j \cdot X_n(t_j)} \right] - E \left[ \prod_{j=0}^{l+1} e^{\mathrm{i}\xi_j \cdot X(t_j)} \right] \right| \\
&\quad \le 6\varepsilon + \overline{\lim_{n \to \infty}} \left| E_n \left[ e^{\mathrm{i}\xi_{l+1} \cdot X_n(1-u_0)} \textstyle\prod_{j=0}^{l} e^{\mathrm{i}\xi_j \cdot X_n(t_j)} \right] - E \left[ e^{\mathrm{i}\xi_{l+1} \cdot X(1-u_0)} \prod_{j=0}^{l} e^{\mathrm{i}\xi_j \cdot X(t_j)} \right] \right| = 6\varepsilon.
\end{aligned}$$

□

**Lemma A.14.** *Let $S_1$ and $S_2$ be Polish spaces. Let $X_n$ and $Y_n$ be random variables defined on $(\Omega_n, \mathcal{F}_n, P_n)$ that take values in $S_1$ and $S_2$, respectively. If $X_n$ and $Y_n$ are independent and $P_n \circ X_n^{-1}$ and $P_n \circ Y_n^{-1}$ converge weakly to probability measures $Q$ on $S_1$ and $R$ on $S_2$, respectively, then $P_n \circ (X_n, Y_n)^{-1}$ converges weakly to the product measure $Q \times R$.*

Proof. First, we prove the tightness of $\{P_n \circ (X_n, Y_n)^{-1}\}_n$. By the assumption and Prohorov's theorem, $\{P_n \circ X_n^{-1}\}_n$ and $\{P_n \circ Y_n^{-1}\}_n$ are tight. Therefore, for $\varepsilon > 0$, we can find compact subsets $K_i$ of $S_i$ such that

$$\inf_{n\in\mathbb{N}} P_n(X_n \in K_1) \geq 1 - \varepsilon, \qquad \inf_{n\in\mathbb{N}} P_n(Y_n \in K_2) \geq 1 - \varepsilon$$

hold. Because $K_1 \times K_2$ is a compact subset of $S_1 \times S_2$ by Tychonoff's theorem and

$$P_n((X_n, Y_n) \in K_1 \times K_2) \geq (1 - \varepsilon)^2 \geq 1 - 2\varepsilon, \qquad n \in \mathbb{N}$$

holds, $\{P_n \circ (X_n, Y_n)^{-1}\}_n$ is tight.

Now, suppose that $P_n \circ (X_n, Y_n)^{-1}$ does not converge weakly to $Q \times R$. Then, we can find an $\mathbb{R}$-valued bounded continuous function $f$ on $S_1 \times S_2$ and a subsequence $\{n_k\}$ such that

$$\lim_{k\to\infty} E_{n_k}[f(X_{n_k}, Y_{n_k})] \text{ exists and } \lim_{k\to\infty} E_{n_k}[f(X_{n_k}, Y_{n_k})] \neq \int_{S_1\times S_2} f(s_1, s_2)(Q \times R)(ds_1 ds_2).$$

On the other hand, combining the tightness of $\{P_n \circ (X_n, Y_n)^{-1}\}_n$ and Prohorov's theorem, taking a subsequence if necessary, we can find the weak limit $\mu$ of $P_{n_k} \circ (X_{n_k}, Y_{n_k})^{-1}$. Because we have

$$\begin{aligned}\int_{S_1\times S_2} f_1(s_1)f_2(s_2)\mu(ds_1 ds_2) &= \lim_{k\to\infty} E_{n_k}[f_1(X_{n_k})f_2(Y_{n_k})] = \lim_{k\to\infty} E_{n_k}[f_1(X_{n_k})]E[f_2(Y_{n_k})] \\ &= \int_{S_1\times S_2} f_1(s_1)f_2(s_2)(Q \times R)(ds_1 ds_2)\end{aligned}$$

for every $\mathbb{R}$-valued bounded continuous function $f_i$ on $S_i$, by an approximation argument, we obtain $\mu(G_1 \times G_2) = (Q \times R)(G_1 \times G_2)$ for open sets $G_i$ in $S_i$. Therefore, because $\mu = Q \times R$ holds, we have

$$\lim_{k\to\infty} E_{n_k}[f(X_{n_k}, Y_{n_k})] = \int_{S_1\times S_2} f(s_1, s_2)\mu(ds_1 ds_2) = \int_{S_1\times S_2} f(s_1, s_2)(Q \times R)(ds_1 ds_2)$$

