Interpretations of quantum mechanics in terms of beable algebras

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Abstract

In terms of beable algebras Halvorson and Clifton [International Journal of Theoretical Physics (1999) **38**, 2441-2484] generalized the uniqueness theorem [Studies in History and Philosophy of Modern Physics (1996) **27**, 181-219] which characterlizes interpretations of quantum mechanics by preferred observables. We examine whether dispersion-free states on beable algebras in the generalized uniqueness theorem can be regarded as truth-value assignments in the case where a preferred observable is the set of all spectral projections of a density operator, and in the case where a preferred observable is the set of all spectral projections of the position operator as well.

Keywords: beable algebra; modal interpretation; truth-value assignment; dispersion-free state; quantum mechanics

1 Introduction

When ψ is a quantum mechanical state of a physical system, ψ gives the probability that a measurement value of a physical quantity A is a. There have been many discussions about whether we can interpret that there are hidden states on which all observational propositions can be assigned truth-value, and that the probability given by ψ is a probability measure on the set of all hidden states. For example, von Neumann, and Jauch and Piron mathematically defined such hidden states, and showed that there was no hidden state in quantum mechanics. But they imposed the condition about incompatible observational propositions on hidden states. Bell (2004) argued that it was not physically proper to impose this condition on hidden states although this is proper to impose quantum mechanical states (pp. 4-6). Then Bell (2004) defined the hidden state which was not imposed the condition about incompatible observational propositions, and showed that there was also no this hidden state in nonrelativistic quantum mechanics (pp. 6-8). In section 2 we will see the fact in detail.

On the other hand, for any observable R there are truth-value assignments to all propositions concerning R. Moreover, given any state ρ, ρ restricted to the set of all propositions concerning R can be expressed as a weighted mixture of truth-value assignments. Then the probability given by ρ restricted to the set of all propositions concerning R can be regarded as a probability on the set of all truth-value assignments, hence we can interpret this probability as the degree of our ignorance. Moreover under some conditions Bub and Clifton (1996) (see also, Bub, Clifton and Goldstein, 2000) determined the maximal set which contains all propositions concerning R, and to which ρ is restricted in order to express as a mixture of truth-value assignments. They called this theorem a uniqueness theorem and R a preferred observable. They showed that each interpretation of quantum mechanics uniquely corresponds to some preferred observable. For example the Kochen-Dieks modal interpretaion corresponds to some density operator in the uniqueness theorem.

Because the uniqueness theorem is proved in a finite dimensional Hilbert space, a position operator cannot be a preferred observable. Halvorson and Clifton (1999) generalized the uniqueness theorem in terms of beable algebras whose definition is given by them, so that we can adopt an observable which has a continuous spectrum as a preferred observable. If the dimension of a Hilbert space is finite, the results of the theorem proved by Halvorson and Clifton coincide with those of the uniqueness theorem. Therefore we call this theorem the generalized uniqueness theorem in the present paper. When the dimension of a Hilbert space is finite, dispersion-free states on beable algebras in the generalized uniqueness theorem coincide with truth-value assignments in the original uniqueness theorem. In section 3 we will show that when the dimension of a Hilbert space is infinite, there appear dispersion-free states which does not exist in a finite dimensional Hilbert space. Then we will point out that these dispersion-free states cannot be regarded as truth-value assignments.

In sections 3 and 4 we will examine whether dispersion-free states on beable algebras in the generalized uniqueness theorem can be regarded as truth-value assignments in the case where a preferred observable is the set of all spectral projections of a density operator, and in the case where a preferred observable is the set of all spectral projections of the position operator as well. In section 3 we will examine the case where a preferred observable is a density operator, and present an interpretation in terms of only dispersion-free states which can be regarded as truth-value assignments. In section 4 we will examine the case where a preferred observable is the set of all spectral projections of the position operator, and point out that all dispersion-free states cannot be regarded as truth-value assignments. Then we will present an interpretation that a physical object exists at some point while dispersion-free states cannot be regarded as truth-value assignments.

2 The generalized uniqueness theorem

In this paper, we use the following notation. Let $\mathcal{H}$ denote a Hilbert space. If $\mathcal{K}$ is a subset of $\mathcal{H}$, let $[\mathcal{K}]$ denote its closed, linear span. If $\mathcal{T}$ is a closed subspace of $\mathcal{H}$, let $P_{\mathcal{T}}$ denote the projection onto $\mathcal{T}$ and let $\mathbb{B}(\mathcal{T})$ denote the set of all bounded operators on $\mathcal{T}$. For a vector $x \in \mathcal{H}$, let P_x denote the projection onto $[x]$. If $\mathbb{S}$ is a subset of $\mathbb{B}(\mathcal{H})$, let $\mathbb{S}'$ denote $\{A \in \mathbb{B}(\mathcal{H}) | AB = BA \text{ for all } B \in \mathbb{S}\}$.

Definition 2.1. *A linear functional ρ on a unital C^*-algebra $\mathfrak{A}$ is called a state if ρ satisfies following conditions:*

1. $\rho(A^*A) \geq 0$ *for any element* $A \in \mathfrak{A}$;
2. $\rho(I) = 1$.

Definition 2.2. *A state ω on a unital C^*-algebra $\mathfrak{A}$ is called a dispersion-free state if $\omega(A^2) = [\omega(A)]^2$ for any self-adjoint element $A \in \mathfrak{A}$.*

Definition 2.3. *A state ρ on a von Neumann algebra $\mathfrak{N}$ is called a normal state if there is a density operator D such that $\rho(A) = tr(DA)$ for any operator $A \in \mathfrak{N}$.*

Definition 2.4 (A finitely additive truth-value assignment)**.** *A mapping μ of the set of all projections in a unital C^*-algebra $\mathfrak{A}$ to $\{0, 1\}$ is called a finitely additive truth-value assignment on $\mathfrak{A}$ if μ satisfies following conditions:*

1. *$\mu(I) = 1$;*

2. *For any mutually orthogonal projections $P, Q \in \mathfrak{A}$, $\mu(P \vee Q) = \mu(P) + \mu(Q)$.*

By Gleason's theorem, the following lemma holds.

Lemma 2.1. *Let $\mathcal{H}$ be a Hilbert space which dimension is at least 3 and finite. Then there is no finitely additive truth-value assignment on $\mathbb{B}(\mathcal{H})$.*

Lemma 2.2 (Hamhalter, 1993, Lemma 5.1)**.** *Let $\mathfrak{N}$ be a properly infinite von Neumann algebra and let μ be a finitely additive truth-value assignment on $\mathfrak{N}$. Then μ can be extended to a dispersion-free state on $\mathfrak{N}$.*

We prove the following theorem, making reference to the proof of Lemma 19 of Doring (2004).

Theorem 2.1. *Let $\mathfrak{N}$ be a properly infinite von Neumann algebra. Then there is no finitely additive truth-value assignment on $\mathfrak{N}$.*

Proof. Suppose that there is a finitely additive truth-value assignment μ on $\mathfrak{N}$. By Lemma 2.2 μ can be extended to a dispersion-free state on $\mathfrak{N}$.

