A remark on the modal interpretation of algebraic quantum field theory

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Abstract

Clifton determined the maximal beable algebra for each faithful normal state in a local algebra [Phys. Lett. A 271 (2000) 167, Proposition 1]. In the present paper we will determine the maximal beable algebra for any normal state under the same conditions as Clifton's.

Keywords: Modal interpretation; Beable algebra; von Neumann algebra; Algebraic quantum field theory

1 Introduction

Clifton characterized the Kochen-Dieks modal interpretation of nonrelativistic quantum mechanics as the interpretation that requires that the maximal set of projections which have simultaneously definite values is determined solely in terms of the state and the structure of the Hilbert space ([1] Section 6). Moreover, to extend the Kochen-Dieks modal interpretation to algebraic quantum field theory, Clifton determined the maximal beable algebra, which is defined in section 2 (Definition 1), for each faithful normal state in a von Neumann algebra under the condition that that algebra was determined solely in terms of this state and the algebraic structure of this von Neumann algebra ([2] Proposition 1). But it seems still possible to extend Clifton's theorem a little further because Clifton's beable algebra is determined only for each faithful normal state.

In Theorem 11 we will determine the maximal beable algebra for any normal state under the same conditions as Clifton's, and thereby show that Clifton's conjecture ([2] p.172) is valid.

2 Generalized Clifton's theorem

In this paper, we use the following notation. If $\mathcal{K}$ is a subset of some Hilbert space $\mathcal{H}$, let $[\mathcal{K}]$ denote its closed, linear span. If $\mathcal{T}$ is a closed subspace of $\mathcal{H}$, let $P_{\mathcal{T}}$ denote the projection onto $\mathcal{T}$. For a vector $x \in \mathcal{H}$, let P_x denote the projection onto $[x]$. Let $\mathbb{B}(\mathcal{H})$ denote the set of all bounded operators on a Hilbert space $\mathcal{H}$.

Definition 1 ([3] p.2447)**.** *Let $\mathfrak{N}$ be a unital C^*-algebra, let $\mathfrak{B}$ be a unital C^*-subalgebra of $\mathfrak{N}$ and let ρ be a state on $\mathfrak{N}$. $\mathfrak{B}$ is a beable algebra for ρ if and only if $\rho|_{\mathfrak{B}}$ is a mixture* *of dispersion-free states, that is, if and only if there is a probability measure* μ *on the space* $\mathbf{S}$ *of dispersion-free states on* $\mathfrak{B}$ *such that*

$$\rho(A) = \int_{\mathbf{S}} \omega_s(A) d\mu(s) \qquad \forall A \in \mathfrak{B}$$

When ρ can be expressed as $\rho(\cdot) = \mathrm{tr}(D\cdot)$ with some density operator D, sometimes we say "$\mathfrak{B}$ is a beable algebra for D" instead "$\mathfrak{B}$ is a beable algebra for ρ".

Definition 2. *Let* ρ *be a normal state on a von Neumann algebra* $\mathfrak{N}$. *Define*

$$\mathfrak{C}_{\rho,\mathfrak{N}} := \{A \in \mathfrak{N} | \rho([A, B]) = 0 \textit{ for all } B \in \mathfrak{N}\}$$

$\mathfrak{C}_{\rho,\mathfrak{N}}$ *is called the centralizer of* ρ *of* $\mathfrak{N}$.

Definition 3. *Let* ρ *be a normal state on a von Neumann algebra* $\mathfrak{N}$. *Define*

$$P_\rho := \sup\{P | P \textit{ is a projection of } \mathfrak{N} \textit{ such that } \rho(P) = 0\}$$

$I - P_\rho$ *is called the support of* ρ.

If S is the support of a normal state ρ on a von Neumann algebra $\mathfrak{N}$, then $\rho(SA) = \rho(AS) = \rho(A)$ and $\rho(S^\perp A) = \rho(AS^\perp) = 0$ for all $A \in \mathfrak{N}$, S belongs to the centralizer $\mathfrak{C}_{\rho,\mathfrak{N}}$ of a state ρ of a von Neumann algebra $\mathfrak{N}$, and $\rho|_{S\mathfrak{N}S}$ is faithful on $S\mathfrak{N}S$ (c.f. [2] p.172).

Lemma 4. *Let* $\mathcal{K}$ *be a closed subspace of a Hilbert space* $\mathcal{H}$ *and let* $\mathfrak{A}$ *be a* C^**-algebra on* $\mathcal{H}$. *If* $Ax \in \mathcal{K}$ *for any self-adjoint operator* $A \in \mathfrak{A}$ *and any vector* $x \in \mathcal{K}$, *then* $P_\mathcal{K} \in \mathfrak{A}'$.