and this is a contradiction. □

**Lemma A.15.** *Let $S$ be a Polish space. Let $X_n$ and $X$ be random variables defined on $(\Omega_n, \mathcal{F}_n, P_n)$ and $(\Omega, \mathcal{F}, P)$ that take values in $S$. Assume that $X_n \xrightarrow{\mathcal{D}} X$ holds and $A \in \mathcal{B}(S)$ satisfies $P(X \in \partial A) = 0$. Then, for every $\mathbb{R}$-valued bounded continuous function $G$ on $S$, we have*

$$\lim_{n\to\infty} E_n[G(X_n)\ ;\ X_n \in A] = E[G(X)\ ;\ X \in A].$$

Proof. By Skorohod's theorem, we may assume that $X_n$ and $X$ are defined on the same probability space and $X_n \to X$ holds almost surely. Then, it follows from $P(X \in \partial A) = 0$ that $1_A(X_n) \to 1_A(X)$ holds almost surely. Therefore, the reverse Fatou's lemma yields

$$\begin{aligned}
&\varlimsup_{n\to\infty} |E[G(X_n)\ ;\ X_n \in A] - E[G(X)\ ;\ X \in A]| \\
&\quad\le \varlimsup_{n\to\infty} E[|G(X_n) - G(X)|] + \sup_{x\in S} |G(x)| \cdot \varlimsup_{n\to\infty} E[|1_{\{X_n\in A\}} - 1_{\{X\in A\}}|] \\
&\quad\le \sup_{x\in S} |G(x)| \cdot E\left[\varlimsup_{n\to\infty} |1_{\{X_n\in A\}} - 1_{\{X\in A\}}|\right] = 0.
\end{aligned}$$

□

**Lemma A.16.** *Let $T > 0$, and let $\mathbb{R}$-valued $C^1$-function $g$ defined on $[0, T]$ that satisfies $\min_{0\le t\le T} g(t) > 0$. Then, for the BES(3)-process $R_{[0,T]}$ starting at $0$ on $[0, T]$ and $b \in (0, g(T))$, we have*

$$P\left(T_g \le T, R_{[0,T]} \in K^-_{[0,T]}(g), R_{[0,T]}(T) \in db\right) = 0.$$

*Here, $T_g$ denotes the hitting time of $R_{[0,T]}$ to $g$.*

Proof. Since $b \in (0, g(T))$, what we must prove is

$$P\left(T_g < T, R_{[0,T]} \in K^-_{[0,T]}(g), R_{[0,T]}(T) \in db\right) = 0.$$

If $W_{[0,T]} = \left(W^{(1)}_{[0,T]}, W^{(2)}_{[0,T]}, W^{(3)}_{[0,T]}\right)$ is a three-dimensional Brownian motion starting at 0, then

$$\begin{aligned}
&P\left(T_g < T, R_{[0,T]} \in K^-_{[0,T]}(g), R_{[0,T]}(T) \in db\right) \\
&\quad= E\left[P\left(\left|(g(s), 0, 0) + W_{[s,T]}\right| \in K^-_{[s,T]}(g),\ \left|(g(s), 0, 0) + W_{[s,T]}(T)\right| \in db\right)\Big|_{s=T_g}\ ;\ T_g < T\right]
\end{aligned}$$

holds. Therefore, we only have to show that

$$P\left(\left|(g(s), 0, 0) + W_{[s,T]}\right| \in K^-_{[s,T]}(g), \left|(g(s), 0, 0) + W_{[s,T]}(T)\right| \in db\right) = 0$$

for $s \in [0, T)$. To prove this, assume that $P\left(\left|(g(s), 0, 0) + W_{[s,T]}\right| \in K^-_{[s,T]}(g)\right) > 0$. Then,

$$P\left(\left|g(s) + W^{(1)}_{[s,T]}\right| \in K^-_{[s,T]}(g)\right) > 0$$

holds. On the other hand, by the law of the iterated logarithm for $W^{(1)}_{[s,T]}$, we can find a sequence $t_n \in (s, T)$ such that

$$W^{(1)}_{[s,T]}(t_n) \ge 0, \qquad \lim_{n\to\infty} \frac{W^{(1)}_{[s,T]}(t_n)}{t_n - s} = \infty, \qquad t_n \downarrow s.$$

Therefore, because

$$\frac{W^{(1)}_{[s,T]}(t_n)}{t_n - s} \le \frac{g(t_n) - g(s)}{t_n - s}, \qquad \text{for } n \in \mathbb{N}$$

holds on $\left\{\left|g(s) + W^{(1)}_{[s,T]}\right| \in K^-_{[s,T]}(g)\right\}$, $g'(s)$ does not exist in $\mathbb{R}$. This contradicts the regularity of $g$. □

