

On the Weak Convergence of Conditioned Bessel Bridges

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Abstract. The purpose of this paper is to introduce the construction of a stochastic process called “ δ -dimensional Bessel house-moving” and its properties. We study the weak convergence of δ -dimensional Bessel bridges conditioned from above, and we refer to this limit as δ -dimensional Bessel house-moving. Applying this weak convergence result, we give the decomposition formula for its distribution and the Radon-Nikodym density for the distribution of the Bessel house-moving with respect to the one of the Bessel process. We also prove that δ -dimensional Bessel house-moving is a δ -dimensional Bessel process hitting a fixed point for the first time at $t = 1$.

1. Introduction and Main Results

The purpose of this paper is to introduce the construction of a stochastic process called “ δ -dimensional Bessel house-moving” and its properties.

Recently, [2] developed a chain rule for Wiener path integrals between two curves that arise in the computation of first-order Greeks of barrier options, demonstrating its effectiveness with some numerical examples. In this chain rule, a 3-dimensional Bessel bridge and a Brownian meander played an important role. We believe higher-order chain rules might be useful in computing higher-order Greeks of barrier options, in which the stochastic process “Brownian house-moving” is expected to play an important role.

The Brownian house-moving is a Brownian bridge that stays between its starting point and its terminal point. In [1], it has been proven that the Brownian house-moving can also be obtained by the weak convergence of 3-dimensional Bessel bridges conditioned from above. In [3], a Monte Carlo sampling technique for Brownian house-moving is studied. Although the existence of the Brownian house-moving was well-known in [7, 12], the weak convergence result for conditioned 3-dimensional Bessel bridges had

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yet to be researched, and this result is necessary for computing higher-order Greeks of barrier options under the Black-Scholes market model.

Furthermore, to compute higher-order Greeks of barrier options under the general market model, we need more general results for the weak convergence of conditioned diffusion bridges. As a preparatory step toward this goal, in this paper we focus on the weak convergence of conditioned δ -dimensional Bessel bridges. We study the weak convergence of δ -dimensional Bessel bridges conditioned from above for all $\delta > 0$ (Theorem 1), and we refer to this limit as “ δ -dimensional Bessel house-moving”.

Since the Brownian house-moving corresponds to the 3-dimensional Bessel house-moving, our results expand on those in [1]. In [1], to prove their result, they use a relation between the one-dimensional Brownian bridge and the 3-dimensional Bessel bridge. However, we are not able to apply the same approach as [1] for $\delta > 0$ because a δ -dimensional Bessel process is abstractly defined as a solution of a stochastic differential equation. For this reason, we prove our result with different approaches, such as by using estimations related to the Fourier-Bessel expansion ([6], [10]).

We also prove that a δ -dimensional Bessel house-moving is a δ -dimensional Bessel process hitting a fixed point for the first time at $t = 1$ (Proposition 1.2). As mentioned above, the first hitting process for one-dimensional diffusion processes already appeared in [7, 12]. However, since we also construct the Bessel house-moving as the weak limit of conditioned Bessel bridges, we can obtain new results on the sample path properties of the Bessel house-moving. For example, applying our weak convergence result, we can obtain the decomposition formula for its distribution (Theorem 2) and the Radon-Nikodym density for the distribution of the Bessel house-moving with respect to one of the Bessel processes (Theorem 3).

1.1. Notations

We start by introducing notations needed for stating our results.

Throughout this paper, we fix $\delta > 0$ and set $\nu := \delta/2 - 1$.

For $0 \leq a < b$, let $R^a = \{R^a(t)\}_{t \geq 0}$ be a δ -dimensional Bessel process (BES(δ) process for short) starting from a . In addition, for $0 \leq t_1 < t_2 \leq 1$, $r_{[t_1, t_2]}^{a \rightarrow b} = \{r_{[t_1, t_2]}^{a \rightarrow b}(t)\}_{t \in [t_1, t_2]}$ denotes a δ -dimensional Bessel bridge (BES(δ) bridge for short) from a to b on $[t_1, t_2]$. We write simply $r^{a \rightarrow b} := r_{[0, 1]}^{a \rightarrow b}$.

For a continuous process X on $[t_1, t_2]$, we denote its maximal value as

$$M_{[t_1, t_2]}(X) = \max_{t_1 \leq u \leq t_2} X(u).$$

In the case that $[t_1, t_2] = [0, t]$, we write $M_t(X) := M_{[0, t]}(X)$. Moreover, in the case that $[t_1, t_2] = [0, 1]$, we write simply $M(X) := M_{[0, 1]}(X)$.

For $\eta > 0$, $0 \leq s < t \leq 1$, and $x, y \in [0, \eta]$, we define

$$\begin{aligned} q_1^{(\eta)}(s, x, t, y) &:= \frac{P(R^x(t-s) \in dy)}{dy} P\left(M_{[s, t]}(r_{[s, t]}^{x \rightarrow y}) \leq \eta\right), \\ q_2^{(\eta)}(t, y) &:= \lim_{\varepsilon \downarrow 0} \frac{\partial}{\partial \varepsilon} q_1^{(\eta + \varepsilon)}(0, y, t, \eta) \\ &= \frac{P(R^y(t) \in d\eta)}{d\eta} \lim_{\varepsilon \downarrow 0} \frac{\partial}{\partial \varepsilon} P\left(M_{[0, t]}(r_{[0, t]}^{y \rightarrow \eta}) \leq \eta + \varepsilon\right). \end{aligned}$$

Let $C([t_1, t_2], \mathbb{R})$ be a class of \mathbb{R} -valued continuous functions defined on $[t_1, t_2]$ and set

$$K_{[t_1, t_2]}^-(c) := \{w \in C([t_1, t_2], \mathbb{R}) \mid w(t) \leq c, t_1 \leq t \leq t_2\}$$

for $c \in \mathbb{R}$. In the case that $[t_1, t_2] = [0, 1]$, we write simply $K^-(c) := K_{[0, 1]}^-(c)$. Let

$$\begin{aligned} d_\infty(w, w') &= \|w - w'\|_{C([t_1, t_2], \mathbb{R})} \\ &:= \sup_{t_1 \leq t \leq t_2} |w(t) - w'(t)| \quad (w, w' \in C([t_1, t_2], \mathbb{R})). \end{aligned}$$

$\mathcal{B}(C([t_1, t_2], \mathbb{R}))$ denotes the Borel σ -algebra with respect to the topology generated by the metric d_∞ . In addition, for $0 \leq s < t \leq 1$, $\pi_{[s, t]} : C([0, 1], \mathbb{R}) \rightarrow C([s, t], \mathbb{R})$ denotes the restriction map.

Assume that $Y : (\Omega, \mathcal{F}, P) \rightarrow (C([0, 1], \mathbb{R}), \mathcal{B}(C([0, 1], \mathbb{R})))$ is a random variable and that $\Lambda \in \mathcal{B}(C([0, 1], \mathbb{R}))$ satisfies $P(Y \in \Lambda) > 0$. Then, we define the probability measure $P_{Y^{-1}(\Lambda)}$ on $(Y^{-1}(\Lambda), Y^{-1}(\Lambda) \cap \mathcal{F})$ as

$$P_{Y^{-1}(\Lambda)}(A) := \frac{P(A)}{P(Y \in \Lambda)}, \quad A \in Y^{-1}(\Lambda) \cap \mathcal{F} := \{Y^{-1}(\Lambda) \cap F \mid F \in \mathcal{F}\}.$$

Throughout this paper, $P_{Y^{-1}(\Lambda)}(Y|_\Lambda \in \Gamma)$ is often written as $P(Y|_\Lambda \in \Gamma)$.

In addition, $X_n \xrightarrow{D} X$ means that $\{X_n\}_{n=1}^\infty$ converges to X in distribution.

1.2. Main results

First, we construct a stochastic process called “ δ -dimensional Bessel house-moving” (BES(δ) house-moving for short) $H^{a \rightarrow b}$ as the weak limit of BES(δ) bridges conditioned from above.

THEOREM 1. *Let $0 \leq a < b$. There exists an \mathbb{R} -valued continuous Markov process $H^{a \rightarrow b} = \{H^{a \rightarrow b}(t)\}_{t \in [0,1]}$ that satisfies*

$$r^{a \rightarrow b} |_{K^-(b+\eta)} \xrightarrow{\mathcal{D}} H^{a \rightarrow b}, \quad \eta \downarrow 0.$$

Moreover, for $0 < s < t < 1$ and $x, y \in (0, b)$, the law of $H^{a \rightarrow b}$ is given by

$$P \left(H^{a \rightarrow b}(t) \in dy \right) = \frac{q_1^{(b)}(0, a, t, y) q_2^{(b)}(1 - t, y)}{q_2^{(b)}(1, a)} dy,$$

$$P \left(H^{a \rightarrow b}(t) \in dy \mid H^{a \rightarrow b}(s) = x \right) = \frac{q_1^{(b)}(s, x, t, y) q_2^{(b)}(1 - t, y)}{q_2^{(b)}(1 - s, x)} dy.$$

Applying Theorem 1, we can prove the decomposition formula for the distribution of the BES(δ) house-moving (Theorem 2). Let $t \in (t_1, t_2)$. For $w_1 \in C([t_1, t], \mathbb{R})$ and $w_2 \in C([t, t_2], \mathbb{R})$ that satisfy $w_1(t) = w_2(t)$, we define $w_1 \oplus_t w_2 \in C([t_1, t_2], \mathbb{R})$ by

$$(w_1 \oplus_t w_2)(s) := \begin{cases} w_1(s), & s \in [t_1, t], \\ w_2(s), & s \in [t, t_2]. \end{cases}$$

THEOREM 2. *Let $0 \leq a < b$. For every bounded continuous function F on $C([0, 1], \mathbb{R})$, it holds that*

$$E \left[F(H^{a \rightarrow b}) \right] = \int_0^b E \left[F(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b)} \oplus_t H_{[t,1]}^{y \rightarrow b}) \right] P \left(H^{a \rightarrow b}(t) \in dy \right),$$

$0 < t < 1,$

where $r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b)}$ and $H_{[t,1]}^{y \rightarrow b}$ are chosen to be independent.

As an application of Theorem 2, we show that the BES(δ) house-moving does not hit b on the time interval $[0, 1)$.

PROPOSITION 1.1. *Let $0 \leq a < b$. For $0 < t < 1$, it holds that*

$$P \left(\max_{0 \leq u \leq t} H^{a \rightarrow b}(u) < b \right) = 1.$$

By using Theorem 1, we can also prove that the distribution of the BES(δ) house-moving is absolutely continuous with respect to the BES(δ) process.

Let P^X denote the measure induced by a continuous process $X = \{X(t)\}_{t \in [0,1]}$. In addition, for $0 < t < 1$, we define

$$P_t^X := P \circ (\pi_{[0,t]} \circ X)^{-1}.$$

THEOREM 3. *Let $0 \leq a < b$ and $t \in (0, 1)$. Then, we have*

$$\frac{dP_t^{H^{a \rightarrow b}}}{dP_t^{R^a}}(w) = \frac{q_2^{(b)}(1-t, w(t))}{q_2^{(b)}(1, a)} \cdot 1_{K_{[0,t]}^-(b)}(w), \quad w \in C([0, t], \mathbb{R}).$$

Let $\tau_{a,b}$ denote the first hitting time of the point b by R^a :

$$\tau_{a,b} := \inf\{r \geq 0 \mid R^a(r) = b\}.$$

PROPOSITION 1.2. *Let $0 \leq a < b$. The BES(δ) house-moving $H^{a \rightarrow b} = \{H^{a \rightarrow b}(t)\}_{t \in [0,1]}$ satisfies*

$$P \left(H^{a \rightarrow b}(t) \in dy \right) = P \left(R^a(t) \in dy \mid \tau_{a,b} = 1 \right),$$

$$P \left(H^{a \rightarrow b}(t) \in dy \mid H^{a \rightarrow b}(s) = x \right) = P \left(R^a(t) \in dy \mid R^a(s) = x, \tau_{a,b} = 1 \right)$$

for $0 < s < t < 1$ and $x, y \in (0, b)$.

Finally, we study the sample path properties of BES(δ) house-moving $H^{a \rightarrow b}$ and establish the regularity of its sample path.

PROPOSITION 1.3. *For every $\gamma \in (0, \frac{1}{2})$, the path of $H^{a \rightarrow b}$ ($0 \leq a < b$) on $[0, 1]$ is locally Hölder-continuous with exponent γ :*

$$P \left(\bigcup_{n=1}^{\infty} \left\{ \sup_{\substack{t, s \in [0, 1] \\ 0 < |t-s| \leq \frac{1}{n}}} \frac{|H^{a \rightarrow b}(t) - H^{a \rightarrow b}(s)|}{|t-s|^\gamma} < \infty \right\} \right) = 1.$$

The remainder of this paper is structured as follows. In Section 2, we introduce some basic facts related to Bessel processes and Bessel bridges, and we prove the results for the distribution of the maximal value of the Bessel bridge. Section 2 is also devoted to proving some inequalities that are used in this paper. In Section 3, we prove Theorem 1, which gives the construction of the Bessel house-moving as the weak limit of conditioned BES(δ) bridges. In Section 4, we prove the decomposition formula for the distribution of the Bessel house-moving (Theorem 2) and use this formula to prove some results, including Proposition 1.1. Section 5 is devoted to proving the absolute continuity of the distribution of the BES(δ) house-moving with respect to the BES(δ) process (Theorem 3). In Section 6, we prove Proposition 1.2 by using the first hitting time of the Bessel process, thus giving us the characterization of the Bessel house-moving. We show sample path properties of the Bessel house-moving in Section 7 and Section 8. Section 7 is devoted to proving the regularity of the sample path of the BES(δ) house-moving (Proposition 1.3). In Section 8, we show that the BES(3) house-moving has the space-time reversal property, and we demonstrate numerical examples for $H^{0 \rightarrow b}$.

2. Preliminaries

2.1. Bessel process and Bessel bridge

The BES(δ) process is a one-dimensional diffusion generated by $\mathcal{L}_\delta := \frac{1}{2} \frac{d^2}{dx^2} + \frac{\delta-1}{2x} \frac{d}{dx}$. Note that the point 0 is an entrance boundary for $\delta \geq 2$ ($\nu \geq 0$) and a regular boundary for $0 < \delta < 2$ ($-1 < \nu < 0$). In the case that $0 < \delta < 2$, we impose the reflecting boundary condition at 0.

In addition, for $0 \leq a < b$, the BES(δ) bridge from a to b on $[0, 1]$ is defined by conditioning the BES(δ) process from a , $R^a = \{R^a(t)\}_{t \geq 0}$, on $R^a(1) = b$.

For $t > 0$ and $x, y \in (0, \infty)$, we set

$$n_t(x) := \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right), \quad A_t^{(\nu)}(x, y) := n_t(x)n_t(y)I_\nu\left(\frac{xy}{t}\right).$$

Let $a, b \geq 0$. For $0 < s < t$ and $x, y > 0$, we have the transition densities of R^a ([9, Chapter XI]):

$$\begin{aligned} P(R^a(t) \in dy) &= 2\pi y \left(\frac{y}{a}\right)^\nu A_t^{(\nu)}(a, y) dy, \\ P(R^a(t) \in dy \mid R^a(s) = x) &= 2\pi y \left(\frac{y}{x}\right)^\nu A_{t-s}^{(\nu)}(x, y) dy. \end{aligned}$$

For $0 < s < t < 1$ and $x, y > 0$, we have the transition densities of the BES(δ) bridge $r^{a \rightarrow b}$ on $[0, 1]$ ([9, Chapter XI]):

$$\begin{aligned} (1) \quad P\left(r^{a \rightarrow b}(t) \in dy\right) &= \frac{P(R^a(t) \in dy) P(R^y(1-t) \in db)}{P(R^a(1) \in db)} \\ &= \frac{2\pi y A_t^{(\nu)}(a, y) A_{1-t}^{(\nu)}(y, b)}{A_1^{(\nu)}(a, b)} dy, \end{aligned}$$

$$\begin{aligned} (2) \quad P\left(r^{a \rightarrow b}(t) \in dy \mid r^{a \rightarrow b}(s) = x\right) &= \frac{P(R^x(t-s) \in dy) P(R^y(1-t) \in db)}{P(R^x(1-s) \in db)} \\ &= \frac{2\pi y A_{t-s}^{(\nu)}(x, y) A_{1-t}^{(\nu)}(y, b)}{A_{1-s}^{(\nu)}(x, b)} dy. \end{aligned}$$

In the next lemma, we express the joint densities of the Bessel bridge and the maximal value of the Bessel process by the maximal values of the Bessel bridge.

