Reichenbach's Common Cause in an Atomless and Complete Orthomodular Lattice

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Abstract

Hofer-Szabo, Redei and Szabo (2000) defined Reichenbach's common cause of two correlated events in an orthomodular lattice. In the present paper it is shown that if logical independent elements in an atomless and complete orthomodular lattice correlate, a common cause of the correlated elements always exists.

1 Introduction

Let $p(A)$ and $p(B)$ be probabilities of the events A and B respectively. When $p(A \wedge B) > p(A)p(B)$, the two events A and B are said to be correlated. Reichenbach (1956) regarded the event C as a common cause of the correlated events A and B when the following conditions were satisfied.

- $0 < p(C) < 1$
- $p(A \wedge B|C) = p(A|C)p(B|C)$
- $p(A \wedge B|\neg C) = p(A|\neg C)p(B|\neg C)$
- $p(A|C) > p(A|\neg C)$
- $p(B|C) > p(B|\neg C)$

When a common cause of the events A and B exists, A and B are correlated (Reichenbach, 1956, pp.159-160). When the events A and B are correlated, does a common cause of A and B exist? Such a problem are investigated by Gyenis and Redei (2004) and Redei and Summers (2002).

Gyenis and Redei (2004) examined whether Reichenbach's common cause of two logically independent and correlated elements in a Boolean algebra exists or not. They showed that uncountably infinite common causes of two logically independent and correlated elements in an atomless Boolean algebra exist (Gyenis and Redei, 2004, Proposition 7).

Algebraic quantum field theory predicts correlations between two events which belong to spacelike separated spacetime regions in many normal states (e.g. Halvorson and Clifton, 2001). Redei (1997) defined Reichenbach's common cause in algebraic quantum field theory, and Summers and Redei (2002) showed that if two events which belong to spacelike separated regions V_1 and V_2 correlate in a faithful normal state, a common cause exists, which locates in the union of the backward light cones of V_1 and V_2 (see also Svozil, Redei and Summers, 2005).

Hofer-Szabo, Redei and Szabo (2000) defined Reichenbach's common cause in an orthomodular lattice. In the present paper we examine whether Reichenbach's common cause of two correlated elements in an orthomodular lattice exsits or not. We show that if logical independent elements in an atomless and complete orthomodular lattice correlate in a completely additive probability measure, a nontrivial common cause of these elements always exists (Proposition 3.9), and that if logically independent elements in an atomless and complete orthomodular lattice correlate in a faithful completely additive probability measure, uncountably infinite common causes of these elements exist (Proposition 3.10).

2 Mathematical preliminaries

Definition 2.1. *A lattice $\mathcal{L}$ is a partially ordered set any two of whose elements a and b have a least upper bound $a \vee b$ and a greatest lower bound $a \wedge b$, which are respectively called the join and the meet of a and b. The least element and the greatest element, if they exist, are denoted by 0 and 1 respectively.*

Definition 2.2. *A lattice $\mathcal{L}$ with 0 and 1 is called orthocomplemented when there is a mapping $a \to a^\perp$ of $\mathcal{L}$ into itself satisfying the following three conditions:*

1. $a \vee a^\perp = 1$ *and* $a \wedge a^\perp = 0$.
2. $a \leq b$ *implies* $a^\perp \geq b^\perp$.
3. $(a^\perp)^\perp = a$ *for every* a.

When $a \leq b^\perp$, we say that a and b are orthogonal.

Definition 2.3. *An orthocomplemented lattice $\mathcal{L}$ is called orthomodular when $a \leq b$ implies $b = a \vee (b \wedge a^\perp)$.*

Definition 2.4. *Let a and b be elements of an orthomodular lattice $\mathcal{L}$. We say that a commutes with b and we write aCb when $a = (a \wedge b) \vee (a \wedge b^\perp)$.*

Definition 2.5. *A complete lattice is a lattice $\mathcal{L}$ in which for every subset S of $\mathcal{L}$, the join $\bigvee\{a|a \in S\}$ and the meet $\bigwedge\{a|a \in S\}$ exists.*

For example, the set of all projections of a von Neumann algebra is a complete orthomodular lattice.

