EPR states and Bell correlated states in algebraic quantum field theory

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Abstract

A mathematical rigorous definition of EPR states has been introduced by Arens and Varadarajan for finite dimensional systems, and extended by Werner to general systems. In the present paper we follow a definition of EPR states due to Werner. Then we show that an EPR state for incommensurable pairs is Bell correlated, and that the set of EPR states for incommensurable pairs is norm dense between two strictly space-like separated regions.

1 Introduction

Einstein, Podolsky and Rosen (EPR) [6] discussed a system consisting of two particles. They have interacted initially and then moved out so that the positions, and the momenta, of the two particles are strictly correlated, respectively. It follows that if one were to measure the position of the first particle, one could predict with certainty the outcome of a position measurement on the second particle; and it is also the case for a momentum measurement. EPR proposed a criterion of reality: "If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity" [6, p.777].

In accordance with this criterion, the position and the momentum of the second particle have simultaneous reality since the measurement on the first particle has not disturbed the second particle. On the other hand, position and momentum cannot have simultaneous reality in any states in quantum mechanics. Therefore EPR regarded quantum-mechanical description as incomplete, and concluded that the quantum mechanical description of a physical system should be supplemented by postulating the existence of "hidden variables," the specification of which would predetermine the result of measuring any observable of the system. Later Bohm simplified EPR original state. It is a unit vector in the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$, represented as

$$\frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right).$$

Bell assumed a complete description in terms of hidden variables and the locality assumptions tacitly assumed in EPR argument, and derived Bell's inequality under these assumptions. Bohm's simplified EPR state, however, violates Bell's inequality. In the

mid 1980s, Summers and Werner obtained a series of deep mathematical results on Bell's inequality in algebraic quantum field theory. For example, in most standard quantum field models, all normal states maximally violate Bell's inequalities across spacelike separated tangent wedges (cf. [17] and [19]).

The relation between EPR states and Bell's inequalities has been also investigated (cf. [9], [11], [12], [13]). For example, Halvorson [9] defined EPR states as a state of canonical commutation relations algebra, and showed that it maximally violates Bell's inequality. Another formulation of EPR states was introduced by Arens and Varadarajan [1], and extended by Werner [21] to general systems.

In the present paper we follow a definition of EPR states due to Werner, and try to clarify the relations between EPR states and Bell correlated states in algebraic quantum theory. We show that an EPR state for incommensurable pairs is Bell correlated (Theorem 2), and that the set of EPR states for incommensurable pairs is norm dense between two strictly space-like separated regions in algebraic quantum field theory (Theorem 3).

2 The relations between EPR states and Bell correlated states

In this paper, we use the following notation. If $\mathbb{A}$ is a set of operators acting on a Hilbert space $\mathcal{H}$, let $\mathbb{A}'$ represent its commutant, the set of all bounded operators on $\mathcal{H}$ which commute with all elements of $\mathbb{A}$. $[X, Y]$ denotes the commutator of X and Y, i.e., $[X, Y] = XY - YX$. We call X a self-adjoint contraction on a Hilbert space $\mathcal{H}$ if X is a self-adjoint operator on $\mathcal{H}$ such that $-I \leq X \leq I$, where I is an identity operator on $\mathcal{H}$. $\mathfrak{N}_1$ and $\mathfrak{N}_2$ denote von Neumann algebras on a Hilbert space $\mathcal{H}$ such that $\mathfrak{N}_1 \subseteq \mathfrak{N}_2'$, and $\mathfrak{N}_1 \vee \mathfrak{N}_2$ denotes a von Neumann algebra generated by $\mathfrak{N}_1$ and $\mathfrak{N}_2$.

In this section, we provide an abstract definition of EPR states. Let $\mathfrak{N}$ be a von Neumann algebra and let A_1 and A_2 be commuting self-adjoint operators in $\mathfrak{N}$. A normal state ω of $\mathfrak{N}$ is called *an EPR state* [21] for (A_1, A_2) if

$$\omega((A_1 - A_2)^2) = 0.$$

For any commuting self-adjoint operators A_1 and A_2, the joint probability distribution $\mu_\omega^{A_1, A_2}$ of A_1, A_2 in φ is defined uniquely to be a probability measure μ on $\mathbb{R}^2$ such that

$$\omega(f(A_1, A_2)) = \int_{\mathbb{R}^2} f(x, y) \, d\mu_\omega^{A_1, A_2}(x, y)$$

for any polynomial $f(x, y)$. Then, ω is an EPR state if and only if $\mu_\omega^{A_1, A_2}$ is concentrated in the diagonal, i.e, $\mu_\omega^{A_1, A_2}(\{(x, x)|x \in \mathbb{R}\}) = 1$. Thus, simultaneous measurements of A_1 and A_2 always give concordant results, and each one of the outcomes would predict with certainty the other (cf. [14]). In other words, A_1 and A_2 are strictly correlated in ω if ω is an EPR state for A_1 and A_2.