Proof. Since $AP_\mathcal{K} = P_\mathcal{K} A P_\mathcal{K}$ for any self-adjoint operator $A \in \mathfrak{A}$, $P_\mathcal{K} A = (AP_\mathcal{K})^* = (P_\mathcal{K} A P_\mathcal{K})^* = P_\mathcal{K} A P_\mathcal{K} = AP_\mathcal{K}$.
Any operator $B \in \mathfrak{A}$ can be expressed in the form $H + iK$, where H and K are self-adjoint operators in $\mathfrak{A}$. Therefore $P_\mathcal{K} \in \mathfrak{A}'$. □

Lemma 5. *Let* $\mathfrak{N}$ *be a von Neumann algebra on a Hilbert space* $\mathcal{H}$ *with a cyclic and separating vector, let* ρ *be a normal state on* $\mathfrak{N}$, *let* S *be the support of* ρ *and let* $\mathcal{S}$ *be the range of* S. *Let* $\mathfrak{B}$ *be a* C^**-subalgebra of* $\mathfrak{N}$.
If $\mathfrak{B}$ *is a beable algebra for* ρ, $\mathfrak{B}$ *is a beable algebra for* P_ψ *where* ψ *is any vector in* $[\mathfrak{B}\mathcal{S}]$.

Proof. Because $\mathfrak{N}$ has a cyclic and separating vector, there exists a vector $x \in \mathcal{H}$ such that $\rho(A) = \langle x, Ax \rangle$ for any operator $A \in \mathfrak{N}$ ([4] Theorem 2.5.31). Since $\rho|_{S\mathfrak{N}S}$ is faithful on $S\mathfrak{N}S$, x is a separating vector for $S\mathfrak{N}S$. A subset $\{Bx/\|Bx\| \; | B \in (S\mathfrak{N}S)', Bx \neq 0\}$ of $\mathcal{S}$ is dense in $\mathcal{S}$ because x is a cyclic vector for $(S\mathfrak{N}S)'$ ([4] Proposition 2.5.3). For any operator $B \in (S\mathfrak{N}S)'$ such that $Bx \neq 0$ and any positive operator $A \in \mathfrak{N}$,

$$\begin{aligned}
\langle Bx, ABx \rangle &= \langle Bx, (S^\perp A S^\perp + SAS^\perp + S^\perp AS + SAS)Bx \rangle \\
&= \langle Bx, (SAS)Bx \rangle = \|B(SAS)^{1/2}x\|^2 \\
&\leq \|B\|^2 \langle x, (SAS)x \rangle = \|B\|^2 \rho(A)
\end{aligned}$$

since $Bx \in \mathcal{S}$ and $(SAS)^{1/2} \in S\mathfrak{N}S$. For any operator $X, Y \in \mathfrak{B}$,

$$\langle \frac{Bx}{\|Bx\|}, [X,Y]^*[X,Y] \frac{Bx}{\|Bx\|} \rangle \leq \frac{\|B\|^2}{\|Bx\|^2} \rho([X,Y]^*[X,Y]) = 0$$

by [3] Proposition 2.2. Then $\mathfrak{B}$ is a beable algebra for $P_{Bx/\|Bx\|}$ by [3] Proposition 2.2. Because a subset $\{Bx/\|Bx\| \; |B \in (S\mathfrak{N}S)', Bx \neq 0\}$ of $\mathcal{S}$ is dense in $\mathcal{S}$, $\mathfrak{B}$ is a beable algebra for P_y where y is any vector in $\mathcal{S}$ by [3] Lemma 2.6 (i). We can prove that $\mathfrak{B}$ is a beable algebra for P_ψ where ψ is any vector in $[\mathfrak{B}\mathcal{S}]$ in a similar way to the proof of [3] Lemma 2.7. □

We will prove Lemma 6, making reference to the proof of [3] Theorem 2.8 (i).

Lemma 6. *Let $\mathfrak{N}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ with a cyclic and separating vector, let ρ be a normal state on $\mathfrak{N}$, let S be the support of ρ and let $\mathcal{S}$ be the range of S. Let $\mathfrak{B}$ be a C^*-subalgebra of $\mathfrak{N}$ and define $\mathcal{T} := [\mathfrak{B}\mathcal{S}]$.*
When $\mathfrak{B}$ is a beable algebra for ρ, $\mathfrak{B}$ can be expressed as $P_{\mathcal{T}}^{\perp}\mathfrak{B}P_{\mathcal{T}}^{\perp} \oplus \mathfrak{A}$ where $\mathfrak{A}$ is an abelian subalgebra of $P_{\mathcal{T}}\mathfrak{N}P_{\mathcal{T}}$.

Proof. Suppose $\mathfrak{B}$ is a beable algebra for ρ. Because $\mathcal{T} = [\mathfrak{B}\mathcal{S}]$ is invariant under $\mathfrak{B}$, $P_{\mathcal{T}} \in \mathfrak{B}'$ by Lemma 4. Therefore $\mathfrak{B}$ can be expressed as $P_{\mathcal{T}}^{\perp}\mathfrak{B}P_{\mathcal{T}}^{\perp} \oplus P_{\mathcal{T}}\mathfrak{B}P_{\mathcal{T}}$.
Next we must show that all operators in $P_{\mathcal{T}}\mathfrak{B}P_{\mathcal{T}}$ commute with each other. We will show that $[P_{\mathcal{T}}XP_{\mathcal{T}}, P_{\mathcal{T}}YP_{\mathcal{T}}] = 0$ for any operators $X, Y \in \mathfrak{B}$. Let x be any vector in $\mathcal{H}$. There are some vectors $y \in \mathcal{T}$ and $y' \in \mathcal{T}^{\perp}$ such that $x = y + y'$. Then we get $(P_{\mathcal{T}}XP_{\mathcal{T}})(P_{\mathcal{T}}YP_{\mathcal{T}})x = XYy$ and $(P_{\mathcal{T}}YP_{\mathcal{T}})(P_{\mathcal{T}}XP_{\mathcal{T}})x = YXy$.
Because $\mathfrak{B}$ is a beable algebra for P_y by Lemma 5, $[X,Y]y = 0$ by [3] Remark 2.5. Therefore $(P_{\mathcal{T}}XP_{\mathcal{T}})(P_{\mathcal{T}}YP_{\mathcal{T}})x = (P_{\mathcal{T}}YP_{\mathcal{T}})(P_{\mathcal{T}}XP_{\mathcal{T}})x$, that is, $[P_{\mathcal{T}}XP_{\mathcal{T}}, P_{\mathcal{T}}YP_{\mathcal{T}}] = 0$. □

