

Reconstructing Bohr's Reply to EPR in Algebraic Quantum Theory

MASANAO OZAWA*

*Graduate School of Information Science, Nagoya University, Nagoya 464-8601,
Japan*

YUICHIRO KITAJIMA†

*College of Industrial Technology, Nihon University, 2-11-1 Shin-ei, Narashino,
Chiba 275-8576, Japan*

Abstract

Halvorson and Clifton have given a mathematical reconstruction of Bohr's reply to Einstein, Podolsky and Rosen (EPR), and argued that this reply is dictated by the two requirements of classicality and objectivity for the description of experimental data, by proving consistency between their objectivity requirement and a contextualized version of the EPR reality criterion which had been introduced by Howard in his earlier analysis of Bohr's reply. In the present paper, we generalize the above consistency theorem, with a rather elementary proof, to a general formulation of EPR states applicable to both non-relativistic quantum mechanics and algebraic quantum field theory; and we clarify the elements of reality in EPR states in terms of Bohr's requirements of classicality and objectivity, in a general formulation of algebraic quantum theory.

KEYWORDS: Bohr, EPR, algebraic quantum theory, modal interpretation

1 Introduction

Einstein, Podolsky and Rosen (EPR) [6] discussed a system consisting of two particles. They have interacted initially and then moved apart so that the positions, and the momenta, of the two particles are strictly correlated. It follows that if one were to measure the position of the first particle, one could predict with certainty the outcome of a position measurement on the second particle; and similarly for a momentum measurement. EPR proposed a "criterion of reality":

If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity. [6, p.777]

*email: ozawa@is.nagoya-u.ac.jp

†email: kitajima.yuichirou@nihon-u.ac.jp

In accordance with this criterion, the position and the momentum of the second particle have simultaneous reality since the measurement on the first particle has not disturbed the second particle. On the other hand, position and momentum cannot have simultaneous reality in any states in quantum mechanics. Therefore, EPR regarded quantum-mechanical description as incomplete.

In reply to EPR, Bohr pronounced that a measurement on the first particle influences the condition which defines elements of reality for the second particle, so that he rejected EPR's conclusion.

[T]here is in a case like that just considered no question of a mechanical disturbance of the system under investigation during the last critical stage of the measuring procedure. But even at this stage there is essentially the question of *an influence on the very conditions which define the possible types of predictions regarding the future behavior of the system*. Since these conditions constitute an inherent element of the description of any phenomenon to which the term "physical reality" can be properly attached, we see that the argumentation of the mentioned authors [EPR] does not justify their conclusion that quantum-mechanical description is essentially incomplete. [3, p.700]

Unfortunately, Bohr's view has prevailed under the Copenhagen interpretation with several improper restatements, and so there have been only a few serious attempts to reconstruct his reply in a rigorous analysis. Here, we shall discuss Howard's original contribution [11, 12] to the reconstruction of Bohr's reply to EPR and its mathematical reformulation due to Halvorson and Clifton [10].

Howard [12] reconstructed Bohr's reply to EPR in terms of Bohr's notion of "classical description" and "a contextualized version of the EPR reality criterion." Bohr stated his view on the classicality requirement of description of experimental data as follows.

[I]t is decisive to recognize that, *however far the phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms*. The argument is simply that by the word "experiment" we refer to a situation where we can tell others what we have done and what we have learned and that, therefore, the account of the experimental arrangement and of the results of the observations must be expressed in unambiguous language with suitable application of the terminology of classical physics. [4, p.209]

To explicate this, Howard [11, 12] considered a "measurement context" (ψ, R) , consisting of a state vector ψ and an observable R , in which the observable R is to be measured in the state ψ . He asked which observables should be considered to possess their values as elements of reality, and claimed that Bohr's notion of classical description is best explicated via the notion of "appropriate mixture" for the measurement context (ψ, R) under consideration [5, pp.190–199].

My claim about the nature of a classical description is that Bohr did not mean simply the application of classical physics — the physics of Newton, Maxwell, Boltzmann, and Einstein — in some combination appropriate

to the occasion. I will argue instead that a classical description, in the sense of “classical” relevant to Bohr’s concerns, is a description in terms of what physicists call “mixtures” (as opposed to what are termed “pure cases”), a formal device that permits us to proceed as if the systems being described were in well-defined, if unknown, intrinsic states, at least with respect to those properties requiring a classical description. The device of mixtures also permits one to give a classical, ignorance interpretation to any statistics that one encounters. Which mixture to employ in a given classical description will depend upon the kind of measurement being performed, the “appropriate mixture” being one constructed out of simultaneous eigenstates of all the observables measurable in a given experimental context. [12, p.203–204]

Halvorson and Clifton [10, p.13] showed that Howard’s notion of “appropriate mixture” can be formulated generally in algebraic quantum theory as a beable subalgebra [9] for a given measurement context (ψ, R) .

However, this notion is not sufficient to select one appropriate beable subalgebra from other candidates. For this purpose, Howard [11, 12] introduced “a contextualized version of the EPR reality criterion.”

Once the experimental context is stipulated, which amounts to the specification of the candidates for real status, our decision as to which particular properties to consider as real will turn on the question of predictability with certainty. [11, p.256]

Halvorson and Clifton [10, pp.14–15] turned down Howard’s contextualized EPR reality criterion¹. For their interpretation of Bohr’s notion of objectivity, they introduced the requirement of invariance under the ‘relevant’ symmetries for the set of observables corresponding to elements of reality. Then they proved the equivalence between Howard’s contextualized EPR reality criterion and the above invariance principle for the case of Bohm’s simplified spin version of the EPR experiment [10, Theorem 1]. For the EPR position-momentum case they proved the consistency between those two requirements [10, Theorem 2].

In the present paper, we follow an abstract definition of the EPR state for a pair of observables due to Werner [19]; and we first show, in Section 2, that such a state exists for incommensurable pairs of observables in both nonrelativistic quantum mechanics and algebraic quantum field theory (Theorem 1 of Section 2). Then it is shown that “contextualized version of the EPR reality criterion” is consistent with the requirements of Halvorson and Clifton for the case of Bohm’s simplified spin version of the EPR experiment, the EPR position-momentum case, and the case of algebraic quantum field theory (Theorems 3, 5, and 6).