**Lemma A.17.** *Let* $0 \le t_1 < t_2 < \infty$ *and* $f, g \in C([t_1, t_2], \mathbb{R})$*. Then*

$$K_{[t_1,t_2]}(f, g) = \{w = \{w(t)\}_{t\in[t_1,t_2]} \in C([t_1, t_2], \mathbb{R}) \mid f(t) \le w(t) \le g(t),\ t_1 \le t \le t_2\}$$

*satisfies*

$$\mathrm{int}(K_{[t_1,t_2]}(f, g)) = \{w \in C([t_1, t_2], \mathbb{R}) \mid f(t) < w(t) < g(t),\ t_1 \le t \le t_2\}\,.$$

Proof. It suffices to show that for any open set $G$ included in $K_{[t_1,t_2]}(f, g)$,

$$G \subset \{w \in C([t_1, t_2], \mathbb{R}) \mid f(t) < w(t) < g(t),\ t_1 \le t \le t_2\}$$

holds. Assume that some $w \in G$ satisfies $w \notin \{w \in C([t_1, t_2], \mathbb{R}) \mid f(t) < w(t) < g(t),\ t_1 \le t \le t_2\}$. Then, we can find $t_0 \in [t_1, t_2]$, for which $w(t_0) = f(t_0)$ or $w(t_0) = g(t_0)$ holds. On the other hand, because $G$ is open, $B(w, \varepsilon) := \{w' \in C([t_1, t_2], \mathbb{R}) \mid d_\infty(w', w) < \varepsilon\} \subset G$ holds for some $\varepsilon > 0$. Thus, $w(t_0) = f(t_0)$ and $w(t_0) = g(t_0)$ cannot happen. This contradiction proves the desired result. □

**Lemma A.18.** *Let* $g$ *be an* $\mathbb{R}$*-valued* $C^1$*-function defined on* $[0, 1]$ *that satisfies* $\min_{0\le t\le 1} g(t) > 0$*. Then, for* $b \in (0, g(1))$*, we have* $P(r^{0\to b} \in \partial K^-(g)) = 0$ *and* $P(W^+ \in \partial K^-(g)) = 0$*.*

Proof. Using Lemma A.17, we obtain

$$\partial K^-(g) = K^-(g) - \mathrm{int}(K^-(g)) = \left\{w \in K^-(g) \;\middle|\; \min_{0\le t\le 1}(g(t) - w(t)) = 0\right\}. \tag{96}$$

Then, (96) and Lemma A.16 imply that $P(r^{0\to b} \in \partial K^-(g)) = 0$ and

$$P(W^+ \in \partial K^-(g)) = \int_0^\infty P(r^{0\to b} \in \partial K^-(g))P(W^+(1) \in db) = 0$$

hold. □

**Proposition A.2.** *Let* $g$ *be an* $\mathbb{R}$*-valued* $C^1$*-function defined on* $[0, 1]$ *that satisfies* $\min_{0\le t\le 1} g(t) > 0$*. Assume that* $\{\eta(\varepsilon)\}_{\varepsilon>0}$ *satisfies*

$$\eta(\varepsilon) \ge 0 \quad (\varepsilon > 0) \quad \textit{and} \quad \eta(\varepsilon) \downarrow 0 \quad (\varepsilon \downarrow 0).$$

*Then, we have*

$$\lim_{\varepsilon\downarrow 0} P(W|_{K^+(-\varepsilon)} \in K^-(g + \eta(\varepsilon))) = P(W^+ \in K^-(g)), \tag{97}$$

$$\lim_{\varepsilon\downarrow 0} P(B^{0\to b}|_{K^+(-\varepsilon)} \in K^-(g + \eta(\varepsilon))) = P(r^{0\to b} \in K^-(g)) \quad (0 \le b < g(1)). \tag{98}$$

*Further, for every* $\mathbb{R}$*-valued bounded continuous function* $F$ *on* $C([0, 1], \mathbb{R})$*, we have*

$$\lim_{\varepsilon\downarrow 0} E[F(W|_{K(-\varepsilon,g+\eta(\varepsilon))})] = E[F(W^+|_{K^-(g)})], \tag{99}$$

$$\lim_{\varepsilon\downarrow 0} E[F(B^{0\to b}|_{K(-\varepsilon,g+\eta(\varepsilon))})] = E[F(r^{0\to b}|_{K^-(g)})] \quad (0 \le b < g(1)). \tag{100}$$