The following lemma can be isolated from the proof of [3] Theorem 4.5.

Lemma 7. *Let $\mathfrak{N}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ with a cyclic and separating vector, let ρ be a normal state on $\mathfrak{N}$, let S be the support of ρ and let $\mathcal{S}$ be the range of S. Let $\mathfrak{B}$ be a von Neumann subalgebra of $\mathfrak{N}$ and let $\mathfrak{B}$ satisfy the following conditions.*

1. *$\mathfrak{B}$ is a beable algebra for ρ.*

2. *$\sigma(\mathfrak{B}) = \mathfrak{B}$ for any weakly continuous automorphism σ on $\mathfrak{N}$ such that $\rho \circ \sigma = \rho$.*

Then $\mathfrak{B}$ can be expressed as $S^{\perp}\mathfrak{B}S^{\perp} \oplus \mathfrak{A}$, where $\mathfrak{A}$ is an abelian von Neumann subalgebra of $S\mathfrak{N}S$.

Proof. Let $\mathfrak{B}$ be a von Neumann subalgebra of $\mathfrak{N}$ and let $\mathfrak{B}$ satisfy Conditions 1 and 2. Define $\mathcal{T}$ as $[\mathfrak{B}\mathcal{S}]$. By Condition 1 and Lemma 6, $\mathfrak{B}$ can be expressed as $P_{\mathcal{T}}^{\perp}\mathfrak{B}P_{\mathcal{T}}^{\perp} \oplus \bar{\mathfrak{A}}$, where $\bar{\mathfrak{A}}$ is an abelian von Neumann subalgebra of $P_{\mathcal{T}}\mathfrak{N}P_{\mathcal{T}}$. We will show that $S \in \mathfrak{B}'$.
Choose $\theta \in \mathbb{R}$ such that $e^{-i\theta} \neq \pm 1$. Observe that $\mathcal{S} \subseteq [\mathfrak{B}\mathcal{S}] = \mathcal{T}$ and define $\bar{V} := P_{\mathcal{T}}^{\perp} \oplus S \oplus (e^{i\theta}P_{\mathcal{T}\wedge\mathcal{S}^{\perp}})$ and $V := S \oplus (e^{i\theta}P_{\mathcal{T}\wedge\mathcal{S}^{\perp}})$. Since $P_{\mathcal{T}} \in \mathfrak{B} \subseteq \mathfrak{N}$ and $S \in \mathfrak{N}$, $\bar{V} \in \mathfrak{N}$. Define a weakly continuous automorphism σ_0 on $\mathfrak{N}$ as $\sigma_0(A) := \bar{V}A\bar{V}^*$ for any operator A in $\mathfrak{N}$. Because $\bar{V} = S \oplus S^{\perp}\bar{V}S^{\perp}$, $\rho(\sigma_0(A)) = \rho(A)$ for any operator $A \in \mathfrak{N}$. Then $\sigma_0(\mathfrak{B}) = \mathfrak{B}$ by Condition 2. Because $\bar{V} = P_{\mathcal{T}}^{\perp} \oplus V$ and $\mathfrak{B} = P_{\mathcal{T}}^{\perp}\mathfrak{B}P_{\mathcal{T}}^{\perp} \oplus \bar{\mathfrak{A}}$, $\sigma_0(\mathfrak{B}) = P_{\mathcal{T}}^{\perp}\mathfrak{B}P_{\mathcal{T}}^{\perp} \oplus V\bar{\mathfrak{A}}V^*$. Therefore $V\bar{\mathfrak{A}}V^* = \bar{\mathfrak{A}}$.
Let $\bar{Q}$ be any projection in $\mathfrak{B}$. There are some projections $Q_0 \in P_{\mathcal{T}}^{\perp}\mathfrak{B}P_{\mathcal{T}}^{\perp}$ and $Q \in \bar{\mathfrak{A}}$ such that $\bar{Q} = Q_0 + Q$. Since $VQV^* \in \bar{\mathfrak{A}}$, $[Q, VQV^*] = 0$. Thus there are mutually orthogonal

projections A, B and C on $\mathcal{T}$ such that $Q = A + B$ and $VQV^* = A + C$.
We will show $B = 0$. Let x be any vector in a range of B. Then