Nocommutativity of operators as well as strict correlation plays an important role in EPR's argument. EPR asserted that "either (1) the quantum-mechanical description of reality given by the wave function is not complete or (2) when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous

reality" [6, p.778]. In this paper, we interpret the second part of this assertion in terms of a beable algebra. Let $\mathfrak{B}$ be a C*-algebra and let ω be a state of $\mathfrak{B}$. $\mathfrak{B}$ is called a beable algebra for a given state ω if there is a probability measure μ on the space $\mathcal{S}_{DF}(\mathfrak{B})$ of dispersion-free states of $\mathfrak{B}$ satisfying

$$\omega(A) = \int_{\mathcal{S}_{DF}(\mathfrak{B})} \omega(A) d\mu(A)$$

for every $A \in \mathfrak{B}$ [8, p.2447]. A dispersion-free state ω of $\mathfrak{B}$ satisfies the condition that $\omega(X^2) = \omega(X)^2$ for any self-adjoint element $X \in \mathfrak{B}$. Roughly speaking, a beable algebra for ω is the set of observable which can be taken to have determinate values statistically distributed in accordance with ω. $\mathfrak{B}$ is a beable algebra for ω if and only if $\omega(|[A,B]|^2) = 0$ for any $A, B \in \mathfrak{B}$ [8, Proposition 2.2]. Thus $\omega(|[A,B]|^2) \neq 0$ means that A and B cannot have simultaneous reality in ω.

Let consider two operators A and B and a unit vector Ψ_0 such that

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \Psi_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then $[A,B] \neq 0$. On the other hand, from a beable algebraic point of view, A and B can have simultaneous reality in Ψ_0 because $A\Psi_0 = B\Psi_0 = \Psi_0$. Therefore $[A,B] \neq 0$ does not always mean that A and B cannot have simultaneous reality.

On the basis of a bealble algebra, we say a normal state ω of $\mathfrak{N}_1 \vee \mathfrak{N}_2$ is *an EPR state for incommensurable pairs* if there exist projections $E_1, F_1 \in \mathfrak{N}_1$ and $E_2, F_2 \in \mathfrak{N}_2$ such that ω is an EPR state for (E_1, E_2) and (F_1, F_2), $\omega(|[E_1, F_1]|^2) \neq 0$ and $\omega(|[E_2, F_2]|^2) \neq 0$ (cf [15]). In this state, if we were to measure E_2, we could predict with certainty the outcome of E_1; and if we were to measure F_2, we could predict with certainty the outcome of F_1. In accordance with EPR's criterion of reality, E_1 and F_1 have simultaneous reality. On the other hand E_1 and F_1 cannot have simultaneous reality since E_1 does not commute with F_1 in ω. Therefore an EPR state for incommensurable pairs allows for similar arguments to EPR's [6].

If there exist self-adjoint contractions $A_1, B_1 \in \mathfrak{N}_1$ and $A_2, B_2 \in \mathfrak{N}_2$ such that

$$\frac{1}{2}|\omega(A_1A_2 + A_1B_2 + B_1A_2 - B_1B_2)| > 1, \tag{1}$$

we say that ω *violates a Bell inequality, or is Bell correlated*. It is known that the left-hand side of (1) cannot exceed $\sqrt{2}$ (cf. [5] and [20]). We say that ω *is strongly maximally correlated* if there exist self-adjoint contractions $A_1, B_1 \in \mathfrak{N}_1$ and $A_2, B_2 \in \mathfrak{N}_2$ such that

$$\frac{1}{2}|\omega(A_1A_2 + A_1B_2 + B_1A_2 - A_2B_2)| = \sqrt{2}.$$

Landau [16] provided a sufficient condition for the strongly maximal Bell correlation in some normal state, and Bohata and Hamhalter [3] [4] characterized strongly maximally Bell correlated states.