LEMMA 2.1. *Let $c \geq 0$ and $0 \leq a, b \leq c$. For $0 < s < t < 1$ and $0 \leq x, y \leq c$, we have*

$$\begin{aligned} &P\left(r^{a \rightarrow b}(t) \in dy, M(r^{a \rightarrow b}) \leq c\right) \\ (3) \quad &= P\left(M_{[0,t]}(r_{[0,t]}^{a \rightarrow y}) \leq c\right) P\left(M_{[t,1]}(r_{[t,1]}^{y \rightarrow b}) \leq c\right) P\left(r^{a \rightarrow b}(t) \in dy\right), \end{aligned}$$

$$\begin{aligned}
& P\left(r^{a \rightarrow b}(t) \in dy, r^{a \rightarrow b}(s) \in dx, M(r^{a \rightarrow b}) \leq c\right) \\
(4) \quad & = P\left(M_{[0,s]}(r_{[0,s]}^{a \rightarrow x}) \leq c\right) P\left(M_{[s,t]}(r_{[s,t]}^{x \rightarrow y}) \leq c\right) P\left(M_{[t,1]}(r_{[t,1]}^{y \rightarrow b}) \leq c\right) \\
& \quad \times P\left(r^{a \rightarrow b}(t) \in dy, r^{a \rightarrow b}(s) \in dx\right).
\end{aligned}$$

PROOF. First, we prove (3). By the Markov property of R^a , we have

$$\begin{aligned}
& P\left(r^{a \rightarrow b}(t) \in dy, M(r^{a \rightarrow b}) \leq c\right) \\
& = \frac{P\left(R^a(t) \in dy, M(R^a) \leq c, R^a(1) \in db\right)}{P\left(R^a(1) \in db\right)} \\
(5) \quad & = \frac{P\left(R^y(1-t) \in db, M_{1-t}(R^y) \leq c\right) \times P\left(R^a(t) \in dy, M_t(R^a) \leq c\right)}{P\left(R^a(1) \in db\right)}.
\end{aligned}$$

Therefore, because

$$\begin{aligned}
& P\left(R^y(1-t) \in db, M_{1-t}(R^y) \leq c\right) \\
& = P\left(M_{[t,1]}(r_{[t,1]}^{y \rightarrow b}) \leq c\right) P\left(R^y(1-t) \in db\right), \\
& P\left(R^a(t) \in dy, M_t(R^a) \leq c\right) = P\left(M_{[0,t]}(r_{[0,t]}^{a \rightarrow y}) \leq c\right) P\left(R^a(t) \in dy\right),
\end{aligned}$$

it follows from (1) and (5) that

$$\begin{aligned}
& P\left(r^{a \rightarrow b}(t) \in dy, M(r^{a \rightarrow b}) \leq c\right) \\
& = P\left(M_{[0,t]}(r_{[0,t]}^{a \rightarrow y}) \leq c\right) \\
& \quad \times P\left(M_{[t,1]}(r_{[t,1]}^{y \rightarrow b}) \leq c\right) \frac{P\left(R^a(t) \in dy\right) P\left(R^y(1-t) \in db\right)}{P\left(R^a(1) \in db\right)} \\
& = P\left(M_{[0,t]}(r_{[0,t]}^{a \rightarrow y}) \leq c\right) P\left(M_{[t,1]}(r_{[t,1]}^{y \rightarrow b}) \leq c\right) P\left(r^{a \rightarrow b}(t) \in dy\right),
\end{aligned}$$

which completes the proof. In a similar manner to the proof of (3), we can obtain (4). \square

2.2. Distribution of the maximal value of the Bessel bridge

In this subsection, we prove the results for the distribution of the maximal value of the Bessel bridge used in this paper.

LEMMA 2.2. *Let $X > 0$. There exist some $\tilde{C}_\nu > 0$ and $N_\nu \in \mathbb{N}$ such that*

$$\begin{aligned} \frac{n\pi}{2} < j_{\nu,n} < 2n\pi, \quad \left| \frac{1}{J_{\nu+1}(j_{\nu,n})} \right| \leq \pi\sqrt{n}, \\ \left| \frac{J_\nu(Xj_{\nu,n})}{J_{\nu+1}(j_{\nu,n})} \right| \vee \left| \frac{J_{\nu+1}(Xj_{\nu,n})}{J_{\nu+1}(j_{\nu,n})} \right| \leq \tilde{C}_\nu \frac{(1+X\pi)^{\frac{1}{2}}}{X}, \quad (n > N_\nu). \end{aligned}$$

PROOF. According to (60), we can find a natural number $N_\nu \geq 2$ that satisfies

$$\frac{n\pi}{2} < j_{\nu,n} < 2n\pi, \quad \left| \frac{1}{J_{\nu+1}(j_{\nu,n})} \right| \leq \pi\sqrt{n} \quad (n \geq N_\nu).$$

In addition, by (58), for $n \geq N_\nu$, the following inequalities hold:

$$\begin{aligned} \left| \frac{J_\nu(Xj_{\nu,n})}{J_{\nu+1}(j_{\nu,n})} \right| &\leq C_\nu \frac{(Xj_{\nu,n})^\nu}{(1+Xj_{\nu,n})^{\nu+\frac{1}{2}}} \pi\sqrt{n} \\ &\leq C_\nu \frac{(2Xn\pi)^\nu}{(1+\frac{Xn\pi}{2})^{\nu+\frac{1}{2}}} \pi\sqrt{n} \leq 2^{2\nu+1/2} C_\nu \frac{(1+X\pi)^{1/2}}{X}, \\ \left| \frac{J_{\nu+1}(Xj_{\nu,n})}{J_{\nu+1}(j_{\nu,n})} \right| &\leq C_{\nu+1} \frac{(Xj_{\nu,n})^{\nu+1}}{(1+Xj_{\nu,n})^{\nu+\frac{3}{2}}} \pi\sqrt{n} \\ &\leq C_{\nu+1} \frac{(2Xn\pi)^{\nu+1}}{(1+\frac{Xn\pi}{2})^{\nu+\frac{3}{2}}} \pi\sqrt{n} \leq 2^{2\nu+5/2} C_{\nu+1} \frac{(1+X\pi)^{1/2}}{X}. \quad \square \end{aligned}$$

THEOREM 4 ([8] (20)). *Let $0 \leq x, y < c$, and $t > 0$, and let $p(t; x, y)$ be the symmetric transition density of a regular one-dimensional diffusion on $[0, \infty)$ $R = \{R(t)\}_{t \geq 0}$ relative to its speed measure. In addition, let $r_{[0,t]}^{x \rightarrow y} = \{r_{[0,t]}^{x \rightarrow y}(s)\}_{s \in [0,t]}$ denote an R -bridge of length t from x to y . Moreover, let φ_λ^\uparrow and $\varphi_\lambda^\downarrow$ denote the increasing and decreasing solutions of $Au = \lambda u$ for A the infinitesimal generator of R , normalized so that*

$$(6) \quad \int_0^\infty e^{-\lambda t} p(t; x, y) dt = \varphi_\lambda^\uparrow(x) \varphi_\lambda^\downarrow(y), \quad 0 \leq x \leq y, \quad \lambda > 0.$$

Then, we have

$$(7) \quad \int_0^\infty e^{-\lambda t} P\left(M_{[0,t]}(r_{[0,t]}^{x \rightarrow y}) > c\right) p(t; x, y) dt = \varphi_\lambda^\uparrow(y) \varphi_\lambda^\downarrow(c) \frac{\varphi_\lambda^\uparrow(x)}{\varphi_\lambda^\uparrow(c)}.$$

REMARK 2.1. In the case of the BES(δ) process, φ_λ^\uparrow and $\varphi_\lambda^\downarrow$ in Theorem 4 are given as follows ([8, (23)]):

$$(8) \quad \varphi_\lambda^\uparrow(x) = I_\nu\left(\sqrt{2\lambda}x\right) x^{-\nu}, \quad \varphi_\lambda^\downarrow(x) = K_\nu\left(\sqrt{2\lambda}x\right) x^{-\nu}, \quad x \geq 0, \lambda > 0.$$

THEOREM 5. Let $c > 0$ and $0 \leq s < t$. For $x, y \in (0, c)$, we have

$$\begin{aligned} & P\left(M_{[s,t]}(r_{[s,t]}^{x \rightarrow y}) \leq c\right) \\ &= \frac{1}{\pi A_{t-s}^{(\nu)}(x, y)} \sum_{n=1}^\infty \frac{J_\nu(xj_{\nu,n}/c) J_\nu(yj_{\nu,n}/c)}{c^2 J_{\nu+1}^2(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2c^2}(t-s)\right). \end{aligned}$$

In addition, for $y \in [0, c)$, we have

$$\begin{aligned} & P\left(M_{[s,t]}(r_{[s,t]}^{0 \rightarrow y}) \leq c\right) \\ &= P\left(M_{[s,t]}(r_{[s,t]}^{y \rightarrow 0}) \leq c\right) \\ &= \frac{2(t-s)^{\nu+\frac{1}{2}}}{\sqrt{2\pi}n_{t-s}(y)} \sum_{n=1}^\infty \left(\frac{j_{\nu,n}}{cy}\right)^\nu \frac{J_\nu(yj_{\nu,n}/c)}{c^2 J_{\nu+1}^2(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2c^2}(t-s)\right). \end{aligned}$$

PROOF. The Laplace transform for a function f is denoted by $L(f)$:

$$L(f)(\lambda) := \int_0^\infty e^{-\lambda s} f(s) ds \quad \lambda > 0.$$

For $0 \leq x \leq y < c$, by (6), (7), and (8), we have

$$\begin{aligned} & L\left(P\left(M_{[0,\cdot]}(r_{[0,\cdot]}^{x \rightarrow y}) \leq c\right) p(\cdot; x, y)\right)(\lambda) \\ &= L\left(\left(1 - P\left(M_{[0,\cdot]}(r_{[0,\cdot]}^{x \rightarrow y}) > c\right)\right) p(\cdot; x, y)\right)(\lambda) \\ &= L(p(\cdot; x, y))(\lambda) - L\left(P\left(M_{[0,\cdot]}(r_{[0,\cdot]}^{x \rightarrow y}) > c\right) p(\cdot; x, y)\right)(\lambda) \end{aligned}$$

$$\begin{aligned}
&= \varphi_\lambda^\uparrow(x) \varphi_\lambda^\downarrow(y) - \varphi_\lambda^\uparrow(y) \varphi_\lambda^\downarrow(x) \frac{\varphi_\lambda^\uparrow(x)}{\varphi_\lambda^\uparrow(c)} \\
&= (xy)^{-\nu} I_\nu(\sqrt{2\lambda}x) \frac{K_\nu(\sqrt{2\lambda}y) I_\nu(\sqrt{2\lambda}c) - I_\nu(\sqrt{2\lambda}y) K_\nu(\sqrt{2\lambda}c)}{I_\nu(\sqrt{2\lambda}c)}, \quad \lambda > 0.
\end{aligned}$$

Here, note that

$$\begin{aligned}
(9) \quad & \frac{I_\nu(XC)}{I_\nu(C)} (I_\nu(C) K_\nu(YC) - I_\nu(YC) K_\nu(C)) \\
&= 2 \sum_{n=1}^{\infty} \frac{J_\nu(Xj_{\nu,n}) J_\nu(Yj_{\nu,n})}{J_{\nu+1}^2(j_{\nu,n})(C^2 + j_{\nu,n}^2)}
\end{aligned}$$

holds for $0 \leq X \leq Y \leq 1$ and $C > 0$ ([8, (161)]). Since we apply this equality for $C = \sqrt{2\lambda}c$, $X = x/c$, and $Y = y/c$, it follows that

$$\begin{aligned}
&L \left(P \left(M_{[0,1]}(r_{[0,\cdot]}^{x \rightarrow y}) \leq c \right) p(\cdot; x, y) \right) (\lambda) \\
&= 2(xy)^{-\nu} \sum_{n=1}^{\infty} \frac{J_\nu(xj_{\nu,n}/c) J_\nu(yj_{\nu,n}/c)}{J_{\nu+1}^2(j_{\nu,n})(2\lambda c^2 + j_{\nu,n}^2)} \\
&= (xy)^{-\nu} \sum_{n=1}^{\infty} \frac{J_\nu(xj_{\nu,n}/c) J_\nu(yj_{\nu,n}/c)}{c^2 J_{\nu+1}^2(j_{\nu,n})} \\
&\quad \times \int_0^\infty \exp \left(- \left(\lambda + \frac{j_{\nu,n}^2}{2c^2} \right) r \right) dr \\
&= \sum_{n=1}^{\infty} \int_0^\infty e^{-\lambda r} (xy)^{-\nu} \frac{J_\nu(xj_{\nu,n}/c) J_\nu(yj_{\nu,n}/c)}{c^2 J_{\nu+1}^2(j_{\nu,n})} \\
(10) \quad &\quad \times \exp \left(- \frac{j_{\nu,n}^2}{2c^2} r \right) dr, \quad \lambda > 0.
\end{aligned}$$

For n , we set

$$f_n(r) := \frac{J_\nu(xj_{\nu,n}/c) J_\nu(yj_{\nu,n}/c)}{c^2 J_{\nu+1}^2(j_{\nu,n})} \exp \left(- \left(\lambda + \frac{j_{\nu,n}^2}{2c^2} \right) r \right), \quad r \geq 0.$$

Then, by Lemma 2.2 and (60), there exist some $\tilde{C}_\nu > 0$ and $N_\nu \in \mathbb{N}$ such that

$$|f_n(r)| \leq \tilde{C}_\nu^2 \frac{\sqrt{(1 + \frac{x\pi}{c})(1 + \frac{y\pi}{c})}}{xy} \exp \left(- \frac{(n\pi)^2}{8c^2} r \right)$$

$$\leq \tilde{C}_\nu^2 \frac{1 + \pi}{xy} \exp\left(-\frac{(n\pi)^2}{8c^2}r\right), \quad n > N_\nu.$$

Therefore, we can see that

$$\sum_{n=N_\nu+1}^\infty \int_0^\infty |f_n(r)|dr \leq \tilde{C}_\nu^2 \frac{1 + \pi}{xy} \sum_{n=N_\nu+1}^\infty \frac{8c^2}{(n\pi)^2} < \infty$$

holds and we can integrate term by term in (10). Hence, it follows that

$$\begin{aligned} &L\left(P\left(M_{[0,\cdot]}(r_{[0,\cdot]}^{x\rightarrow y}) \leq c\right) p(\cdot; x, y)\right)(\lambda) \\ &= \int_0^\infty e^{-\lambda r} (xy)^{-\nu} \sum_{n=1}^\infty \frac{J_\nu(xj_{\nu,n}/c) J_\nu(yj_{\nu,n}/c)}{c^2 J_{\nu+1}^2(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2c^2}r\right) dr \\ &= L\left((xy)^{-\nu} \sum_{n=1}^\infty \frac{J_\nu(xj_{\nu,n}/c) J_\nu(yj_{\nu,n}/c)}{c^2 J_{\nu+1}^2(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2c^2}(\cdot)\right)\right)(\lambda) \quad (\lambda > 0). \end{aligned}$$

By the inverse Laplace transform of this identity, we obtain the following expression:

$$\begin{aligned} &P\left(M_{[0,t]}(r_{[0,t]}^{x\rightarrow y}) \leq c\right) \\ &= \frac{(xy)^{-\nu}}{\pi (xy)^{-\nu} A_t^{(\nu)}(x, y)} \sum_{n=1}^\infty \frac{J_\nu(xj_{\nu,n}/c) J_\nu(yj_{\nu,n}/c)}{c^2 J_{\nu+1}^2(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2c^2}t\right) \\ &= \frac{1}{\pi A_t^{(\nu)}(x, y)} \sum_{n=1}^\infty \frac{J_\nu(xj_{\nu,n}/c) J_\nu(yj_{\nu,n}/c)}{c^2 J_{\nu+1}^2(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2c^2}t\right). \end{aligned}$$

Because the right-hand side of (9) is symmetric for X and Y , we can see that this result holds for $0 < y \leq x < c$.

Finally, for $0 \leq y < c$, we can calculate the following:

$$\begin{aligned} &P\left(M_{[0,t]}(r_{[0,t]}^{0\rightarrow y}) \leq c\right) \\ &= \frac{2t^{\nu+\frac{1}{2}}}{\sqrt{2\pi}n_t(y)} \sum_{n=1}^\infty \left(\frac{j_{\nu,n}}{cy}\right)^\nu \frac{J_\nu(yj_{\nu,n}/c)}{c^2 J_{\nu+1}^2(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2c^2}t\right). \quad \square \end{aligned}$$

PROPOSITION 2.1. Let $\eta > 0$ and $0 \leq s < t$. For $x, y \in (0, \eta)$, we have

$$\begin{aligned} & \frac{\partial}{\partial \eta} P \left(M_{[s,t]}(r_{[s,t]}^{x \rightarrow y}) \leq \eta \right) \\ &= \frac{1}{\pi A_{t-s}^{(\nu)}(x, y)} \sum_{n=1}^{\infty} \frac{1}{J_{\nu+1}^2(j_{\nu,n})} \exp \left(-\frac{j_{\nu,n}^2}{2\eta^2}(t-s) \right) \\ & \quad \times \left\{ \left(-\frac{2\nu+2}{\eta^3} + \frac{j_{\nu,n}^2}{\eta^5}(t-s) \right) J_{\nu}(xj_{\nu,n}/\eta) J_{\nu}(yj_{\nu,n}/\eta) \right. \\ & \quad + \frac{xj_{\nu,n}}{\eta^4} J_{\nu+1}(xj_{\nu,n}/\eta) J_{\nu}(yj_{\nu,n}/\eta) \\ & \quad \left. + \frac{yj_{\nu,n}}{\eta^4} J_{\nu+1}(yj_{\nu,n}/\eta) J_{\nu}(xj_{\nu,n}/\eta) \right\}, \\ & \frac{\partial}{\partial \eta} P \left(M_{[s,t]}(r_{[s,t]}^{0 \rightarrow y}) \leq \eta \right) \\ &= \frac{2(t-s)^{\nu+\frac{1}{2}}}{\sqrt{2\pi n_{t-s}}(y)} \sum_{n=1}^{\infty} y^{-\nu} \frac{(j_{\nu,n}/\eta)^{\nu+2}}{J_{\nu+1}^2(j_{\nu,n})} \frac{1}{\eta^3} \exp \left(-\frac{j_{\nu,n}^2}{2\eta^2}(t-s) \right) \\ & \quad \times \left\{ \left(t-s - \frac{2\eta^2(\nu+1)}{j_{\nu,n}^2} \right) J_{\nu}(yj_{\nu,n}/\eta) \right. \\ & \quad \left. + \frac{y\eta}{j_{\nu,n}} J_{\nu+1}(yj_{\nu,n}/\eta) \right\}. \end{aligned}$$

PROOF. Let $\eta > 0$ and let $0 < x, y < \eta$ be fixed. For n , we set

$$\begin{aligned} f_n(\eta, x, y) &= \frac{1}{J_{\nu+1}^2(j_{\nu,n})} \exp \left(-\frac{j_{\nu,n}^2}{2\eta^2}(t-s) \right) \\ & \quad \times \left\{ \left(-\frac{2\nu+2}{\eta^3} + \frac{j_{\nu,n}^2}{\eta^5}(t-s) \right) J_{\nu}(xj_{\nu,n}/\eta) J_{\nu}(yj_{\nu,n}/\eta) \right. \\ & \quad + \frac{xj_{\nu,n}}{\eta^4} J_{\nu+1}(xj_{\nu,n}/\eta) J_{\nu}(yj_{\nu,n}/\eta) \\ & \quad \left. + \frac{yj_{\nu,n}}{\eta^4} J_{\nu+1}(yj_{\nu,n}/\eta) J_{\nu}(xj_{\nu,n}/\eta) \right\}. \end{aligned}$$