Since $\mathfrak{N}$ is a properly infinite von Neumann algebra, there is a projection Q in $\mathfrak{N}$ such that for some partial isometry $V \in \mathfrak{N}$, $Q = VV^*$ and $Q^\perp = V^*V$ by Lemma 6.3.3 of Kadison and Ringrose (1997). By Lemma 2 of Misra (1967) $\omega(Q) = \omega(VV^*) = \omega(V)\omega(V^*) = \omega(V^*)\omega(V) = \omega(V^*V) = \omega(Q^\perp)$. Since $1 = \omega(Q + Q^\perp) = \omega(Q) + \omega(Q^\perp)$, $\omega(Q) = 1/2$. This contradicts that $\omega(Q) = 0$ or 1. Therefore there is no finitely additive truth-value assignment on $\mathfrak{N}$. □

If $\mathcal{H}$ is an infinite dimensional Hilbert space, $\mathbb{B}(\mathcal{H})$ is a properly infinite von Neumann algebra. Then by Lemma 2.1 and Theorem 2.1 we get:

Theorem 2.2. *Let $\mathcal{H}$ be a Hilbert space which dimension is at least 3. Then there is no finitely additive truth-value assignment on $\mathbb{B}(\mathcal{H})$.*

Therefore finitely additive truth-values cannot be assigned simultaneously to all projections in quantum mechanics. Moreover, finitely additive truth-values cannot be assigned to all projections which belong to each local algebra in algebraic quantum field theory because any local algebra is a properly infinite von Neumann algebra (Baumgartel, 1995, Corollary 1.11.6). But we do not deal with interpretations of algebraic quantum field theory in the present paper (see e.g. Clifton, 2000 and Kitajima, 2004).

For any state ρ on $\mathbb{B}(\mathcal{H})$, Halvorson and Clifton defined the C*-algebra on which ρ can be expressed as a mixture of finitely additive truth-value assignments, and called this C*-algebra a beable algebra after terminology due to Bell (see Bell, 2004, chapters 5, 7 and 19).

Definition 2.5 (Halvorson and Clifton, 1999, p.2447)**.** *Let $\mathfrak{A}$ be a unital C*-algebra, let $\mathfrak{B}$ be a unital C*-subalgebra of $\mathfrak{A}$ and let ρ be a state on $\mathfrak{A}$. $\mathfrak{B}$ is a beable algebra for ρ if and only if $\rho|_{\mathfrak{B}}$ is a mixture of dispersion-free states, that is, if and only if there is a probability measure μ on the space $\mathbf{S}$ of dispersion-free states on $\mathfrak{B}$ such that*

$$\rho(A) = \int_{\mathbf{S}} \omega_s(A) d\mu(s) \quad (\forall A \in \mathfrak{B}).$$

Halvorson and Clifton (1999) proved the following theorem in terms of beable algebras.

Theorem 2.3 (Halvorson and Clifton, 1999, Theorem 4.5)**.** *Let D be a density operator on $\mathcal{H}$, let $\mathcal{D}$ be the range of D and let ρ be the state on $\mathbb{B}(\mathcal{H})$ such that $\rho(A) = tr(DA)$ for any operator $A \in \mathbb{B}(\mathcal{H})$. Let $\mathbb{P}$ be a set of mutually commuting self-adjoint operators and let $\mathcal{S}$ be $[\mathbb{P}''\mathcal{D}]$. We call $\mathbb{P}$ a preferred observable. Let $\mathfrak{B}$ be a C*-subalgebra of $\mathbb{B}(\mathcal{H})$ and let $\mathfrak{B}$ satisfy the following conditions:*

1. *$\mathfrak{B}$ is a beable algebra for ρ;*
2. *$\mathbb{P} \subseteq \mathfrak{B}$;*
3. *$U\mathfrak{B}U^* = \mathfrak{B}$ for any unitary operator $U \in \mathbb{B}(\mathcal{H})$ such that $U \in \mathbb{P}'$ and $U \in \{D\}'$;*
4. *$\mathfrak{B}$ is a maximal with respect to conditions 1, 2 and 3.*

Then $\mathfrak{B}$ is $\mathbb{B}(\mathcal{S}^{\perp}) \oplus \mathfrak{N}$ where $\mathfrak{N}$ is a maximal Abelian von Neumann subalgebra of $(\mathbb{P} \cup \{D\})''P_{\mathcal{S}}$ such that $\mathbb{P}''P_{\mathcal{S}} \subseteq \mathfrak{N}$.

If $\mathcal{H}$ is a finite dimensional Hilbert space and D is an one dimensional projection, $\mathfrak{B}$ is uniquely determined and the set of all projections in $\mathfrak{B}$ coincides with the set of definite projections in the original uniqueness theorem proved by Bub and Clifton (1996) (see Halvorson and Clifton, 1999, Corollary 4.6 (ii) and Remark 4.7). Then we call Theorem 2.3 the generalized uniqueness theorem.

3 The case where a preferred observable is the set of all spectral projections of a density operator

In this section we adopt the set of all spectral projections of a density operator as a preferred observable in the generalized uniqueness theorem (Theorem 2.3).

Corollary 3.1 (Halvorson and Clifton, 1999, Corollary 4.6 (i))**.** *Let D be a density operator on $\mathcal{H}$, let $\mathcal{D}$ be the range of D and let ρ be the state on $\mathbb{B}(\mathcal{H})$ such that $\rho(A) = tr(DA)$ for any operator $A \in \mathbb{B}(\mathcal{H})$. Let $\mathbb{P}$ be the set of all spectral projections of D. Let $\mathfrak{B}$ be a C*-subalgebra of $\mathbb{B}(\mathcal{H})$ and let $\mathfrak{B}$ satisfy the following conditions:*

1. $\mathfrak{B}$ *is a beable algebra for* ρ*;*

2. $\mathbb{P} \subseteq \mathfrak{B}$*;*

3. $U\mathfrak{B}U^* = \mathfrak{B}$ *for any unitary operator* $U \in \mathbb{B}(\mathcal{H})$ *such that* $U \in \{D\}'$*;*

4. $\mathfrak{B}$ *is maximal with respect to conditions 1, 2 and 3.*

Then $\mathfrak{B}$ *is* $\mathbb{B}(\mathcal{D}^{\perp}) \oplus \{D\}'' P_{\mathcal{D}}$*.*

As easily seen, the set of all projections in $\mathbb{B}(\mathcal{D}^{\perp}) \oplus \{D\}'' P_{\mathcal{D}}$ coincides with the set ($\mathrm{Def}_{\mathrm{KD}}(W)$ in Clifton (1995)) of all projections which have simultaneously definite values under the Kochen-Dieks modal interpretation. Therefore Corollary 3.1 can be regarded as one of the theorems that motivate the Kochen-Dieks modal interpretation of quantum mechanics (cf. Clifton, 1995, Section 6 and Halvorson and Clifton, 1999, Remark 4.7).