As it is shown in Theorem 1 and Corollary 1, a strongly maximally Bell correlated normal state is an EPR state for incommensurable pairs. The following theorem can be proven in a similar way of the proof of [20, Theorem 2.1 (2b)].

Theorem 1. *Let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be von Neumann algebras on a Hilbert space $\mathcal{H}$ such that $\mathfrak{N}_1 \subseteq \mathfrak{N}_2'$, let ω be a normal state of $\mathfrak{N}_1 \vee \mathfrak{N}_2$ and let S_1 and S_2 be the support of $\omega|_{\mathfrak{N}_1}$ and $\omega|_{\mathfrak{N}_2}$ respectively, where $\omega|_{\mathfrak{N}_i}$ is the restriction of ω to $\mathfrak{N}_i$. Then the following conditions are equivalent.*

1. *ω is strongly maximally Bell correlated.*

2. *There are projections $E_1, F_1 \in S_1\mathfrak{N}_1 S_1$ and $E_2, F_2 \in S_2\mathfrak{N}_2 S_2$ such that*

$$\omega((E_1 - E_2)^2) = \omega((F_1 - F_2)^2) = 0,$$

$$(2E_1 - S_1)(2F_1 - S_1) + (2F_1 - S_1)(2E_1 - S_1) = 0.$$

Proof. Let $(\pi, \mathcal{H}_\omega, \Omega)$ be GNS representation of $\mathfrak{N}_1 \vee \mathfrak{N}_2$ induced by ω. Then $\pi(S_1)\Omega = \pi(S_2)\Omega = \Omega$.

$1 \Rightarrow 2$ Since ω is strongly maximally correlated, there exists self-adjoint contractions $A_1, B_1 \in \mathfrak{N}_1$ and $A_2, B_2 \in \mathfrak{N}_2$ such that $\frac{1}{2}|\omega(A_1 A_2 + A_1 B_2 + B_1 A_2 - B_1 B_2)| = \sqrt{2}$. Let $X_1 := (1/2)(S_1 A_1 S_1 + i S_1 B_1 S_1)$ and $X_2 := (1/2\sqrt{2})(S_2 A_2 S_2 + S_2 B_2 S_2 + i(S_2 A_2 S_2 - S_2 B_2 S_2))$. Then

$$X_1^* X_1 + X_1 X_1^* = \frac{1}{2}(S_1 A_1 S_1)^2 + \frac{1}{2}(S_1 B_1 S_1)^2 \leq S_1 \tag{2}$$

and

$$X_2^* X_2 + X_2 X_2^* = \frac{1}{2}(S_2 A_2 S_2)^2 + \frac{1}{2}(S_2 B_2 S_2)^2 \leq S_2. \tag{3}$$

Since $\pi(S_1)\Omega = \pi(S_2)\Omega = \Omega$,

$$\begin{aligned}
\sqrt{2} &\times \frac{1}{2}|\omega(A_1 A_2 + A_1 B_2 + B_1 A_2 - B_1 B_2)| \\
&= \frac{\sqrt{2}}{2}|\langle\Omega, \pi((S_1 A_1 S_1)(S_2 A_2 S_2) + (S_1 A_1 S_1)(S_2 B_2 S_2) \\
&\qquad\qquad + (S_1 B_1 S_1)(S_2 A_2 S_2) - (S_1 B_1 S_1)(S_2 B_2 S_2))\Omega\rangle| \\
&= 4|\mathrm{Re}\langle\Omega, \pi(X_1^* X_2)\Omega\rangle| \\
&\leq 2|\mathrm{Re}\langle\pi(X_1)\Omega, \pi(X_2)\Omega\rangle| + 2|\mathrm{Re}\langle\pi(X_2^*)\Omega, \pi(X_1^*)\Omega\rangle| \\
&= |\|\pi(X_1)\Omega\|^2 + \|\pi(X_2)\|^2 - \|\pi(X_1 - X_2)\Omega\|^2| \\
&\qquad + |\|\pi(X_2^*)\Omega\|^2 + \|\pi(X_1^*)\Omega\|^2 - \|\pi(X_2^* - X_1^*)\|^2| \\
&\leq \langle\Omega, \pi(X_1^* X_1 + X_1 X_1^* + X_2^* X_2 + X_2 X_2^*)\Omega\rangle \\
&\leq 2.
\end{aligned} \tag{4}$$