$$Qx = x \tag{1}$$

$$VQV^*x = 0 \tag{2}$$

There are some vectors $y \in \mathcal{S}$ and $y' \in \mathcal{S}^\perp \wedge \mathcal{T}$ such that $x = y + y'$. By equation (2), $QV^*(y + y') = 0$. By $V^* = S \oplus (e^{-i\theta} P_{\mathcal{T}\wedge\mathcal{S}^\perp})$,

$$Q(y + e^{-i\theta} y') = 0 \tag{3}$$

Then $Q(x - y' + e^{-i\theta} y') = 0$ since $y = x - y'$. Using equation (1) and $e^{-i\theta} \neq 1$,

$$Qy' = (1 - e^{-i\theta})^{-1} x \tag{4}$$

By equation (2) and equation (4), $VQV^*Qy' = (1 - e^{-i\theta})^{-1} VQV^*x = 0$. Because $[Q, VQV^*] = 0$ on $\mathcal{T}$ and $y' \in \mathcal{T} \wedge \mathcal{S}^\perp \subseteq \mathcal{T}$, $QVQV^*y' = 0$. Then $QVQy' = 0$ using $V^* = S \oplus (e^{-i\theta} P_{\mathcal{T}\wedge\mathcal{S}^\perp})$, $y' \in \mathcal{T} \wedge \mathcal{S}^\perp$ and $e^{-i\theta} \neq 0$. By equation (4) and $(1 - e^{-i\theta})^{-1} \neq 0$, $QVx = 0$. By the definition of V and $x = y + y'$,

$$Q(y + e^{i\theta} y') = 0 \tag{5}$$

Because $Q(y + e^{-i\theta} y' - (y + e^{i\theta} y')) = 0$ by equation (3) and (5), $Q((e^{-i\theta} - e^{i\theta}) y') = 0$. $e^{-i\theta} - e^{i\theta} \neq 0$ since $e^{-i\theta} \neq \pm 1$. Then

$$Qy' = 0 \tag{6}$$

By equation (5), $Qy = 0$. Using equation (6)

$$Qx = Q(y + y') = 0 \tag{7}$$

By equation (1) and (7), $x = 0$. Thus $B = 0$.
Let x' be any vector in a range of C. Then $VQV^*x' = x', V^*(VQV^*)Vx' = 0$. We can show $C = 0$ in a similar way to prove $B = 0$. Therefore $VQV^* = Q$, that is, $[V, Q] = 0$. Because $V = S + e^{i\theta}(P_{\mathcal{T}} - S) = (1 - e^{i\theta})S + e^{i\theta} P_{\mathcal{T}}$, $S = (1 - e^{i\theta})^{-1}(V - e^{i\theta} P_{\mathcal{T}})$. So $[S, Q] = 0$ by $[V, Q] = 0$. $(I - P_{\mathcal{T}})S = 0$ since $\mathcal{S} \subseteq \mathcal{T}$. Then

$$\bar{Q}S = [(I - P_{\mathcal{T}})\bar{Q}(I - P_{\mathcal{T}}) + Q]S = QS = SQ = S\bar{Q}$$

Because $\mathfrak{B}$ is a von Neumann algebra, $S \in \mathfrak{B}'$. Thus $\mathfrak{B}$ can be expressed as $S^\perp \mathfrak{B} S^\perp \oplus S\mathfrak{B}S$. Since $S\mathfrak{B}S \subseteq \bar{\mathfrak{A}}$, $S\mathfrak{B}S$ is an abelian von Neumann subalgebra of $S\mathfrak{N}S$. □

Lemma 8. *Let $\mathfrak{N}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, let ρ be a normal state on $\mathfrak{N}$, let S be the support of ρ, let $\mathfrak{C}_{\rho,\mathfrak{N}}$ be the centralizer of ρ of $\mathfrak{N}$ and let $\mathfrak{Z}(\mathfrak{C}_{\rho,\mathfrak{N}})$ be the center of $\mathfrak{C}_{\rho,\mathfrak{N}}$. Then*

1. $S\mathfrak{C}_{\rho,\mathfrak{N}}S = \mathfrak{C}_{\rho|_{S\mathfrak{N}S}, S\mathfrak{N}S}$.

2. $\mathfrak{C}_{\rho,\mathfrak{N}}$ *and* $\mathfrak{C}'_{\rho,\mathfrak{N}}S$ *is a von Neumann algebra.*

Proof. 1. Let A be any operator in $S\mathfrak{C}_{\rho,\mathfrak{N}}S$. Because A belongs to $\mathfrak{C}_{\rho,\mathfrak{N}}$, $\rho(AB) = \rho(BA)$ for any operator B in $S\mathfrak{N}S$. Thus A belongs to $\mathfrak{C}_{\rho|_{S\mathfrak{N}S},S\mathfrak{N}S}$.

Let X be any operator in $\mathfrak{C}_{\rho|_{S\mathfrak{N}S},S\mathfrak{N}S}$. $X = SXS$ since $X \in S\mathfrak{N}S$. For any operator $Y \in \mathfrak{N}$, $\rho(XY) = \rho(YX)$ since $Y = SYS + SYS^{\perp} + S^{\perp}YS + S^{\perp}YS^{\perp}$ and $\rho([SXS, SYS]) = 0$. Then $X \in \mathfrak{C}_{\rho,\mathfrak{N}}$. Since $X = SXS$, $X \in S\mathfrak{C}_{\rho,\mathfrak{N}}S$.