Definition 2.6. *An orthomodular lattice is called atomless when for any nonzero element $a \in \mathcal{L}$ there exists $b \in \mathcal{L}$ such that $0 < b < a$.*

For example, the set of all projections of a von Neumann algebra without direct summand of type I is an atomless orthomodular lattice.

In the present paper, $\mathbb{N}$ denotes $\{1, 2, 3, \dots\}$. For mutually orthogonal elements $x_1, \dots, x_n$ in an orthomodular lattice, $x_1 + \cdots + x_n$ denotes $x_1 \vee \cdots \vee x_n$.

Definition 2.7. *The mapping μ of an orthomodular lattice $\mathcal{L}$ to $[0, 1]$ is called a completely additive probability measure on $\mathcal{L}$ when μ satisfies the following conditions.*

1. $\mu(1) = 1$
2. *For any mutually orthogonal elements* $\{x_a\}_{a \in A}$, *if* $\sum_{a \in A} x_a$ *exists,*

$$\mu\left(\sum_{a \in A} x_a\right) = \sum_{a \in A} \mu(x_a)$$

For example, a normal state on a von Neumann algebra is a completely additive probability measure on the set of all projections of a von Neumann algebra.

Definition 2.8. *Let μ be a completely additive probability measure on an orthomodular lattice $\mathcal{L}$. μ is called faithful when $\mu(x) \neq 0$ for any nonzero element $x \in \mathcal{L}$.*

Lemma 2.9. *Let μ be a completely additive probability measure on an orthomodular lattice $\mathcal{L}$ and let x and y be elements in $\mathcal{L}$.*

1. $\mu(x^\perp) = 1 - \mu(x)$.
2. *If x commutes with y,* $\mu(x) = \mu(x \wedge y) + \mu(x \wedge y^\perp)$.
3. *If* $x \leq y$, $\mu(x) \leq \mu(y)$.
4. *If μ is faithful and $x \leq y$, $x = y$ is equivalent to $\mu(x) = \mu(y)$.*

Proof. 1. Since $1 = \mu(1) = \mu(x + x^\perp) = \mu(x) + \mu(x^\perp)$, $\mu(x^\perp) = 1 - \mu(x)$.

2. Since xCy, $x = (x \wedge y) + (x \wedge y^\perp)$. Therefore $\mu(x) = \mu(x \wedge y) + \mu(x \wedge y^\perp)$.

3. Since $y = x + (x^\perp \wedge y)$, $\mu(y) = \mu(x) + \mu(x^\perp \wedge y) \geq \mu(x)$.

4. When $\mu(x) = \mu(y)$, $\mu(x^\perp \wedge y) = 0$ because $\mu(y) = \mu(x) + \mu(x^\perp \wedge y)$. Since μ is faithful, $x^\perp \wedge y = 0$, which implies $x = y$.

□

3 Existence of Reichenbach's common cause in an atomless and complete orthomodular lattice

Definition 3.1 (Hofer-Szabo, Redei and Szabo, 2000, p.914)**.** *Let $\mathcal{L}$ be an orthomodular lattice, let μ be a completely additive probability measure on $\mathcal{L}$ and let a and b be elements of $\mathcal{L}$. $c \in \mathcal{L}$ is called a common cause of a and b in μ when cCa, cCb, $0 < \mu(c) < 1$ and c satisfies the following equations.*

$$\frac{\mu(a \wedge b \wedge c)}{\mu(c)} = \frac{\mu(a \wedge c)}{\mu(c)} \frac{\mu(b \wedge c)}{\mu(c)} \tag{1}$$

$$\frac{\mu(a \wedge b \wedge c^\perp)}{\mu(c^\perp)} = \frac{\mu(a \wedge c^\perp)}{\mu(c^\perp)} \frac{\mu(b \wedge c^\perp)}{\mu(c^\perp)} \tag{2}$$

$$\frac{\mu(a \wedge c)}{\mu(c)} > \frac{\mu(a \wedge c^\perp)}{\mu(c^\perp)} \tag{3}$$

$$\frac{\mu(b \wedge c)}{\mu(c)} > \frac{\mu(b \wedge c^\perp)}{\mu(c^\perp)} \tag{4}$$