Since $(1/2)\omega(A_1 A_2 + A_1 B_2 + B_1 A_2 - B_1 B_2) = \sqrt{2}$, equality obtains in (4). By Equations (2) and (3),

$$\langle\Omega, \pi(X_1^* X_1 + X_1 X_1^*)\Omega\rangle = 1, \tag{5}$$

$$\langle\Omega, \pi(X_2^* X_2 + X_2 X_2^*)\Omega\rangle = 1, \tag{6}$$

$$\|\pi(X_1 - X_2)\Omega\|^2 = \|\pi(X_1^* - X_2^*)\Omega\|^2 = 0. \tag{7}$$

Equations (2) and (5) imply $\langle \Omega, (S_1 A_1 S_1)^2 \Omega \rangle = \langle \Omega, (S_1 B_1 S_1)^2 \Omega \rangle = \langle \Omega, S_1 \Omega \rangle$. Thus

$$(S_1 A_1 S_1)^2 = (S_1 B_1 S_1)^2 = S_1. \tag{8}$$

Similarly Equations (3) and (6) imply

$$(S_2 A_2 S_2)^2 = (S_2 B_2 S_2)^2 = S_2. \tag{9}$$

Equation (7) entails

$$\pi(S_1 A_1 S_1)\Omega = \pi\left(\frac{1}{\sqrt{2}}(S_2 A_2 S_2 + S_2 B_2 S_2)\right)\Omega, \tag{10}$$

$$\pi(S_1 B_1 S_1)\Omega = \pi\left(\frac{1}{\sqrt{2}}(S_2 A_2 S_2 - S_2 B_2 S_2)\right)\Omega. \tag{11}$$

By Equations (7), (8) and (9)

$$\begin{aligned}
\pi(S_1 A_1 S_1 S_1 B_1 S_1 + S_1 B_1 S_1 S_1 A_1 S_1)\Omega &= (2/i)\pi(X_1^2 - X_1^{*2})\Omega \\
&= (2/i)\pi(X_2^2 - X_2^{*2})\Omega \\
&= 0,
\end{aligned} \tag{12}$$

which implies

$$S_1 A_1 S_1 S_1 B_1 S_1 + S_1 B_1 S_1 S_1 A_1 S_1 = 0. \tag{13}$$

Let

$$E_1 := \frac{1}{2}\left(\frac{1}{\sqrt{2}}(S_1 A_1 S_1 + S_1 B_1 S_1) + S_1\right), \quad E_2 := \frac{S_2 A_2 S_2 + S_2}{2},$$

$$F_1 := \frac{1}{2}\left(\frac{1}{\sqrt{2}}(S_1 A_1 S_1 - S_1 B_2 S_1) + S_1\right), \quad F_2 := \frac{S_2 B_2 S_2 + S_2}{2}.$$

By Equations (8), (9) and (13), $E_1, F_1 \in S_1 \mathfrak{N}_1 S_1$ and $E_2, F_2 \in S_2 \mathfrak{N}_2 S_2$ are projections, and $(2E_1 - S_1)(2F_1 - S_1) + (2F_1 - S_1)(2E_1 - S_1) = 0$. Equations (10) and (11) imply $\pi(E_1)\Omega = \pi(E_2)\Omega$ and $\pi(F_1)\Omega = \pi(F_2)\Omega$. Therefore $\omega((E_1 - E_2)^2) = \omega((F_1 - F_2)^2) = 0$.

$2 \Rightarrow 1$ Let define $A_1 := 1/\sqrt{2}(2E_1 - S_1) + 1/\sqrt{2}(2F_1 - S_1)$, $B_1 := 1/\sqrt{2}(2E_1 - S_1) - 1/\sqrt{2}(2F_1 - S_1)$, $A_2 := 2E_2 - S_2$ and $B_2 := 2F_2 - S_2$.