2. To show that $\mathfrak{C}_{\rho,\mathfrak{N}}$ is a von Neumann algebra, we will prove that the closed unit ball of $\mathfrak{C}_{\rho,\mathfrak{N}}$ is σ-strongly* closed ([4] Theorem 2.4.11 (7a)). Let a net $\{A_\alpha\}$ in the closed unit ball of $\mathfrak{C}_{\rho,\mathfrak{N}}$ converge σ-strongly* to A. Because A belongs to the closed unit ball of $\mathfrak{N}$ by [4] Theorem 2.4.11 (7a), we will show that A belongs to $\mathfrak{C}_{\rho,\mathfrak{N}}$. Because $(A_\alpha - A)^*(A_\alpha - A) + (A_\alpha - A)(A_\alpha - A)^*$ converges σ-weakly to 0 (the proof of [5] Lemma II 2.5) and ρ is σ-weakly continuous ([4] Theorem 2.4.21), $\rho((A_\alpha - A)^*(A_\alpha - A) + (A_\alpha - A)(A_\alpha - A)^*)$ converges to 0. Both $\rho((A_\alpha - A)^*(A_\alpha - A))$ and $\rho((A_\alpha - A)(A_\alpha - A)^*)$ are positive, so they converge to 0. For any operator $B \in \mathfrak{N}$, $|\rho([A, B])| \leq |\rho((A - A_\alpha)B)| + |\rho(B(A_\alpha - A))| \leq \rho((A_\alpha - A)(A_\alpha - A)^*)^{1/2}\rho(B^*B)^{1/2} + \rho(BB^*)^{1/2}\rho((A_\alpha - A)^*(A_\alpha - A))^{1/2} \to 0$. Therefore $A \in \mathfrak{C}_{\rho,\mathfrak{N}}$.

Since $S \in \mathfrak{C}_{\rho,\mathfrak{N}}$, $S\mathfrak{C}_{\rho,\mathfrak{N}}S$ is a von Neumann algebra with commutant $\mathfrak{C}'_{\rho,\mathfrak{N}}S$ by [5] Proposition II 3.10.

□

Lemma 9. *Let $\mathfrak{N}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ with a cyclic and separating vector, let ρ be a normal state on $\mathfrak{N}$ and let S be the support of ρ. Let $\mathfrak{B}$ be a C*-subalgebra of $\mathfrak{N}$ and let $\mathfrak{B}$ satisfy the following conditions.*

1. *$\mathfrak{B}$ is a beable algebra for ρ.*
2. *$\sigma(\mathfrak{B}) = \mathfrak{B}$ for any weakly continuous automorphism σ on $\mathfrak{N}$ such that $\rho \circ \sigma = \rho$.*

Then $\mathfrak{B} \subseteq S^{\perp}\mathfrak{N}S^{\perp} \oplus \mathfrak{Z}(\mathfrak{C}_{\rho,\mathfrak{N}})S$, where $\mathfrak{Z}(\mathfrak{C}_{\rho,\mathfrak{N}})$ is the center of the centralizer $\mathfrak{C}_{\rho,\mathfrak{N}}$ of ρ of $\mathfrak{N}$.

Proof. Let $\mathfrak{B}$ be a C*-subalgebra of $\mathfrak{N}$, let $\mathfrak{B}$ satisfy Conditions 1 and 2 and let $\mathfrak{B}^-$ be the closure of $\mathfrak{B}$ in the weak topology. By [3] Corollary 2.9 (ii), $\mathfrak{B}^-$ is also a beable algebra for ρ. Because $\sigma(\mathfrak{B}^-) = \sigma(\mathfrak{B})^- = \mathfrak{B}^-$ for any weakly continuous automorphism σ on $\mathfrak{N}$, $\mathfrak{B}^-$ also satisfies Condition 2. Therefore we can assume that $\mathfrak{B}$ is a von Neumann algebra.
By Lemma 7, $\mathfrak{B}$ can be expressed as $S^{\perp}\mathfrak{B}S^{\perp} \oplus \mathfrak{A}$, where $\mathfrak{A}$ is an abelian von Neumann subalgebra of $S\mathfrak{N}S$. Therefore it suffices to show that $\mathfrak{A} \subseteq \mathfrak{Z}(\mathfrak{C}_{\rho,\mathfrak{N}})S$.

1. We will show $\mathfrak{A} \subseteq \mathfrak{C}'_{\rho,\mathfrak{N}}S$, making reference to the proof of [2] Proposition 1. Because both $\mathfrak{A}$ and $\mathfrak{C}'_{\rho,\mathfrak{N}}S$ are von Neumann algebras by Lemma 8 (2), it suffices to show that any projection in $\mathfrak{A}$ is contained in $\mathfrak{C}'_{\rho,\mathfrak{N}}S$. Let R be any projection in $\mathfrak{A}$.