If we regard dispersion-free states on $\mathbb{B}(\mathcal{D}^{\perp}) \oplus \{D\}'' P_{\mathcal{D}}$ as truth-value assignments, it is natural to think that for any dispersion-free state ω, $\vee_{i \in \mathbb{N}} P_i$ is false ($\omega(\vee_{i \in \mathbb{N}} P_i) = 0$) whenever all projections in a set $\{P_i | i \in \mathbb{N}\}$ of mutually orthogonal projections in $\mathbb{B}(\mathcal{D}^{\perp}) \oplus \{D\}'' P_{\mathcal{D}}$ are false ($\omega(P_i) = 0$ for any $i \in \mathbb{N}$). If ω is a normal state, this holds. But when $\{D\}'' P_{\mathcal{D}}$ contains a set $\{P_i | i \in \mathbb{N}\}$ of mutually orthogonal countably infinite non-zero projections, there is a dispersion-free state ω' on $\mathbb{B}(\mathcal{D}^{\perp}) \oplus \{D\}'' P_{\mathcal{D}}$ such that $\omega'(\vee_{i \in \mathbb{N}} P_i) = 1$ and $\omega'(P_i) = 0$ for any $i \in \mathbb{N}$ as shown below (Proposition 3.1). Then $\vee_{i \in \mathbb{N}} P_i$ is true and P_i is false for any $i \in \mathbb{N}$. Therefore we cannot regard this state as a truth-value assignment.

Lemma 3.1. *Let* $\mathfrak{A}$ *be an Abelian von Neumann algebra on* $\mathcal{H}$*, let* P *be a non-zero projection in* $\mathfrak{A}$ *and let* J_0 *be a proper ideal in* $\mathfrak{A}$ *which does not contain* P*. Then there is a dispersion-free state* ω *on* $\mathfrak{A}$ *such that* $\omega(P) = 1$ *and* $\omega(X) = 0$ *for any operator* $X \in J_0$*.*

Proof. Let $\mathcal{J}$ be the set of all proper ideals in $\mathfrak{A}$. Define $\bar{\mathcal{J}}$ as $\{J \in \mathcal{J} | J_0 \subseteq J \text{ and } P \notin J\}$. $\bar{\mathcal{J}}$ is partially ordered by set inclusion. Every linearly ordered subset $\bar{\mathcal{J}}'$ of $\bar{\mathcal{J}}$ has an upper bound in $\bar{\mathcal{J}}$ because $\cup \bar{\mathcal{J}}'$ is a proper ideal such that $J_0 \subseteq \cup \bar{\mathcal{J}}'$ and $P \notin \cup \bar{\mathcal{J}}'$. $\bar{\mathcal{J}}'$ is not empty because $J_0 \in \bar{\mathcal{J}}'$. Zorn's lemma implies that $\bar{\mathcal{J}}$ has a maximal element $\bar{J}$.

Define $\bar{\bar{\mathcal{J}}}$ as $\{J \in \mathcal{J} | \bar{J} \subseteq J\}$. Similarly we can show that $\bar{\bar{\mathcal{J}}}$ has a maximal element $\bar{\bar{J}}$ by Zorn's lemma. Any proper ideal J' such that $\bar{\bar{J}} \subseteq J'$ contains $\bar{J}$, so $J' \in \bar{\bar{\mathcal{J}}}$. Because $\bar{\bar{J}}$ is a maximal element of $\bar{\bar{\mathcal{J}}}$, $J' = \bar{\bar{J}}$. Therefore $\bar{\bar{J}}$ is a maximal ideal in $\mathfrak{A}$.

We will show that $\bar{J} = \bar{\bar{J}}$. Suppose that $P \in \bar{\bar{J}}$. If $P^{\perp} \in \bar{\bar{J}}$, then $I = P + P^{\perp} \in \bar{\bar{J}}$. This contradicts that $\bar{\bar{J}}$ is a proper ideal, so $P^{\perp} \notin \bar{\bar{J}}$. Since $\bar{J} \subseteq \bar{\bar{J}}$, $P^{\perp} \notin \bar{J}$. Define J_1 as $\{AP^{\perp} + B | A \in \mathfrak{A} \text{ and } B \in \bar{J}\}$. J_1 is an ideal and $\bar{J} \subset J_1$. Because $\bar{J}$ is a maximal element of $\bar{\mathcal{J}}$, $J_1 \notin \bar{\mathcal{J}}$. Since $J_0 \subset J_1$, $P \in J_1$. Then there are $A \in \mathfrak{A}$ and $B \in \bar{J}$ such that $P = AP^{\perp} + B$. Since $B \in \bar{J}$, $P = BP \in \bar{J}$. This is a contradiction. Therefore $P \notin \bar{\bar{J}}$. $J_0 \subseteq \bar{\bar{J}}$, so $\bar{\bar{J}} \in \bar{\mathcal{J}}$. Because $\bar{J} \subseteq \bar{\bar{J}}$ and $\bar{J}$ is a maximal element of $\bar{\mathcal{J}}$, $\bar{J} = \bar{\bar{J}}$.

By Exercise 4.6.29 (iii) of Kadison and Ringrose (1997), there is a dispersion-free state ω on $\mathfrak{A}$ such that $\bar{J} = \{A \in \mathfrak{A} | \omega(A) = 0\}$. Therefore $\omega(P) = 1$ and $\omega(X) = 0$ for any operator $X \in J_0$. □

Proposition 3.1. *Let* $\mathcal{K}$ *be a closed subspace of* $\mathcal{H}$ *and let* $\mathfrak{A}$ *be an Abelian von Neumann algebra on* $\mathcal{K}$ *which contains a set* $\{P_i | i \in \mathbb{N}\}$ *of mutually orthogonal countably infinite non-zero projections. Then there is a dispersion-free state* ω *on* $\mathbb{B}(\mathcal{K}^{\perp}) \oplus \mathfrak{A}$ *such that* $\omega(\sum_{i \in \mathbb{N}} P_i) = 1$ *and* $\omega(P_k) = 0$ *for any* $k \in \mathbb{N}$*.*

Proof. Let $\mathfrak{A}''$ be the von Neumann algebra on $\mathcal{H}$ which is generated by $\mathfrak{A}$. Define

$$J_0 := \{A \in \mathfrak{A}'' | \exists \text{finite elements } P_{i_1}, \ldots, P_{i_n} \in \{P_i | i \in \mathbb{N}\}, \forall x \in \mathcal{H}$$
$$[(P_{i_1} + \cdots + P_{i_n})x = 0 \wedge \left(\sum_{i \in \mathbb{N}} P_i\right) x = x] \to Ax = 0\}.$$

Then J_0 is a proper ideal of $\mathfrak{A}''$. Any projection P_k in $\{P_i | i \in \mathbb{N}\}$ belongs to J_0 and $\sum_{i \in \mathbb{N}} P_i$ does not belong to J_0. By Lemma 3.1, there is a dispersion-free state ω on $\mathfrak{A}''$ such that $\omega(\sum_{i \in \mathbb{N}} P_i) = 1$ and $\omega(P_k) = 0$ for any $k \in \mathbb{N}$.

By Theorem 4.3.13 (ii) of Kadison and Ringrose (1997), there is a state ω on $\mathbb{B}(\mathcal{K}^\perp) \oplus \mathfrak{A}$ which is an extention of ω_0. Since $\sum_{i \in \mathbb{N}} P_i \le P_{\mathcal{K}}$, $\omega(P_{\mathcal{K}}^\perp) = 0$. For any operator B in $\mathbb{B}(\mathcal{K}^\perp)$, $|\omega(B)| = |\omega(P_{\mathcal{K}}^\perp B P_{\mathcal{K}}^\perp)| \le \omega(P_{\mathcal{K}}^\perp)\omega((BP_{\mathcal{K}}^\perp)^*(BP_{\mathcal{K}}^\perp)) = 0$. Therefore ω is a dispersion-free state on $\mathbb{B}(\mathcal{K}^\perp) \oplus \mathfrak{A}$. □

Although there is a dispersion-free state on $\mathbb{B}(\mathcal{D}^\perp) \oplus \{D\}'' P_{\mathcal{D}}$ which is not normal, the normal state $\mathrm{tr}(D\cdot)$ on $\mathbb{B}(\mathcal{D}^\perp) \oplus \{D\}'' P_{\mathcal{D}}$ can be expressed as a mixture of dispersion-free normal states as shown below (Theorem 3.1).