Since $(2E_1 - S_1)(2F_1 - S_1) + (2F_1 - S_1)(2E_1 - S_1) = 0$, $A_1^2 = B_1^2 = S_1$ and $A_2^2 = B_2^2 = S_2$. Thus A_1, B_1, A_2 and B_2 are self-adjoint contractions. $\pi(E_1)\Omega = \pi(E_2)\Omega$ and $\pi(F_1)\Omega = \pi(F_2)\Omega$ imply

$$\frac{1}{2}\omega(A_1 A_2 + A_1 B_2 + B_1 A_2 - B_1 B_2) = \sqrt{2}.$$

$\square$

Corollary 1. *Let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be von Neumann algebras on a Hilbert space $\mathcal{H}$ such that $\mathfrak{N}_1 \subseteq \mathfrak{N}_2'$, let ω be a normal state of $\mathfrak{N}_1 \vee \mathfrak{N}_2$. If ω is strongly maximally Bell correlated, then ω is an EPR state for incommensurable pairs.*

Proof. Let $(\pi, \mathcal{H}_\omega, \Omega)$ be GNS representation of $\mathfrak{N}_1 \vee \mathfrak{N}_2$ induced by ω, and let S_1 and S_2 be the support of $\omega|_{\mathfrak{N}_1}$ and $\omega|_{\mathfrak{N}_2}$ respectively. By Theorem 1, there are projections $E_1, F_1 \in S_1\mathfrak{N}_1 S_1$ and $E_2, F_2 \in S_2\mathfrak{N}_2 S_2$ such that $\omega((E_1 - E_2)^2) = \omega((F_1 - F_2)^2) = 0$ and $(2E_1 - S_1)(2F_1 - S_1) + (2F_1 - S_1)(2E_1 - S_1) = 0$.

Let $A_1 := 2E_1 - S_1$ and $B_1 := 2F_1 - S_1$. Then $\omega([A_1, B_1]^*[A_1, B_1]) = 4$ and $[E_1, F_1]^*[E_1, F_1] = 1/16[A_1, B_1]^*[A_1, B_1]$ since $A_1, B_1 \in S_1\mathfrak{N}_1 S_1$, $A_1 B_1 + B_1 A_1 = 0$ and $A_1^2 = B_1^2 = S_1$. It follows that $\omega([E_1, F_1]^*[E_1, F_1]) = 1/4$.

$\pi(E_1)\Omega = \pi(E_2)\Omega$ and $\pi(F_1)\Omega = \pi(F_2)\Omega$ imply $\pi([E_1, F_1])\Omega = -\pi([E_2, F_2])\Omega$. Thus $\omega([E_2, F_2]^*[E_2, F_2]) = \omega([E_1, F_1]^*[E_1, F_1]) = 1/4$. Therefore ω is an EPR state for incommensurable pairs. $\square$

The converse of Corollary 1 does not hold. Let consider a vector state induced by Ψ_1.

$$\Psi_1 := \frac{1}{3}\begin{pmatrix}1\\0\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\\0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}0\\1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{\sqrt{7}}{3}\begin{pmatrix}0\\0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Then this state is an EPR state for

$$\left(\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} \otimes \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \otimes \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} \right)$$

and

$$\left(\frac{1}{2}\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix} \otimes \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \otimes \begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix} \right),$$

so it is an EPR state for incommensurable pairs.

Let define

$$\mathbb{A}_1 := \{A \otimes I \in \mathbb{B}(\mathcal{H}_3) \otimes I | \text{there exists } I \otimes B \in I \otimes \mathbb{B}(\mathcal{H}_3) \text{ such that}$$
$$(A \otimes I)\Psi_1 = (I \otimes B)\Psi_1\},$$

where $\mathbb{B}(\mathcal{H}_3)$ is the set of all operators on the 3-dimensional Hilbert space. Then any projection in $\mathbb{A}_1$ can be expressed as

$$\begin{pmatrix}a_{11} & a_{12} & 0\\a_{21} & a_{22} & 0\\0 & 0 & a_{33}\end{pmatrix} \otimes I,$$

where $a_{33} = 0, 1$. Thus for any projections $E, F \in \mathbb{A}_1$, $(2E-I)(2F-I)+(2F-I)(2E-I) \neq 0$. Therefore the vector state induced by Ψ_1 is not strongly maximally Bell correlated state by Theorem 1 although it is an EPR state for incommensurable pairs.

In the following theorem, we examine whether an EPR state for incommensurable pairs is Bell correlated or not.