If a commutes with b, $\mu(a) = \mu(a \wedge b) + \mu(a \wedge b^\perp)$ and $\mu(b) = \mu(a \wedge b) + \mu(a^\perp \wedge b)$ by Lemma 2.9. So for any elements a and b such that aCb and $\mu(a \wedge b) > \mu(a)\mu(b)$,

$$\frac{\mu(a \wedge b)}{\mu(b)} > \frac{\mu(a \wedge b^\perp)}{\mu(b^\perp)} \quad \text{and} \quad \frac{\mu(b \wedge a)}{\mu(a)} > \frac{\mu(b \wedge a^\perp)}{\mu(a^\perp)}$$

since $\mu(a \wedge b) > \mu(a)\mu(b)$ implies

$$\frac{\mu(a \wedge b)}{\mu(b)} > \frac{\mu(a) - \mu(a \wedge b)}{1 - \mu(b)} \quad \text{and} \quad \frac{\mu(b \wedge a)}{\mu(a)} > \frac{\mu(b) - \mu(a \wedge b)}{1 - \mu(a)}.$$

Therefore if a commutes with b and $\mu(a \wedge b) > \mu(a)\mu(b)$, both a and b are always common causes of a and b in μ.

Definition 3.2. *Let a, b and c be elements in an orthomodular lattice $\mathcal{L}$. If c is a common cause of a and b, and differs from both a and b, we call the element c a nontrivial common cause.*

Definition 3.3 (Redei, 1995a and 1995b)**.** *Let a and b be elements in an orthomodular lattice $\mathcal{L}$. a and b are called logically independent if the following conditions hold.*

$$a \wedge b \neq 0 \qquad a^\perp \wedge b \neq 0 \qquad a \wedge b^\perp \neq 0 \qquad a^\perp \wedge b^\perp \neq 0$$

We examine whether a nontrivial common cause of two correlated elements in an atomless and complete orthomodular lattice exsits or not (Proposition 3.9 and Proposition 3.10).

We prove the following lemma in reference to Lemma 3 in Redei and Summers (2002).

Lemma 3.4. *Let μ be a completely additive probability measure on an orthomodular lattice $\mathcal{L}$ and let a and b be elements in $\mathcal{L}$ such that $\mu(a \wedge b) > \mu(a)\mu(b)$. Then $1 - \mu(a) - \mu(b) + \mu(a \wedge b) > 0$ and the following facts hold.*

1. *If* $\mu(a) > \mu(a \wedge b)$*, then* $\dfrac{\mu(a \wedge b) - \mu(a)\mu(b)}{1 - \mu(a) - \mu(b) + \mu(a \wedge b)} < \mu(a \wedge b)$*.*

2. *If* $\mu(a) = \mu(a \wedge b)$*, then* $\dfrac{\mu(a \wedge b) - \mu(a)\mu(b)}{1 - \mu(a) - \mu(b) + \mu(a \wedge b)} = \mu(a \wedge b)$*.*

Proof. Since $\mu(a \wedge b) > \mu(a)\mu(b)$, $\mu(a) < 1$ and $\mu(b) < 1$. Therefore

$$1 - \mu(a) - \mu(b) + \mu(a \wedge b) > 1 - \mu(a) - \mu(b) + \mu(a)\mu(b) = (1 - \mu(a))(1 - \mu(b)) > 0.$$

1. If $\mu(a) > \mu(a \wedge b)$, then

$$\begin{aligned}\mu(a)\mu(b) &= \mu(a)((\mu(b) - \mu(a \wedge b)) + \mu(a \wedge b)) = \mu(a)(\mu(b) - \mu(a \wedge b)) + \mu(a)\mu(a \wedge b) \\ &> \mu(a \wedge b)(\mu(b) - \mu(a \wedge b)) + \mu(a)\mu(a \wedge b) = (\mu(a) + \mu(b) - \mu(a \wedge b))\mu(a \wedge b).\end{aligned}$$