Theorem 3.1. *Let D be a density operator on $\mathcal{H}$ and let $\mathcal{D}$ be the range of D. Define ϕ as the state on $\mathbb{B}(\mathcal{D}^\perp) \oplus \{D\}'' P_{\mathcal{D}}$ such that $\phi(A) = tr(DA)$ for any operator A in $\mathbb{B}(\mathcal{D}^\perp) \oplus \{D\}'' P_{\mathcal{D}}$. Let $\mathbf{S}_n$ be the set of all dispersion-free normal states on $\mathbb{B}(\mathcal{D}^\perp) \oplus \{D\}'' P_{\mathcal{D}}$. Then ϕ is a mixture of dispersion-free states in $\mathbf{S}_n$, that is, there is a probability measure μ on $\mathbf{S}_n$ such that*

$$\phi(A) = \int_{\mathbf{S}_n} \omega_s(A) d\mu(s) \quad (\forall A \in \mathbb{B}(\mathcal{D}^\perp) \oplus \{D\}'' P_{\mathcal{D}}).$$

Proof. Let $D = \sum_i d_i P_i$ be the spectral resolution of D where $\{P_i\}$ is the set of spectral projections of D. Any non-zero spectral projection P_k of D is a minimal projection of $\mathbb{B}(\mathcal{D}^\perp) \oplus \{D\}'' P_{\mathcal{D}}$ and belongs to the center of $\mathbb{B}(\mathcal{D}^\perp) \oplus \{D\}'' P_{\mathcal{D}}$. By Lemma 3.3 of Plymen (1968), a dispersion-free normal state ω_k on $\mathbb{B}(\mathcal{D}^\perp) \oplus \{D\}'' P_{\mathcal{D}}$ uniquely exists whose support is P_k. Since $P_k \le P_{\mathcal{D}}$, $\omega_k(P_{\mathcal{D}}^\perp) = 0$. Then $|\omega_k(B)| = |\omega_k(P_{\mathcal{D}}^\perp B P_{\mathcal{D}}^\perp)| \le \omega_k(P_{\mathcal{D}}^\perp)\omega_k((BP_{\mathcal{D}}^\perp)^*(BP_{\mathcal{D}}^\perp)) = 0$ for any operator B in $\mathbb{B}(\mathcal{D}^\perp)$.

Any operator in $\mathbb{B}(\mathcal{D}^\perp) \oplus \{D\}'' P_{\mathcal{D}}$ can be written as $A_0 \oplus \sum_i a_i P_i$ where A_0 is an operator in $\mathbb{B}(\mathcal{D}^\perp)$, a_i is a complex number and $\{P_i\}$ is the set of spectral projections of D. Since $\phi|_{\{D\}'' P_{\mathcal{D}}}$ and $\omega_i|_{\{D\}'' P_{\mathcal{D}}}$ ($\forall i \in \mathbb{N}$) are weakly continuous (Kadison and Ringrose, 1997, Exercise 7.6.4), $\phi(A_0 \oplus \sum_i a_i P_i) = \phi(\sum_i a_i P_i) = \sum_i a_i \phi(P_i)$ and $\sum_i d_j \omega_j (A_0 \oplus \sum_i a_i P_i) = \sum_j \phi(P_j)\omega_j(A_0 \oplus \sum_i a_i P_i) = \sum_j \phi(P_j)\omega_j(\sum_i a_i P_i) = \sum_j a_j \phi(P_j)$. Therefore $\phi = \sum_j d_j \omega_j$. □

Due to Theorem 3.1, we can interpret the state ϕ as follows.

The ignorance interpretation of ϕ Some dispersion-free state in $\mathbf{S}_n$ is the real state. But because of our ignorance we cannot tell which state is real and a probability measure on $\mathbf{S}_n$ represents a degree of our ignorance.

4 The case where a preferred observable is the set of all spectral projections of the position operator

The position operator Q and its domain $\mathcal{D}(Q)$ on the Hilbert space $L^2(\mathbb{R})$ are defined as

$$\mathcal{D}(Q) = \{f \in L^2(\mathbb{R}) | \int_{\mathbb{R}} |xf(x)|^2 dx < \infty\} \quad (Qf)(x) = xf(x) \ (\forall f \in \mathcal{D}(Q)).$$

Let $E(S)$ be a spectral projection of Q corresponding to a Borel set S.

In this section we adopt $\{E(S)|S \text{ is a Borel set}\}$ as a preferred observable in the generalized uniqueness theorem (Theorem 2.3).

Corollary 4.1. *Let D be a density operator on $L^2(\mathbb{R})$, let $\mathcal{D}$ be a range of D and let ρ be the state on $\mathbb{B}(\mathcal{H})$ such that $\rho(A) = tr(DA)$ for any operator $A \in \mathbb{B}(\mathcal{H})$. Let $\mathbb{P}$ be $\{E(S)|S$ is a Borel set$\}$ and let $\mathcal{S}$ be $[\mathbb{P}''\mathcal{D}]$. Let $\mathfrak{B}$ be a C^*-subalgebra of $\mathbb{B}(\mathcal{H})$ and let $\mathfrak{B}$ satisfy the following conditions:*

1. *$\mathfrak{B}$ is a beable algebra for ρ;*
2. *$\mathbb{P} \subseteq \mathfrak{B}$;*
3. *$U\mathfrak{B}U^* = \mathfrak{B}$ for any unitary operator $U \in \mathbb{B}(\mathcal{H})$ such that $U \in \mathbb{P}'$ and $U \in \{D\}'$;*
4. *$\mathfrak{B}$ is a maximal with respect to conditions 1, 2 and 3.*

Then $\mathfrak{B}$ is $\mathbb{B}(\mathcal{S}^{\perp}) \oplus \mathbb{P}''P_{\mathcal{S}}$.

Proof. Since $\mathcal{S}$ is invariant on $\mathbb{P}''$, $P_{\mathcal{S}} \in (\mathbb{P}'')'$ (Kitajima, 2004, Lemma 4). By Proposition 5.5.6 of Kadison and Ringrose (1997), $(\mathbb{P}''P_{\mathcal{S}})' = P_{\mathcal{S}}(\mathbb{P}'')'P_{\mathcal{S}} = (\mathbb{P}'')'P_{\mathcal{S}}$. Because $\mathbb{P}''$ is a maximal Abelian von Neumann algebra on $\mathcal{H}$ (Kadison and Ringrose, 1997, Example 5.1.6), $(\mathbb{P}'')' = \mathbb{P}''$. Then $(\mathbb{P}''P_{\mathcal{S}})' = \mathbb{P}''P_{\mathcal{S}}$, that is, $\mathbb{P}''P_{\mathcal{S}}$ is a maximal Abelian von Neumann algebra on $\mathcal{S}$. By Theorem 2.3, $\mathfrak{B}$ is $\mathbb{B}(\mathcal{S}^{\perp}) \oplus \mathbb{P}''P_{\mathcal{S}}$. □

If $\mathcal{D}$ in Corollary 4.1 is $\mathcal{H}$, then $\mathfrak{B}$ is $\mathbb{P}''$. We will examine such a case. Let $\mathfrak{M}$ be $\mathbb{P}''$ and let $\mathbf{S}_{\mathfrak{M}}$ be the set of all dispersion-free states on $\mathfrak{M}$.