¹ The reason for their rejection of Howard’s contextualized version of the EPR reality criterion is obscure. They argued that the EPR reality criterion is best construed as a version of ‘inference to the best explanation’ following Redhead [18, p.72] and that we cannot expect Bohr to be persuaded by such inference to the best explanation since he is not a classical scientific realist [10, p.11]. This is plausible, and yet it is not clear whether Bohr would contend against the contextualized version either.

2 EPR states

In this section, we provide an abstract definition of EPR states and show that such a state exists in both nonrelativistic quantum mechanics and algebraic quantum field theory (Theorem 1).

Let \mathfrak{A} be a unital \mathbf{C}^* -algebra. A *commuting pair* in \mathfrak{A} is a pair (A, B) of commuting self-adjoint elements $A, B \in \mathfrak{A}_{sa}$. A state φ of \mathfrak{A} is called an *EPR state* [1, 19] for a commuting pair (A, B) if $\varphi((A - B)^2) = 0$. For any commuting pair (A, B) , the *joint probability distribution* $\mu_\varphi^{A,B}$ of A, B in φ is defined uniquely to be a probability measure μ on \mathbf{R}^2 such that

$$\varphi(f(A, B)) = \int_{\mathbf{R}^2} f(x, y) d\mu_\varphi^{A,B}(x, y) \quad (1)$$

for any polynomial $f(A, B)$. Then, φ is an EPR state if and only if $\mu_\varphi^{A,B}$ is concentrated on the diagonal, i.e., $\mu_\varphi^{A,B}(\{(x, x) \mid x \in \mathbf{R}\}) = 1$. Thus, simultaneous measurements of A and B always give concordant results, and each one of the outcomes would predict with certainty the other. In Ref. [16] one of the present authors introduced a general notion of “quantum perfect correlation”, which can be applicable to any pair of not necessarily bounded observables A, B that do not necessarily commute, and showed that this notion can be equivalently characterized by several mathematical conditions, including the condition that they have the joint probability distribution concentrated on the diagonal; a quantum logical reconstruction of this notion has been given in Ref. [17]. According to the above, an EPR state φ for (A, B) realizes a “quantum perfect correlation” between commuting A and B in the state φ .

Let \mathfrak{A}_1 and \mathfrak{A}_2 be mutually commuting \mathbf{C}^* -subalgebra of \mathfrak{A} . Commuting pairs (A_1, A_2) and (B_1, B_2) in \mathfrak{A} are called *incommensurable pairs* from $(\mathfrak{A}_1, \mathfrak{A}_2)$ for a state φ of \mathfrak{A} if $A_j, B_j \in \mathfrak{A}_j$ and there are integers n_j, m_j such that $\varphi(|[A_j^{n_j}, B_j^{m_j}]|^2) \neq 0$ for $j = 1, 2$, where $[A, B]$ denotes the commutator of A and B , i.e., $[A, B] = AB - BA$. Here, we note that the incommensurability here is related to the state-dependent notion of non-commutativity. We say that observables A and B *commute in a state* φ if there exists a joint probability distribution $\mu_\varphi^{A,B}$ of A, B in φ satisfying Eq. (1) for any polynomial $f(A, B)$. Then, it can be seen that A and B commute in φ if and only if $\varphi(|[A^n, B^m]|^2) = 0$ for any integers n, m (See, for instance, Ref. [16]). Thus, commuting pairs (A_1, A_2) and (B_1, B_2) from $(\mathfrak{A}_1, \mathfrak{A}_2)$ are incommensurable for φ if and only if A_j and B_j do not commute in φ for $j = 1, 2$.

Let φ be an EPR state for two incommensurable pairs (A_1, A_2) and (B_1, B_2) . Then $\mu_\varphi^{A_1, A_2}(\{(x, x) \mid x \in \mathbf{R}\}) = 1$ and $\mu_\varphi^{B_1, B_2}(\{(x, x) \mid x \in \mathbf{R}\}) = 1$. It follows that if we were to measure A_1 in φ , we could predict with certainty the outcome of A_2 ; and if we were to measure B_1 in φ , we could predict with certainty the outcome of B_2 .

Let us consider Bohm’s spin version of the EPR experiment. Suppose that we have prepared an ensemble of spin-1/2 particles in the singlet state

$$\Psi = \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right),$$

and let φ be the vector state of $\mathbb{B}(\mathcal{H}_2) \otimes \mathbb{B}(\mathcal{H}_2)$ induced by Ψ , where $\mathbb{B}(\mathcal{H}_2)$ is the space of operators on the 2-dimensional Hilbert space $\mathcal{H}_2 = \mathbf{C}^2$. Then, φ is an EPR

state for incommensurable pairs

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes I, I \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

and

$$\left(\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes I, I \otimes \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right)$$

from $(\mathbb{B}(\mathcal{H}_2) \otimes \mathbf{CI}, \mathbf{CI} \otimes \mathbb{B}(\mathcal{H}_2))$.

In the rest of this section, we shall consider algebraic quantum field theory. In algebraic quantum field theory, each bounded open region \mathcal{O} in Minkowski space is associated with a von Neumann algebra $\mathfrak{N}(\mathcal{O})$. Such a von Neumann algebra is called a *local algebra*. We say that bounded open regions \mathcal{O}_1 and \mathcal{O}_2 are *strictly space-like separated* if there is a neighborhood \mathcal{V} of the origin of Minkowski space such that $\mathcal{O}_1 + \mathcal{V}$ and \mathcal{O}_2 are space-like separated.