So $\mu(a)\mu(b)/\mu(a \wedge b) > \mu(a) + \mu(b) - \mu(a \wedge b)$, which implies

$$1 - \frac{\mu(a)\mu(b)}{\mu(a \wedge b)} < 1 - \mu(a) - \mu(b) + \mu(a \wedge b).$$

Because $1 - \mu(a) - \mu(b) + \mu(a \wedge b) > 0$,

$$\frac{\mu(a \wedge b) - \mu(a)\mu(b)}{1 - \mu(a) - \mu(b) + \mu(a \wedge b)} < \mu(a \wedge b).$$

2. It can be proved in a similar way to the proof of 1.

□

We prove the following lemma in reference to Lemma 3 in Redei and Summers (2002).

Lemma 3.5. *Let* μ *be a completely additive probability measure on an orthomodular lattice* $\mathcal{L}$ *and let* a*,* b *and* c *be elements in* $\mathcal{L}$ *such that* $\mu(a \wedge b) > \mu(a)\mu(b)$*,* $c \leq a \wedge b$ *and*

$$\mu(c) = \frac{\mu(a \wedge b) - \mu(a)\mu(b)}{1 - \mu(a) - \mu(b) + \mu(a \wedge b)}.$$

Then c *is a common cause of* a *and* b *in* μ*.*

Proof. Since $c \leq a$ and $c \leq b$, cCa and cCb (Maeda and Maeda, 1970, Lemma (36.3)). Elementary algebraic calculation shows that

$$\mu(c) = \frac{\mu(a \wedge b) - \mu(a)\mu(b)}{1 - \mu(a) - \mu(b) + \mu(a \wedge b)}$$

is equivalent to

$$(1 - \mu(c))(\mu(a \wedge b) - \mu(c)) = (\mu(a) - \mu(c))(\mu(b) - \mu(c)).$$

Because $a \wedge b \geq c$ and $\mu(a \wedge b) \neq 1$, $\mu(c) \neq 1$. So,

$$\frac{\mu(a \wedge b) - \mu(c)}{1 - \mu(c)} = \frac{\mu(a) - \mu(c)}{1 - \mu(c)} \frac{\mu(b) - \mu(c)}{1 - \mu(c)}$$

Since $a \wedge b \geq c$, one can write

$$x = c + (x \wedge c^{\perp}) \qquad (x = a \wedge b, a, b),$$

which implies

$$\mu(x) = \mu(c) + \mu(x \wedge c^{\perp}) \qquad (x = a \wedge b, a, b).$$

Therefore the equation (2) holds. Since $c \leq a \wedge b$, the equation (1) holds trivially.

Since $\mu(a \wedge b) > \mu(a)\mu(b)$, $\mu(a) < 1$ and $\mu(b) < 1$, which implies $\mu(a) - \mu(c) < 1 - \mu(c)$ and $\mu(b) - \mu(c) < 1 - \mu(c)$. So

$$\frac{\mu(a \wedge c)}{\mu(c)} = \frac{\mu(c)}{\mu(c)} = 1 > \frac{\mu(a) - \mu(c)}{1 - \mu(c)} = \frac{\mu(a \wedge c^{\perp})}{\mu(c^{\perp})},$$

$$\frac{\mu(b \wedge c)}{\mu(c)} = \frac{\mu(c)}{\mu(c)} = 1 > \frac{\mu(b) - \mu(c)}{1 - \mu(c)} = \frac{\mu(b \wedge c^{\perp})}{\mu(c^{\perp})}.$$

Therefore the equations (3) and (4) hold. □

Lemma 3.6. *Let μ be a completely additive probability measure on an atomless and complete orthomodular lattice $\mathcal{L}$ and let r be an element in $\mathcal{L}$ such that $\mu(r) \neq 0$. For any real number α such that $0 < \alpha < \mu(r)$ there exists $c \in \mathcal{L}$ such that $c < r$ and $\mu(c) = \alpha$.*

Proof. Let $\mathbb{S}$ be the set of all subsets of $\mathcal{L}$ and define $\mathbb{T}$ as

$$\Bigg\{ T \in \mathbb{S} \Bigg| \text{For any finite elements } t_1, \ldots, t_k \in T, \text{ they are mutually orthogonal,} \\ \sum_{i=1}^{k} \mu(t_i) \leq \alpha \text{ and } \sum_{i=1}^{k} t_i < r. \Bigg\}.$$

$\mathbb{T}$ is partially ordered by set inclusion. Since $\{0\} \in \mathbb{T}$, $\mathbb{T}$ is not empty.