By Lemma 2.2, there exist some $\tilde{C}_\nu > 0$ and $N_\nu \in \mathbb{N}$ such that

$$\begin{aligned}
& |f_n(\eta, x, y)| \\
& \leq \exp\left(-\frac{\pi^2(t-s)}{8\eta^2}n^2\right) \\
& \quad \times \left\{ \left(\frac{2\nu+2}{\eta^3} + \frac{(2n\pi)^2}{\eta^5}(t-s) \right) \eta^2 \tilde{C}_\nu^2 \frac{\left(\left(1 + \frac{x\pi}{\eta}\right) \left(1 + \frac{y\pi}{\eta}\right) \right)^{1/2}}{xy} \right. \\
& \quad \quad + \frac{x(2n\pi)}{\eta^4} \eta \tilde{C}_\nu \frac{\left(1 + \frac{x\pi}{\eta}\right)^{1/2}}{x} \cdot \eta \tilde{C}_\nu \frac{\left(1 + \frac{y\pi}{\eta}\right)^{1/2}}{y} \\
& \quad \quad \left. + \frac{y(2n\pi)}{\eta^4} \eta \tilde{C}_\nu \frac{\left(1 + \frac{y\pi}{\eta}\right)^{1/2}}{y} \cdot \eta \tilde{C}_\nu \frac{\left(1 + \frac{x\pi}{\eta}\right)^{1/2}}{x} \right\} \\
& \leq \exp\left(-\frac{\pi^2(t-s)}{8\eta^2}n^2\right) \\
& \quad \times \left\{ \frac{2\nu+2}{\eta} + \frac{(2n\pi)^2}{\eta^3}(t-s) + \frac{2\pi(x+y)}{\eta^2}n \right\} \frac{\tilde{C}_\nu^2(1+\pi)}{xy}, \quad n > N_\nu.
\end{aligned}$$

Therefore, we can differentiate term by term the first identity of Theorem 5 in some neighborhood of η . Similarly, for n , we set

$$\begin{aligned}
f_n(\eta, y) &= \frac{(j_{\nu,n}/\eta)^{\nu+2}}{J_{\nu+1}^2(j_{\nu,n})} \frac{1}{\eta^3} \exp\left(-\frac{j_{\nu,n}^2}{2\eta^2}(t-s)\right) \\
& \quad \times \left\{ \left(t - s - \frac{2\eta^2(\nu+1)}{j_{\nu,n}^2} \right) J_\nu(yj_{\nu,n}/\eta) + \frac{y\eta}{j_{\nu,n}} J_{\nu+1}(yj_{\nu,n}/\eta) \right\}.
\end{aligned}$$

By Lemma 2.2, there exist some $\tilde{C}_\nu > 0$ and $N_\nu \in \mathbb{N}$ such that

$$\begin{aligned}
|f_n(\eta, y)| &\leq \frac{\sqrt{\pi(1+\pi)}}{y\eta^{\nu+4}\sqrt{2}} \tilde{C}_\nu (2n\pi)^{\nu+\frac{5}{2}} \\
& \quad \times \left\{ t - s + \frac{8\eta^2(\nu+1)}{(n\pi)^2} + \frac{2y\eta}{n\pi} \right\} \exp\left(-\frac{\pi^2(t-s)}{8\eta^2}n^2\right), \quad n > N_\nu.
\end{aligned}$$

Therefore, we can differentiate term by term the second identity of Theorem 5 in some neighborhood of η . \square

According to Proposition 2.1 and Lebesgue’s dominated convergence theorem, we can obtain the next corollary.

COROLLARY 1. *Let $b > 0$. For $0 \leq s < t$ and $y \in (0, b)$, we have*

$$\begin{aligned} & \lim_{\eta \downarrow b} \frac{\partial}{\partial \eta} P \left(M_{[s,t]}(r^{y \rightarrow b}) \leq \eta \right) \\ &= \frac{1}{\pi A_{t-s}^{(\nu)}(y, b)} \sum_{n=1}^{\infty} \frac{j_{\nu,n} J_{\nu}(yj_{\nu,n}/b)}{b^3 J_{\nu+1}(j_{\nu,n})} \exp \left(-\frac{j_{\nu,n}^2}{2b^2}(t-s) \right), \\ & \lim_{\eta \downarrow b} \frac{\partial}{\partial \eta} P \left(M(r_{[s,t]}^{0 \rightarrow b}) \leq \eta \right) \\ &= \frac{2(t-s)^{\nu+\frac{1}{2}}}{\sqrt{2\pi} n_{t-s}(b)} \sum_{n=1}^{\infty} \frac{j_{\nu,n}^{\nu+1}}{b^{2\nu+3} J_{\nu+1}(j_{\nu,n})} \exp \left(-\frac{j_{\nu,n}^2}{2b^2}(t-s) \right). \end{aligned}$$

2.3. Some inequalities

We prepare the following inequalities:

LEMMA 2.3. *Let $0 \leq a < b$ and $\eta \in (0, 1]$. There exists some $C_{\nu,b} > 0$ such that*

$$(11) \quad q_1^{(b+\eta)}(s, x, t, y) \leq \frac{C_{\nu,b}}{(t-s)^{\nu+1}} n_{t-s}(y-x),$$

$$0 \leq s < t \leq 1, \quad x, y \in [0, b+\eta),$$

$$(12) \quad q_1^{(b+\eta)}(r, z, 1, b) \leq \frac{C_{\nu,b}}{(1-r)^{\nu+1}} \left(1 \wedge \frac{2\eta(b+\eta)}{1-r} \right) n_{1-r}(z-b),$$

$$0 < r < 1, \quad z \in (0, b+\eta).$$

PROOF. First, we prove inequality (11). By (59), there exists some $C_{\nu} > 0$ such that

$$\left(\frac{xy}{t-s} \right)^{-\nu} I_{\nu} \left(\frac{xy}{t-s} \right) \leq \frac{C_{\nu}}{\left(1 + \frac{xy}{t-s} \right)^{\nu+\frac{1}{2}}} \exp \left(\frac{xy}{t-s} \right).$$

Thus, by this inequality, it follows that

$$\begin{aligned} q_1^{(b+\eta)}(s, x, t, y) &\leq \frac{P(R^x(t-s) \in dy)}{dy} \\ &\leq 2\pi y^{1+\nu} \left(\frac{y}{t-s}\right)^\nu n_{t-s}(x)n_{t-s}(y) \frac{C_\nu}{\left(1 + \frac{xy}{t-s}\right)^{\nu+\frac{1}{2}}} \exp\left(\frac{xy}{t-s}\right) \\ &\leq \frac{\widehat{C}_{\nu,b}}{(t-s)^{\nu+1}} n_{t-s}(y-x), \end{aligned}$$

where $\widehat{C}_{\nu,b} := \sqrt{2\pi}C_\nu(b+1)^{2\nu+1}\sqrt{1+(b+1)^2}$. Next, we prove inequality (12). According to [6], there exists some $\widetilde{C}_\nu > 0$ such that

$$\begin{aligned} 2(xy)^{-\nu} \sum_{n=1}^\infty \frac{J_\nu(j_{\nu,n}x)J_\nu(j_{\nu,n}y)}{J_{\nu+1}^2(j_{\nu,n})} \exp(-j_{\nu,n}^2 t) \\ \leq \widetilde{C}_\nu \frac{(1+t)^{\nu+2}}{(t+xy)^{\nu+1/2}} \left(1 \wedge \frac{(1-x)(1-y)}{t}\right) \frac{1}{\sqrt{t}} \exp\left(-\frac{(x-y)^2}{4t} - j_{\nu,1}^2 t\right), \\ x, y \in (0, 1), t > 0. \end{aligned}$$

Using this inequality and Theorem 5, we can obtain the following estimate:

$$\begin{aligned} q_1^{(b+\eta)}(r, z, 1, b) &= \frac{P(R^z(1-r) \in db)}{db} P\left(M_{[r,1]}(r \overset{z}{\rightarrow} b) \leq b + \eta\right) \\ &= \frac{b^{2\nu+1}}{(b+\eta)^{2\nu+2}} 2 \left(\frac{z}{b+\eta} \frac{b}{b+\eta}\right)^{-\nu} \\ &\quad \times \sum_{n=1}^\infty \frac{J_\nu(zj_{\nu,n}/(b+\eta)) J_\nu(bj_{\nu,n}/(b+\eta))}{J_{\nu+1}^2(j_{\nu,n})} \exp\left(-j_{\nu,n}^2 \frac{1-r}{2(b+\eta)^2}\right) \\ &\leq \widetilde{C}_\nu \frac{b^{2\nu+1}}{(b+\eta)^{2\nu+2}} \left(1 + \frac{1-r}{2(b+\eta)^2}\right)^{\nu+2} \\ &\quad \times \frac{\left(\frac{(1-r)+2bz}{2(b+\eta)^2}\right)^{1/2}}{\left(\frac{(1-r)+2bz}{2(b+\eta)^2}\right)^{\nu+1}} \left(1 \wedge \frac{2\eta(b+\eta-z)}{1-r}\right) \\ &\quad \times \exp\left(-j_{\nu,1}^2 \frac{1-r}{2(b+\eta)^2}\right) 2\sqrt{\pi}(b+\eta)n_{1-r}(z-b) \end{aligned}$$

$$\begin{aligned} &\leq \tilde{C}_\nu \frac{b^{2\nu+1}}{(b+\eta)^{2\nu+2}} \left(\frac{2(b+1)^2+1}{2(b+\eta)^2} \right)^{\nu+2} \\ &\quad \times \frac{\left(\frac{1+2(b+1)^2}{2(b+\eta)^2} \right)^{1/2}}{\left(\frac{1-r}{2(b+\eta)^2} \right)^{\nu+1}} \left(1 \wedge \frac{2\eta(b+\eta)}{1-r} \right) 2\sqrt{\pi}(b+\eta)n_{1-r}(z-b) \\ &\leq \frac{\tilde{C}_{\nu,b}}{(1-r)^{\nu+1}} \left(1 \wedge \frac{2\eta(b+\eta)}{1-r} \right) n_{1-r}(z-b), \end{aligned}$$

where $\tilde{C}_{\nu,b} := \tilde{C}_\nu \sqrt{\frac{\pi}{2}} \frac{(1+2(b+1)^2)^{\nu+5/2}}{b^3}$. Since we set $C_{\nu,b} := \widehat{C}_{\nu,b} \vee \tilde{C}_{\nu,b}$, we can obtain our assertions. \square

LEMMA 2.4. *Let $b > 0$. For $0 < t \leq 1$ and $y \in [0, b)$, we have*

$$q_2^{(b)}(t, y) > 0.$$

PROOF. According to [10, Theorem 3.3], for all $x \in [0, 1)$ and $t > 0$, there exists a constant $C_\nu > 0$ such that

$$\begin{aligned} &x^{-\nu} \sum_{n=1}^{\infty} \frac{j_{\nu,n} J_\nu(j_{\nu,n}x)}{J_{\nu+1}(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2}t\right) \\ &\geq C_\nu \frac{(1-x)(1+t)^{\nu+2}}{(x+t)^{\nu+\frac{1}{2}}t^{\frac{3}{2}}} \exp\left(-\frac{(1-x)^2}{2t} - \frac{1}{2}j_{\nu,1}^2t\right). \end{aligned}$$

Hence, by Corollary 1, we can prove the assertion as follows:

$$\begin{aligned} q_2^{(b)}(t, y) &= 2 \left(\frac{b}{y} \right)^\nu \sum_{n=1}^{\infty} \frac{j_{\nu,n} J_\nu(yj_{\nu,n}/b)}{b^2 J_{\nu+1}(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2b^2}t\right) \\ &\geq C_\nu \frac{2}{b^2} \frac{(1-\frac{y}{b})(1+\frac{t}{b^2})^{\nu+2}}{(\frac{y}{b} + \frac{t}{b^2})^{\nu+\frac{1}{2}}(\frac{t}{b^2})^{\frac{3}{2}}} \exp\left(-\frac{(b-y)^2}{2t} - \frac{j_{\nu,1}^2}{2b^2}t\right) > 0. \quad \square \end{aligned}$$

3. Proof of Theorem 1

In this section, we prove Theorem 1, which gives the construction of the Bessel house-moving as the weak limit of the conditioned BES(δ) bridges.

LEMMA 3.1. *Let $0 \leq a < b$ and $\eta > 0$. For $0 < s < t < 1$ and $x, y \in (0, b + \eta)$, we have*

$$(13) \quad \begin{aligned} & P\left(r^{a \rightarrow b}|_{K^-(b+\eta)}(t) \in dy\right) \\ &= \frac{q_1^{(b+\eta)}(0, a, t, y)q_1^{(b+\eta)}(t, y, 1, b)}{q_1^{(b+\eta)}(0, a, 1, b)} dy, \end{aligned}$$

$$(14) \quad \begin{aligned} & P\left(r^{a \rightarrow b}|_{K^-(b+\eta)}(t) \in dy \mid r^{a \rightarrow b}|_{K^-(b+\eta)}(s) = x\right) \\ &= \frac{q_1^{(b+\eta)}(s, x, t, y)q_1^{(b+\eta)}(t, y, 1, b)}{q_1^{(b+\eta)}(s, x, 1, b)} dy. \end{aligned}$$

PROOF. By Lemma 2.1, we obtain

$$(15) \quad \begin{aligned} & P\left(r^{a \rightarrow b}|_{K^-(b+\eta)}(u) \in dy\right) \\ &= \frac{P\left(r^{a \rightarrow b}(u) \in dy, M(r^{a \rightarrow b}) \leq b + \eta\right)}{P\left(M(r^{a \rightarrow b}) \leq b + \eta\right)} \\ &= \frac{P\left(M_{[0,u]}(r^{a \rightarrow y}) \leq b + \eta\right) P\left(M_{[u,1]}(r^{y \rightarrow b}) \leq b + \eta\right)}{P\left(M(r^{a \rightarrow b}) \leq b + \eta\right)} \\ &\quad \times P\left(r^{a \rightarrow b}(u) \in dy\right), \quad 0 < u < 1. \end{aligned}$$

It holds from (15) and (1) that

$$\begin{aligned} & P\left(r^{a \rightarrow b}|_{K^-(b+\eta)}(t) \in dy\right) \\ &= \frac{P\left(M_{[0,t]}(r^{a \rightarrow y}) \leq b + \eta\right) P\left(M_{[t,1]}(r^{y \rightarrow b}) \leq b + \eta\right)}{P\left(M(r^{a \rightarrow b}) \leq b + \eta\right)} \\ &\quad \times \frac{P\left(R^a(t) \in dy\right) P\left(R^y(1-t) \in db\right)}{P\left(R^a(1) \in db\right)} \\ &= \frac{q_1^{(b+\eta)}(0, a, t, y)q_1^{(b+\eta)}(t, y, 1, b)}{q_1^{(b+\eta)}(0, a, 1, b)} dy. \end{aligned}$$

Hence, (13) holds.

Next, we prove (14). By Lemma 2.1, we have

$$\begin{aligned}
 & P\left(r^{a \rightarrow b} |_{K^{-(b+\eta)}}(t) \in dy, r^{a \rightarrow b} |_{K^{-(b+\eta)}}(s) \in dx\right) \\
 &= \frac{P\left(r^{a \rightarrow b}(t) \in dy, r^{a \rightarrow b}(s) \in dx, M(r^{a \rightarrow b}) \leq b + \eta\right)}{P\left(M(r^{a \rightarrow b}) \leq b + \eta\right)} \\
 &= \frac{P\left(M_{[s,t]}(r_{[s,t]}^{x \rightarrow y}) \leq b + \eta\right) P\left(M_{[t,1]}(r_{[t,1]}^{y \rightarrow b}) \leq b + \eta\right) P\left(M_{[0,s]}(r_{[0,s]}^{a \rightarrow x}) \leq b + \eta\right)}{P\left(M(r^{a \rightarrow b}) \leq b + \eta\right)} \\
 (16) \quad & \times P\left(r^{a \rightarrow b}(t) \in dy, r^{a \rightarrow b}(s) \in dx\right).
 \end{aligned}$$

Therefore, combining (15), (16) and (2), we obtain

$$\begin{aligned}
 & P\left(r^{a \rightarrow b} |_{K^{-(b+\eta)}}(t) \in dy \mid r^{a \rightarrow b} |_{K^{-(b+\eta)}}(s) = x\right) \\
 &= \frac{P\left(r^{a \rightarrow b} |_{K^{-(b+\eta)}}(t) \in dy, r^{a \rightarrow b} |_{K^{-(b+\eta)}}(s) \in dx\right)}{P\left(r^{a \rightarrow b} |_{K^{-(b+\eta)}}(s) \in dx\right)} \\
 &= \frac{P\left(M_{[s,t]}(r_{[s,t]}^{x \rightarrow y}) \leq b + \eta\right) P\left(M_{[t,1]}(r_{[t,1]}^{y \rightarrow b}) \leq b + \eta\right)}{P\left(M_{[s,1]}(r_{[s,1]}^{x \rightarrow b}) \leq b + \eta\right)} \\
 & \quad \times P\left(r^{a \rightarrow b}(t) \in dy \mid r^{a \rightarrow b}(s) = x\right) \\
 &= \frac{P\left(M_{[s,t]}(r_{[s,t]}^{x \rightarrow y}) \leq b + \eta\right) P\left(M_{[t,1]}(r_{[t,1]}^{y \rightarrow b}) \leq b + \eta\right)}{P\left(M_{[s,1]}(r_{[s,1]}^{x \rightarrow b}) \leq b + \eta\right)} \\
 & \quad \times \frac{P(R^x(t-s) \in dy) P(R^y(1-t) \in db)}{P(R^x(1-s) \in db)} \\
 &= \frac{q_1^{(b+\eta)}(s, x, t, y) q_1^{(b+\eta)}(t, y, 1, b)}{q_1^{(b+\eta)}(s, x, 1, b)} dy.
 \end{aligned}$$