In the present paper, we make the following assumptions. *For any bounded open region \mathcal{O} in Minkowski space, $\mathfrak{N}(\mathcal{O})$ is not abelian. If \mathcal{O}_1 and \mathcal{O}_2 are space-like separated, then $[A_1, A_2] = 0$ for any $A_1 \in \mathfrak{N}(\mathcal{O}_1)$ and $A_2 \in \mathfrak{N}(\mathcal{O}_2)$. If \mathcal{O}_1 and \mathcal{O}_2 are strictly space-like separated, then $A_1 A_2 \neq 0$ for any nonzero operators $A_1 \in \mathfrak{N}(\mathcal{O}_1)$ and $A_2 \in \mathfrak{N}(\mathcal{O}_2)$ (cf. [2, Theorem 1.12.3]).*

The following theorem shows that there is an EPR state of $\mathfrak{N}(\mathcal{O}_1) \vee \mathfrak{N}(\mathcal{O}_2)$ for incommensurable pairs from $(\mathfrak{N}(\mathcal{O}_1), \mathfrak{N}(\mathcal{O}_2))$ if \mathcal{O}_1 and \mathcal{O}_2 are strictly space-like separated, where $\mathfrak{N}(\mathcal{O}_1) \vee \mathfrak{N}(\mathcal{O}_2)$ is the von Neumann algebra generated by $\mathfrak{N}(\mathcal{O}_1)$ and $\mathfrak{N}(\mathcal{O}_2)$.

Theorem 1. *Let \mathfrak{N}_1 and \mathfrak{N}_2 be mutually commuting non-abelian von Neumann algebras on a Hilbert space \mathcal{H} such that $A_1 A_2 \neq 0$ for any nonzero operators $A_1 \in \mathfrak{N}_1$ and $A_2 \in \mathfrak{N}_2$. Then, there exists a vector state φ of $\mathfrak{N}_1 \vee \mathfrak{N}_2$ which is an EPR state for incommensurable pairs (E_1, E_2) and (F_1, F_2) of projections from $(\mathfrak{N}_1, \mathfrak{N}_2)$.*

Proof. Since \mathfrak{N}_1 is not abelian, there exists projections R and S such that $[R, S] \neq 0$. Define $T \in \mathfrak{N}_1$ by $T = (I - R)SR$. Then $T \neq 0$. Let $T = V|T|$ be the polar decomposition of T . It follows from the construction of T that VV^* is orthogonal to V^*V and $V^2 = 0$ (cf. [13, Lemma]). Similarly, there exists a partial isometry $W \in \mathfrak{N}_2$ such that WW^* is orthogonal to W^*W and $W^2 = 0$. Let $E_1 = VV^*$, $F_1 = \frac{1}{2}(VV^* + V^*V + V + V^*)$, $E_2 = WW^*$, and $F_2 = \frac{1}{2}(WW^* + W^*W + W + W^*)$. Then F_1 and F_2 are projections in \mathfrak{N}_1 and \mathfrak{N}_2 , respectively. Since $E_1 \in \mathfrak{N}_1$ and $E_2 \in \mathfrak{N}_2$, we have $E_1 E_2 \neq 0$ by hypothesis. Let Ψ_1 be a unit vector such that $\Psi_1 \in E_1 E_2 \mathcal{H}$. Since $VW(V^*W^*)\Psi_1 = \Psi_1$, we have $V^*W^*\Psi_1 \neq 0$. Define $\Psi \in \mathcal{H}$ by $\Psi = \frac{1}{\sqrt{2}}(\Psi_1 + V^*W^*\Psi_1)$. Then Ψ is a unit vector. Let φ be the vector state of $\mathfrak{N}_1 \vee \mathfrak{N}_2$ induced by Ψ . Then

$$\varphi(E_1) = \varphi(E_2) = \varphi(E_1 E_2) = \varphi(F_1) = \varphi(F_2) = \varphi(F_1 F_2) = \frac{1}{2}.$$

Since $E_1 F_1 \Psi = \frac{1}{2\sqrt{2}}(\Psi_1 + W^*\Psi_1)$, $F_1 E_1 \Psi = \frac{1}{2\sqrt{2}}\Psi_1$ and $W^*\Psi_1 \neq 0$, we have $\varphi(|[E_1, F_1]|^2) \neq 0$. Similarly $\varphi(|[E_2, F_2]|^2) \neq 0$. Therefore, the vector state φ of $\mathfrak{N}_1 \vee \mathfrak{N}_2$ is an EPR state for the incommensurable pairs (E_1, E_2) and (F_1, F_2) from $(\mathfrak{N}_1, \mathfrak{N}_2)$. \square

3 Bohr's requirements of classicality and objectivity

Halvorson and Clifton followed Howard's interpretation of Bohr's notion of "classical description," and they reformulated this notion in algebraic quantum theory as follows [10, p.13]. Let \mathfrak{B} be a unital C*-subalgebra of a unital C*-algebra \mathfrak{A} with a state φ . A state ω of \mathfrak{B} is said to be dispersion-free if $\omega(A^*A) = |\omega(A)|^2$ for any $A \in \mathfrak{B}$. We say that φ is *classical* on \mathfrak{B} or that \mathfrak{B} is *beable* for φ if there is a probability measure μ on the space $\mathcal{S}_{DF}(\mathfrak{B})$ of dispersion-free states of \mathfrak{B} satisfying

$$\varphi(A) = \int_{\mathcal{S}_{DF}(\mathfrak{B})} \omega(A) d\mu(\omega) \quad (2)$$

for every $A \in \mathfrak{B}$.

A *measurement context* is defined to be a pair (φ, A) of a state φ of \mathfrak{A} and a self-adjoint element $A \in \mathfrak{A}$. If $A \in \mathfrak{A}$ is an observable being measured in state φ , then A must be contained in a beable subalgebra \mathfrak{B} determined by the measurement context (φ, A) , since the outcome of the A measurement must be described classically by the classical probability distribution μ of possible values $\omega(A)$ with $\omega \in \mathcal{S}_{DF}(\mathfrak{B})$. Then, the subalgebra \mathfrak{B} contains only observables for which we can speak of their values in terms of classical language.

As discussed in Section 1, we need some proposal to uniquely determine an appropriate subalgebra \mathfrak{B} . Howard's strategy to single out such a subalgebra is to collect all the observables whose value can be predictable from the value of A without any further disturbance of the system. This can be done if φ is an EPR state for (A, B) . Thus, Howard's strategy is to collect all observables of the form $f(B)$ such that φ is an EPR state for (A, B) . This strategy was successful to reconstruct Bohr's reply in Bohm's modified spin version of the EPR argument. Halvorson and Clifton called the above doctrine a *contextualized version of the EPR reality criterion*, since that assigns an element of reality to an observable the value of which can be predicted without disturbing the system, provided it can be described by a classical language together with the value of measured observable A .