   Let U be any unitary operator in $\mathfrak{C}_{\rho,\mathfrak{N}}$. Define a weakly continuous automorphism σ_1 on $\mathfrak{N}$ as $\sigma_1(A) := UAU^*$ for any operator $A \in \mathfrak{N}$. Since $\rho(\sigma_1(A)) = \rho(UAU^*) = \rho(AU^*U) = \rho(A)$ for any operator $A \in \mathfrak{N}$, $\sigma_1(\mathfrak{B}) = \mathfrak{B}$ by Condition 2. There are operators $R_0 \in S^{\perp}\mathfrak{N}S^{\perp}$ and $R_1 \in \mathfrak{A}$ such that $URU^* = R_0 + R_1$ since $URU^* \in \mathfrak{B}$. Then $R(URU^*) = RR_1 = R_1R = (URU^*)R$. Because $\mathfrak{C}_{\rho,\mathfrak{N}}$ is a von Neumann algebra by Lemma 8 (2), $\mathfrak{C}_{\rho,\mathfrak{N}} = \mathfrak{C}''_{\rho,\mathfrak{N}}$. By [3] Lemma 4.2, $R \in \mathfrak{C}'_{\rho,\mathfrak{N}}$. Because $R \in \mathfrak{A} \subseteq S\mathfrak{N}S$, $R = SRS$. Therefore $R \in S\mathfrak{C}'_{\rho,\mathfrak{N}}S = \mathfrak{C}'_{\rho,\mathfrak{N}}S$.

2. We will show $\mathfrak{A} \subseteq S\mathfrak{C}_{\rho,\mathfrak{N}}S$. Let X be any operator X in $\mathfrak{A}$. We assume that $S \neq I$. When $S = I$, we can prove it in a similar way to Proof (2)(b) or the proof of [2] Proposition 1.

   (a) Since $\mathfrak{N}$ has a cyclic and separating vector, there is a vector $x \in \mathcal{H}$ such that $\rho(A) = \langle x, Ax\rangle$ for any operator $A \in \mathfrak{N}$ ([4] Theorem 2.5.31). Since $\rho|_{S\mathfrak{N}S}$ is faithful on $S\mathfrak{N}S$, x is a separating vector for $S\mathfrak{N}S$. Let x' be a separating vector for $\mathfrak{N}$. Observe $S^\perp x' \neq 0$ and define $x^\perp$ as $S^\perp x'/\|S^\perp x'\|$. Then $x^\perp$ is a separating vector for $S^\perp\mathfrak{N}S^\perp$. Define a normal state $\bar{\rho}$ on $\mathfrak{N}$ as

$$\begin{aligned}\bar{\rho}(A) &:= \mathrm{tr}((wP_x + (1-w)P_{x^\perp})A)\\ &= w\langle x, Ax\rangle + (1-w)\langle x^\perp, Ax^\perp\rangle\end{aligned}$$

   for any operator $A \in \mathfrak{N}$, where w is a real number such that $0 < w < 1$. We will show that $\bar{\rho}$ is a faithful normal state on $\mathfrak{N}$.
   Let Q' be the support of $\bar{\rho}$ and let Q be $I - Q'$. Then $\bar{\rho}(Q) = 0$.

$$\begin{aligned}&\langle x, (SQS)^{1/2}(SQS)^{1/2}x\rangle\\ &= \langle x^\perp, (S^\perp QS^\perp)^{1/2}(S^\perp QS^\perp)^{1/2}x^\perp\rangle = 0\end{aligned}$$

   since
$$\bar{\rho}(Q) = w\langle x, SQSx\rangle + (1-w)\langle x^\perp, S^\perp QS^\perp x^\perp\rangle,$$
   $SQS = (QS)^*(QS)$ and $S^\perp QS^\perp = (QS^\perp)^*(QS^\perp)$. Then $SQS = S^\perp QS^\perp = 0$. So,
$$Q = SQS + S^\perp QS + SQS^\perp + S^\perp QS^\perp = S^\perp QS + SQS^\perp.$$
   Because $Q = Q^2$,

$$Q = (S^\perp QS + SQS^\perp)(S^\perp QS + SQS^\perp) = SQS^\perp QS + S^\perp QSQS^\perp.$$

$$\begin{aligned}&\langle x, (SQS^\perp QS)^{1/2}(SQS^\perp QS)^{1/2}x\rangle\\ &= \langle x^\perp, (S^\perp QSQS^\perp)^{1/2}(S^\perp QSQS^\perp)^{1/2}x^\perp\rangle = 0\end{aligned}$$

   since
$$\bar{\rho}(Q) = w\langle x, SQS^\perp QSx\rangle + (1-w)\langle x^\perp, S^\perp QSQS^\perp x^\perp\rangle,$$
   $SQS^\perp QS = (S^\perp QS)^* S^\perp QS$ and $S^\perp QSQS^\perp = (SQS^\perp)^* SQS^\perp$. Then $SQS^\perp QS = S^\perp QSQS^\perp = 0$.
   Since $Q = SQS^\perp QS + S^\perp QSQS^\perp$, $Q = 0$, that is, $Q' = I$. Therefore $\bar{\rho}$ is a faithful normal state on $\mathfrak{N}$.