Let $\mathbb{T}'$ be any linearly ordered subset of $\mathbb{T}$. For any finite elements $t'_1, \ldots, t'_l$ in $\bigcup \mathbb{T}'$, there exists $T' \in \mathbb{T}'$ such that $t'_1, \ldots, t'_l \in T'$ since $\mathbb{T}'$ is a linear ordered subset of $\mathbb{T}$. So $t'_1, \ldots, t'_l$ are mutually orthogonal elements such that $\sum_{i=1}^{l} \mu(t'_i) \leq \alpha$ and $\sum_{i=1}^{l} t'_i < r$. Therefore $\bigcup \mathbb{T}' \in \mathbb{T}$. By Zorn's lemma $\mathbb{T}$ has a maximal element U. Since $\mathcal{L}$ is complete, $\bigvee\{u | u \in U\}$ exists. Define c as $\bigvee\{u | u \in U\}$. Since $u < r$ for any element $u \in U$, $c \leq r$.

Let $\{u_a | a \in A\}$ denote U. Then

$$\mu(c) = \mu\left(\sum_{a \in A} u_a\right) = \sum_{a \in A} \mu(u_a).$$

Since $\sum_{a \in A'} \mu(u_a) \leq \alpha$ for any finite subset A' of A, $\mu(c) \leq \alpha$.

Suppose that $\mu(c) < \alpha$. Because $\mu(c) < \alpha < \mu(r)$ and $c \leq r$, $c < r$. So $c^{\perp} \wedge r \neq 0$. Let v denote $c^{\perp} \wedge r$. Define the countbaly infinite set $\{v_i\}_{i \in \mathbb{N}}$ of mutually orthogonal nonzero elements in $\mathcal{L}$ such that $v > \sum_{i=1}^{n} v_i$ for any $n \in \mathbb{N}$ as follows. Since $\mathcal{L}$ is atomless, there exists an element $v' \in \mathcal{L}$ such that $0 < v' < v$. Define v_1 as v'. If mutually orthogonal nonzero elements $v_1, \ldots, v_k$ such that $v > \sum_{i=1}^{k} v_i$ are defined, then

$$v = \sum_{i=1}^{k} v_i + \left(\left(\sum_{i=1}^{k} v_i\right)^{\perp} \wedge v\right)$$

and $\left(\sum_{i=1}^{k} v_i\right)^{\perp} \wedge v \neq 0$. There exists $v'' \in \mathcal{L}$ such that $\left(\sum_{i=1}^{k} v_i\right)^{\perp} \wedge v > v'' > 0$ since $\mathcal{L}$ is atomless. Define v_{k+1} as v''. Since $v_1^{\perp} \wedge \cdots \wedge v_k^{\perp} = \left(\sum_{i=1}^{k} v_i\right)^{\perp} > v_{k+1}$, any elements in $\{v_1, \ldots, v_{k+1}\}$ are mutually orthogonal and

$$v > \sum_{i=1}^{k} v_i + v_{k+1} = \sum_{i=1}^{k+1} v_i.$$

Because $\mathcal{L}$ is complete, $\sum_{i=1}^{\infty} v_i$ exists and $v \geq \sum_{i=1}^{\infty} v_i$.