Proof. Combining Lemma A.18 and the fact that $W|_{K^+(-\varepsilon)}$ converges weakly to $W^+$ ([2]), we obtain

$$P(W^+ \in K^-(g)) = \lim_{\varepsilon\downarrow 0} P(W|_{K^+(-\varepsilon)} \in K^-(g)) \le \varliminf_{\varepsilon\downarrow 0} P(W|_{K^+(-\varepsilon)} \in K^-(g+\eta(\varepsilon))), \tag{101}$$

$$\varlimsup_{\varepsilon\downarrow 0} P(W|_{K^+(-\varepsilon)} \in K^-(g+\eta(\varepsilon))) \le \lim_{\varepsilon\downarrow 0} P(W|_{K^+(-\varepsilon)} \in K^-(g+\delta)) = P(W^+ \in K^-(g+\delta)) \quad (\delta > 0). \tag{102}$$

Then, it follows from (101), (102) and $\bigcap_{\delta>0} K^-(g+\delta) = K^-(g)$ that (97) holds. Similarly, combining Lemma A.18 and Theorem 6, we can also deduce that (98) holds.

Because $W|_{K^+(-\varepsilon)}$ converges weakly to $W^+$ ([2]), Lemmas A.15 and A.18 imply that

$$\begin{aligned} E[F(W|_{K(-\varepsilon,g)})] &= \frac{E[F(W|_{K^+(-\varepsilon)})\ ;\ W|_{K^+(-\varepsilon)} \in K^-(g)]}{P(W|_{K^+(-\varepsilon)} \in K^-(g))} \\ &\to \frac{E[F(W^+)\ ;\ W^+ \in K^-(g)]}{P(W^+ \in K^-(g))} = E[F(W^+|_{K^-(g)})] \qquad (\varepsilon \downarrow 0). \end{aligned} \tag{103}$$

On the other hand, because we have

$$\begin{aligned} \varDelta(\varepsilon) :=& |E[F(W|_{K(-\varepsilon,g+\eta(\varepsilon))})] - E[F(W|_{K(-\varepsilon,g)})]| \\ \le& \left| \frac{E[F(W|_{K^+(-\varepsilon)})\ ;\ W|_{K^+(-\varepsilon)} \in K^-(g+\eta(\varepsilon)) \setminus K^-(g)]}{P(W|_{K^+(-\varepsilon)} \in K^-(g+\eta(\varepsilon)))} \right| \\ &+ \left| \frac{E[F(W|_{K^+(-\varepsilon)})\ ;\ W|_{K^+(-\varepsilon)} \in K^-(g)]}{P(W|_{K^+(-\varepsilon)} \in K^-(g))} \left( \frac{P(W|_{K^+(-\varepsilon)} \in K^-(g))}{P(W|_{K^+(-\varepsilon)} \in K^-(g+\eta(\varepsilon)))} - 1 \right) \right| \\ \le& 2\|F\|_\infty \frac{P(W|_{K^+(-\varepsilon)} \in K^-(g+\eta(\varepsilon))) - P(W|_{K^+(-\varepsilon)} \in K^-(g))}{P(W|_{K^+(-\varepsilon)} \in K^-(g))} \qquad (\varepsilon > 0) \end{aligned}$$

for $\|F\|_\infty := \sup_{w\in C([0,1],\mathbb{R})} |F(w)|$, we can deduce that

$$\varlimsup_{\varepsilon\downarrow 0} |\varDelta(\varepsilon)| \le 2\|F\|_\infty \frac{P(W^+ \in K^-(g)) - P(W^+ \in K^-(g))}{P(W^+ \in K^-(g))} = 0 \tag{104}$$

by (97) and Corollary 7. Therefore, (103) and (104) imply (99). Similarly, combining Lemmas A.15, A.18 and Theorem 6, we can deduce that (100) holds. □

We can find the following proposition in [2], which is stated there without proof.