The fact that there is no normal dispersion-free state on $\mathfrak{M}$ was proved in Proposition 1 of Halvorson (2001). It is also derived from the following proposition. We prove this proposition, making reference to the proof of Theorem 1 of Ishigaki (2001).

Proposition 4.1. *For any dispersion-free state ω on $\mathfrak{M}$ there is a set $\{S_i|i \in \mathbb{N}\}$ of mutually disjoint Borel sets on $\mathbb{R}$ such that $\omega(E(\cup_{i=0}^{\infty} S_i)) = 1$ and $\omega(E(S_i)) = 0$ for all $i \in \mathbb{N}$.*

Proof. Let ω be a dispersion-free state on $\mathfrak{M}$. Then

$$1 = \omega(I) = \omega(E(\mathbb{R})) = \omega(E(\cup_{n \in \mathbb{Z}}[n, n+1))).$$

If $\omega(E([n, n+1))) = 0$ for all $n \in \mathbb{Z}$, the proof is completed. We will consider the case where there is $n_0 \in \mathbb{Z}$ such that $\omega(E([n_0, n_0+1))) = 1$. We define S_0 as $[n_0, n_0+1)$. Since $\omega(E([n_0, n_0+1))) = \omega(E([n_0, n_0+\frac{1}{2}))) + \omega(E([n_0+\frac{1}{2}, n_0+1)))$, either $\omega(E([n_0, n_0+\frac{1}{2})))$

or $\omega(E([n_0+\frac{1}{2},n_0+1)))$ equals to 1. We define $S_1=[\lambda_1,\lambda_1+\frac{1}{2})$ as the set which satisfies $\omega(E(S_1))=1$ and is either $[n_0,n_0+\frac{1}{2})$ or $[n_0+\frac{1}{2},n_0+1)$. We define T_1 as $S_0\setminus S_1$. Then

$$S_0=S_1+T_1 \quad \omega(E(S_1))=1 \quad \omega(E(T_1))=0.$$

Since $\omega(E(S_1))=\omega(E([\lambda_1,\lambda_1+\frac{1}{2^2})))+\omega(E([\lambda_1+\frac{1}{2^2},\lambda+\frac{1}{2})))$, either $\omega(E([\lambda_1,\lambda_1+\frac{1}{2^2})))$ or $\omega(E([\lambda_1+\frac{1}{2^2},\lambda+\frac{1}{2})))$ equals to 1. We define S_2 as the set which satisfies $\omega(E(S_2))=1$ and is either $[\lambda_1,\lambda_1+\frac{1}{2^2})$ or $[\lambda_1+\frac{1}{2^2},\lambda_1+\frac{1}{2})$. We define T_2 as $S_1\setminus S_2$. Then

$$S_1=S_2+T_2 \quad \omega(E(S_2))=1 \quad \omega(E(T_2))=0.$$

When we repeat the similar operation, we get

$$S_k=S_{k+1}+T_{k+1} \quad \omega(E(S_{k+1}))=1 \quad \omega(E(T_{k+1}))=0$$

for any $k\in\mathbb{N}$.

By the principle of successive division there is a real number μ such that $\cap_{k\in\mathbb{N}}S_k=\{\mu\}$ or $\cap_{k\in\mathbb{N}}S_k$ is empty, so $E(\cap_{k\in\mathbb{N}}S_k)=0$. Because $S_0=\sum_{k\in\mathbb{N}}T_k+\cap_{k\in\mathbb{N}}S_k$, $E(S_0)=E(\sum_{k\in\mathbb{N}}T_k)$. Therefore $\omega(E(\sum_{k\in\mathbb{N}}T_k))=1$ and $\omega(E(T_k))=0$ for all $k\in\mathbb{N}$. □

If we interpret any spectral projection $E(S)$ as the statement 'a physical object exists in S' and any dispersion-free state ω on $\mathfrak{M}$ as a truth-value assignment, there is a Borel set $\cup_{k\in\mathbb{N}}S_k$ such that a physical object exists in $\cup_{k\in\mathbb{N}}S_k$ and a physical object does not exist in S_k for any $k\in\mathbb{N}$ by Proposition 4.1. Therefore we do not interpret $E(S)$ as a statement 'a physical object exists in S' and ω as a truth-value assignment. Then we will not investigate how large a beable algebra containing $\mathbb{P}$ can be. We take $\mathfrak{M}$ as the beable algebra for any normal state on $L^2(\mathbb{R})$.

Let $\mathcal{B}_f$ be the set of all bounded Borel sets of $\mathbb{R}$. Define $\mathbf{S}_f$ and $\mathbf{S}_i$ as

$$\mathbf{S}_f:=\{\omega\in\mathbf{S}_{\mathfrak{M}}|\exists S\in\mathcal{B}_f\ \omega(E(S))=1\},$$

$$\mathbf{S}_i:=\{\omega\in\mathbf{S}_{\mathfrak{M}}|\forall S\in\mathcal{B}_f\ \omega(E(S))=0\}.$$

Then $\mathbf{S}_{\mathfrak{M}}=\mathbf{S}_f\cup\mathbf{S}_i$ and $\mathbf{S}_f\cap\mathbf{S}_i=\emptyset$. $\mathbf{S}_f$ is not empty by Exercise 4.6.29 (iv) of Kadison and Ringrose (1996). We will show that $\mathbf{S}_i$ is not empty in Proposition 4.4.

For any point $\lambda\in\mathbb{R}$, we define $\mathbf{S}_\lambda$ as

$$\mathbf{S}_\lambda:=\{\omega\in\mathbf{S}_{\mathfrak{M}}|\forall\epsilon>0\ \omega(E((\lambda-\epsilon,\lambda+\epsilon)))=1\}.$$

We will show that $\mathbf{S}_\lambda$ is not empty for any $\lambda\in\mathbb{R}$ in Proposition 4.3.

Proposition 4.2. $\mathbf{S}_f=\bigcup_{\lambda\in\mathbb{R}}\mathbf{S}_\lambda$ *and* $\mathbf{S}_\lambda\cap\mathbf{S}_{\lambda'}=\emptyset$ *when* $\lambda\neq\lambda'$.