Hence, (14) holds. \square

PROPOSITION 3.1. *Let $0 \leq a < b$. For $0 < s < t < 1$ and $x, y \in (0, b)$, we have*

$$(17) \quad \lim_{\eta \downarrow 0} P\left(r^{a \rightarrow b} |_{K^{-(b+\eta)}}(t) \in dy\right) = \frac{q_1^{(b)}(0, a, t, y) q_2^{(b)}(1-t, y)}{q_2^{(b)}(1, a)} dy,$$

$$\begin{aligned}
 (18) \quad & \lim_{\eta \downarrow 0} P \left(r^{a \rightarrow b} |_{K^{-(b+\eta)}}(t) \in dy \mid r^{a \rightarrow b} |_{K^{-(b+\eta)}}(s) = x \right) \\
 &= \frac{q_1^{(b)}(s, x, t, y) q_2^{(b)}(1-t, y)}{q_2^{(b)}(1-s, x)} dy.
 \end{aligned}$$

PROOF. By Lemma 3.1 and L'Hôpital's rule, we obtain our assertion. \square

Let $b > 0$. For $0 \leq s < t \leq 1$ and $x, y \in [0, b]$, we define

$$(19) \quad h_b(s, x, t, y) := \frac{q_1^{(b)}(s, x, t, y) q_2^{(b)}(1-t, y)}{q_2^{(b)}(1-s, x)}.$$

PROPOSITION 3.2. *Let $b > 0$. For $0 \leq s < t \leq 1$ and $x \in [0, b]$, we have*

$$\int_0^b h_b(s, x, t, y) dy = 1.$$

PROOF. By (19), it suffices to show the following identity:

$$q_2^{(b)}(1-s, x) = \int_0^b q_1^{(b)}(s, x, t, y) q_2^{(b)}(1-t, y) dy.$$

Here, using Lemma 3.1, it holds that

$$(20) \quad \frac{q_1^{(b+\eta)}(s, x, 1, b)}{\eta} = \int_0^{b+\eta} q_1^{(b+\eta)}(s, x, t, y) \frac{q_1^{(b+\eta)}(t, y, 1, b)}{\eta} dy.$$

According to L'Hôpital's rule, we obtain

$$(21) \quad \lim_{\eta \downarrow 0} \frac{q_1^{(b+\eta)}(s, x, 1, b)}{\eta} = q_2^{(b)}(1-s, x).$$

On the other hand, by Lemma 2.3, for $\eta \in (0, 1)$ and $y \in (0, b + \eta)$, we have the following estimate:

$$\begin{aligned}
 & q_1^{(b+\eta)}(s, x, t, y) \frac{q_1^{(b+\eta)}(t, y, 1, b)}{\eta} \\
 & \leq \frac{1}{\eta} \frac{C_{\nu, b}}{(t-s)^{\nu+1}} n_{t-s}(y-x) \frac{C_{\nu, b}}{(1-t)^{\nu+1}} \left(1 \wedge \frac{2\eta(b+\eta)}{1-t} \right) n_{1-t}(y-b) \\
 (22) \quad & \leq \frac{C_{\nu, b}^2(b+1)}{\pi(t-s)^{\nu+3/2}(1-t)^{\nu+5/2}} < \infty.
 \end{aligned}$$

Again, using L'Hôpital's rule, it holds that

$$(23) \quad \lim_{\eta \downarrow 0} q_1^{(b+\eta)}(s, x, t, y) \frac{q_1^{(b+\eta)}(t, y, 1, b)}{\eta} = q_1^{(b)}(s, x, t, y) q_2^{(b)}(1 - t, y)$$

for $y \in (0, b)$. Therefore, by (22), (23), and Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned} \lim_{\eta \downarrow 0} \frac{1}{\eta} \int_0^{b+\eta} q_1^{(b+\eta)}(s, x, t, y) q_1^{(b+\eta)}(t, y, 1, b) dy \\ = \int_0^b q_1^{(b)}(s, x, t, y) q_2^{(b)}(1 - t, y) dy. \end{aligned}$$

By this equality and (21), taking the limit $\eta \downarrow 0$ in (20) allows us to prove the assertion. \square

The following proposition implies that $h_b(s, x, t, y)$ satisfies the Chapman–Kolmogorov identity.

PROPOSITION 3.3. *Let $b > 0$. For $0 < s < t < u < 1$ and $x, z \in (0, b)$, we have*

$$h_b(s, x, u, z) = \int_0^b h_b(s, x, t, y) h_b(t, y, u, z) dy.$$

PROOF. By (19), it suffices to show the following identity:

$$q_1^{(b)}(s, x, u, z) = \int_0^b q_1^{(b)}(s, x, t, y) q_1^{(b)}(t, y, u, z) dy.$$

According to Lemma 2.1 and (2), we can prove the assertion as follows:

$$\begin{aligned} & q_1^{(b)}(s, x, u, z) \\ &= \frac{P(R^x(u - s) \in dz)}{dz} \int_0^b P\left(r_{[s,u]}^{x \rightarrow z}(t) \in dy, M_{[s,u]}(r_{[s,u]}^{x \rightarrow z}) \leq b\right) \\ &= \frac{P(R^x(u - s) \in dz)}{dz} \\ &\quad \times \int_0^b P\left(M_{[s,t]}(r_{[s,t]}^{x \rightarrow y}) \leq b\right) P\left(M_{[t,u]}(r_{[t,u]}^{y \rightarrow z}) \leq b\right) P\left(r_{[s,u]}^{x \rightarrow z}(t) \in dy\right) \\ &= \int_0^b q_1^{(b)}(s, x, t, y) q_1^{(b)}(t, y, u, z) dy. \quad \square \end{aligned}$$

By Proposition 3.2 and Proposition 3.3, the right sides of (17) and (18) determine the continuous Markov process $H^{a \rightarrow b} = \{H^{a \rightarrow b}(t)\}_{t \in [0,1]}$. Then, by Proposition 3.1 and Lemma A.1, we obtain the convergence $r^{a \rightarrow b}|_{K^-(b+\eta)} \rightarrow H^{a \rightarrow b}$ as $\eta \downarrow 0$ in the finite-dimensional distributional sense. Therefore, all that remains in proving Theorem 1 is the tightness of the family $\{r^{a \rightarrow b}|_{K^-(b+\eta)}\}_{0 < \eta < \eta_0}$ for some $\eta_0 > 0$. By

$$\lim_{\eta \downarrow 0} q_2^{(b+\eta)}(1, a) = q_2^{(b)}(1, a),$$

we can take $\eta_1 > 0$ so that $q_2^{(b+\eta)}(1, a) > q_2^{(b)}(1, a)/2$ holds for $\eta \in (0, \eta_1)$. Throughout this section, we fix η_1 in this fashion and denote

$$(24) \quad \eta_0 := \min\{\eta_1, 1\}.$$

LEMMA 3.2. *Let $0 \leq a < b$ and let $0 < \eta < \eta_0$ be fixed. We have*

$$q_1^{(b+\eta)}(0, a, 1, b) > \eta \frac{q_2^{(b)}(1, a)}{2}.$$

PROOF. According to Taylor's theorem, we can find $\theta \in (0, 1)$ so that

$$q_1^{(b+\eta)}(0, a, 1, b) = \eta q_2^{(b+\theta\eta)}(1, a) > \eta \frac{q_2^{(b)}(1, a)}{2}. \quad \square$$

Using Lemmas 2.3 and 3.2, we obtain the following moment inequalities:

LEMMA 3.3. *Let $0 \leq a < b$. For each $\alpha > 0$, we can find a constant $C_{\alpha, \nu, a, b} > 0$ such that*

$$(25) \quad \sup_{0 < \eta < \eta_0} E \left[\left| r^{a \rightarrow b}|_{K^-(b+\eta)}(r) - r^{a \rightarrow b}|_{K^-(b+\eta)}(0) \right|^{2\alpha} \right] \\ \leq \frac{C_{\alpha, \nu, a, b}}{r^{\nu+1-\alpha}(1-r)^{\nu+\frac{5}{2}}}, \quad r \in (0, 1),$$

$$(26) \quad \sup_{0 < \eta < \eta_0} E \left[\left| r^{a \rightarrow b}|_{K^-(b+\eta)}(1-r) - r^{a \rightarrow b}|_{K^-(b+\eta)}(1) \right|^{2\alpha} \right] \\ \leq \frac{C_{\alpha, \nu, a, b}}{r^{\nu+2-\alpha}(1-r)^{\nu+\frac{3}{2}}}, \quad r \in (0, 1),$$

$$(27) \quad \sup_{0 < \eta < \eta_0} E \left[\left| r^{a \rightarrow b} |_{K^-(b+\eta)}(t) - r^{a \rightarrow b} |_{K^-(b+\eta)}(s) \right|^{2\alpha} \right] \\ \leq \frac{C_{\alpha, \nu, a, b}}{(t-s)^{\nu+1-\alpha} s^{\nu+\frac{3}{2}} (1-t)^{\nu+\frac{5}{2}}}, \quad s, t \in (0, 1).$$

PROOF. By Lemmas 2.3 and 3.2, we have

$$P \left(r^{a \rightarrow b} |_{K^-(b+\eta)}(u) \in dz \right) \\ = \frac{q_1^{(b+\eta)}(0, a, u, z) q_1^{(b+\eta)}(u, z, 1, b)}{q_1^{(b+\eta)}(0, a, 1, b)} 1_{[0, b+\eta]}(z) dz \\ \leq \frac{2}{\eta q_2^{(b)}(1, a)} \left(\frac{C_{\nu, b}}{u^{\nu+1}} n_u(z-a) \right) \\ \times \left(\frac{C_{\nu, b}}{(1-u)^{\nu+1}} \left(1 \wedge \frac{2\eta(b+\eta)}{1-u} \right) n_{1-u}(z-b) \right) dz \\ \leq \frac{4(b+\eta) C_{\nu, b}^2}{q_2^{(b)}(1, a)} \frac{1}{u^{\nu+1} (1-u)^{\nu+2}} n_u(z-a) n_{1-u}(z-b) dz$$

for $0 < u < 1$. On the other hand, for each $c \in \mathbb{R}$,

$$\int_0^{b+\eta} |z-c|^{2\alpha} n_r(z-c) dz \leq 2 \int_0^\infty w^{2\alpha} n_r(w) dw = \frac{(2r)^\alpha}{\sqrt{\pi}} \Gamma \left(\alpha + \frac{1}{2} \right)$$

holds. Hence, because we have

$$E \left[\left| r^{a \rightarrow b} |_{K^-(b+\eta)}(r) - r^{a \rightarrow b} |_{K^-(b+\eta)}(0) \right|^{2\alpha} \right] \\ \leq \frac{4(b+\eta) C_{\nu, b}^2}{\sqrt{2\pi} q_2^{(b)}(1, a)} \frac{1}{r^{\nu+1} (1-r)^{\nu+5/2}} \int_0^{b+\eta} |z-a|^{2\alpha} n_r(z-a) dz \\ \leq \frac{2^{\alpha+2} (b+\eta) C_{\nu, b}^2 \Gamma \left(\alpha + \frac{1}{2} \right)}{\sqrt{2\pi} q_2^{(b)}(1, a)} \frac{1}{r^{\nu+1-\alpha} (1-r)^{\nu+5/2}}$$

and

$$E \left[\left| r^{a \rightarrow b} |_{K^-(b+\eta)}(1-r) - r^{a \rightarrow b} |_{K^-(b+\eta)}(1) \right|^{2\alpha} \right] \\ \leq \frac{4(b+\eta) C_{\nu, b}^2}{\sqrt{2\pi} q_2^{(b)}(1, a)} \frac{1}{(1-r)^{\nu+3/2} r^{\nu+2}} \int_0^{b+\eta} |z-b|^{2\alpha} n_r(z-b) dz$$

$$\leq \frac{2^{\alpha+2}(b+\eta)C_{\nu,b}^2\Gamma\left(\alpha+\frac{1}{2}\right)}{\sqrt{2}\pi q_2^{(b)}(1,a)} \frac{1}{(1-r)^{\nu+3/2}r^{\nu+2-\alpha}},$$

we obtain inequalities (25) and (26) as follows:

$$\begin{aligned} & \sup_{0<\eta<\eta_0} E \left[\left| r^{a-b}|_{K^-(b+\eta)}(r) - r^{a-b}|_{K^-(b+\eta)}(0) \right|^{2\alpha} \right] \\ & \leq \frac{2^{\alpha+2}(b+1)C_{\nu,b}^2\Gamma\left(\alpha+\frac{1}{2}\right)}{\sqrt{2}\pi q_2^{(b)}(1,a)} \frac{1}{r^{\nu+1-\alpha}(1-r)^{\nu+5/2}}, \\ & \sup_{0<\eta<\eta_0} E \left[\left| r^{a-b}|_{K^-(b+\eta)}(1-r) - r^{a-b}|_{K^-(b+\eta)}(1) \right|^{2\alpha} \right] \\ & \leq \frac{2^{\alpha+2}(b+1)C_{\nu,b}^2\Gamma\left(\alpha+\frac{1}{2}\right)}{\sqrt{2}\pi q_2^{(b)}(1,a)} \frac{1}{r^{\nu+2-\alpha}(1-r)^{\nu+\frac{3}{2}}}. \end{aligned}$$

Next, we prove (27). We note that

$$\begin{aligned} & P\left(r^{a-b}|_{K^-(b+\eta)}(t) \in dy, r^{a-b}|_{K^-(b+\eta)}(s) \in dx\right) \\ & = P\left(r^{a-b}|_{K^-(b+\eta)}(t) \in dy \mid r^{a-b}|_{K^-(b+\eta)}(s) = x\right) \\ & \quad \times P\left(r^{a-b}|_{K^-(b+\eta)}(s) \in dx\right) \\ & = \frac{q_1^{(b+\eta)}(0,a,s,x)q_1^{(b+\eta)}(t,y,1,b)}{q_1^{(b+\eta)}(0,a,1,b)} q_1^{(b+\eta)}(s,x,t,y) dx dy, \\ & \quad 0 < x, y < b + \eta. \end{aligned}$$

By Lemmas 2.3 and 3.2, we have

$$\begin{aligned} & P\left(r^{a-b}|_{K^-(b+\eta)}(t) \in dy, r^{a-b}|_{K^-(b+\eta)}(s) \in dx\right) \\ & \leq \frac{2}{\eta q_2^{(b)}(1,a)} \cdot \frac{C_{\nu,b}}{s^{\nu+1}} n_s(x-a) \cdot \frac{C_{\nu,b}}{(1-t)^{\nu+1}} \frac{2\eta(b+\eta)}{1-t} n_{1-t}(y-b) \\ & \quad \cdot \frac{C_{\nu,b}}{(t-s)^{\nu+1}} n_{t-s}(y-x) dx dy \\ & \leq \frac{2(b+\eta)C_{\nu,b}^3}{\pi q_2^{(b)}(1,a)} \cdot \frac{1}{(t-s)^{\nu+1} s^{\nu+3/2} (1-t)^{\nu+5/2}} \cdot n_{t-s}(y-x) dx dy. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \iint_{(0, b+\eta)^2} |y-x|^{2\alpha} n_{t-s}(y-x) dx dy \\ &= \int_0^{b+\eta} \left(\int_0^{b+\eta} |y-x|^{2\alpha} n_{t-s}(y-x) dy \right) dx \\ &\leq (b+\eta) \frac{(2(t-s))^\alpha}{\sqrt{\pi}} \Gamma\left(\alpha + \frac{1}{2}\right) \end{aligned}$$

holds. Hence, we obtain

$$\begin{aligned} & E \left[\left| r^{a \rightarrow b} |_{K^-(b+\eta)}(t) - r^{a \rightarrow b} |_{K^-(b+\eta)}(s) \right|^{2\alpha} \right] \\ &= \iint_{(0, b+\eta)^2} |y-x|^{2\alpha} P \left(r^{a \rightarrow b} |_{K^-(b+\eta)}(t) \in dy, r^{a \rightarrow b} |_{K^-(b+\eta)}(s) \in dx \right) \\ &\leq \frac{2(b+\eta) C_{\nu, b}^3}{\pi q_2^{(b)}(1, a)} \cdot \frac{1}{(t-s)^{\nu+1} s^{\nu+3/2} (1-t)^{\nu+5/2}} \\ &\quad \times \iint_{(0, b+\eta)^2} |y-x|^{2\alpha} n_{t-s}(y-x) dx dy \\ &\leq \frac{2^{\alpha+1} (b+\eta)^2 C_{\nu, b}^3 \Gamma\left(\alpha + \frac{1}{2}\right)}{\pi \sqrt{\pi} q_2^{(b)}(1, a)} \cdot \frac{1}{(t-s)^{\nu+1-\alpha} s^{\nu+3/2} (1-t)^{\nu+5/2}}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \sup_{0 < \eta < \eta_0} E \left[\left| r^{a \rightarrow b} |_{K^-(b+\eta)}(t) - r^{a \rightarrow b} |_{K^-(b+\eta)}(s) \right|^{2\alpha} \right] \\ &\leq \frac{2^{\alpha+1} (b+1)^2 C_{\nu, b}^3 \Gamma\left(\alpha + \frac{1}{2}\right)}{\pi \sqrt{\pi} q_2^{(b)}(1, a)} \cdot \frac{1}{(t-s)^{\nu+1-\alpha} s^{\nu+3/2} (1-t)^{\nu+5/2}}, \end{aligned}$$

and inequality (27) is proved. \square

COROLLARY 2. *Let $0 \leq a < b$. For each $u \in (0, \frac{1}{2})$, the family $\{\pi_{[u, 1-u]} \circ r^{a \rightarrow b} |_{K^-(b+\eta)}\}_{\eta \in (0, \eta_0)}$ is tight.*