   (b) Since $\mathfrak{N}$ has a cyclic and separating vector and $\bar{\rho}$ is a faithful normal state on $\mathfrak{N}$, there is a cyclic and separating vector $y \in \mathcal{H}$ such that $\bar{\rho}(A) = \langle y, Ay\rangle$ for any operator $A \in \mathfrak{N}$ ([4] Theorem 2.5.31 and Proposition 2.5.30). There is a modular automorphism σ_t associated with y such that $\sigma_t(A) = \Delta^{it}A\Delta^{-it}$

for any operator $A \in \mathfrak{N}$, where Δ is the modular operator associated with y ([4] Definition 2.5.15). Then σ_t is a weakly continuous automorphism on $\mathfrak{N}$. $\bar{\rho} \circ \sigma_t = \bar{\rho}$ by [4] Proposition 5.3.3 and $\sigma_t(S) = S$ by [4] Proposition 5.3.28 since S belongs to the centralizer $\mathfrak{C}_{\bar{\rho},\mathfrak{N}}$ of $\bar{\rho}$. Then

$$\begin{aligned}\rho(\sigma_t(A)) &= \langle x, \sigma_t(A)x\rangle = \langle x, S\sigma_t(A)Sx\rangle \\ &= \frac{1}{w}\bar{\rho}(S\sigma_t(A)S) = \frac{1}{w}\bar{\rho}(\sigma_t(SAS)) = \frac{1}{w}\bar{\rho}(SAS) \\ &= \langle x, SASx\rangle = \rho(A)\end{aligned}$$

for any operator $A \in \mathfrak{N}$. Therefore $\sigma_t(\mathfrak{B}) = \mathfrak{B}$ by Condition 2.

By [5] Theorem IX 4.2, there exists a mapping $\mathcal{E}$ of $\mathfrak{N}$ onto $\mathfrak{B}$ such that $\mathcal{E}(B_1AB_2) = B_1\mathcal{E}(A)B_2$ and $\bar{\rho}(\mathcal{E}(A)) = \bar{\rho}(A)$ for any operator $A \in \mathfrak{N}$ and $B_1, B_2 \in \mathfrak{B}$. Let Y be any operator in $S\mathfrak{N}S$. Since $\mathcal{E}(Y) \in \mathfrak{B}$, there are operators $Y_0 \in S^\perp\mathfrak{B}S^\perp$ and $Y_1 \in \mathfrak{A}$ such that $\mathcal{E}(Y) = Y_0 + Y_1$. $X\mathcal{E}(Y) = X(Y_0 + Y_1) = XY_1 = Y_1X = \mathcal{E}(Y)X$ since $\mathfrak{A}$ is an abelian von Neumann algebra. Because $\mathcal{E}(XY) = \mathcal{E}(YX)$, $\bar{\rho}(\mathcal{E}(XY)) = \bar{\rho}(\mathcal{E}(YX))$. Then $\bar{\rho}(XY) = \bar{\rho}(YX)$.

Since $\bar{\rho}(A) = w\rho(A)$ for any operator $A \in S\mathfrak{N}S$, $\rho(XY) = \rho(YX)$. Therefore $X \in \mathfrak{C}_{\rho|_{S\mathfrak{N}S}, S\mathfrak{N}S}$. By Lemma 8 (1), $X \in S\mathfrak{C}_{\rho,\mathfrak{N}}S$.

□

Lemma 10. *Let $\mathfrak{N}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, let ρ be a normal state on $\mathfrak{N}$ and let S be the support of ρ.*
Then $\sigma(S) = S$ and $\sigma(S^\perp\mathfrak{N}S^\perp) = S^\perp\mathfrak{N}S^\perp$ for any automorphism σ on $\mathfrak{N}$ such that $\rho \circ \sigma = \rho$.

Proof. Let σ be an automorphism on $\mathfrak{N}$ such that $\rho \circ \sigma = \rho$. Then $\rho(\sigma(S)) = \rho(S) = 1$ and $\rho(\sigma^{-1}(S)) = \rho(\sigma \circ \sigma^{-1}(S)) = \rho(S) = 1$. Since both $\sigma(S)$ and $\sigma^{-1}(S)$ are projections and S is the support of ρ, $S \leq \sigma(S)$ and $S \leq \sigma^{-1}(S)$. Therefore $S = \sigma(S)$. Since $\sigma(S^\perp) = \sigma(I - S) = I - \sigma(S) = I - S = S^\perp$, $\sigma(S^\perp\mathfrak{N}S^\perp) = S^\perp\mathfrak{N}S^\perp$. □

Theorem 11 (Generalized Clifton's theorem)**.** *Let $\mathfrak{N}$ be a σ-finite von Neumann algebra*[1] *on a Hilbert space, let ρ be a normal state on $\mathfrak{N}$ and let S be the support of ρ. Let $\mathfrak{B}$ be a C^*-subalgebra of $\mathfrak{N}$ and let $\mathfrak{B}$ satisfy the following conditions.*

1. *$\mathfrak{B}$ is a beable algebra for ρ.*
2. *$\sigma(\mathfrak{B}) = \mathfrak{B}$ for any automorphism σ on $\mathfrak{N}$ such that $\rho \circ \sigma = \rho$.*[2]
3. *$\mathfrak{B}$ is maximal with respect to Conditions 1 and 2.*

Then $\mathfrak{B}$ can be uniquely expressed as $S^\perp\mathfrak{N}S^\perp \oplus \mathfrak{Z}(\mathfrak{C}_{\rho,\mathfrak{N}})S$, where $\mathfrak{Z}(\mathfrak{C}_{\rho,\mathfrak{N}})$ is the center of the centralizer $\mathfrak{C}_{\rho,\mathfrak{N}}$ of ρ of $\mathfrak{N}$.