If $\mu(v_i) \geq \alpha - \mu(c)$ for any $i \in \mathbb{N}$, $\lim_{n \to \infty} \sum_{i=1}^{n} \mu(v_i) \to \infty$. Because $\mu(v) \geq \sum_{i=1}^{\infty} \mu(v_i)$, this is a contradiction. So there is $v_k \in \{v_i\}_{i \in \mathbb{N}}$ such that $\mu(v_k) < \alpha - \mu(c)$, which implies $\mu(c + v_k) < \alpha$. Therefore for any finite elements $u'_1, \ldots, u'_m$ in $U \cup \{v_k\}$, they are mutually orthogonal, $\sum_{i=1}^{m} \mu(u'_i) \leq \alpha$ and $\sum_{i=1}^{m} u'_i < r$. Therefore $U \cup \{v_k\} \in \mathbb{T}$. This is a contradiction because $v_k \notin U$ and U is a maximal element of $\mathbb{T}$. Therefore $\mu(c) = \alpha$. □

We will prove the following lemma in reference to the proof of Proposition 7 in Gyenis and Redei (2004).

Lemma 3.7. *Let μ be a faithful completely additive probability measure on an atomless and complete orthomodular lattice $\mathcal{L}$, let r be a nonzero element in $\mathcal{L}$ and let α be a real number such that $0 < \alpha < \mu(r)$. Then the set $\{c \in \mathcal{L} | c < r \text{ and } \mu(c) = \alpha\}$ is an uncountably infinite set.*

Proof. By Lemma 3.6, there exists $c \in \mathcal{L}$ such that $\mu(c) = \alpha$ and $c < r$. Since $c < r$, $r \wedge c^{\perp} \neq 0$, that is, $\mu(r \wedge c^{\perp}) \neq 0$. Let β be any real number such that $0 < \beta < \min\{\mu(r \wedge c^{\perp}), \alpha\}$. By Lemma 3.6, there exist $c_{\beta 1}, c_{\beta 2} \in \mathcal{L}$ such that $c_{\beta 1} < c$, $c_{\beta 2} < r \wedge c^{\perp}$ and $\mu(c_{\beta 1}) = \mu(c_{\beta 2}) = \beta$.

Define c_β as $(c \wedge c_{\beta 1}^{\perp}) + c_{\beta 2}$. Then $c_\beta < c \vee (r \wedge c^{\perp}) < r$ and

$$\mu(c_\beta) = \mu(c \wedge c_{\beta 1}^{\perp}) + \mu(c_{\beta 2}) = \mu(c) - \mu(c_{\beta 1}) + \mu(c_{\beta 2}) = \alpha.$$

For any real numbers β and γ such that $0 < \beta < \gamma < \min\{\mu(r \wedge c^{\perp}), \alpha\}$, $c_{\beta 2} \neq c_{\gamma 2}$ since μ is a faithful completely additive probability measure. Therefore $c_\beta \neq c_\gamma$. □

Reichenbach (1956) proved the following lemma (pp.159-160).

Lemma 3.8. *Let μ be a completely additive probability measure on an orthomodular lattice $\mathcal{L}$ and let a, b and c be elements in $\mathcal{L}$. If c is a common cause of a and b in μ, then $\mu(a \wedge b) > \mu(a)\mu(b)$.*

Proof. Since aCc and bCc, $a \wedge bCc$ (Maeda and Maeda, 1970, Lemma (36.4)). So $\mu(a) = \mu(a \wedge c) + \mu(a \wedge c^{\perp})$, $\mu(b) = \mu(b \wedge c) + \mu(b \wedge c^{\perp})$ and $\mu(a \wedge b) = \mu(a \wedge b \wedge c) + \mu(a \wedge b \wedge c^{\perp})$ by Lemma 2.9. Then

$$\mu(a \wedge b) - \mu(a)\mu(b) = \mu(c)(1 - \mu(c)) \left(\frac{\mu(a \wedge c)}{\mu(c)} - \frac{\mu(a \wedge c^{\perp})}{\mu(c^{\perp})} \right) \left(\frac{\mu(b \wedge c)}{\mu(c)} - \frac{\mu(b \wedge c^{\perp})}{\mu(c^{\perp})} \right)$$

by the equations (1) and (2). By the equations (3) and (4), $\mu(a \wedge b) > \mu(a)\mu(b)$. □

Proposition 3.9. *Let μ be a completely additive probability measure on an atomless and complete orthomodular lattice $\mathcal{L}$ and let a and b be elements in $\mathcal{L}$ such that $a \wedge b^{\perp} \neq 0$ and $a^{\perp} \wedge b \neq 0$. Then the following conditions are equivalent.*

1. $\mu(a \wedge b) > \mu(a)\mu(b)$
2. *There exists a nontrivial common cause of a and b in μ.*

Proof. $2 \Rightarrow 1$ It holds by Lemma 3.8.