PROOF. Using inequalities (25) and (27) for $\alpha = \frac{1}{2}$ and $\alpha = \nu + 3$,

respectively, we obtain

$$\begin{aligned} & \sup_{0 < \eta < \eta_0} E \left[\left| r^{a \rightarrow b} |_{K^-(b+\eta)}(u) \right| \right] \\ & \leq \sup_{0 < \eta < \eta_0} \left(E \left[\left| r^{a \rightarrow b} |_{K^-(b+\eta)}(u) - r^{a \rightarrow b} |_{K^-(b+\eta)}(0) \right| \right] \right. \\ & \quad \left. + E \left[\left| r^{a \rightarrow b} |_{K^-(b+\eta)}(0) \right| \right] \right) \\ & \leq C_{1/2, \nu, a, b} (1-u)^{-\nu - \frac{5}{2}} u^{-\nu - \frac{1}{2}} + a < \infty \end{aligned}$$

and

$$\begin{aligned} & \sup_{0 < \eta < \eta_0} E \left[\left| r^{a \rightarrow b} |_{K^-(b+\eta)}(t) - r^{a \rightarrow b} |_{K^-(b+\eta)}(s) \right|^{2(\nu+3)} \right] \\ & \leq C_{\nu+3, \nu, a, b} s^{-\nu - \frac{3}{2}} (1-t)^{-\nu - \frac{5}{2}} (t-s)^2 \\ & \leq C_{\nu+3, \nu, a, b} u^{-2\nu-4} (t-s)^2 \end{aligned}$$

for $u \leq s < t \leq 1-u$. Hence, by Lemma A.2, we establish the assertion. \square

PROPOSITION 3.4. *Let $0 \leq a < b$. For $\xi > 0$, we have*

$$\begin{aligned} & \lim_{u \downarrow 0} \sup_{\eta \in (0, \eta_0)} P \left(\sup_{0 \leq t \leq u} |r^{a \rightarrow b} |_{K^-(b+\eta)}(t) - r^{a \rightarrow b} |_{K^-(b+\eta)}(0)| > \xi \right) = 0, \\ & \lim_{u \downarrow 0} \sup_{\eta \in (0, \eta_0)} P \left(\sup_{1-u \leq t \leq 1} |r^{a \rightarrow b} |_{K^-(b+\eta)}(t) - r^{a \rightarrow b} |_{K^-(b+\eta)}(1)| > \xi \right) = 0. \end{aligned}$$

PROOF. Applying (25), (26) and (27) for $\alpha = 3\nu + 7$ and $t, s, r \in (0, 1)$ with $s < t$, we have

$$\begin{aligned} (28) \quad & \sup_{0 < \eta < \eta_0} E \left[\left| r^{a \rightarrow b} |_{K^-(b+\eta)}(r) - r^{a \rightarrow b} |_{K^-(b+\eta)}(0) \right|^{2(3\nu+7)} \right] \\ & \leq C_{3\nu+7, \nu, a, b} \frac{r^{2\nu+6}}{(1-r)^{\nu + \frac{5}{2}}}, \end{aligned}$$

$$\begin{aligned} (29) \quad & \sup_{0 < \eta < \eta_0} E \left[\left| r^{a \rightarrow b} |_{K^-(b+\eta)}(t) - r^{a \rightarrow b} |_{K^-(b+\eta)}(s) \right|^{2(3\nu+7)} \right] \\ & \leq C_{3\nu+7, \nu, a, b} \frac{|t-s|^{2\nu+6}}{s^{\nu + \frac{3}{2}} (1-t)^{\nu + \frac{5}{2}}}, \end{aligned}$$

$$(30) \quad \sup_{0 < \eta < \eta_0} E \left[\left| r^{a \rightarrow b} |_{K^-(b+\eta)}(1-r) - r^{a \rightarrow b} |_{K^-(b+\eta)}(1) \right|^{2(3\nu+7)} \right] \\ \leq C_{3\nu+7, \nu, a, b} \frac{r^{2\nu+5}}{(1-r)^{\nu+\frac{3}{2}}}.$$

Let $\gamma = \frac{1}{4\alpha} = \frac{1}{4(3\nu+7)}$, $0 < \eta < \eta_0$, and fix $n \in \mathbb{N}$. We define

$$F_n^\eta = \left\{ \max_{1 \leq k \leq 2^{n-1}} \left| r^{a \rightarrow b} |_{K^-(b+\eta)} \left(\frac{k-1}{2^n} \right) - r^{a \rightarrow b} |_{K^-(b+\eta)} \left(\frac{k}{2^n} \right) \right| \geq 2^{-n\gamma} \right\}, \\ \widetilde{F}_n^\eta = \left\{ \max_{2^{n-1} \leq k \leq 2^n} \left| r^{a \rightarrow b} |_{K^-(b+\eta)} \left(\frac{k-1}{2^n} \right) - r^{a \rightarrow b} |_{K^-(b+\eta)} \left(\frac{k}{2^n} \right) \right| \geq 2^{-n\gamma} \right\}, \\ a(n, k, \eta) = P \left(\left| r^{a \rightarrow b} |_{K^-(b+\eta)} \left(\frac{k-1}{2^n} \right) - r^{a \rightarrow b} |_{K^-(b+\eta)} \left(\frac{k}{2^n} \right) \right| \geq 2^{-n\gamma} \right), \\ 1 \leq k \leq 2^n.$$

Then, by Chebyshev's inequality, we have

$$(31) \quad a(n, k, \eta) \leq 2^{\frac{n}{2}} E \left[\left| r^{a \rightarrow b} |_{K^-(b+\eta)} \left(\frac{k-1}{2^n} \right) - r^{a \rightarrow b} |_{K^-(b+\eta)} \left(\frac{k}{2^n} \right) \right|^{2(3\nu+7)} \right], \\ 1 \leq k \leq 2^n.$$

Therefore, using (28), (29), (30), and (31), we have

$$a(n, 1, \eta) \leq 2^{\frac{n}{2}} C_{3\nu+7, \nu, a, b} \left(\frac{2^n}{2^n - 1} \right)^{\nu+\frac{5}{2}} \left(\frac{1}{2^n} \right)^{\alpha-\nu-1} \\ \leq C_{3\nu+7, \nu, a, b} 2^{-n(\nu+3)} \leq C_{3\nu+7, \nu, a, b} 2^{-\frac{3}{2}n}, \\ a(n, k, \eta) \leq 2^{\frac{n}{2}} C_{3\nu+7, \nu, a, b} \left(\frac{2^n}{k-1} \right)^{\nu+\frac{3}{2}} \left(\frac{2^n}{k} \right)^{\nu+\frac{5}{2}} \left(\frac{1}{2^n} \right)^{2\nu+6} \\ \leq C_{3\nu+7, \nu, a, b} 2^{-\frac{3}{2}n}, \quad (2 \leq k \leq 2^n - 1), \\ a(n, 2^n, \eta) \leq 2^{\frac{n}{2}} C_{3\nu+7, \nu, a, b} \left(\frac{2^n}{2^n - 1} \right)^{\nu+\frac{3}{2}} \left(\frac{1}{2^n} \right)^{2\nu+5} \\ \leq C_{3\nu+7, \nu, a, b} 2^{-n(\nu+3)} \leq C_{3\nu+7, \nu, a, b} 2^{-\frac{3}{2}n}.$$

Thus, it follows that

$$P(F_n^\eta) \leq \sum_{k=1}^{2^{n-1}} a(n, k, \eta) \leq C_{3\nu+7, \nu, a, b} 2^{-\frac{n}{2}},$$

$$P(\tilde{F}_n^\eta) \leq \sum_{k=2^{n-1}}^{2^n} a(n, k, \eta) \leq C_{3\nu+7, \nu, a, b} 2^{-\frac{n}{2}}.$$

Therefore, Lemmas A.3 and A.4 prove the desired results. \square

By Corollary 2 and Proposition 3.4, we can apply Theorem 8 to $\{r^{a \rightarrow b}|_{K^-(b+\eta)}\}_{0 < \eta < \eta_0}$ and obtain the tightness of this family.

4. Decomposition Formula and Sample Path Properties

In this section, we prove the decomposition formula for the distribution of the BES(δ) house-moving (Theorem 2). In addition, applying this result, we study sample path properties of the BES(δ) house-moving.

First, we prove Theorem 2. By Theorem 1, because $r^{a \rightarrow b}|_{K^-(b+\eta)} \xrightarrow{\mathcal{D}} H^{a \rightarrow b} (\eta \downarrow 0)$ holds,

$$E \left[F(H^{a \rightarrow b}) \right] = \lim_{\eta \downarrow 0} E \left[F(r^{a \rightarrow b}|_{K^-(b+\eta)}) \right]$$

for every bounded continuous function F on $C([0, 1], \mathbb{R})$. We calculate the numerator of

$$E \left[F(r^{a \rightarrow b}|_{K^-(b+\eta)}) \right] = \frac{E \left[F(r^{a \rightarrow b}) ; r^{a \rightarrow b} \in K^-(b + \eta) \right]}{P(r^{a \rightarrow b} \in K^-(b + \eta))}$$

as

$$\begin{aligned} & E \left[F(r^{a \rightarrow b}) ; r^{a \rightarrow b} \in K^-(b + \eta) \right] \\ &= \int_0^\infty E \left[F(r^{a \rightarrow b}) ; r^{a \rightarrow b} \in K^-(b + \eta), r^{a \rightarrow b}(t) \in dy \right] \\ &= \int_0^\infty E \left[F(r_{[0,t]}^{a \rightarrow y}|_{K_{[0,t]}^-(b+\eta)} \oplus_t r_{[t,1]}^{y \rightarrow b}|_{K_{[t,1]}^-(b+\eta)}) \right] \\ &\quad \times P \left(r^{a \rightarrow b} \in K^-(b + \eta), r^{a \rightarrow b}(t) \in dy \right) \end{aligned}$$

$$\begin{aligned}
&= \int_0^\infty E \left[F(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b+\eta)} \oplus_t r_{[t,1]}^{y \rightarrow b} |_{K_{[t,1]}^-(b+\eta)}) \right] \\
&\quad \times P \left(r^{a \rightarrow b} |_{K^-(b+\eta)}(t) \in dy \right) P \left(r^{a \rightarrow b} \in K^-(b+\eta) \right).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
E \left[F(r^{a \rightarrow b} |_{K^-(b+\eta)}) \right] &= \int_0^\infty E \left[F(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b+\eta)} \oplus_t r_{[t,1]}^{y \rightarrow b} |_{K_{[t,1]}^-(b+\eta)}) \right] \\
&\quad \times P \left(r^{a \rightarrow b} |_{K^-(b+\eta)}(t) \in dy \right).
\end{aligned}$$

Then, it suffices to show that

$$\begin{aligned}
(32) \quad &\lim_{\eta \downarrow 0} \int_0^\infty E \left[F(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b+\eta)} \oplus_t r_{[t,1]}^{y \rightarrow b} |_{K_{[t,1]}^-(b+\eta)}) \right] \\
&\quad \times P \left(r^{a \rightarrow b} |_{K^-(b+\eta)}(t) \in dy \right) \\
&= \int_0^\infty E \left[F(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b)} \oplus_t H_{[t,1]}^{y \rightarrow b}) \right] P \left(H^{a \rightarrow b}(t) \in dy \right).
\end{aligned}$$

We obtain the following estimate:

$$\begin{aligned}
&\left| \int_0^\infty E \left[F(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b+\eta)} \oplus_t r_{[t,1]}^{y \rightarrow b} |_{K_{[t,1]}^-(b+\eta)}) \right] P \left(r^{a \rightarrow b} |_{K^-(b+\eta)}(t) \in dy \right) \right. \\
&\quad \left. - \int_0^\infty E \left[F(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b)} \oplus_t H_{[t,1]}^{y \rightarrow b}) \right] P \left(H^{a \rightarrow b}(t) \in dy \right) \right| \\
&\leq \int_0^\infty \left| E \left[F(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b+\eta)} \oplus_t r_{[t,1]}^{y \rightarrow b} |_{K_{[t,1]}^-(b+\eta)}) \right. \right. \\
&\quad \left. \left. - F(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b)} \oplus_t H_{[t,1]}^{y \rightarrow b}) \right] \right| P \left(r^{a \rightarrow b} |_{K^-(b+\eta)}(t) \in dy \right) \\
&\quad + \left| \int_0^\infty E \left[F(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b)} \oplus_t H_{[t,1]}^{y \rightarrow b}) \right] \right. \\
&\quad \left. \times \left(P \left(r^{a \rightarrow b} |_{K^-(b+\eta)}(t) \in dy \right) - P \left(H^{a \rightarrow b}(t) \in dy \right) \right) \right| \\
&=: I_t^{(1)}(\eta) + I_t^{(2)}(\eta).
\end{aligned}$$

Then, if $I_t^{(1)}(\eta), I_t^{(2)}(\eta) \rightarrow 0$ as $\eta \downarrow 0$, we can prove (32). First, consider

$$I_t^{(1)}(\eta) = \int_0^\infty \left| E \left[F(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b+\eta)} \oplus_t r_{[t,1]}^{y \rightarrow b} |_{K_{[t,1]}^-(b+\eta)}) \right. \right. \\ \left. \left. - F(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b)} \oplus_t H_{[t,1]}^{y \rightarrow b}) \right] \right| \\ \times \frac{q_1^{(b+\eta)}(0, a, t, y) q_1^{(b+\eta)}(t, y, 1, b)}{q_1^{(b+\eta)}(0, a, 1, b)} \mathbf{1}_{[0, b+\eta]}(y) dy.$$

We have

$$(33) \quad \sup_{\substack{\eta > 0 \\ y \in (0, b+\eta)}} \left| E \left[F(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b+\eta)} \oplus_t r_{[t,1]}^{y \rightarrow b} |_{K_{[t,1]}^-(b+\eta)}) \right. \right. \\ \left. \left. - F(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b)} \oplus_t H_{[t,1]}^{y \rightarrow b}) \right] \right| \\ \leq 2 \sup_{w \in C([0,1], \mathbb{R})} |F(w)| < \infty.$$

By Theorem 1 and Lemma A.5, it holds that

$$(34) \quad \lim_{\eta \downarrow 0} E \left[F(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b+\eta)} \oplus_t r_{[t,1]}^{y \rightarrow b} |_{K_{[t,1]}^-(b+\eta)}) \right] \\ = E \left[F(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b)} \oplus_t H_{[t,1]}^{y \rightarrow b}) \right].$$

In addition, by Lemmas 2.3 and 3.2, and for $\eta \in (0, \eta_0)$ and $y \in [0, b + \eta]$, we obtain

$$(35) \quad \frac{q_1^{(b+\eta)}(0, a, t, y) q_1^{(b+\eta)}(t, y, 1, b)}{q_1^{(b+\eta)}(0, a, 1, b)} \\ \leq \frac{2}{\eta q_2^{(b)}(1, a)} \left(\frac{C_{\nu, b}}{t^{\nu+1}} n_t(y - a) \right) \\ \times \left(\frac{C_{\nu, b}}{(1-t)^{\nu+1}} \left(1 \wedge \frac{2\eta(b+\eta)}{1-t} \right) n_{1-t}(y - b) \right) \\ \leq \frac{2(b+1)C_{\nu, b}^2}{\pi t^{\nu+3/2} (1-t)^{\nu+5/2} q_2^{(b)}(1, a)},$$

and by Proposition 3.1, it holds that

$$(36) \quad \lim_{\eta \downarrow 0} \frac{q_1^{(b+\eta)}(0, a, t, y) q_1^{(b+\eta)}(t, y, 1, b)}{q_1^{(b+\eta)}(0, a, 1, b)} = \frac{q_1^{(b)}(0, a, t, y) q_2^{(b)}(1-t, y)}{q_2^{(b)}(1, a)}.$$

Therefore, according to (33), (34), (35), (36), and Lebesgue's dominated convergence theorem,

$$I_t^{(1)}(\eta) \rightarrow 0, \quad \eta \downarrow 0.$$

Next, we consider

$$I_t^{(2)}(\eta) = \left| \int_0^\infty E \left[F(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b)} \oplus_t H_{[t,1]}^{y \rightarrow b}) \right] \times \left(P \left(r^{a \rightarrow b} |_{K^{-(b+\eta)}}(t) \in dy \right) - P \left(H^{a \rightarrow b}(t) \in dy \right) \right) \right|.$$

We have

$$\sup_{y > 0} \left| E \left[F(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b)} \oplus_t H_{[t,1]}^{y \rightarrow b}) \right] \right| \leq \sup_{w \in C([0,1], \mathbb{R})} |F(w)| < \infty.$$

Then, by (35), (36), and Lebesgue's dominated convergence theorem,

$$\begin{aligned} \lim_{\eta \downarrow 0} \int_0^\infty E \left[F(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b)} \oplus_t H_{[t,1]}^{y \rightarrow b}) \right] P \left(r^{a \rightarrow b} |_{K^{-(b+\eta)}}(t) \in dy \right) \\ = \int_0^\infty E \left[F(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b)} \oplus_t H_{[t,1]}^{y \rightarrow b}) \right] P \left(H^{a \rightarrow b}(t) \in dy \right). \end{aligned}$$

Therefore, it follows that

$$I_t^{(2)}(\eta) \rightarrow 0, \quad \eta \downarrow 0.$$

Thus, we prove (32) and the proof is completed. \square

LEMMA 4.1. *Let $B \in \mathcal{B}(C([0,1], \mathbb{R}))$. Then, Theorem 2 holds true for $F = 1_B$.*

PROOF. Let A be a closed subset of $C([0,1], \mathbb{R})$ and let

$$\phi(x) := 1 - \int_0^1 1_{(-\infty, x]}(u) du, \quad x \in \mathbb{R}.$$

Then, for $w \in C([0, 1], \mathbb{R})$, we have

$$F_n(w) := \phi(nd_\infty(w, A)) \downarrow 1_A(w), \quad n \rightarrow \infty,$$

where

$$d_\infty(w, A) := \inf\{\|w - v\|_{C([0,1],\mathbb{R})} \mid v \in A\}.$$

Therefore, by Lebesgue’s dominated convergence theorem and Dynkin’s π - λ theorem, we can obtain our assertion. \square

LEMMA 4.2. *Let $0 \leq a < b$. For $0 < z \leq x \leq b$ and $t \in (0, 1)$,*

$$P\left(\max_{u \in [0,t]} H^{a \rightarrow b}(u) = x\right) = 0,$$

$$P\left(\max_{u \in [0,t]} H^{a \rightarrow b}(u) \leq x, H^{a \rightarrow b}(t) \leq z\right) = \int_0^z \frac{q_1^{(x)}(0, a, t, y) q_2^{(b)}(1 - t, y)}{q_2^{(b)}(1, a)} dy.$$