Proof. $\bigcup_{\lambda\in\mathbb{R}}\mathbf{S}_\lambda\subseteq\mathbf{S}_f$ is trivial. We will show that $\mathbf{S}_f\subseteq\bigcup_{\lambda\in\mathbb{R}}\mathbf{S}_\lambda$. Let ω be any dispersion-free state which belongs to $\mathbf{S}_f$. By the assumption, there is a bounded closed set S such that $\omega(E(S))=1$. Suppose that $\forall\lambda\in S,\exists\epsilon_\lambda>0\ \omega(E((\lambda-\epsilon_\lambda,\lambda+\epsilon_\lambda)))=0$. Because $S\subseteq\cup_{\lambda\in S}(\lambda-\epsilon_\lambda,\lambda+\epsilon_\lambda)$ and S is compact, there are finite points $\lambda_1,\ldots,\lambda_n$ such that $S\subseteq\cup_{k=1}^n(\lambda_k-\epsilon_{\lambda_k},\lambda_k+\epsilon_{\lambda_k})$. Then

$$\begin{aligned}1&=\omega(E(S))=\omega\left(E\left(\cup_{k=1}^n(\lambda_k-\epsilon_{\lambda_k},\lambda_k+\epsilon_{\lambda_k})\right)\right)\\&\leq\omega\left(\sum_{k=1}^n E((\lambda_k-\epsilon_{\lambda_k},\lambda_k+\epsilon_{\lambda_k}))\right)=\sum_{k=1}^n\omega(E((\lambda_k-\epsilon_{\lambda_k},\lambda_k+\epsilon_{\lambda_k})))=0.\end{aligned}$$

This is a contradiction, so $\exists \lambda \in S, \forall \epsilon > 0\ \omega\left(E\left((\lambda-\epsilon, \lambda+\epsilon)\right)\right) = 1$. Therefore $\mathbf{S}_f \subseteq \bigcup_{\lambda \in \mathbb{R}} \mathbf{S}_\lambda$.

Suppose that there are two points λ and λ' such that $\lambda \neq \lambda'$ and $\mathbf{S}_\lambda \cap \mathbf{S}_{\lambda'} \neq \emptyset$, then there is a dispersion-free state ω in $\mathbf{S}_\lambda \cap \mathbf{S}_{\lambda'}$ such that $\omega\left(E\left((\lambda-\epsilon, \lambda+\epsilon)\right)\right) = 1, \omega\left(E\left((\lambda'-\epsilon, \lambda'+\epsilon)\right)\right) = 1$ and $(\lambda-\epsilon, \lambda+\epsilon) \cap (\lambda'-\epsilon, \lambda'+\epsilon) = \emptyset$ for some real number $\epsilon > 0$. Since $\omega\left(E\left((\lambda-\epsilon, \lambda+\epsilon)\right)\right) = 1$, $\omega\left(E\left((\lambda'-\epsilon, \lambda'+\epsilon)\right)\right) = 0$. This is a contradiction. Therefore $\mathbf{S}_\lambda \cap \mathbf{S}_{\lambda'} = \emptyset$ when $\lambda \neq \lambda'$.

□

Proposition 4.2 shows that there is a point $\lambda \in \mathbb{R}$ such that $\mathbf{S}_\lambda \neq \emptyset$. But it does not show whether $\mathbf{S}_\lambda$ is empty or not for any $\lambda \in \mathbb{R}$. Halvorson showed that there are countably infinite dispersion-free states in $\mathbf{S}_\lambda$ for any $\lambda \in \mathbb{R}$ (Halvorson, 2001, Proposition 2). We prove it in another way.

Lemma 4.1. *For any point $\lambda \in \mathbb{R}$ there is a set $\{S_k | k \in \mathbb{N}\}$ of mutually disjoint sets such that for any $k \in \mathbb{N}$, λ belongs to a closure of S_k and $S_k \subset (\lambda, \lambda+1)$.*

Proof. We will prove the case where $\lambda = 0$. Define

$$S_k := \bigcup_{n \in \mathbb{N}} \left(\frac{2^k+1}{2^{2n+k}}, \frac{2^k+2}{2^{2n+k}} \right) \quad (\forall k \in \mathbb{N}).$$

0 belongs to a closure of S_k for any $k \in \mathbb{N}$ because $\frac{2^k+(3/2)}{2^{2n+k}}$ belongs to S_k for any $k, n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{2^k+(3/2)}{2^{2n+k}} = 0$. S_k is contained in $(0, 1)$.

We will show that $S_k \cap S_{k'} = \emptyset$ when $k \neq k'$. We assume that $k < k'$. Because

$$S_k \cap S_{k'} = \cup_{n \in \mathbb{N}} \cup_{n' \in \mathbb{N}} \left(\left(\frac{2^k+1}{2^{2n+k}}, \frac{2^k+2}{2^{2n+k}} \right) \cap \left(\frac{2^{k'}+1}{2^{2n'+k'}}, \frac{2^{k'}+2}{2^{2n'+k'}} \right) \right),$$

it is sufficient to show that

$$\left(\frac{2^k+1}{2^{2n+k}}, \frac{2^k+2}{2^{2n+k}} \right) \cap \left(\frac{2^{k'}+1}{2^{2n'+k'}}, \frac{2^{k'}+2}{2^{2n'+k'}} \right) = \emptyset$$

for any $n, n' \in \mathbb{N}$.

When $n \leq n'$,

$$\begin{aligned} \frac{2^k+1}{2^{2n+k}} - \frac{2^{k'}+2}{2^{2n'+k'}} &\geq \frac{2^k+1}{2^{2n'+k}} - \frac{2^{k'}+2}{2^{2n'+k'}} \quad (\because n' \geq n) \\ &= \frac{2^{k'-k}-2}{2^{2n'+k'}} \geq 0 \quad (\because k'-k \geq 1). \end{aligned}$$

When $n > n'$,

$$\begin{aligned} \frac{2^{k'}+1}{2^{2n'+k'}} - \frac{2^k+2}{2^{2n+k}} &= \frac{2^{2(n-n')}(1+2^{-k'}) - 1 - 2^{-k+1}}{2^{2n}} \\ &\geq \frac{2^2(1+2^{-k'}) - 1 - 2^0}{2^{2n}} \quad (\because n-n' \geq 1, k \geq 1) \\ &= \frac{2^{-k'+2}+2}{2^{2n}} > 0. \end{aligned}$$

Therefore $S_k \cap S_{k'} = \emptyset$ when $k \neq k'$. □

For any $\lambda \in \mathbb{R}$ we define

$$\mathbf{S}_{\lambda,+} := \{\omega \in \mathbf{S}_\lambda | \forall \epsilon > 0 \ \ \omega(E((\lambda, \lambda+\epsilon))) = 1\},$$

$$\mathbf{S}_{\lambda,-} := \{\omega \in \mathbf{S}_\lambda | \forall \epsilon > 0 \ \ \omega(E((\lambda-\epsilon, \lambda))) = 1\}.$$

Then $\mathbf{S}_\lambda = \mathbf{S}_{\lambda,+} \cup \mathbf{S}_{\lambda,-}$ and $\mathbf{S}_{\lambda,+} \cap \mathbf{S}_{\lambda,-} = \emptyset$.

Proposition 4.3. *Let λ be any point in $\mathbb{R}$. The power $\aleph_{\lambda,+}$ of $\mathbf{S}_{\lambda,+}$ and the power $\aleph_{\lambda,-}$ of $\mathbf{S}_{\lambda,-}$ are greater than or equal to $\aleph_0$.*

Proof. Let λ be any point in $\mathbb{R}$. By Lemma 4.1 there is a set $\{S_k | k \in \mathbb{N}\}$ of mutually disjoint sets such that λ belongs to a closure of S_k and $S_k \subset (\lambda, \lambda+1)$ for any $k \in \mathbb{N}$. Let S_i be any set in $\{S_k | k \in \mathbb{N}\}$. Define

$$J_0 := \{X \in \mathfrak{M} | \exists \epsilon > 0, \forall f \in L^2(\mathbb{R}) \ [E((\lambda-\epsilon, \lambda+\epsilon) \cap S_i)f = f \to Xf = 0]\}.$$

J_0 is a proper ideal in $\mathfrak{M}$. $E(S_i) \notin J_0$ and $E((\lambda-\epsilon, \lambda+\epsilon)^c) \in J_0$ for any $\epsilon > 0$. By Lemma 3.1 there is a dispersion-free state ω_i on $\mathfrak{M}$ such that $\omega_i(E(S_i)) = 1$ and $\omega_i(E((\lambda-\epsilon, \lambda+\epsilon))) = 1$ for any $\epsilon > 0$. Because $\omega_i(E(S_i^c)) = 0$, $\omega_i(E(S_k)) = 0$ when $i \neq k$. Since $\omega_i(E(S_i)) = 1$, $\omega_i \in \mathbf{S}_{\lambda,+}$. Therefore $\aleph_{\lambda,+} \geq \aleph_0$.