Halvorson and Clifton [10, pp.14–15] proposed an alternative approach to single out a suitable set of observables to possess their values in the given measurement context. According to them, a feature of a system is "objective" by being "invariant under the 'relevant' group of symmetries" [10, p.11], as follows. Let \mathfrak{B} be a unital C*-subalgebra of a unital C*-algebra \mathfrak{A} with a self-adjoint element A and a state φ . We say that \mathfrak{B} is *definable in the measurement context* (φ, A) if $U^*\mathfrak{B}U = \mathfrak{B}$ for any unitary $U \in \mathfrak{A}$ such that $[A, U] = 0$ and $\varphi_U = \varphi$, where φ_U is defined by $\varphi_U(X) = \varphi(U^*XU)$ for every $X \in \mathfrak{A}$. Here, we have used a notion of implicit definability, in the sense that \mathfrak{B} can be considered to be implicitly defined by φ and A if the membership of \mathfrak{B} is not affected by any automorphism of \mathfrak{A} that leaves φ and A invariant; this notion is widely used in foundational studies, for example, by Malament [15, p.297].

Recall from Section 1 that Halvorson and Clifton proved consistency between the above two approaches to single out an appropriate subset of observables — Howard's contextualized version of the EPR reality criterion and the invariance under the relevant group of symmetry — for the position-momentum case. Now, we shall prove that the above two approaches are consistent even in a general formulation

of algebraic quantum theory. To formulate the statement, we say that a unital C^* -subalgebra \mathfrak{B} of \mathfrak{A} is a *beable subalgebra* for (φ, A) if it satisfies following conditions:

- (i) (Beable) φ is classical on \mathfrak{B} .
- (ii) (A -Priv) $A \in \mathfrak{B}$.
- (iii) (Def) \mathfrak{B} is definable in (φ, A) .

The following theorem shows that the beable for the measurement context (φ, A) may include an observable perfectly correlated with A but excludes observables non-commuting with observables perfectly correlated with A .

Theorem 2. *Let (A, B) be a pair of commuting self-adjoint elements in a unital C^* -algebra \mathfrak{A} . Let φ be a state of \mathfrak{A} and let \mathfrak{B} be a C^* -subalgebra of \mathfrak{A} . If φ is an EPR state for (A, B) and \mathfrak{B} is beable for (φ, A) , we have*

$$\varphi(|[B^n, Z]|^2) = 0$$

for all $Z \in \mathfrak{B}$ and $n = 1, 2, \dots$

Proof. Let $V_t = \exp\{-it(A - B)\}$, where $t \in \mathbf{R}$. Then, we have $[A, V_t] = 0$. Let $(\pi, \mathcal{H}, \Omega)$ be the GNS representation of \mathfrak{A} induced by φ . Since φ is an EPR state for (A, B) , we have $\pi(A - B)\Omega = 0$, and consequently $\pi(V_t)\Omega = \Omega$. Thus, we have $\varphi_{V_t} = \varphi$. From (Def), we conclude $V_t^*\mathfrak{B}V_t = \mathfrak{B}$. Now let $Z \in \mathfrak{B}$. From (A -Priv) we have $[A, Z] \in \mathfrak{B}$. Note here that $V_t^*ZV_t$ is uniformly continuous in t , since $A - B$ is bounded. Thus, we also have $[A - B, Z] = -i\frac{d}{dt}V_t^*ZV_t|_{t=0} \in \mathfrak{B}$, so that we obtain $[B, Z] \in \mathfrak{B}$. Let ω be a dispersion-free state on \mathfrak{B} . By the Hahn-Banach theorem, ω can be extended to a state $\tilde{\omega}$ of \mathfrak{A} . Then, we have $\tilde{\omega}(ZB) = \tilde{\omega}(BZ) = \omega(Z)\tilde{\omega}(B)$. Thus, we have $\omega([B, Z]) = 0$, so that $\omega(|[B, Z]|^2) = |\omega([B, Z])|^2 = 0$. Since $\varphi|_{\mathfrak{B}}$ is classical, there is a probability distribution μ of dispersion-free states satisfying

$$\varphi(|[B, Z]|^2) = \int \omega(|[B, Z]|^2) d\mu(\omega) = 0.$$

This completes the proof for $n = 1$. The proof for $n > 1$ is analogously obtained by replacing $A - B$ by $A^n - B^n$. \square

Let φ be an EPR state for incommensurable pairs (A_1, A_2) and (B_1, B_2) . Suppose that A_1 is measured in the state φ . Then, the measurement result on A_1 is considered real. By Bohr's requirement of a classical description, B_1 cannot have the elements of reality. Since A_2 and B_2 are not commuting in φ , they have no simultaneous reality, so that the problem is on what ground to choose either A_2 or B_2 . According to Howard, Bohr regarded elements predictable with certainty as real [11, p.256] plausibly, once given such a context. Therefore, A_2 is an element of reality if A_1 is measured in the state φ . Since A_2 and B_2 cannot be ascribed simultaneous reality, B_2 is not an element of reality in this case. This interpretation is consistent with the objectivity requirement, implicit definability of the beables \mathfrak{B} , or invariance under the relevant group of symmetries, as shown by the following theorem.

Theorem 3. *Let φ be an EPR state of a unital C^* -algebra \mathfrak{A} for incommensurable pairs (A_1, A_2) and (B_1, B_2) in \mathfrak{A} . Then, neither B_1 nor B_2 can be an element of any beable subalgebra for (φ, A_1) .*

Proof. Let \mathfrak{B} be a beable subalgebra for (φ, A_1) . Suppose $B_1 \in \mathfrak{B}$. By [9, Proposition 2.2], $\varphi(|[A_1^n, B_1^m]|^2) = 0$ for all n, m . This contradicts the incommensurability of (A_1, A_2) and (B_1, B_2) , so that $B_1 \notin \mathfrak{B}$. Suppose $B_2 \in \mathfrak{B}$. It follows from Theorem 2 that $\varphi(|[A_2^n, Z]|^2) = 0$ with $Z = B_2^m \in \mathfrak{B}$ for all n, m . This contradicts the incommensurability of (A_1, A_2) and (B_1, B_2) , so that $B_2 \notin \mathfrak{B}$. \square

4 Unbounded observables

In the previous section, we have extended the Halvorson-Clifton consistency theorem to EPR states in algebraic quantum theory. However, this formulation is limited to the case where observables to be correlated by EPR states are bounded. Since the original EPR argument involves unbounded observables such as position and momentum, a further extension of the formulation is necessary to include any EPR states for unbounded observables. This section is devoted to such an extension of the preceding result. In the next section, we shall show that the result obtained in this section reconstructs Bohr's reply to the original EPR argument.