**Proposition A.3.** *Let $(T, \mathcal{T})$ be a measurable space and $(\Omega, \mathcal{F}, P)$ be a probability space, and let $Y = \{Y(t), \mathcal{F}_t^Y, 0 \le t \le 1\}$ be a $T$-valued Markov process on $(\Omega, \mathcal{F}, P)$. For $\Lambda \in \mathcal{F}$ with $P(\Lambda) > 0$, we define a new probability space $(\Lambda, \Lambda \cap \mathcal{F}, P_\Lambda)$ by $\Lambda \cap \mathcal{F} := \{\Lambda \cap F \mid F \in \mathcal{F}\}$ and*

$$P_\Lambda(F) := \frac{P(\Lambda \cap F)}{P(\Lambda)}.$$

*Assume that for $t \in [0,1]$ there exist $A_t \in \mathcal{F}_t^Y$ and $B_t \in \sigma(Y(s) \mid t \le s \le 1)$ that satisfy $\Lambda = A_t \cap B_t$. If we write the restriction $Y$ to $(\Lambda, \Lambda \cap \mathcal{F}, P_\Lambda)$ as $Y_\Lambda$, then $Y_\Lambda = \{Y_\Lambda(t), \mathcal{F}_t^{Y_\Lambda}, 0 \le t \le 1\}$ is a $T$-valued Markov process on $(\Lambda, \Lambda \cap \mathcal{F}, P_\Lambda)$.*

Proof. For $0 < t < s \leq 1$ and $\Gamma \in \mathcal{T}$, we must show that $P_\Lambda(Y_\Lambda(s) \in \Gamma \,|\, \mathcal{F}_t^{Y_\Lambda})$ has a $\sigma(Y_\Lambda(t))$-measurable version. Let $0 = t_0 < t_1 < \cdots < t_n = t < s \leq 1$, $K_1, \ldots, K_n, \Gamma \in \mathcal{B}$ be given. We define a measure $\mu$ on $(T^n, \mathcal{T}^n)$ by

$$\mu(C) := P(\{(Y(t_1), \ldots, Y(t_n)) \in C\} \cap A_t), \qquad C \in \mathcal{T}^n.$$

Using the Markov property of $Y$, we obtain

$$\begin{aligned} &P(\{Y(t_1) \in K_1, \ldots, Y(t_n) \in K_n, Y(s) \in \Gamma\} \cap \Lambda) \\ &\quad = E[P(\{Y(s) \in \Gamma\} \cap B_t \,|\, \mathcal{F}_t^Y) \,;\, \{Y(t_1) \in K_1, \ldots, Y(t_n) \in K_n\} \cap A_t] \\ &\quad = \int_{K_1 \times \cdots \times K_n} P(\{Y(s) \in \Gamma\} \cap B_t \,|\, Y(t_n) = x_n) \mu(dx). \end{aligned} \tag{105}$$

On the other hand, for any $\mathcal{T}/\mathcal{B}([0, \infty))$-measurable function $f : T \to [0, \infty)$, we have

$$\begin{aligned} &E[f(Y(t_n)) \,;\, \{Y(t_1) \in K_1, \ldots, Y(t_n) \in K_n\} \cap \Lambda] \\ &\quad = E[P(B_t \,|\, \mathcal{F}_t^Y) f(Y(t_n)) \,;\, \{Y(t_1) \in K_1, \ldots, Y(t_n) \in K_n\} \cap A_t] \\ &\quad = \int_{K_1 \times \cdots \times K_n} P(B_t \,|\, Y(t_n) = x_n) f(x_n) \mu(dx). \end{aligned} \tag{106}$$

Therefore, applying (106) for

$$f(x_n) := \begin{cases} \dfrac{P(\{Y(s) \in \Gamma\} \cap B_t \,|\, Y(t_n) = x_n)}{P(B_t \,|\, Y(t_n) = x_n)}, & \text{for } P(B_t \,|\, Y(t_n) = x_n) > 0, \\ 0, & \text{otherwise,} \end{cases}$$

we can obtain

$$P(\{Y(t_1) \in K_1, \ldots, Y(t_n) \in K_n, Y(s) \in \Gamma\} \cap \Lambda) = E[f(Y(t_n)) \,;\, \{Y(t_1) \in K_1, \ldots, Y(t_n) \in K_n\} \cap \Lambda]$$

by (105). Dividing by $P(\Lambda)$, we obtain

$$P_\Lambda(Y_\Lambda(t_1) \in K_1, \ldots, Y_\Lambda(t_n) \in K_n, Y_\Lambda(s) \in \Gamma) = E_\Lambda[f(Y_\Lambda(t_n)) \,;\, Y_\Lambda(t_1) \in K_1, \ldots, Y_\Lambda(t_n) \in K_n],$$

and, hence, Dynkin's $\pi$-$\lambda$ theorem yields $P_\Lambda(Y_\Lambda(s) \in \Gamma \,|\, \mathcal{F}_t^{Y_\Lambda}) = f(Y_\Lambda(t_n)) = f(Y_\Lambda(t))$. □

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