Similarly we can prove that $\aleph_{\lambda,-} \geq \aleph_0$. □

Next we examine whether $\mathbf{S}_i$ is empty or not.

Proposition 4.4. *The power $\aleph_i$ of $\mathbf{S}_i$ satisfies $\aleph_i \geq 2^{\aleph_0}$.*

Proof. Let λ be a real number contained in $[0,1)$ and let $\mathcal{B}_f$ be a set of all bounded Borel sets. Define

$$J_\lambda := \{X \in \mathfrak{M} | \exists S \in \mathcal{B}_f, \exists \epsilon \in (0,1), \forall f \in L^2(\mathbb{R}) \\ [E(S \cup (\cup_{n\in\mathbb{Z}}[\lambda+2n+1-\epsilon, \lambda+2n+1+\epsilon]))f = 0 \to Xf = 0]\}.$$

J_λ is a proper ideal. $E(S) \in J_\lambda$ for any bounded Borel set S and $E(\cup_{n\in\mathbb{Z}}[\lambda+2n+1-\epsilon, \lambda+2n+1+\epsilon]) \in J_\lambda$ for any real number $\epsilon \in (0,1)$.

By Lemma 3.1 there is a dispersion-free state ω_λ on $\mathfrak{M}$ such that $\omega_\lambda(X) = 0$ for any operator $X \in J_\lambda$. Then $\omega_\lambda(E(S)) = 0$ for any bounded Borel set S. Because $(\cup_{n\in\mathbb{Z}}[\lambda+2n+1-\epsilon, \lambda+2n+1+\epsilon])^c = \cup_{n\in\mathbb{Z}}(\lambda+2n-(1-\epsilon), \lambda+2n+(1-\epsilon))$, $\omega_\lambda(E(\cup_{n\in\mathbb{Z}}(\lambda+2n-\epsilon, \lambda+2n+\epsilon))) = 1$ for any real number $\epsilon \in (0,1)$.

Similarly for any real number $\lambda' \in [0,1)$ which is not λ there is a dispersion-free state $\omega_{\lambda'}$ such that $\omega_{\lambda'}(E(S)) = 0$ for any bounded Borel set S and $\omega_{\lambda'}(E(\cup_{n\in\mathbb{Z}}(\lambda'+2n-\epsilon, \lambda'+2n+\epsilon))) = 1$ for any real number $\epsilon \in (0,1)$. There is a real number $\epsilon' \in (0,1)$ such that $(\cup_{n\in\mathbb{Z}}(\lambda+2n-\epsilon', \lambda+2n+\epsilon')) \cap (\cup_{n\in\mathbb{Z}}(\lambda'+2n-\epsilon', \lambda'+2n+\epsilon')) = \emptyset$. $\omega_\lambda \neq \omega_{\lambda'}$ because $\omega_{\lambda'}(E(\cup_{n\in\mathbb{Z}}(\lambda+2n-\epsilon', \lambda+2n+\epsilon'))) = 0$ and $\omega_\lambda(E(\cup_{n\in\mathbb{Z}}(\lambda+2n-\epsilon', \lambda+2n+\epsilon'))) = 1$. Therefore $\aleph_i \geq 2^{\aleph_0}$.

□

Although $\mathbf{S}_i$ is not empty, the following fact holds.

Theorem 4.1. *Let ψ be any normal state on $\mathfrak{M}$. ψ is a mixture of dispersion-free states in $\bigcup_{\lambda\in\mathbb{R}}\mathbf{S}_\lambda$, that is, there is a probability measure μ on $\bigcup_{\lambda\in\mathbb{R}}\mathbf{S}_\lambda$ such that*

$$\psi(A) = \int_{\bigcup_{\lambda\in\mathbb{R}}\mathbf{S}_\lambda} \omega_s(A)d\mu(s) \quad (\forall A\in\mathfrak{M}).$$

Proof. Let $\{S_k|k\in\mathbb{N}\}$ be a set of bounded Borel sets such that $\mathbb{R}=\bigcup_{k=1}^\infty S_k$ and $S_j\cap S_k = \emptyset$ when $j\neq k$. When X is any operator in $\mathfrak{M}$, $X = XE(\bigcup_{k=1}^\infty S_k) = X(\sum_{k=1}^\infty E(S_k)) = \sum_{k=1}^\infty(XE(S_k))$. Since there is a probability measure μ on the space $\mathbf{S}_\mathfrak{M}$ on the dispersion-free state on $\mathfrak{M}$ such that

$$\psi(A) = \int_{\mathbf{S}_\mathfrak{M}} \omega_s(A)d\mu(s) \quad (\forall A\in\mathfrak{M})$$

by Proposition 2.2 of Halvorson and Clifton (1999) and ψ is weakly continuous (Kadison and Ringrose, 1997, Exercise 7.6.4 (i)),

$$\psi(X) = \sum_{k=1}^\infty \psi(XE(S_k)) = \sum_{k=1}^\infty \int_{\mathbf{S}_\mathfrak{M}} \omega_s(XE(S_k))d\mu(s).$$

Because

$$\begin{aligned}
\sum_{k=1}^\infty \int_{\mathbf{S}_\mathfrak{M}} |\omega_s(XE(S_k))|d\mu(s) &= \sum_{k=1}^\infty \int_{\mathbf{S}_\mathfrak{M}} \sqrt{\overline{\omega_s(XE(S_k))}\omega_s(XE(S_k))}d\mu(s)\\
&= \sum_{k=1}^\infty \int_{\mathbf{S}_\mathfrak{M}} \sqrt{\omega_s((XE(S_k))^*)\omega_s(XE(S_k))}d\mu(s)\\
&= \sum_{k=1}^\infty \int_{\mathbf{S}_\mathfrak{M}} \sqrt{\omega_s((\sqrt{X^*X}E(S_k))^2)}d\mu(s)\\
&= \sum_{k=1}^\infty \int_{\mathbf{S}_\mathfrak{M}} \omega_s(\sqrt{X^*X}E(S_k))d\mu(s)\\
&= \sum_{k=1}^\infty \psi(\sqrt{X^*X}E(S_k))\\
&= \psi(\sqrt{X^*X}) < \infty,
\end{aligned}$$

$$\psi(X) = \int_{\mathbf{S}_\mathfrak{M}} \sum_{k=1}^\infty \omega_s(XE(S_k))d\mu(s) = \int_{\mathbf{S}_\mathfrak{M}} \sum_{k=1}^\infty (\omega_s(X)\omega_s(E(S_k)))d\mu(s).$$

If $\omega_s(E(S_k)) = 0$ for any $k\in\mathbb{N}$, then $\sum_{k=1}^\infty(\omega_s(X)\omega_s(E(S_k))) = 0$. If $\omega_s(E(S_i)) = 1$ for some $i\in\mathbb{N}$, $\sum_{k=1}^\infty(\omega_s(X)\omega_s(E(S_k))) = \omega_s(X)$. Therefore ψ is a mixture of $\mathbf{S}_f$. By Proposition 4.2, ψ is a mixture of $\bigcup_{\lambda\in\mathbb{R}}\mathbf{S}_\lambda$. □

Due to Theorem 4.1, we can interpret the state ψ as follows.