In algebraic quantum theory, any quantum system \mathbf{S} is represented by a C^* -algebra \mathfrak{A} . Obviously, \mathfrak{A} does not contain any unbounded observables. Thus, \mathfrak{A} "describes" an unbounded observable A of \mathbf{S} through a class of bounded functions $f(A)$ of A . Here, we assume that the C^* -algebra \mathfrak{A} describes an unbounded observable A of \mathbf{S} if and only if \mathfrak{A} contains the exponential function e^{itA} of A for any real number t , or equivalently if and only if \mathfrak{A} contains every almost periodic functions $f(A)$ of A . This assumption is relevant to CCR algebras or Weyl algebras as discussed in the next section.

Denote by $\text{CB}(\mathbf{R})$ the space of complex-valued bounded continuous functions on the real line \mathbf{R} . We say that a function $f \in \mathbf{R}$ is *almost periodic* ($f \in \text{AP}(\mathbf{R})$) if its orbit under translations $O(f) = \{f_t | t \in \mathbf{R}\}$ is relatively compact in $\text{CB}(\mathbf{R})$, where $f_t(x) = f(x - t)$ for all $x \in \mathbf{R}$ [7]. Denote by χ_t for a real $t \in \mathbf{R}$ the character on \mathbf{R} defined by $\chi_t(x) = e^{itx}$ for all $x \in \mathbf{R}$. The space $\text{AP}(\mathbf{R})$ is characterized as the C^* -algebra generated by the set of all characters χ_t with $t \in \mathbf{R}$, and is $*$ -isomorphic to the C^* -algebra $C(b\mathbf{R})$ of continuous functions on the Bohr compactification $b\mathbf{R}$ of \mathbf{R} [14].

Let \mathfrak{A} be a unital C^* -algebra and let φ be a state of \mathfrak{A} . An *observable* of \mathfrak{A} is defined to be a $*$ -homomorphism from the abelian C^* -algebra $\text{AP}(\mathbf{R})$ of almost periodic functions on \mathbf{R} into \mathfrak{A} . Denote by $\mathcal{O}(\mathfrak{A})$ the space of observables of \mathfrak{A} . The range $A(\text{AP}(\mathbf{R}))$ of an observable A is denoted by $C^*(A)$, which is a unital C^* -subalgebra of \mathfrak{A} . For any observable $A \in \mathcal{O}(\mathfrak{A})$, we define $U_t^A \in \mathfrak{A}$ by $U_t^A = A(\chi_t)$ for $t \in \mathbf{R}$ and $f(A) \in \mathfrak{A}$ by $f(A) = A(f)$ for $f \in \text{AP}(\mathbf{R})$. Then, the mapping $t \mapsto U_t^A$ is a (not necessarily continuous) group homomorphism from \mathbf{R} to $\mathcal{U}(\mathfrak{A})$, where $\mathcal{U}(\mathfrak{A})$ is the group of unitary elements of \mathfrak{A} . Conversely, for every group homomorphism U_t from \mathbf{R} to $\mathcal{U}(\mathfrak{A})$, there is a $*$ -homomorphism π from $\text{AP}(\mathbf{R})$ into $C^*(A)$ such that $\pi(f) = \sum_{k=1}^n \alpha_k U_{t_k}^A$ if $f(x) = \sum_{k=1}^n \alpha_k e^{it_k x}$. An observable A is said to be *bounded*, if $t \mapsto U_t^A$ is norm continuous, i.e., $\lim_{t \rightarrow 0} \|U_t^A - 1\| = 0$. For every bounded observable A , there is some $\hat{A} \in \mathfrak{A}_{sa}$ uniquely such that $U_t^A = e^{it\hat{A}}$. Conversely, every $\hat{A} \in \mathfrak{A}_{sa}$ has the corresponding bounded observable A so that bounded observables and self-adjoint elements of \mathfrak{A} are in one-to-one correspondence. Observables $A, B \in \mathcal{O}(\mathfrak{A})$ are said to be *commuting* if $C^*(A)$ and $C^*(B)$ are commuting subalgebras. Let

φ be a state of \mathfrak{A} . For any commuting $A, B \in \mathcal{O}(\mathfrak{A})$, we say that φ is an *EPR state* for (A, B) if $\varphi(U_t^A U_{-t}^B) = 1$ for any $t \in \mathbf{R}$. This relation is equivalent to the relation $\varphi((\hat{A} - \hat{B})^2) = 0$, if A, B are bounded, and is equivalent to the “perfect correlation” between A and B , if A and B are represented by unbounded self-adjoint operators and φ is represented by a density operator on a Hilbert space on which \mathfrak{A} acts [16, Theorem 5.5].

Now, we extend the notion of a measurement context to any pair (φ, A) of a state φ of \mathfrak{A} and an unbounded observable A of \mathfrak{A} . A C^* -subalgebra \mathfrak{B} of \mathfrak{A} is said to be *beable* for a measurement context (φ, A) if it satisfies the following conditions.

- (i) (Beable) φ is classical on \mathfrak{B} .
- (ii) (A -Prev) $C^*(A) \subseteq \mathfrak{B}$.
- (iii) (Def) For any unitary $U \in \mathfrak{A}$ leaving (φ, A) invariant, the subalgebra \mathfrak{B} is globally invariant under U ; namely, $U^* \mathfrak{B} U = \mathfrak{B}$, if $[U, U_t^A] = 0$ for every $t \in \mathbf{R}$ and $\varphi_U = \varphi$.