Theorem 2. *Let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be von Neumann algebras on a Hilbert space $\mathcal{H}$ such that $\mathfrak{N}_1 \subseteq \mathfrak{N}_2'$ and let ω be a normal state of $\mathfrak{N}_1 \vee \mathfrak{N}_2$. If ω is an EPR state for incommensurable pairs, then ω is Bell correlated.*

Proof. Let ω is an EPR state for incommensurable pair. Then there exist projections $E_1, F_1 \in \mathfrak{N}_1$ and $E_2, F_2 \in \mathfrak{N}_2$ such that ω is an EPR state for incommensurable pairs (E_1, E_2) and (F_1, F_2).

Let $(\pi, \mathcal{H}_\omega, \Omega)$ be a GNS representation of $\mathfrak{N}_1 \vee \mathfrak{N}_2$ induced by ω and let

$$c := \langle \pi(E_1^\perp F_1 E_1)\Omega, \pi(E_1^\perp F_1 E_1)\Omega \rangle + \langle \pi(E_1 F_1 E_1^\perp)\Omega, \pi(E_1 F_1 E_1^\perp)\Omega \rangle.$$

Since $\omega([E_1, F_1]^*[E_1, F_1]) \neq 0$, $\pi([E_1, F_1])\Omega \neq 0$. It implies that $\pi(E_1^\perp F_1 E_1)\Omega \neq 0$ or $\pi(E_1 F_1 E_1^\perp)\Omega \neq 0$. Thus $c \neq 0$.

Let

$$A_1 := E_1 - E_1^\perp,$$

$$B_1 := E_1 F_1 E_1^\perp + E_1^\perp F_1 E_1,$$

$$A_2 := \frac{1}{\sqrt{1+c^2}}(E_2 - E_2^\perp) + \frac{c}{\sqrt{1+c^2}}(E_2 F_2 E_2^\perp + E_2^\perp F_2 E_2),$$

$$B_2 := \frac{1}{\sqrt{1+c^2}}(E_2 - E_2^\perp) - \frac{c}{\sqrt{1+c^2}}(E_2 F_2 E_2^\perp + E_2^\perp F_2 E_2).$$

Then A_1, B_1, A_2 and B_2 are self-adjoint contractions because $A_1^2, B_1^2, A_2^2, B_2^2 \leq I$, where I is an identity operator on $\mathcal{H}$.

By $\pi(E_1)\Omega = \pi(E_2)\Omega$ and $\pi(F_1)\Omega = \pi(F_2)\Omega$,

$$\begin{aligned}
\omega((E_1 - E_1^\perp)(E_2 - E_2^\perp)) &= \langle \Omega, \pi((E_1 - E_1^\perp)(E_2 - E_2^\perp))\Omega \rangle \\
&= \langle \Omega, \pi((E_1 - E_1^\perp)(E_1 - E_1^\perp))\Omega \rangle \\
&= 1,
\end{aligned}$$

$$\begin{aligned}
&\omega((E_1 F_1 E_1^\perp + E_1^\perp F_1 E_1)(E_2 F_2 E_2^\perp + E_2^\perp F_2 E_2)) \\
&= \langle \Omega, \pi((E_1 F_1 E_1^\perp + E_1^\perp F_1 E_1)(E_2 F_2 E_2^\perp + E_2^\perp F_2 E_2))\Omega \rangle \\
&= \langle \Omega, \pi((E_1 F_1 E_1^\perp + E_1^\perp F_1 E_1)(E_1^\perp F_1 E_1 + E_1 F_1 E_1^\perp))\Omega \rangle \\
&= \langle \Omega, \pi((E_1 F_1 E_1^\perp F_1 E_1 + E_1^\perp F_1 E_1 F_1 E_1^\perp))\Omega \rangle \\
&= c.
\end{aligned}$$

Therefore

$$\frac{1}{2}\omega(A_1 A_2 + A_1 B_2 + B_1 A_2 - B_1 B_2) = \sqrt{1+c^2} > 1.$$

$\square$

The converse of Theorem 2 does not hold. Let consider a vector state induced by Ψ_2.

$$\Psi_2 := \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\sqrt{3}}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

A vector state induced by Ψ_2 is a Bell correlated state [7].