The ignorance interpretation of ψ Some dispersion-free state in $\bigcup_{\lambda\in\mathbb{R}}\mathbf{S}_\lambda$ is the real state. But because of our ignorance we cannot tell which state is real and a probability measure on $\bigcup_{\lambda\in\mathbb{R}}\mathbf{S}_\lambda$ represents a degree of our ignorance.

Although $\mathbf{S}_i$ is not empty (Proposition 4.4), we will examine the interpretation of those dispersion-free states in $\bigcup_{\lambda\in\mathbb{R}}\mathbf{S}_\lambda$.

Let ω be a dispersion-free state in $\bigcup_{\lambda\in\mathbb{R}}\mathbf{S}_\lambda$. When K is a density operator, $\mathrm{tr}(KE(S))$ is interpreted as the probability that a physical object is detected in a region S. We intepret $\omega(E(S))$ in the same way. But ω is not a normal state (Proposition 4.1) while $\mathrm{tr}(K\cdot)$ is a normal state. Then we restrict regions in which a physical object can be measured to finitely additive class $\mathcal{F}$ generated by the set of all open intervals in $\mathbb{R}$. For example, sets defined in the proof of Lemma 4.1 cannot be regarded as regions in which a physical object can be measured. This restriction excludes these sets from the set of regions in which a physical object can be measured.

The interpretation on measurements of a physical object Let ω be any dispersion-free state in $\bigcup_{\lambda\in\mathbb{R}}\mathbf{S}_\lambda$ and F be any set in $\mathcal{F}$. $\omega(E(F))$ is a probability that a physical object is detected in the region F.

Let $\omega_{\lambda,+}$ and $\omega'_{\lambda,+}$ be different dispersion-free states in $\mathbf{S}_{\lambda,+}$. Since $\omega_{\lambda,+}(E((a,b))) = \omega'_{\lambda,+}(E((a,b)))$ for any open interval (a,b), $\omega_{\lambda,+}(E(F)) = \omega'_{\lambda,+}(E(F))$ for any set F in $\mathcal{F}$ by mathematical induction. Therefore no measurement can distinguish $\omega_{\lambda,+}$ from $\omega'_{\lambda,+}$.

It is natural to think that a physical object exists at a point λ when a probability that a physical object is detected in $(\lambda-\epsilon,\lambda+\epsilon)$ is 1 for any real number $\epsilon > 0$.

The interpretation on the existence of a physical object Let ω be any dispersion-free state in $\bigcup_{\lambda\in\mathbb{R}}\mathbf{S}_\lambda$. ω belongs to $\mathbf{S}_\lambda$ if and only if a physical object exists at a point λ.

Let $\omega_{\lambda,+}$ be a dispersion-free state in $\mathbf{S}_{\lambda,+}$ and let $\omega_{\lambda,-}$ be a dispersion-free state in $\mathbf{S}_{\lambda,-}$. $\omega_{\lambda,+}$ and $\omega_{\lambda,-}$ are the same state in terms of the existence of a physical object although $\omega_{\lambda,+}(E((\lambda-\epsilon,\lambda))) \neq \omega_{\lambda,-}(E((\lambda-\epsilon,\lambda)))$ for any real number $\epsilon > 0$.

5 Summary

We have examined whether dispersion-free states on beable algebras in the generalized uniqueness theorem can be regarded as truth-value assignments in the case where a preferred observable is the set of all spectral projections of a density operator, and in the case where a preferred observable is the set of all spectral projections of the position operator as well.

If a preferred observable in Theorem 2.3 is the set of all spectral projections of a density operator D, $\mathfrak{B}$ in Theorem 2.3 is $\mathbb{B}(\mathcal{D}^\perp)\oplus\{D\}''P_{\mathcal{D}}$ (Corollary 3.1). When $\{D\}''P_{\mathcal{D}}$ contains a set $\{P_i|i\in\mathbb{N}\}$ of mutually orthogonal countably infinite non-zero projections, there is a dispersion-free state ω' on $\mathbb{B}(\mathcal{D}^\perp)\oplus\{D\}''P_{\mathcal{D}}$ such that $\omega'(\vee_{i\in\mathbb{N}}P_i) = 1$ and $\omega'(P_i) = 0$ for any $i\in\mathbb{N}$ (Proposition 3.1). If we interpret this state as a truth-value assignment, $\vee_{i\in\mathbb{N}}P_i$ is true and P_i is false for any $i\in\mathbb{N}$. Therefore we cannot regard this state as a truth-value assignment. But a normal state $\mathrm{tr}(D\cdot)$ on $\mathbb{B}(\mathcal{D}^\perp)\oplus\{D\}''P_{\mathcal{D}}$ can be expressed as a mixture of dispersion-free normal states (Theorem 3.1). Due to this theorem, we can interpret that some dispersion-free normal state which can be regarded as truth-value assignment is the real state.

If a preferred observable in Theorem 2.3 is the set $\{E(S)|S \text{ is a Borel set}\}$ of all spectral projections of the position operator, $\mathfrak{B}$ in Theorem 2.3 is $\{E(S)|S \text{ is a Borel set}\}''$ in the case where $\mathcal{D} = \mathcal{H}$ (Corollary 4.1). Let $\mathfrak{M}$ be $\{E(S)|S \text{ is a Borel set}\}''$. If we interpret a dispersion-free state on $\mathfrak{M}$ as a truth-value assignment, there is a set $\{S_i|i \in \mathbb{N}\}$ of Borel sets such that a physical object exists in $\cup_{i\in\mathbb{N}}S_i$ and does not exist in S_i for any $i \in \mathbb{N}$ (Propsotion 4.1). Therefore we cannot regard any dispersion-free state on $\mathfrak{M}$ as a truth-value assignment. Then we interpret $\omega(E(S))$ as a probability that a physical object is detected in S where ω is a dispersion-free state on $\mathfrak{M}$ and $E(S)$ is a spectral projection of the position operator. If we interpret a dispersion-free state in $\mathbf{S}_i$ as real, $\omega(E(S'))$ is the probability that a physical object is detected in S' for any finite region S'. Then the probability that a physical object is detected in any finite region is 0. Therefore this state cannot be regarded as real. Although $\mathbf{S}_i$ is not empty (Proposition 4.4), any normal state on $\mathfrak{M}$ can be expressed as a mixture of dispersion-free states in $\bigcup_{\lambda\in\mathbb{R}} \mathbf{S}_\lambda$ (Theorem 4.1). Then we can regard some dispersion-free state in $\bigcup_{\lambda\in\mathbb{R}} \mathbf{S}_\lambda$ as real. Under this interpretation a physical object exists at some point.

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