Proposition 1. *Let A, B be commuting observables of \mathfrak{A} . A state φ of \mathfrak{A} is an EPR state for $(f(A), f(B))$ for every $f \in \text{AP}(\mathbf{R})$ if it is an EPR state for (A, B) .*

Proof. Suppose that φ is an EPR state for (A, B) . Let $(\pi, \mathcal{H}, \Omega)$ be a GNS representation of \mathfrak{A} induced by φ . By assumption,

$$\|\pi(U_t^A)\Omega - \pi(U_t^B)\Omega\|^2 = 2 - 2\Re\langle \pi(U_t^A)\Omega, \pi(U_t^B)\Omega \rangle = 2 - 2\varphi(U_{-t}^A U_t^B) = 0.$$

Thus, $\pi(U_t^A)\Omega = \pi(U_t^B)\Omega$. It follows from linearity that if

$$f(x) = \sum_{k=1}^n \alpha_k e^{it_k x}$$

then $\pi(f(A))\Omega = \pi(f(B))\Omega$, and this holds for every $f \in \text{AP}(\mathbf{R})$ by continuity. Therefore, we have $\varphi((f(A) - f(B))^2) = 0$ for all $f \in \text{AP}(\mathbf{R})$. \square

The following theorem extends Theorem 2 to unbounded observables.

Theorem 4. *Let \mathfrak{A} be a unital C^* -algebra and let φ be a state of \mathfrak{A} . Let $A, B \in \mathcal{O}(\mathfrak{A})$. If φ is an EPR state for (A, B) and \mathfrak{B} is beable for (φ, A) , then we have*

$$\varphi(|[f(B), Z]|^2) = 0$$

for all $Z \in \mathfrak{B}$ and $f \in \text{AP}(\mathbf{R})$.

Proof. The proof is obtained from the proof of Theorem 2 by replacing $A - B$ by $f(A) - f(B)$ using Proposition 1 to show $\varphi_{v_t} = \varphi$ if f is real-valued. Then, the assertion follows easily. \square

Let φ be a state of \mathfrak{A} . Observables A and B of \mathfrak{A} are said to *commute in φ* if $\varphi(|[U_t^A, U_s^B]|^2) = 0$ for all $t, s \in \mathbf{R}$. This definition is equivalent to the previous one for bounded observables A and B . Let $\mathfrak{A}_1, \mathfrak{A}_2$ be commuting subalgebras of \mathfrak{A} . Then, pairs (A_1, A_2) and (B_1, B_2) with $A_j, B_j \in \mathfrak{A}_j$ for $j = 1, 2$ are called *incommensurable pairs* from $\mathfrak{A}_1, \mathfrak{A}_2$ for φ if $A_j, B_j \in \mathcal{O}(\mathfrak{A}_j)$ for $j = 1, 2$ and $\varphi(|[U_t^{A_j}, U_s^{B_j}]|^2) \neq 0$ for some $t, s \in \mathbf{R}$.

The following theorem extends Theorem 3 to unbounded observables.

Theorem 5. *Let $\mathfrak{A}_1, \mathfrak{A}_2$ be commuting subalgebras of a unital C^* -algebra \mathfrak{A} . Let φ be an EPR state of \mathfrak{A} for incommensurable pairs (A_1, A_2) and (B_1, B_2) from $(\mathfrak{A}_1, \mathfrak{A}_2)$. Then, neither B_1 nor B_2 can be an observable of any beable subalgebra for (φ, A_1) .*

Proof. Let \mathfrak{B} be a beable subalgebra for (φ, A_1) . Suppose $B_1 \in \mathcal{O}(\mathfrak{B})$. By [9, Proposition 2.2], $\varphi(|[U_t^{A_1}, U_s^{B_1}]|^2) = 0$ for any $t, s \in \mathbf{R}$. This contradicts the incommensurability of (A_1, A_2) and (B_1, B_2) , so that $B_1 \notin \mathfrak{B}$. Suppose $B_2 \in \mathfrak{B}$. It follows from Theorem 4 that $\varphi(|[U_t^{A_2}, Z]|^2) = 0$ with $Z = U_s^{B_2} \in \mathfrak{B}$. This contradicts the incommensurability of (A_1, A_2) and (B_1, B_2) , so that $B_2 \notin \mathfrak{B}$. \square

5 EPR argument

In this section, we formulate the original EPR argument following Halvorson and Clifton [10], and explicitly show that the general theorem obtained in the preceding section can reconstruct Bohr's reply to the original EPR argument.

Let \hat{Q}, \hat{P} be self-adjoint operators corresponding to position and momentum, respectively, in the Schrödinger representation on $L^2(\mathbf{R})$ of one-dimensional motion. Let $\mathfrak{A}[\mathbf{R}^2]$ be the Weyl algebra, namely, $\mathfrak{A}[\mathbf{R}^2]$ be the C^* -algebra generated by $f(\hat{Q}), f(\hat{P})$ for all $f \in \text{AP}(\mathbf{R})$. We define observables Q, P of $\mathfrak{A}[\mathbf{R}^2]$ by $Q(f) = f(\hat{Q})$ and $P(f) = f(\hat{P})$ for all $f \in \text{AP}(\mathbf{R})$. Then, $C^*(\hat{Q})$ and $C^*(\hat{P})$ are $*$ -isomorphic to $\text{AP}(\mathbf{R})$. For any observable X of $\mathfrak{A}[\mathbf{R}^2]$, we denote the corresponding observables in $\mathfrak{A}[\mathbf{R}^2] \otimes \mathbf{C}I$ and $\mathbf{C}I \otimes \mathfrak{A}[\mathbf{R}^2]$ by $X_1 = X \otimes I$ and $X_2 = I \otimes X$, respectively.