PROOF. Let A_i ($i = 1, 2$) be closed subsets of $C([0, 1], \mathbb{R})$ given by

$$A_1 := \left\{ w \in C([0, 1], \mathbb{R}) \mid \max_{u \in [0,t]} w(u) = x \right\},$$

$$A_2 := \left\{ w \in C([0, 1], \mathbb{R}) \mid \max_{u \in [0,t]} w(u) \leq x, w(t) \leq z \right\}.$$

Lemma 4.1 implies that Theorem 2 can be applied for $F = 1_{A_i}$ ($i = 1, 2$). Thus, we obtain

$$(37) \quad P\left(M_t(H^{a \rightarrow b}) = x\right) = \int_0^x P\left(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b)} \in \partial K_{[0,t]}^-(x)\right) P\left(H^{a \rightarrow b}(t) \in dy\right),$$

$$(38) \quad P\left(M_t(H^{a \rightarrow b}) \leq x, H^{a \rightarrow b}(t) \leq z\right) = \int_0^z P\left(r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b)} \in K_{[0,t]}^-(x), r_{[0,t]}^{a \rightarrow y} |_{K_{[0,t]}^-(b)}(t) \leq z\right) \times P\left(H^{a \rightarrow b}(t) \in dy\right).$$

By Proposition 2.1 and (37), we obtain

$$P\left(M_t(H^{a \rightarrow b}) = x\right) = \int_0^x \frac{P\left(r_{[0,t]}^{a \rightarrow y} \in \partial K_{[0,t]}^-(x)\right)}{P\left(r_{[0,t]}^{a \rightarrow y} \in K_{[0,t]}^-(b)\right)} P\left(H^{a \rightarrow b}(t) \in dy\right) = 0.$$

Furthermore, (38) implies that

$$\begin{aligned} & P\left(M_t(H^{a \rightarrow b}) \leq x, H^{a \rightarrow b}(t) \leq z\right) \\ &= \int_0^z \frac{P\left(r_{[0,t]}^{a \rightarrow y} \in K_{[0,t]}^-(x), r_{[0,t]}^{a \rightarrow y}(t) \leq z\right)}{P\left(r_{[0,t]}^{a \rightarrow y} \in K_{[0,t]}^-(b)\right)} \frac{q_1^{(b)}(0, a, t, y) q_2^{(b)}(1-t, y)}{q_2^{(b)}(1, a)} dy \\ &= \int_0^z \frac{P\left(r_{[0,t]}^{a \rightarrow y} \in K_{[0,t]}^-(x)\right)}{P\left(r_{[0,t]}^{a \rightarrow y} \in K_{[0,t]}^-(b)\right)} \frac{q_1^{(b)}(0, a, t, y) q_2^{(b)}(1-t, y)}{q_2^{(b)}(1, a)} dy \\ &= \int_0^z \frac{q_1^{(x)}(0, a, t, y) q_2^{(b)}(1-t, y)}{q_2^{(b)}(1, a)} dy. \quad \square \end{aligned}$$

4.1. Proof of Proposition 1.1

Let $t \in (0, 1)$. Lemma 4.2 implies that

$$\begin{aligned} P\left(M_t(H^{a \rightarrow b}) = b\right) &= 0, \\ P\left(M_t(H^{a \rightarrow b}) \leq b\right) &= P\left(M_t(H^{a \rightarrow b}) \leq b, H^{a \rightarrow b}(t) \leq b\right) \\ &= \int_0^b P\left(H^{a \rightarrow b}(t) \in dy\right) = 1. \end{aligned}$$

Therefore, $P(M_t(H^{a \rightarrow b}) < b) = 1$ holds and Proposition 1.1 is obtained. Proposition 1.1 implies that Bessel house-moving $H^{a \rightarrow b}$ does not hit b on the time interval $[0, 1)$.

5. Proof of Theorem 3

For $t > 0$ and $x, y \in [0, \infty)$, we set

$$p(t; x, y) := \frac{P(R^x(t) \in dy)}{dy}.$$

In addition, we denote the expectation with respect to a probability Q by E^Q .

First, we prepare two lemmas.

LEMMA 5.1. *Let $0 \leq a < b$. For $t \in (0, 1)$, we have*

$$\frac{dP_t^{r^{a \rightarrow b}}}{dP_t^{R^a}}(w) = \frac{p(1-t; w(t), b)}{p(1; a, b)}, \quad w \in C([0, t], \mathbb{R}).$$

PROOF. Let $A \in \mathcal{B}(C([0, t], \mathbb{R}))$ be fixed. By the Markov property of R^a , we obtain the assertion as follows:

$$\begin{aligned} P_t^{r^{a \rightarrow b}}(A) &= \frac{P^{R^a}(\pi_{[0,t]}^{-1}(A), w(1) \in db)}{P^{R^a}(w(1) \in db)} \\ &= \frac{E^{R^a} \left[\mathbf{1}_{\pi_{[0,t]}^{-1}(A)}(w) \cdot P^{R^a}(w(1) \in db \mid w(t)) \right]}{P^{R^a}(w(1) \in db)} \\ &= \int_{\pi_{[0,t]}^{-1}(A)} \frac{P^{R^a}(w(1) \in db \mid w(t))}{P^{R^a}(w(1) \in db)} P^{R^a}(dw) \\ &= \int_A \frac{p(1-t; w(t), b)}{p(1; a, b)} P_t^{R^a}(dw). \quad \square \end{aligned}$$

LEMMA 5.2. *Let $0 \leq a < b$ and $t \in (0, 1)$. For every bounded continuous functions F on $C([0, t], \mathbb{R})$, it holds that*

$$E^{P_t^{H^{a \rightarrow b}}}[F] = \int_{C([0,t], \mathbb{R})} F(w) \cdot \frac{q_2^{(b)}(1-t, w(t))}{q_2^{(b)}(1, a)} \cdot \mathbf{1}_{K_{[0,t]}^-(b)}(w) P_t^{R^a}(dw).$$

PROOF. By the Markov property of $r^{a \rightarrow b}$ and Lemma 5.1, for $A \in \mathcal{B}(C([0, t], \mathbb{R}))$, it holds that

$$\begin{aligned} &P^{r^{a \rightarrow b}}(\pi_{[0,t]}^{-1}(A) \mid K^-(b + \eta)) \\ &= \frac{P^{r^{a \rightarrow b}}(\pi_{[0,t]}^{-1}(A) \cap \pi_{[0,t]}^{-1}(K_{[0,t]}^-(b + \eta)) \cap \pi_{[t,1]}^{-1}(K_{[t,1]}^-(b + \eta)))}{P^{r^{a \rightarrow b}}(K^-(b + \eta))} \end{aligned}$$

$$\begin{aligned}
&= \int_{\pi_{[0,t]}^{-1}(A)} \frac{P^{r^{a \rightarrow b}} \left(\pi_{[t,1]}^{-1}(K_{[t,1]}^-(b+\eta)) \mid w(t) \right)}{P^{r^{a \rightarrow b}}(K^-(b+\eta))} \\
&\quad \times 1_{K_{[0,t]}^-(b+\eta)}(\pi_{[0,t]} \circ w) P^{r^{a \rightarrow b}}(dw) \\
&= \int_{\pi_{[0,t]}^{-1}(A)} \frac{P^{r^{a \rightarrow b}} \left(\pi_{[t,1]}^{-1}(K_{[t,1]}^-(b+\eta)) \mid w(t) \right) p(1-t; w(t), b)}{P^{r^{a \rightarrow b}}(K^-(b+\eta)) p(1; a, b)} \\
&\quad \times 1_{K_{[0,t]}^-(b+\eta)}(\pi_{[0,t]} \circ w) P^{R^a}(dw) \\
&= \int_A \frac{q_1^{(b+\eta)}(t, w(t), 1, b)}{q_1^{(b+\eta)}(0, a, 1, b)} 1_{K_{[0,t]}^-(b+\eta)}(w) P_t^{R^a}(dw), \quad \eta > 0.
\end{aligned}$$

Then, for a bounded continuous function F on $C([0, t], \mathbb{R})$, we obtain

$$\begin{aligned}
&\int_{C([0,1], \mathbb{R})} F(\pi_{[0,t]} \circ w) P^{r^{a \rightarrow b}}(dw \mid K^-(b+\eta)) \\
&= \int_{C([0,t], \mathbb{R})} F(w) \frac{q_1^{(b+\eta)}(t, w(t), 1, b)}{q_1^{(b+\eta)}(0, a, 1, b)} 1_{K_{[0,t]}^-(b+\eta)}(w) P_t^{R^a}(dw), \quad \eta > 0.
\end{aligned}$$

By Lemmas 2.3 and 3.2, and for $\eta \in (0, \eta_0)$, we obtain

$$\frac{q_1^{(b+\eta)}(t, w(t), 1, b)}{q_1^{(b+\eta)}(0, a, 1, b)} \leq \frac{4(b+\eta)C_{\nu,b}}{\sqrt{2\pi}q_2^{(b)}(1, a)} \frac{1}{(1-t)^{\nu+5/2}}, \quad w \in C([0, t], [0, b+\eta]).$$

In addition, it holds that

$$\begin{aligned}
&\lim_{\eta \downarrow 0} \frac{q_1^{(b+\eta)}(t, w(t), 1, b)}{q_1^{(b+\eta)}(0, a, 1, b)} 1_{K_{[0,t]}^-(b+\eta)}(w) \\
&= \frac{q_2^{(b)}(1-t, w(t))}{q_2^{(b)}(1, a)} 1_{K_{[0,t]}^-(b)}(w), \quad w \in C([0, t], [0, \infty)).
\end{aligned}$$

Therefore, Lebesgue's dominated convergence theorem implies

$$\begin{aligned}
&\lim_{\eta \downarrow 0} \int_{C([0,1], \mathbb{R})} F(\pi_{[0,t]} \circ w) P^{r^{a \rightarrow b}}(dw \mid K^-(b+\eta)) \\
&= \int_{C([0,t], \mathbb{R})} F(w) \frac{q_2^{(b)}(1-t, w(t))}{q_2^{(b)}(1, a)} 1_{K_{[0,t]}^-(b)}(w) P_t^{R^a}(dw).
\end{aligned}$$

According to this equality and Theorem 1, it follows that

$$\begin{aligned} E^{P_t^{H^{a \rightarrow b}}} [F] &= \int_{C([0,1],\mathbb{R})} F(\pi_{[0,t]} \circ w) P^{H^{a \rightarrow b}}(dw) \\ &= \lim_{\eta \downarrow 0} \int_{C([0,1],\mathbb{R})} F(\pi_{[0,t]} \circ w) P^{r^{a \rightarrow b}}(dw \mid K^-(b + \eta)) \\ &= \int_{C([0,t],\mathbb{R})} F(w) \frac{q_2^{(b)}(1-t, w(t))}{q_2^{(b)}(1, a)} 1_{K_{[0,t]}^-(b)}(w) P_t^{R^a}(dw). \end{aligned}$$

Thus, the proof is completed. \square

Now, we prove Theorem 3. Let A be a closed subset of $C([0, t], \mathbb{R})$. In a similar manner to the proof of Lemma 4.1, by Lemma 5.2 and Lebesgue’s dominated convergence theorem, it holds that

$$(39) \quad E^{P_t^{H^{a \rightarrow b}}} [1_A] = \int_{C([0,t],\mathbb{R})} 1_A(w) \frac{q_2^{(b)}(1-t, w(t))}{q_2^{(b)}(1, a)} 1_{K_{[0,t]}^-(b)}(w) P_t^{R^a}(dw).$$

Using (39) and Dynkin’s π - λ theorem, we can prove the assertion completely. \square

6. Proof of Proposition 1.2

In this section, we prove Proposition 1.2, which gives the characterization of the Bessel house-moving by using the first hitting time of the Bessel process.

LEMMA 6.1. *Let $b > 0$. For $t > 0$ and $y \in (0, b)$, we have*

$$(40) \quad \frac{P(\tau_{y,b} \in dt)}{dt} = \frac{q_2^{(b)}(t, y)}{2},$$

$$(41) \quad \frac{P(\tau_{0,b} \in dt)}{dt} = \frac{q_2^{(b)}(t, 0)}{2}.$$

PROOF. First, we prove (40). It holds that

$$\begin{aligned} \frac{P(\tau_{y,b} \in dt)}{dt} &= -\frac{\partial}{\partial t} P(M_t(R^y) < b) \\ &= -\frac{\partial}{\partial t} \int_0^b P(M_t(R^y) \leq b, R^y(t) \in dx) \end{aligned}$$

$$= -\frac{\partial}{\partial t} \int_0^b P \left(M_{[0,t]}(r_{[0,t]}^{y \rightarrow x}) \leq b \right) P(R^y(t) \in dx).$$

For each n , we set

$$f_n(t, x) := \frac{J_\nu(xj_{\nu,n}/b) J_\nu(yj_{\nu,n}/b)}{b^2 J_{\nu+1}^2(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2b^2}t\right), \quad t \geq 0, \quad x \in (0, b).$$

Then, by Theorem 5, we have

$$q_1^{(b)}(0, y, t, x) = 2x \left(\frac{x}{y}\right)^\nu \sum_{n=1}^\infty f_n(t, x).$$

Let $T > 0$ be fixed. By Lemma 2.2 and (60), there exist some $\tilde{C}_\nu > 0$ and $N_\nu \in \mathbb{N}$ such that

$$\begin{aligned} \left| \frac{\partial}{\partial t} f_n(t, x) \right| &= \left| \frac{J_\nu(xj_{\nu,n}/b) J_\nu(yj_{\nu,n}/b) \frac{j_{\nu,n}^2}{2b^2} \exp\left(-\frac{j_{\nu,n}^2}{2b^2}t\right)}{b^2 J_{\nu+1}^2(j_{\nu,n})} \right| \\ &\leq 2\tilde{C}_\nu^2 \frac{\sqrt{(1 + \frac{x\pi}{b})(1 + \frac{y\pi}{b})}}{xy} (n\pi)^2 \exp\left(-\frac{(n\pi)^2}{8b^2}t\right) \end{aligned}$$

holds for $n > N_\nu$ and $t \in (T, \infty)$. Since

$$\sum_{n=N_\nu+1}^\infty (n\pi)^2 \exp\left(-\frac{(n\pi)^2}{8b^2}T\right) < \infty$$

holds, we have

$$\frac{\partial}{\partial t} q_1^{(b)}(0, y, t, x) = 2x \left(\frac{x}{y}\right)^\nu \sum_{n=1}^\infty \frac{\partial}{\partial t} f_n(t, x) \quad t \in (T, \infty),$$

by Lebesgue's dominated convergence theorem. Thus, we obtain

$$\begin{aligned} (42) \quad &\sup_{t \in (T, \infty)} \left| \frac{\partial}{\partial t} q_1^{(b)}(0, y, t, x) \right| \\ &\leq y^{-\nu} \sum_{n=1}^{N_\nu} \frac{|x^{\nu+1} J_\nu(xj_{\nu,n}/b) J_\nu(yj_{\nu,n}/b)|}{b^2 J_{\nu+1}^2(j_{\nu,n})} \frac{j_{\nu,n}^2}{b^2} \exp\left(-\frac{j_{\nu,n}^2}{2b^2}T\right) \end{aligned}$$

$$\begin{aligned}
 &+ 4\tilde{C}_\nu^2 \left(\frac{x}{y}\right)^\nu \frac{\sqrt{\left(1 + \frac{x\pi}{b}\right)\left(1 + \frac{y\pi}{b}\right)}}{y} \\
 &\times \sum_{n=N_\nu+1}^\infty (n\pi)^2 \exp\left(-\frac{(n\pi)^2}{8b^2}T\right), \quad x \in (0, b).
 \end{aligned}$$

By (58), because there exists $C_\nu > 0$ such that

$$\begin{aligned}
 |x^{\nu+1} J_\nu(xj_{\nu,n}/b)| &\leq C_\nu \left(\frac{j_{\nu,n}}{b}\right)^\nu \frac{x^{2\nu+1}}{\left(1 + \frac{xj_{\nu,n}}{b}\right)^{\nu+\frac{1}{2}}} \\
 &\leq C_\nu \left(\frac{j_{\nu,n}}{b}\right)^{-1} x^\nu \left(1 + \frac{xj_{\nu,n}}{b}\right), \quad x \in (0, b)
 \end{aligned}$$

holds, the functions $x^{\nu+1} J_\nu(xj_{\nu,n}/b)$, $n = 1, \dots, N_\nu$ in the first term on the right-hand side of (42) are integrable with respect to x on $[0, b]$. In addition, since

$$x^\nu \sqrt{1 + \frac{x\pi}{b}} \leq x^\nu \left(1 + \frac{x\pi}{b}\right) \quad (x \in (0, b))$$

holds, the function $x^\nu \sqrt{1 + \frac{x\pi}{b}}$ in the second term on the right-hand side of (42) is integrable with respect to x on $[0, b]$. Therefore, by Lebesgue’s dominated convergence theorem,

$$\frac{\partial}{\partial t} \int_0^b q_1^{(b)}(0, y, t, x) dx = \int_0^b \frac{\partial}{\partial t} q_1^{(b)}(0, y, t, x) dx.$$

Recall that $\nu = \delta/2 - 1$, and let $m(x)dx = 2x^{2\nu+1}dx$ be the speed measure of the BES(δ) process. Then, we obtain

$$\begin{aligned}
 m(x) \frac{\partial}{\partial t} \left(\frac{q_1^{(b)}(0, y, t, x)}{m(x)}\right) &= m(x) \mathcal{L}_\delta \left(\frac{q_1^{(b)}(0, y, t, x)}{m(x)}\right) \\
 &= \frac{1}{2} \frac{\partial}{\partial x} \left(m(x) \frac{\partial}{\partial x} \left(\frac{q_1^{(b)}(0, y, t, x)}{m(x)}\right)\right),
 \end{aligned}$$

where \mathcal{L}_δ is the infinitesimal generator of the BES(δ) process. So, we get

$$\begin{aligned} \frac{P(\tau_{y,b} \in dt)}{dt} &= -\frac{1}{2} \int_0^b \frac{\partial}{\partial x} \left(m(x) \frac{\partial}{\partial x} \left(\frac{q_1^{(b)}(0, y, t, x)}{m(x)} \right) \right) dx \\ &= -\frac{1}{2} \left[m(x) \frac{\partial}{\partial x} \left(\frac{q_1^{(b)}(0, y, t, x)}{m(x)} \right) \right]_{x=0}^{x=b}. \end{aligned}$$