$1 \Rightarrow 2$ If $\mu(a \wedge b) = \mu(a)$,

$$\frac{\mu(a \wedge b) - \mu(a)\mu(b)}{1 - \mu(a) - \mu(b) + \mu(a \wedge b)} = \mu(a \wedge b)$$

by Lemma 3.4. By Lemma 3.5 $a \wedge b$ is a common cause of a and b in μ. If $a \wedge b = a$ or $a \wedge b = b$, $a \wedge b^{\perp} = 0$ or $a^{\perp} \wedge b = 0$. This contradicts the assumption. So $a \wedge b$ is a nontrivial common cause of a and b in μ.

If $\mu(a \wedge b) \neq \mu(a)$,

$$\frac{\mu(a \wedge b) - \mu(a)\mu(b)}{1 - \mu(a) - \mu(b) + \mu(a \wedge b)} < \mu(a \wedge b)$$

by Lemma 3.4. By Lemma 3.6 there exists $c \in \mathcal{L}$ such that

$$c < a \wedge b \text{ and } \mu(c) = \frac{\mu(a \wedge b) - \mu(a)\mu(b)}{1 - \mu(a) - \mu(b) + \mu(a \wedge b)}.$$

By Lemma 3.5 c is a nontrivial common cause of a and b in μ.

□

Proposition 3.10. *Let μ be a faithful completely additive probability measure on an atomless and complete orthomodular lattice $\mathcal{L}$ and let a and b be elements in $\mathcal{L}$ such that $a \wedge b^{\perp} \neq 0$. Then the following conditions are equivalent.*

1. $\mu(a \wedge b) > \mu(a)\mu(b)$

2. *There exist uncountably infinite nontrivial common causes of a and b in μ.*

Proof. $2 \Rightarrow 1$ It holds by Lemma 3.8.

$1 \Rightarrow 2$ If $\mu(a \wedge b) = \mu(a)$, $a \wedge b = a$ by Lemma 2.9. Then $a \wedge b^{\perp} = a \wedge b \wedge b^{\perp} = 0$. This contradicts the assumption. So $\mu(a \wedge b) < \mu(a)$. By Lemma 3.4

$$\frac{\mu(a \wedge b) - \mu(a)\mu(b)}{1 - \mu(a) - \mu(b) + \mu(a \wedge b)} < \mu(a \wedge b).$$

By Lemma 3.7 the set

$$\left\{ c \in \mathcal{L} \middle| c < a \wedge b \text{ and } \mu(c) = \frac{\mu(a \wedge b) - \mu(a)\mu(b)}{1 - \mu(a) - \mu(b) + \mu(a \wedge b)} \right\}$$

is an uncountably infinite set. By Lemma 3.5 the element in the set is a nontrivial common cause of a and b in μ.

□

4 Concluding remarks

The common cause of a and b in Proposition 3.9 is a lower bound of $a \wedge b$. Redei (1997) called such a common cause a strong common cause (p.1312). There is another common cause c such that $c \not\leq a$ and $c \not\leq b$. Redei (1997) called such a common cause a genuinely probabilistic common cause.

If $\mu(a \wedge b) > \mu(a)\mu(b)$ implies $1 - \mu(a)\mu(b)/\mu(a \wedge b) < \mu(a^{\perp} \wedge b^{\perp})$ in Lemma 3.4, it can be proved that there exists a genuinely probabilistic common cause of two correlated elements in an atomless and complete orthomodular lattice as well as the proof of Proposition 3.9 (cf. Gyenis and Redei, 2004, Propositin 7). When a commutes with b, $\mu(a \wedge b) > \mu(a)\mu(b)$ implies $1 - \mu(a)\mu(b)/\mu(a \wedge b) < \mu(a^{\perp} \wedge b^{\perp})$. But it is not obvious whether the inequality holds in the case where a does not commute with b.

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