We define observables $Q_1 - Q_2$ and $P_1 + P_2$ of $\mathfrak{A}[\mathbf{R}^2] \otimes \mathfrak{A}[\mathbf{R}^2]$ by $Q_1 - Q_2(f) = f(\hat{Q}_1 - \hat{Q}_2)$, $P_1 + P_2(f) = f(\hat{P}_1 + \hat{P}_2)$ for all $f \in \text{AP}(\mathbf{R})$. Denote by $C^*(Q_1 - Q_2, P_1 + P_2)$ the C^* -algebra generated by $C^*(Q_1 - Q_2)$ and $C^*(P_1 + P_2)$. In what follows, let $\mathfrak{A} = \mathfrak{A}[\mathbf{R}^2] \otimes \mathfrak{A}[\mathbf{R}^2]$.

Proposition 2. *$C^*(Q_1 - Q_2, P_1 + P_2)$ is $*$ -isomorphic to $\text{AP}(\mathbf{R}^2)$.*

Proof. Let \mathcal{F} be the Fourier transform on $L^2(\mathbf{R})$. Let $\alpha_1(X) = (1 \otimes \mathcal{F}^*)X(1 \otimes \mathcal{F})$. For every $f \in \text{AP}(\mathbf{R})$, we have $\alpha_1(f(\hat{Q}_1)) = f(\hat{Q}_1)$ and $\alpha_1(f(\hat{Q}_2)) = f(\hat{P}_2)$. Let $U = e^{-i\hat{P}_1\hat{Q}_2/\hbar}$ and $\alpha_2(X) = U^*XU$. Then, α_2 is a $*$ -automorphism of \mathfrak{A} . We have $\alpha_2(f(\hat{Q}_1)) = f(\hat{Q}_1 - \hat{Q}_2)$ and $\alpha_2(f(\hat{P}_2)) = f(\hat{P}_1 + \hat{P}_2)$ for every $f \in \text{AP}(\mathbf{R})$. Thus, we have $\alpha_2 \circ \alpha_1(f(\hat{Q}_1, \hat{Q}_2)) = f(\hat{Q}_1 - \hat{Q}_2, \hat{P}_1 + \hat{P}_2)$ for all $f \in \text{AP}(\mathbf{R}^2)$. The correspondence $\iota : f(x, y) \mapsto f(\hat{Q}_1, \hat{Q}_2)$ is $*$ -isomorphism from $\text{AP}(\mathbf{R}^2)$ to $C^*(Q_1, Q_2)$, and hence $C^*(Q_1 - Q_2, P_1 + P_2)$ is $*$ -isomorphic to $\text{AP}(\mathbf{R}^2)$. \square

For every $(u, v) \in \mathbf{R}^2$, a *canonical (u, v) -EPR state* is a state φ of \mathfrak{A} satisfying

$$\varphi(f(\hat{Q}_1 - \hat{Q}_2, \hat{P}_1 + \hat{P}_2)) = f(u, v) \quad (3)$$

for all $f \in \text{AP}(\mathbf{R})$.

Canonical EPR states can be constructed by the following standard argument. By the $*$ -isomorphism between $C^*(Q_1 - Q_2, P_1 + P_2)$ and $\text{AP}(\mathbf{R}^2)$, every point $(u, v) \in \mathbf{R}^2$ defines a state φ of $C^*(Q_1 - Q_2, P_1 + P_2)$ satisfying Eq. (3). By the Hahn-Banach theorem, this state extends to a canonical (u, v) -EPR state of \mathfrak{A} . Halvorson [8] proved the uniqueness of the canonical (u, v) -EPR state for every $u, v \in \mathbf{R}$.

Proposition 3. *The canonical (u, v) -EPR state is an EPR state of \mathfrak{A} for $(Q_1, u1 + Q_2)$ and $(P_1, v1 - P_2)$.*

Proof. Let φ be the canonical (u, v) -EPR state. We have

$$\varphi(U_t^{Q_1} U_{-t}^{u1+Q_2}) = \varphi(e^{it\hat{Q}_1} e^{-it(u1+\hat{Q}_2)}) = e^{-itu} \varphi(e^{it(\hat{Q}_1-\hat{Q}_2)}) = 1.$$

Thus, φ is an EPR state for $(Q_1, u1 + Q_2)$, and similarly it can be shown that it is an EPR state for $(P_1, v1 - P_2)$. \square

According to Bohr, a measurement on the first particle in an EPR state influences the condition which defines elements of reality for the second particle. The following theorem mathematically reconstructs Bohr's reply to EPR.

Theorem 6. *Let φ be the canonical (u, v) -EPR state of \mathfrak{A} for $u, v \in \mathbf{R}$. Then, neither P_1 nor P_2 can be an observable of any beable subalgebra \mathfrak{B} for (φ, Q_1) .*

Proof. Let \mathfrak{B} be a beable subalgebra for (φ, Q_1) . Since P_1 does not commute with Q_1 , it is obvious that $P_1 \notin \mathcal{O}(\mathfrak{B})$. Suppose $P_2 \in \mathcal{O}(\mathfrak{B})$. Since φ is an EPR state for $(Q_1, uI + Q_2)$ and \mathfrak{B} is beable for (φ, Q_1) , from Theorem 4 with $Z = e^{i\pi P_2} \in \mathfrak{B}$ we have

$$\varphi(|[e^{iQ_2}, e^{i\pi P_2}]|^2) = 0.$$

Let $(\mathcal{H}, \varpi, \Omega)$ be the GNS construction of \mathfrak{A} induced by φ . Then, we have

$$\varpi(e^{iQ_2} e^{i\pi P_2})\Omega = \varpi(e^{i\pi P_2} e^{iQ_2})\Omega.$$

By the Weyl commutation relation, we have

$$\varpi(e^{iQ_2} e^{i\pi P_2})\Omega = -\varpi(e^{i\pi P_2} e^{iQ_2})\Omega.$$

Thus, we have

$$2\varpi(e^{iQ_2} e^{i\pi P_2})\Omega = 0.$$

This is a contradiction, since $\varpi(e^{iQ_2} e^{i\pi P_2})$ is unitary, so that we conclude $P_2 \notin \mathcal{O}(\mathfrak{B})$. \square

6 Concluding Remarks

Let \mathcal{O}_1 and \mathcal{O}_2 be strictly space-like separated regions in Minkowski space. Then there exists a vector state φ of $\mathfrak{N}(\mathcal{O}_1) \vee \mathfrak{N}(\mathcal{O}_2)$ which is an EPR state for incommensurable pairs (E_1, E_2) and (F_1, F_2) of projections from $(\mathfrak{N}(\mathcal{O}_1), \mathfrak{N}(\mathcal{O}_2))$ by Theorem 1. If we were to measure E_1 in \mathcal{O}_1 in φ , we could predict with certainty the outcome of E_2 ; and if we were to measure F_1 in \mathcal{O}_1 in ψ , we could predict with certainty the outcome of F_2 .