Inequality (58) and Lemma 2.2 imply the following inequality:

(43)

$$\begin{aligned} &\sum_{n=1}^\infty \sup_{x \in (0, \infty)} \left| \frac{j_{\nu,n} x^{-(\nu+1)} J_{\nu+1}(x j_{\nu,n}/b) J_\nu(y j_{\nu,n}/b)}{J_{\nu+1}^2(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2b^2} t\right) \right| \\ &\leq C_{\nu+1} \sum_{n=1}^\infty j_{\nu,n} \left(\frac{j_{\nu,n}}{b}\right)^{\nu+1} \left| \frac{J_\nu(y j_{\nu,n}/b)}{J_{\nu+1}^2(j_{\nu,n})} \right| \exp\left(-\frac{j_{\nu,n}^2}{2b^2} t\right) \\ &\leq C_{\nu+1} \sum_{n=1}^{N_\nu} j_{\nu,n} \left(\frac{j_{\nu,n}}{b}\right)^{\nu+1} \left| \frac{1}{J_{\nu+1}(j_{\nu,n})} \right| \left| \frac{J_\nu(y j_{\nu,n}/b)}{J_{\nu+1}(j_{\nu,n})} \right| \exp\left(-\frac{j_{\nu,n}^2}{2b^2} t\right) \\ &\quad + \frac{C_{\nu+1} \tilde{C}_\nu \pi (1 + y\pi/b)^{\frac{1}{2}}}{y b^\nu} \sum_{n=N_\nu+1}^\infty \sqrt{n} (2n\pi)^{\nu+2} \exp\left(-\frac{(n\pi)^2}{8b^2} t\right). \end{aligned}$$

Then Lebesgue’s dominated convergence theorem and the inequality (43) show that

$$\begin{aligned} &m(x) \frac{\partial}{\partial x} \left(\frac{q_1^{(b)}(0, y, t, x)}{m(x)} \right) \\ &= 2x^{2\nu+1} \frac{\partial}{\partial x} \left((xy)^{-\nu} \sum_{n=1}^\infty \frac{J_\nu(x j_{\nu,n}/b) J_\nu(y j_{\nu,n}/b)}{b^2 J_{\nu+1}^2(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2b^2} t\right) \right) \\ &= -2x^{2\nu+2} y^{-\nu} \sum_{n=1}^\infty \frac{j_{\nu,n} x^{-(\nu+1)} J_{\nu+1}(x j_{\nu,n}/b) J_\nu(y j_{\nu,n}/b)}{b^3 J_{\nu+1}^2(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2b^2} t\right) \end{aligned}$$

and

$$\begin{aligned} &\left(m(x) \frac{\partial}{\partial x} \left(\frac{q_1^{(b)}(0, y, t, x)}{m(x)} \right) \right) \Big|_{x=b} = -q_2^{(b)}(t, y), \\ &\left(m(x) \frac{\partial}{\partial x} \left(\frac{q_1^{(b)}(0, y, t, x)}{m(x)} \right) \right) \Big|_{x=0} = 0 \end{aligned}$$

hold. Thus, we have

$$-\frac{1}{2} \left[m(x) \frac{\partial}{\partial x} \left(\frac{q_1^{(b)}(0, y, t, x)}{m(x)} \right) \right]_{x=0}^{x=b} = \frac{q_2^{(b)}(t, y)}{2},$$

and (40) is proved. By (40), Theorem 5, and Corollary 1, we easily obtain (41). \square

REMARK 6.1. It is well-known that

$$P(\tau_{a,b} \leq t) = 1 - 2 \left(\frac{b}{a} \right)^\nu \sum_{n=1}^\infty \frac{J_\nu(a j_{\nu,n}/b)}{j_{\nu,n} J_{\nu+1}(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2b^2} t\right),$$

$$P(\tau_{0,b} \leq t) = 1 - \frac{1}{2^{\nu-1} \Gamma(\nu+1)} \sum_{n=1}^\infty \frac{j_{\nu,n}^{\nu-1}}{J_{\nu+1}(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2b^2} t\right)$$

hold for $0 < a < b$ and $t > 0$ ([5]). By differentiating these identities, we obtain

$$(44) \quad P(\tau_{a,b} \in dt) = \left(\frac{b}{a} \right)^\nu \sum_{n=1}^\infty \frac{j_{\nu,n} J_\nu(a j_{\nu,n}/b)}{b^2 J_{\nu+1}(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2b^2} t\right) dt,$$

$$(45) \quad P(\tau_{0,b} \in dt) = \frac{1}{2^\nu \Gamma(\nu+1)} \sum_{n=1}^\infty \frac{j_{\nu,n}^{\nu+1}}{b^2 J_{\nu+1}(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2b^2} t\right) dt.$$

By using (44) and (45), we can also prove Lemma 6.1.

THEOREM 6. Let $0 \leq a < b$. For $0 < s < t < 1$ and $x, y \in (0, b)$, we have

$$(46) \quad P(R^a(t) \in dy \mid \tau_{a,b} = 1) = \frac{q_1^{(b)}(0, a, t, y) q_2^{(b)}(1-t, y)}{q_2^{(b)}(1, a)} dy,$$

$$(47) \quad P(R^a(t) \in dy \mid R^a(s) = x, \tau_{a,b} = 1) = \frac{q_1^{(b)}(s, x, t, y) q_2^{(b)}(1-t, y)}{q_2^{(b)}(1-s, x)} dy.$$

PROOF. Using the Markov property of R^a , for $0 < t < u$, it holds that

$$\begin{aligned} P(R^a(t) \in dy, \tau_{a,b} > u) &= P(R^a(t) \in dy, M_u(R^a) < b) \\ &= P(R^a(t) \in dy, M_t(R^a) < b) P(M_{u-t}(R^y) < b) \\ &= P(R^a(t) \in dy, M_t(R^a) < b) P(\tau_{y,b} > u-t). \end{aligned}$$

Since the density of $\tau_{a,b}$ is a derivative of the distribution function, we obtain

$$\begin{aligned}
 (48) \quad & P(R^a(t) \in dy, \tau_{a,b} \in du) \\
 &= -\frac{d}{du} P(R^a(t) \in dy, \tau_{a,b} > u) \\
 &= P(R^a(t) \in dy, M_t(R^a) < b) P(\tau_{y,b} \in du - t).
 \end{aligned}$$

We can calculate the first term of the right-hand side of (48) as

$$\begin{aligned}
 (49) \quad & P(R^a(t) \in dy, M_t(R^a) < b) \\
 &= P(R^a(t) \in dy) \frac{P(R^a(t) \in dy, M_t(R^a) < b)}{P(R^a(t) \in dy)} \\
 &= P(R^a(t) \in dy) P\left(M_{[0,t]}(r_{[0,t]}^{a \rightarrow y}) < b\right) \\
 &= q_1^{(b)}(0, a, t, y) dy.
 \end{aligned}$$

Therefore, by (48), (49), (40), (41), and L'Hôpital's rule, we can prove (46) as follows:

$$\begin{aligned}
 & P(R^a(t) \in dy \mid \tau_{a,b} = 1) \\
 &= P(R^a(t) \in dy, M_t(R^a) < b) \frac{P(\tau_{y,b} \in du - t)}{P(\tau_{a,b} \in du)} \Big|_{u=1} \\
 &= \frac{q_1^{(b)}(0, a, t, y) q_2^{(b)}(1-t, y)}{q_2^{(b)}(1, a)} dy.
 \end{aligned}$$

Next, we prove (47). Using the Markov property of R^a , for $0 < s < t < u$, it holds that

$$\begin{aligned}
 & P(R^a(t) \in dy, R^a(s) \in dx, \tau_{a,b} > u) \\
 &= P(R^a(t) \in dy, R^a(s) \in dx, M_u(R^a) < b) \\
 &= P(R^a(s) \in dx, M_s(R^a) < b) P(R^x(t-s) \in dy, M_{t-s}(R^x) < b) \\
 &\quad \times P(M_{u-(t-s)}(R^y) < b) \\
 &= P(R^a(s) \in dx, M_s(R^a) < b) P(R^x(t-s) \in dy, M_{t-s}(R^x) < b) \\
 &\quad \times P(\tau_{y,b} > u - (t-s)).
 \end{aligned}$$

Thus, it follows that

$$\begin{aligned}
 & P(R^a(t) \in dy, R^a(s) \in dx, \tau_{a,b} \in du) \\
 &= -\frac{d}{du} P(R^a(t) \in dy, R^a(s) \in dx, \tau_{a,b} > u)
 \end{aligned}$$

$$= P(R^a(s) \in dx, M_s(R^a) < b) P(R^x(t-s) \in dy, M_{t-s}(R^x) < b) \\ \times P(\tau_{y,b} \in du - (t-s)).$$

On the other hand, by (48), we obtain

$$P(R^a(s) \in dx, \tau_{a,b} \in du) = P(R^a(s) \in dx, M_s(R^a) < b) P(\tau_{x,b} \in du - s).$$

Combining this equality, (49), (40), and L'Hôpital's rule, we can prove (47) as follows:

$$P(R^a(t) \in dy \mid R^a(s) = x, \tau_{a,b} = 1) \\ = \frac{P(R^a(s) \in dx, M_s(R^a) < b) P(R^x(t-s) \in dy, M_{t-s}(R^x) < b) P(\tau_{y,b} \in du - (t-s))}{P(R^a(s) \in dx, M_s(R^a) < b) P(\tau_{x,b} \in du - s)} \Big|_{u=1} \\ = \frac{P(R^x(t-s) \in dy, M_{t-s}(R^x) < b) P(\tau_{y,b} \in du - (t-s))}{P(\tau_{x,b} \in du - s)} \Big|_{u=1} \\ = \frac{q_1^{(b)}(s, x, t, y) q_2^{(b)}(1-t, y)}{q_2^{(b)}(1-s, x)} dy. \quad \square$$

According to Theorem 1, the right sides of (46) and (47) are the transition densities of $H^{a \rightarrow b}$. Therefore, the proof of Proposition 1.2 is completed.

7. Proof of Proposition 1.3

The proof is similar to that in Chapter 2, Theorem 2.8 in [4]. We fix $\gamma \in (0, \frac{1}{2})$. Then, we can find $m_0 \in \mathbb{N}$ so that $\gamma < \frac{m_0 - 3\nu - 6}{2m_0}$ holds. For this m_0 , combining Theorem 1, Skorohod's theorem, Fatou's lemma, and Lemma 3.3, we can take a positive number $C_{m_0, \nu, a, b}$ that satisfies

$$E \left[\left| H^{a \rightarrow b}(r) - a \right|^{2m_0} \right] \leq \frac{C_{m_0, \nu, a, b}}{r^{\nu+1-m_0} (1-r)^{\nu+\frac{5}{2}}}, \\ E \left[\left| H^{a \rightarrow b}(1-r) - b \right|^{2m_0} \right] \leq \frac{C_{m_0, \nu, a, b}}{r^{\nu+2-m_0} (1-r)^{\nu+\frac{3}{2}}}, \\ E \left[\left| H^{a \rightarrow b}(t) - H^{a \rightarrow b}(s) \right|^{2m_0} \right] \leq \frac{C_{m_0, \nu, a, b}}{(t-s)^{\nu+1-m_0} s^{\nu+\frac{3}{2}} (1-t)^{\nu+\frac{5}{2}}}$$

for $t, s, r \in (0, 1)$ with $s < t$. Now, for $n \in \mathbb{N}$, we define

$$F_n = \left\{ \max_{1 \leq k \leq 2^n} \left| H^{a \rightarrow b} \left(\frac{k-1}{2^n} \right) - H^{a \rightarrow b} \left(\frac{k}{2^n} \right) \right| \geq 2^{-n\gamma} \right\},$$

$$a(n, k) = P \left(\left| H^{a \rightarrow b} \left(\frac{k-1}{2^n} \right) - H^{a \rightarrow b} \left(\frac{k}{2^n} \right) \right| \geq 2^{-n\gamma} \right), \quad 1 \leq k \leq 2^n.$$

Then, Chebyshev's inequality yields

$$\begin{aligned} a(n, 1) &\leq 2^{2nm_0\gamma} E \left[\left| H^{a \rightarrow b}(1/2^n) - a \right|^{2m_0} \right] \leq C_{m_0, \nu, a, b} 2^{-n(m_0 - 2m_0\gamma - 2\nu - \frac{7}{2})}, \\ a(n, 2^n) &\leq 2^{2nm_0\gamma} E \left[\left| H^{a \rightarrow b}(1 - 1/2^n) - b \right|^{2m_0} \right] \\ &\leq C_{m_0, \nu, a, b} 2^{-n(m_0 - 2m_0\gamma - 2\nu - \frac{7}{2})}, \end{aligned}$$

and, for $2 \leq k \leq 2^n - 1$,

$$\begin{aligned} a(n, k) &\leq 2^{2nm_0\gamma} E \left[\left| H^{a \rightarrow b}((k-1)/2^n) - H^{a \rightarrow b}(k/2^n) \right|^{2m_0} \right] \\ &\leq C_{m_0, \nu, a, b} 2^{-n(m_0 - 2m_0\gamma - 3\nu - 5)}. \end{aligned}$$

Therefore, $P(F_n) \leq C_{m_0, \nu, a, b} \times 2^{-n(m_0 - 2m_0\gamma - 3\nu - 6)}$, and because $m_0 - 2m_0\gamma - 3\nu - 6 > 0$, we have $P(\liminf_{n \rightarrow \infty} F_n^c) = 1$ by the first Borel-Cantelli lemma. If $\omega \in \liminf_{n \rightarrow \infty} F_n^c$, then there exists $n^*(\omega) \in \mathbb{N}$ such that $\omega \in \bigcap_{n \geq n^*(\omega)} F_n^c$. For $n \geq n^*(\omega)$, we can deduce that

$$\begin{aligned} \left| H^{a \rightarrow b}(t) - H^{a \rightarrow b}(s) \right| &\leq 2 \sum_{j=n+1}^{\infty} 2^{-\gamma j} \\ &= \frac{2}{1 - 2^{-\gamma}} 2^{-(n+1)\gamma}, \quad 0 < t - s < 2^{-n}. \end{aligned}$$

Now, let $t, s \in [0, 1]$ satisfy $0 < t - s < 2^{-n^*(\omega)}$ and choose $n \geq n^*(\omega)$ so that $2^{-(n+1)} \leq t - s < 2^{-n}$. Then, the above inequality yields

$$\left| H^{a \rightarrow b}(t) - H^{a \rightarrow b}(s) \right| \leq \frac{2}{1 - 2^{-\gamma}} |t - s|^\gamma.$$

Hence, $H^{a \rightarrow b}$ is locally Hölder-continuous with exponent γ for $\omega \in \liminf_{n \rightarrow \infty} F_n^c$. \square

8. The Space-Time Reversal Property of the BES(3) House-Moving and Numerical Examples

In this section, we show that the BES(3) house-moving has the space-time reversal property. Furthermore, we demonstrate numerical examples for $H^{0 \rightarrow b}$.

Although the following proposition is showed by [9, Proposition 4.8], we prove it based on our setting for completeness.

PROPOSITION 8.1. *Let $\delta = 3$ ($\nu = \frac{1}{2}$) and $b > 0$. For $0 < t < 1$ and $y \in (0, b)$, we have*

$$P\left(H^{0 \rightarrow b}(t) \in dy\right) = P\left(H^{0 \rightarrow b}(1-t) \in b - dy\right).$$

PROOF. The Fourier expansion of the heat kernel shows that the following equality

$$(50) \quad \sum_{k=-\infty}^{\infty} (n_t(y-x+2k) - n_t(y+x+2k)) \\ = 2 \sum_{n=1}^{\infty} \sin xn\pi \sin yn\pi \exp\left(-\frac{(n\pi)^2}{2}t\right)$$

holds for $x, y \in \mathbb{R}$ and $t > 0$. So, we obtain

$$(51) \quad 2 \sum_{k=-\infty}^{\infty} \frac{y+2k}{t} n_t(y+2k) \\ = \lim_{x \downarrow 0} \sum_{k=-\infty}^{\infty} \frac{1}{x} (n_t(y-x+2k) - n_t(y+x+2k)) \\ = \lim_{x \downarrow 0} 2 \sum_{n=1}^{\infty} \frac{\sin xn\pi}{x} \sin yn\pi \exp\left(-\frac{(n\pi)^2}{2}t\right) \\ = 2 \sum_{n=1}^{\infty} n\pi \sin yn\pi \exp\left(-\frac{(n\pi)^2}{2}t\right), \quad y \in \mathbb{R}, t > 0.$$

Also, in [11], we have

$$(52) \quad J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \quad J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x\right), \\ j_{\frac{1}{2},n} = n\pi \quad (n \in \mathbb{N}).$$

Using (50), (51), (52), and Theorem 5, for $0 < s < t < 1$, $0 < x, y < \eta$, we can obtain the expressions for $q_1^{(\eta)}(s, x, t, y)$ and $q_1^{(\eta)}(0, 0, t, y)$ in the case of $\nu = \frac{1}{2}$:

$$(53) \quad q_1^{(\eta)}(s, x, t, y) = \frac{y}{x} \sum_{k=-\infty}^{\infty} (n_{t-s}(y-x+2k\eta) - n_{t-s}(y+x+2k\eta)),$$

$$(54) \quad q_1^{(\eta)}(0, 0, t, y) = y \sum_{k=-\infty}^{\infty} 2 \frac{y+2k\eta}{t} n_t(y+2k\eta).$$

On the other hand, according to (53), we obtain

$$(55) \quad \begin{aligned} q_1^{(\eta+\varepsilon)}(t, y, 1, \eta) &= \frac{\eta}{y} \sum_{k=-\infty}^{\infty} (n_{1-t}(\eta-y+2k(\eta+\varepsilon)) - n_{1-t}(y+(2k+1)\eta+2k\varepsilon)) \\ &= \frac{\eta}{y} \sum_{k=-\infty}^{\infty} (n_{1-t}(\eta-y+2k(\eta+\varepsilon)) - n_{1-t}(\eta-y-2(k+1)\eta-2k\varepsilon)) \\ &= \frac{\eta}{y} \sum_{k=-\infty}^{\infty} (n_{1-t}(\eta-y+2k(\eta+\varepsilon)) - n_{1-t}(\eta-y+2k\eta+2(k+1)\varepsilon)) \end{aligned}$$

for $\varepsilon > 0$. Using (54) and (55), we get

$$\begin{aligned} q_2^{(\eta)}(1-t, y) &= \lim_{\varepsilon \downarrow 0} \frac{\partial}{\partial \varepsilon} q_1^{(\eta+\varepsilon)}(t, y, 1, \eta) \\ &= \frac{\eta}{y} \sum_{k=-\infty}^{\infty} \left(-2k \frac{\eta-y+2k\eta}{1-t} n_{1-t}(\eta-y+2k\eta) \right. \\ &\quad \left. + 2(k+1) \frac{\eta-y+2k\eta}{1-t} n_{1-t}(\eta-y+2k\eta) \right) \\ &= \frac{\eta}{y} \sum_{k=-\infty}^{\infty} 2 \frac{\eta-y+2k\eta}{1-t} n_{1-t}(\eta-y+2k\eta) \\ &= \frac{\eta}{y(\eta-y)} q_1^{(\eta)}(0, 0, 1-t, \eta-y). \end{aligned}$$

Thus, it holds that

$$\begin{aligned} P\left(H^{0\rightarrow b}(t) \in dy\right) &= \frac{q_1^{(b)}(0, 0, t, y)q_2^{(b)}(1-t, y)}{q_2^{(b)}(1, 0)} dy \\ &= \frac{b}{y(b-y)} \cdot \frac{q_1^{(b)}(0, 0, t, y)q_1^{(b)}(0, 0, 1-t, b-y)}{q_2^{(b)}(1, 0)} dy \\ &= P\left(H^{0\rightarrow b}(1-t) \in b-dy\right), \end{aligned}$$

and the proof is completed. \square

The densities of $H^{0\rightarrow b}(k/10)$ ($1 \leq k \leq 9$) are shown in Figs 1, 2, 3, and 4.