Let define

$$\mathbb{A}_2 := \{A \otimes I \in \mathbb{B}(\mathcal{H}_2) \otimes I \,|\, \text{there exists } I \otimes B \in I \otimes \mathbb{B}(\mathcal{H}_2) \text{ such that}$$
$$(A \otimes I)\Psi_2 = (I \otimes B)\Psi_2\},$$

where $\mathbb{B}(\mathcal{H}_2)$ is the set of all operators on the 2-dimensional Hilbert space. Then any projection in $\mathbb{A}_2$ can be expressed as

$$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \otimes I.$$

Thus for any operators $E \otimes I, F \otimes I \in \mathbb{A}_2$, $[E \otimes I, F \otimes I]\Psi_2 = 0$. Therefore the vector state induced by Ψ_2 is not an EPR state for incommensurable pairs although it is a Bell correlated state.

3 Existence of EPR states for incommensurable pairs in algebraic quantum field theory

In this section, we shall consider algebraic quantum field theory. In algebraic quantum field theory, each bounded open region $\mathcal{O}$ in the Minkowski space is associated with a von Neumann algebra $\mathfrak{N}(\mathcal{O})$. Such a von Neumann algebra is called a local algebra. We say that bounded open regions $\mathcal{O}_1$ and $\mathcal{O}_2$ are strictly space-like separated if there is a neighborhood $\mathcal{V}$ of the origin of the Minkowski space such that $\mathcal{O}_1 + \mathcal{V}$ and $\mathcal{O}_2$ are space-like separated.

In the present paper, we make the following assumptions. For any bounded open region $\mathcal{O}$ in the Minkowski space, $\mathfrak{N}(\mathcal{O})$ is properly infinite [2, Corollary 1.11.6]. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are space-like separated, then $[X_1, X_2] = 0$ for any $X_1 \in \mathfrak{N}(\mathcal{O}_1)$ and $X_2 \in \mathfrak{N}(\mathcal{O}_2)$. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are strictly space-like separated, then $X_1 X_2 \neq 0$ for any nonzero operators $X_1 \in \mathfrak{N}(\mathcal{O}_1)$ and $X_2 \in \mathfrak{N}(\mathcal{O}_2)$ [2, Theorem 1.12.3].

The following theorem shows that there is a dense set of EPR states for incommensurable pairs between two strictly space-like separated regions.

Theorem 3. *Let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be properly infinite von Neumann algebras on a Hilbert space $\mathcal{H}$ such that $\mathfrak{N}_1 \subseteq \mathfrak{N}_2'$ and $X_1 X_2 \neq 0$ for any nonzero operators $X_1 \in \mathfrak{N}_1$ and $X_2 \in \mathfrak{N}_2$.*

For any unit vector Ψ and any real number $\epsilon > 0$, there exists a unit vector Ψ' such that $\|\Psi - \Psi'\| < \epsilon$ and a vector state induced by Ψ' is an EPR state for incommensurable pairs.

Proof. Since $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are properly infinite, there are a countably infinite families $\{E_{1,i} \in \mathfrak{N}_1 | i \in \mathbb{N}\}$ and $\{E_{2,i} \in \mathfrak{N}_2 | i \in \mathbb{N}\}$ of mutually orthogonal projections and families $\{V_{1,i} \in \mathfrak{N}_1 | i \in \mathbb{N}\}$ and $\{V_{2,i} \in \mathfrak{N}_2 | i \in \mathbb{N}\}$ of partial isometries such that $\sum_{i=0}^{\infty} E_{1,i} = \sum_{i=0}^{\infty} E_{2,i} = I$, $E_{1,i} = V_{1,i}^* V_{1,i}$, $E_{1,i+1} = V_{1,i} V_{1,i}^*$, $E_{2,i} = V_{2,i}^* V_{2,i}$ and $E_{2,i+1} = V_{2,i} V_{2,i}^*$ for any $i \in \mathbb{N}$, where I is an identity operator on $\mathcal{H}$ [18, Proposition 2.2.4].

By an assumption, $E_{1,n} E_{2,n} \neq 0$ for any $n \in \mathbb{N}$. Thus there is a unit vector Φ_n such that $E_{1,n} E_{2,n} \Phi_n = \Phi_n$. Let Ψ be a unit vector in a Hilbert space $\mathcal{H}$ and

$$\Psi_n := \left(1 - \frac{1}{n}\right)^{1/2} \frac{(I - E_{1,n} - E_{1,n+1})(I - E_{2,n} - E_{2,n+1})}{\|(I - E_{1,n} - E_{1,n+1})(I - E_{2,n} - E_{2,n+1})\|} \Psi + \left(\frac{1}{2n}\right)^{1/2} (\Phi_n + V_{1,n} V_{2,n} \Phi_n).$$