If we accepted the criterion of reality which was proposed by EPR, E_2 and F_2 would simultaneously be elements of reality. However, quantum theory has no theoretical counter part of them, since they do not commute. Therefore algebraic quantum field theory would be incomplete with this criterion.

On the other hand, Bohr proposed another criterion of reality. According to him, a measurement in \mathcal{O}_1 influences the condition which defines elements of reality in

\mathcal{O}_2 . Howard [12] reconstructed Bohr’s reply to EPR in terms of Bohr’s notion of “a classical description” and “a contextualized version of the EPR reality criterion”. According to this criterion, “our decision as to which particular properties to consider as real will turn on the question of predictability with certainty” [11, p.256]. Therefore F_1 is an element of reality if E_1 is measured in φ . Since F_1 and F_2 cannot be ascribed simultaneous reality, F_2 is not an element of reality.

Halvorson and Clifton showed that Bohr’s reply to EPR is consistent with the following requirements [10, pp.16-17]: (1) Empirical Adequacy: When an observable is measured, it possesses a determinate value distributed in accordance with the probabilities determined by the quantum state. (2) Classical Description: Properties P and P' can be simultaneously real in a quantum state only if that state can be represented as a joint classical probability distribution over P and P' . (3) Objectivity: Elements of reality must be invariants of those symmetries that preserve the defining features of the measurement context. For the EPR position-momentum case they proved the consistency between those two requirements [10, Theorem 2], by showing that given the measurement context (φ, Q_1) with an EPR state and the position Q_1 of the particle I, Q_1 has the value as an element of reality, the value of P_1 might be ascribed reality, but the value of P_2 cannot be ascribed reality. In Theorem 6 we have reconstructed this consistency theorem through an rather elementary arguments in a general formulation of algebraic quantum theory.

It is shown that Howard’s contextualized reality criterion is consistent with the above three requirements in Theorems 2 and 3 for the case of bounded observables in algebraic quantum field theory, and in Theorems 4 and 5 for the case of unbounded observables. Thus, Howard’s reconstruction of Bohr’s reply based on a contextualized version of the EPR reality criterion has now acquired independent grounds from the objectivity requirement represented by an invariance principle under relevant symmetries even in a general formulation of algebraic quantum theory.

Acknowledgements

The authors thank Jeremy Butterfield for helpful comments and suggestions for an earlier version of this paper. M.O. thanks Andreas Doering for his warm hospitality at the Department of Computer Science, Oxford University, where the final part of this work has been done. M.O. is supported in part by the JSPS KAKENHI, No.21244007 and No.22654013. Y.K. is supported by the JSPS KAKENHI, No.23701009.

References

- [1] Arens, R., Varadarajan, V.S.: On the concept of Einstein-Podolsky-Rosen states and their structure. *J. Math. Phys.* **41**, 638–651 (2000)
- [2] Baumgärtel, H.: *Operatoralgebraic Methods in Quantum Field Theory. A series of Lectures.* Akademie Verlag, Berlin (1995)
- [3] Bohr, N.: Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.* **48**, 696–702 (1935)

- [4] Bohr, N.: Discussion with Einstein on epistemological problems in atomic physics. In: P.A. Shilpp (ed.) *Albert Einstein: Philosopher-Scientist*, *The Library of Living Philosophers*, vol. VII, pp. 200–241. Northwestern University, Evanston (1949)
- [5] Bub, J.: *Interpreting the Quantum World*. Cambridge University Press, Cambridge (1997)
- [6] Einstein, A., Podolsky, B., Rosen, N.: Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.* **47**, 777–780 (1935)
- [7] Greenleaf, F.P.: *Invariant Means on Topological Groups*. Van Nostrand, New York (1969)
- [8] Halvorson, H.: The Einstein-Podolsky-Rosen state maximally violates Bell’s inequalities. *Lett. Math. Phys.* **53**, 321–329 (2000)
- [9] Halvorson, H., Clifton, R.: Maximal beable subalgebras of quantum mechanical observables. *Int. J. Theor. Phys.* **38**, 2441–2484 (1999)
- [10] Halvorson, H., Clifton, R.: Reconsidering Bohr’s reply to EPR. In: T. Placek, J. Butterfield (eds.) *Non-locality and Modality*, pp. 3–18. Kluwer, Dordrecht (2002)
- [11] Howard, D.: *Complementarity and ontology: Niels Bohr and the problem of scientific realism in quantum physics*. Ph.D. thesis, Boston University (1979)
- [12] Howard, D.: What makes a classical concept classical? In: J. Faye, H.J. Folse (eds.) *Niels Bohr and Contemporary Philosophy*, pp. 201–229. Kluwer, Dordrecht (1994)
- [13] Landau, L.J.: On the violation of Bell’s inequality in quantum theory. *Phys. Lett. A* **120**, 54–56 (1987)
- [14] Loomis, L.H.: *An Introduction to Abstract Harmonic Analysis*. Van Nostrand, Princeton, NJ (1953)
- [15] Malament, D.: Causal theories of time and the conventionality of simultaneity. *Noûs* **11**, 293–300 (1977)
- [16] Ozawa, M.: Quantum perfect correlations. *Ann. Phys. (N.Y.)* **321**, 744–769 (2006)
- [17] Ozawa, M.: Quantum reality and measurement: A quantum logical approach. *Found. Phys.* **41**, 592–607 (2011)
- [18] Redhead, M.: *Incompleteness, Nonlocality, and Realism: A Prolegomenon to the Philosophy of Quantum Mechanics*. Oxford University Press, Oxford (1987)
- [19] Werner, R.F.: EPR states for von Neumann algebras (1999). ePrint arXiv:quant-ph/9910077v1