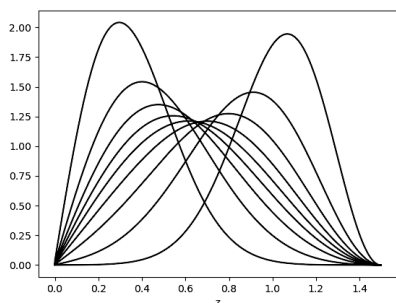


Fig. 1. The densities of $\{H^{0\rightarrow b}(k/10)\}_{k=1}^9$ for $b = 1.5$ and $\delta = 2$.

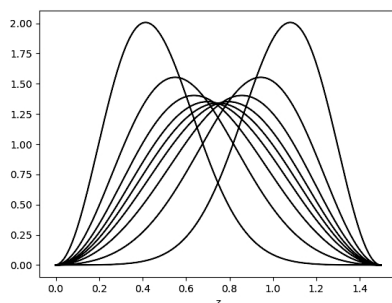


Fig. 2. The densities of $\{H^{0\rightarrow b}(k/10)\}_{k=1}^9$ for $b = 1.5$ and $\delta = 3$.

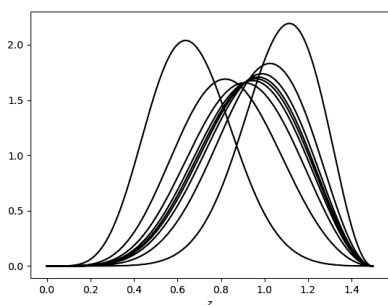


Fig. 3. The densities of $\{H^{0\rightarrow b}(k/10)\}_{k=1}^9$ for $b = 1.5$ and $\delta = 6$.

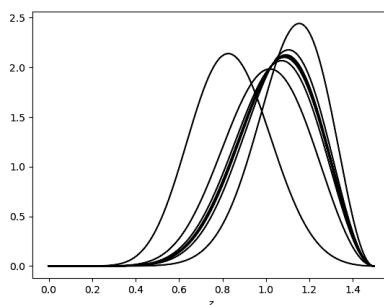


Fig. 4. The densities of $\{H^{0\rightarrow b}(k/10)\}_{k=1}^9$ for $b = 1.5$ and $\delta = 10$.

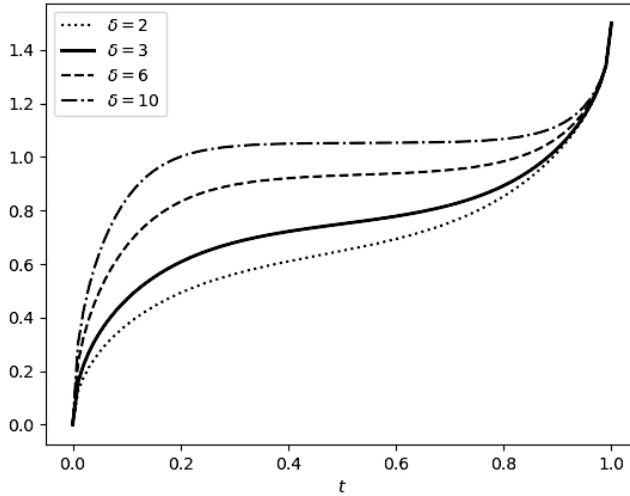


Fig. 5. The graphs of $\{E[H^{0 \rightarrow b}(t)]\}_{t \in [0,1]}$ for $b = 1.5$ and $\delta = 2, 3, 6, 10$.

Further, the graphs of $\{E[H^{0 \rightarrow b}(t)]\}_{t \in [0,1]}$ are shown in Fig 5.

We can also see the space-time reversal property of the BES(3) house-moving in Fig 2 and Fig 5. Figs 1-5 were created using Python.

9. Future Work

We are interested in finding the stochastic differential equations for the BES(δ) house-moving. We are currently investigating this problem by using Theorem 3.

In addition, let $R = \{R(t)\}_{t \geq 0}$ be a regular one-dimensional diffusion on $[0, \infty)$. For an R -bridge $r^{0 \rightarrow b} = \{r^{0 \rightarrow b}(s)\}_{s \in [0,1]}$ ($b > 0$) from 0 to b on $[0, 1]$, we are also interested in the weak convergence of $r^{0 \rightarrow b}|_{K^{-(b+\eta)}}$ as $\eta \downarrow 0$.

Appendix A.

A.1 Bessel functions

Let $J_\alpha(z)$ and $I_\alpha(z)$ denote the Bessel function and modified Bessel function of the first kind with index $\alpha \in \mathbb{R}$, respectively. They are defined

as

$$J_\alpha(z) = \left(\frac{1}{2}z\right)^\alpha \sum_{k \in \mathbb{Z}_+} \frac{\left(-\frac{1}{4}z^2\right)^k}{k! \Gamma(\alpha + k + 1)},$$

$$I_\alpha(z) = \left(\frac{1}{2}z\right)^\alpha \sum_{k \in \mathbb{Z}_+} \frac{\left(\frac{1}{4}z^2\right)^k}{k! \Gamma(\alpha + k + 1)}$$

for $z \in \mathbb{C} \setminus \mathbb{R}_-$. In addition, for $z \in \mathbb{C} \setminus \mathbb{R}_-$, we define

$$K_\alpha(z) := \frac{\pi (I_{-\alpha}(z) - I_\alpha(z))}{2 \sin(\alpha\pi)},$$

when $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, and

$$K_\alpha(z) := \lim_{\beta \rightarrow \alpha} K_\beta(z)$$

when $\alpha \in \mathbb{Z}$. $K_\alpha(z)$ is called the modified Bessel function with index α of the second kind. Moreover, the values of $z^{-\alpha} J_\alpha(z)$ and $z^{-\alpha} I_\alpha(z)$ at zero are written as

$$z^{-\alpha} J_\alpha(z)|_{z=0} = \frac{1}{2^\alpha \Gamma(\alpha + 1)}, \quad z^{-\alpha} I_\alpha(z)|_{z=0} = \frac{1}{2^\alpha \Gamma(\alpha + 1)}.$$

We obtain the following derivatives:

$$(56) \quad \frac{d}{dz} (z^\alpha J_\alpha(z)) = z^\alpha J_{\alpha-1}(z), \quad \frac{d}{dz} (z^{-\alpha} J_\alpha(z)) = -z^{-\alpha} J_{\alpha+1}(z),$$

$$z \in \mathbb{C} \setminus \mathbb{R}_-,$$

$$(57) \quad \frac{d}{dz} (z^\alpha I_\alpha(z)) = z^\alpha I_{\alpha-1}(z), \quad \frac{d}{dz} (z^{-\alpha} I_\alpha(z)) = z^{-\alpha} I_{\alpha+1}(z),$$

$$z \in \mathbb{C} \setminus \mathbb{R}_-.$$

Moreover, using (56) and (57), we have

$$\frac{d}{dz} J_\alpha(z) = J_{\alpha-1}(z) - \frac{\alpha}{z} J_\alpha(z) = -J_{\alpha+1}(z) + \frac{\alpha}{z} J_\alpha(z),$$

$$\frac{d}{dz} I_\alpha(z) = I_{\alpha-1}(z) - \frac{\alpha}{z} I_\alpha(z) = I_{\alpha+1}(z) + \frac{\alpha}{z} I_\alpha(z).$$

In the rest of this section, we assume that $\alpha > -1$. According to [10, (2.2), and (2.8)], there exists $C_\alpha > 0$ such that

$$(58) \quad z^{-\alpha}|J_\alpha(z)| \leq C_\alpha \frac{1}{(1+z)^{\alpha+\frac{1}{2}}}, \quad z \geq 0,$$

$$(59) \quad z^{-\alpha}I_\alpha(z) \leq C_\alpha \frac{1}{(1+z)^{\alpha+\frac{1}{2}}}e^z, \quad z \geq 0.$$

The sequence of positive zeros of the Bessel function J_α is denoted by $\{j_{\alpha,n}\}_{n=1}^\infty$. According to [11], we have

$$J_{\alpha+1}(j_{\alpha,n}) \neq 0 \quad \text{and} \quad j_{\alpha,n} < j_{\alpha+1,n} < j_{\alpha,n+1} \quad (n = 1, 2, \dots).$$

In addition, from [11] we find the following asymptotics as $n \rightarrow \infty$:

$$(60) \quad j_{\alpha,n} \sim n\pi, \quad J_{\alpha+1}(j_{\alpha,n}) \sim (-1)^{n-1} \sqrt{\frac{2}{\pi j_{\alpha,n}}} \sim (-1)^{n-1} \frac{1}{\pi} \sqrt{\frac{2}{n}}.$$

A.2 General results on continuous processes

In this subsection, we introduce some general results used in this paper. The proofs of them are found in [1].

THEOREM 7 ([4, Chapter 2, Theorem 4.15]). *Let $\{X_n\}_{n=1}^\infty$ be the family of $C([0, 1], \mathbb{R}^d)$ -valued random variables. If the family $\{X_n\}_{n=1}^\infty$ is tight and the finite-dimensional distribution of X_n converges to that of some X , then $X_n \xrightarrow{\mathcal{D}} X$ holds.*

LEMMA A.1 ([1, Appendix]). *Let $a, b \in \mathbb{R}^d$, and let X_n and X are \mathbb{R}^d -valued Markovian bridges from a to b on $[0, 1]$ for $n \in \mathbb{N}$. Let X_n and X have the respective transition densities*

$$\begin{aligned} P(X_n(t) \in dy) &= q_n(t, y)dy, \\ P(X_n(t) \in dy \mid X_n(s) = x) &= q_n(s, x, t, y)dy, \\ P(X(t) \in dy) &= q(t, y)dy, \quad P(X(t) \in dy \mid X(s) = x) = q(s, x, t, y)dy \end{aligned}$$

for $0 < s < t < 1, x, y \in \mathbb{R}^d$, and $n \in \mathbb{N}$. If

$$\begin{aligned} \lim_{n \rightarrow \infty} q_n(t, y) &= q(t, y), \quad \text{a.e. } y \in \mathbb{R}^d, \\ \lim_{n \rightarrow \infty} q_n(s, x, t, y) &= q(s, x, t, y), \quad \text{a.e. } (x, y) \in \mathbb{R}^d \times \mathbb{R}^d, \end{aligned}$$

for $0 < s < t < 1$, then the finite-dimensional distribution of X_n converges to that of X as $n \rightarrow \infty$.

THEOREM 8 ([1, Appendix]). For $\varepsilon \in \mathcal{E}$, $X^{(\varepsilon)}$ is a $(C([0, 1], \mathbb{R}^d), \mathcal{B}(C([0, 1], \mathbb{R}^d)))$ -valued random variable defined on $(\Omega^{(\varepsilon)}, \mathcal{F}^{(\varepsilon)}, P^{(\varepsilon)})$. Assume that $\{X^{(\varepsilon)}(0)\}_{\varepsilon \in \mathcal{E}}$ is uniformly integrable and that the following conditions hold:

- (1) For each $u \in (0, \frac{1}{2})$, $\{\pi_{[u, 1-u]} \circ X^{(\varepsilon)}\}_{\varepsilon \in \mathcal{E}}$ is tight.
- (2) For each $\xi > 0$, it holds that

$$\limsup_{u \downarrow 0} \sup_{\varepsilon \in \mathcal{E}} P^{(\varepsilon)} \left(\sup_{0 \leq t \leq u} |X^{(\varepsilon)}(t) - X^{(\varepsilon)}(0)| > \xi \right) = 0,$$

$$\limsup_{u \downarrow 0} \sup_{\varepsilon \in \mathcal{E}} P^{(\varepsilon)} \left(\sup_{1-u \leq t \leq 1} |X^{(\varepsilon)}(t) - X^{(\varepsilon)}(1)| > \xi \right) = 0.$$

Then, the family $\{X^{(\varepsilon)}\}_{\varepsilon \in \mathcal{E}}$ is tight.

LEMMA A.2 (Chapter 2, Problem 4.11 in [4]). For $\varepsilon \in \mathcal{E}$, $X^{(\varepsilon)}$ is a $(C([0, 1], \mathbb{R}^d), \mathcal{B}(C([0, 1], \mathbb{R}^d)))$ -valued random variable defined on $(\Omega^{(\varepsilon)}, \mathcal{F}^{(\varepsilon)}, P^{(\varepsilon)})$. Assume that $\{X^{(\varepsilon)}\}_{\varepsilon \in \mathcal{E}}$ satisfies the following conditions:

- (1) There exists some $\nu > 0$ that satisfies

$$\sup_{\varepsilon \in \mathcal{E}} E^{(\varepsilon)} \left[\left| X^{(\varepsilon)}(0) \right|^\nu \right] < \infty.$$

- (2) There exist $\alpha, \beta, C > 0$ that satisfy

$$\sup_{\varepsilon \in \mathcal{E}} E^{(\varepsilon)} \left[\left| X^{(\varepsilon)}(t) - X^{(\varepsilon)}(s) \right|^\alpha \right] \leq C |t - s|^{1+\beta}, \quad t, s \in [0, 1].$$

Then $\{X^{(\varepsilon)}\}_{\varepsilon \in \mathcal{E}}$ is tight.

LEMMA A.3 ([1, Appendix]). Let $\gamma > 0$. For $\varepsilon \in \mathcal{E}$, $X^{(\varepsilon)}$ is a $(C([0, 1], \mathbb{R}^d), \mathcal{B}(C([0, 1], \mathbb{R}^d)))$ -valued random variable defined on $(\Omega^{(\varepsilon)}, \mathcal{F}^{(\varepsilon)}, P^{(\varepsilon)})$. Assume that

$$F_l^\varepsilon := \left\{ \max_{1 \leq k \leq 2^{l-1}} \left| X^{(\varepsilon)} \left(\frac{k-1}{2^l} \right) - X^{(\varepsilon)} \left(\frac{k}{2^l} \right) \right| \geq 2^{-l\gamma} \right\} \in \mathcal{F}^{(\varepsilon)},$$

$$\varepsilon \in \mathcal{E}, \quad l = 1, 2, \dots$$

satisfy $\sum_{l=1}^{\infty} \sup_{\varepsilon \in \mathcal{E}} P^{(\varepsilon)}(F_l^\varepsilon) < \infty$, then we have

$$\limsup_{u \downarrow 0} \sup_{\varepsilon \in \mathcal{E}} P^{(\varepsilon)} \left(\sup_{0 \leq t \leq u} |X^{(\varepsilon)}(t) - X^{(\varepsilon)}(0)| > \xi \right) = 0, \quad \xi > 0.$$

LEMMA A.4 ([1, Appendix]). Under the same assumption of Lemma A.3, if

$$\begin{aligned} \tilde{F}_l^\varepsilon = \left\{ \max_{2^{l-1} \leq k \leq 2^l} \left| X^{(\varepsilon)} \left(\frac{k-1}{2^l} \right) - X^{(\varepsilon)} \left(\frac{k}{2^l} \right) \right| \geq 2^{-l\gamma} \right\} \in \mathcal{F}^{(\varepsilon)}, \\ \varepsilon \in \mathcal{E}, \quad l = 1, 2, \dots \end{aligned}$$

satisfy $\sum_{l=1}^{\infty} \sup_{\varepsilon \in \mathcal{E}} P^{(\varepsilon)}(\tilde{F}_l^\varepsilon) < \infty$, then we have

$$\limsup_{u \downarrow 0} \sup_{\varepsilon \in \mathcal{E}} P^{(\varepsilon)} \left(\sup_{0 \leq t \leq u} |X^{(\varepsilon)}(1-t) - X^{(\varepsilon)}(1)| > \xi \right) = 0, \quad \xi > 0.$$

LEMMA A.5 ([1, Appendix]). Let S_1 and S_2 be Polish spaces, and let X_n and Y_n be random variables defined on $(\Omega_n, \mathcal{F}_n, P_n)$ that take values in S_1 and S_2 , respectively. If X_n and Y_n are independent and $P_n \circ X_n^{-1}$ and $P_n \circ Y_n^{-1}$ converge to probability measures Q on S_1 and R on S_2 , respectively, then $P_n \circ (X_n, Y_n)^{-1}$ converges to the product measure $Q \times R$.

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