[1] Since local algebras have cyclic and separating vectors by the Reeh-Schlieder theorem, these algebras are σ-finite von Neumann algebras ([4] Proposition 2.5.6).

[2] To represent that $\mathfrak{B}$ is determined solely in terms of ρ and algebraic structure of $\mathfrak{N}$, Clifton assumed this condition in [2] Proposition 1.

Proof. By [3] Proposition 2.2, $S^{\perp}\mathfrak{N}S^{\perp} \oplus \mathfrak{Z}(\mathfrak{C}_{\rho,\mathfrak{N}})S$ is a beable algebra for ρ. Since $\sigma(\mathfrak{Z}(\mathfrak{C}_{\rho,\mathfrak{N}})) = \mathfrak{Z}(\mathfrak{C}_{\rho,\mathfrak{N}})$ for any automorphism σ on $\mathfrak{N}$ such that $\rho \circ \sigma = \rho$, $\sigma(S^{\perp}\mathfrak{N}S^{\perp} \oplus \mathfrak{Z}(\mathfrak{C}_{\rho,\mathfrak{N}})S) = S^{\perp}\mathfrak{N}S^{\perp} \oplus \mathfrak{Z}(\mathfrak{C}_{\rho,\mathfrak{N}})S$ for any automorphism σ on $\mathfrak{N}$ such that $\rho \circ \sigma = \rho$ by Lemma 10. We will show that any C*-subalgebra $\mathfrak{B}$ which satisfies Conditions 1 and 2 is contained in $S^{\perp}\mathfrak{N}S^{\perp} \oplus \mathfrak{Z}(\mathfrak{C}_{\rho,\mathfrak{N}})S$.
Because $\mathfrak{N}$ is σ-finite, there is an isomorphism π such that $\pi(\mathfrak{N})$ is a von Neumann algebra which has a cyclic and separating vector and $\rho \circ \pi^{-1}$ is a normal state on $\pi(\mathfrak{N})$ ([4] Theorem 2.4.24 and Proposition 2.5.6). $\pi(S)$ is the support of $\rho \circ \pi^{-1}$.
Because $\rho \circ \pi^{-1}([\pi(A), \pi(B)]^*[\pi(A), \pi(B)]) = 0$ for any operator $A, B \in \mathfrak{B}$, $\pi(\mathfrak{B})$ is a beable algebra for $\rho \circ \pi^{-1}$ by [3] Proposition 2.2. For any weakly continuous automorphism σ on $\pi(\mathfrak{N})$ such that $\rho \circ \pi^{-1} \circ \sigma = \rho \circ \pi^{-1}$, then $\sigma \circ \pi(\mathfrak{B}) = \pi(\mathfrak{B})$ since $\pi^{-1} \circ \sigma \circ \pi(\mathfrak{B}) = \mathfrak{B}$ by Condition 2.
Because $\pi(\mathfrak{N})$ has a cyclic and separating vector, $\pi(\mathfrak{B}) \subseteq \pi(S)^{\perp}\pi(\mathfrak{N})\pi(S)^{\perp} \oplus \mathfrak{Z}(\mathfrak{C}_{\rho\circ\pi^{-1},\pi(\mathfrak{N})})\pi(S)$ by Lemma 9. Therefore $\mathfrak{B} \subseteq S^{\perp}\mathfrak{N}S^{\perp} \oplus \mathfrak{Z}(\mathfrak{C}_{\rho,\mathfrak{N}})S$. □

3 Concluding remarks

We will consider connection between Theorem 11 and the Kochen-Dieks modal interpretation of nonrelativistic quantum mechanics. As a corollary of Theorem 11, we have:

Corollary 12. *Let D be a density operator on a separable Hilbert space $\mathcal{H}$ and let $\mathcal{D}$ be the range of D. Let $\mathfrak{B}$ be a C*-subalgebra of $\mathbb{B}(\mathcal{H})$ and $\mathfrak{B}$ satisfy the following conditions.*

1. *$\mathfrak{B}$ is a beable algebra for D.*
2. *$\sigma(\mathfrak{B}) = \mathfrak{B}$ for any automorphism σ on $\mathbb{B}(\mathcal{H})$ such that $tr(D\sigma(A)) = tr(DA)$ for any operator $A \in \mathbb{B}(\mathcal{H})$.*
3. *$\mathfrak{B}$ is maximal with respect to Conditions 1 and 2.*

Then $\mathfrak{B}$ can be uniquely expressed as $\mathbb{B}(\mathcal{D}^{\perp}) \oplus \{D\}'' P_{\mathcal{D}}$.

As easily seen, the set of all projections in $\mathbb{B}(\mathcal{D}^{\perp}) \oplus \{D\}'' P_{\mathcal{D}}$ coincides with the set ($\mathrm{Def}_{\mathrm{KD}}(W)$ in [1] p. 42) of all projections which have simultaneously definite values under the Kochen-Dieks modal interpretation. Therefore Corollary 12 can be regarded as one of the theorems that motivate the Kochen-Dieks modal interpretation of nonrelativistic quantum mechanics (cf. [1] Section 6).

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