Then Ψ_n is a unit vector and $\lim_{n \to \infty} \Psi_n = \Psi$ since $\sum_{i=0}^{n-1} E_{1,i} \leq I - E_{1,n} - E_{1,n+1}$, $\sum_{i=0}^{n-1} E_{2,i} \leq I - E_{2,n} - E_{2,n+1}$, and $\sum_{i=0}^{\infty} E_{1,i} = \sum_{i=0}^{\infty} E_{2,i} = I$. Let

$$F_{1,n} := \frac{1}{2}(E_{1,n} + E_{1,n+1} + V_{1,n} + V_{1,n}^*),$$

$$F_{2,n} := \frac{1}{2}(E_{2,n} + E_{2,n+1} + V_{2,n} + V_{2,n}^*).$$

Then $F_{1,n}$ and $F_{2,n}$ are projections in $\mathfrak{N}_1$ and $\mathfrak{N}_2$ respectively, and

$$E_{1,n}F_{1,n}\Psi_n = \frac{1}{2}\left(\frac{1}{2n}\right)^{1/2}(\Phi_n + V_{2,n}\Phi_n),$$

$$F_{1,n}E_{1,n}\Psi_n = \frac{1}{2}\left(\frac{1}{2n}\right)^{1/2}(\Phi_n + V_{1,n}\Phi_n).$$

Since $\|V_{1,n}\Phi_n - V_{2,n}\Phi_n\|^2 = \langle V_{1,n}\Phi_n, V_{1,n}\Phi_n\rangle + \langle V_{2,n}\Phi_n, V_{2,n}\Phi_n\rangle - \langle V_{1,n}\Phi_n, V_{2,n}\Phi_n\rangle - \langle V_{2,n}\Phi_n, V_{1,n}\Phi_n\rangle = 2$, $V_{1,n}\Phi_n \neq V_{2,n}\Phi_n$. Thus $[E_{1,n}, F_{1,n}]\Psi_n \neq 0$. Similarly $[E_{2,n}, F_{2,n}]\Psi_n \neq 0$. Because

$$E_{1,n}\Psi_n = E_{2,n}\Psi_n = \left(\frac{1}{2n}\right)^{1/2}\Phi_n,$$

$$F_{1,n}\Psi_n = F_{2,n}\Psi_n = \frac{1}{2}\left(\frac{1}{2n}\right)^{1/2}(\Phi_n + V_{1,n}\Phi_n + V_{2,n}\Phi_n + V_{1,n}V_{2,n}\Phi_n),$$

the vector state of $\mathfrak{N}_1 \vee \mathfrak{N}_2$ induced by Ψ_n is an EPR state for incommensurable pairs $(E_{1,n}, E_{2,n})$ and $(F_{1,n}, F_{2,n})$.

$\square$

Theorem 2 and Theorem 3 entail the following corollary. This is a theorem of Halvorson and Clifton [10, Proposition 1].

Corollary 2 (Halvorson-Clifton)**.** *Let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be properly infinite von Neumann algebras on a Hilbert space $\mathcal{H}$ such that $\mathfrak{N}_1 \subseteq \mathfrak{N}_2'$ and $X_1 X_2 \neq 0$ for any nonzero operators $X_1 \in \mathfrak{N}_1$ and $X_2 \in \mathfrak{N}_2$.*

For any unit vector Ψ and any real number $\epsilon > 0$, there exists a unit vector Ψ' such that $\|\Psi - \Psi'\| < \epsilon$ and a vector state induced by Ψ' is Bell correlated.

4   Concluding remarks

In this paper, we have defined EPR states for incommensurable pairs which allow for similar arguments to EPR's, and clarified the logical relations between EPR states for incommensurable pairs and Bell correlated states. Let ω be a normal state of $\mathfrak{N}_1 \vee \mathfrak{N}_2$. Then the following relations hold (Corollary 1 and Theorem 2).

ω is strongly maximally Bell correlated.
$\Downarrow \not\Uparrow$
ω is an EPR state for incommensurable pairs.
$\Downarrow \not\Uparrow$
ω is Bell correlated.

In Theorem 3 it is shown that there is a norm dense set of EPR states for incommensurable pairs between two strictly space-like separated regions. This theorem entails that there is a norm dense set of Bell correlated states between two strictly space-like separated regions (Corollary 2).

Acknowledgements

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