Categorical Universal Logic: LT Topology and Logical Translation

Yoshihiro Maruyama

Quantum Group
Dept. of Computer Science
University of Oxford

ALGI, Fukuoka, Sep 6, 2012

What I have been working on

My primary area of research:

- Theory of categorical dualities: e.g., Stone Duality, Isbell Duality, Alg-Coalg duality, Hilbert Duality, Gelfand Duality.
  - Introductory article: Y. Maruyama, “The Logic of Categorical Dualities", *Suugaku Seminar*, May 2012.

My current project in Oxford:

- Find nice applications of duality theory in foundations of quantum mechanics and information.
  - E.g., I showed state-observable duality leads us to a purely coalgebraic characterisation of quantum symmetries.
- This talk is relevant to applications of duality to categorical logic, including categorical quantum logic.
  - Introductory article: Y. Maruyama, “Categorical Logic: Beyond Topos Theory", *Suugaku Seminar*, June 2012.

Applications are Interpretations for Understandings

Why are applications of theories important? (Are they evil?)

- To convince people of practical values of theories?
  - Crucial when people object to theories themselves.
- Yet another idea: applications are often involved in interpretations for understandings of what theories represent. They do improve our understanding of theories.

To exemplify what I mean, think of:

- Abramsky's application of Isbell Duality to program semantics, which interprets spaces as denotations of programs, and locales as observable properties of programs.
  - State-observable duality in OQM arises from a similar idea.
- Hilbert space formalism for quantum mechanics.

Thus, applications are interpretations for understandings.

End of Duality: Deconstructing Duality qua Dichotomy

Consider the issue of interpreting or applying duality.

- Duality arises b/w the formal and the real.
  As such, it represents a dichotomy.
  - E.g.: syntax vs. semantics; polynomials vs. varieties; observables vs. states; properties vs. systems.
  - Lawvere: duality b/w the formal and the conceptual.

I argue the idea does not make so much sense in categ. logic.

- Syntax and semantics are just two faces of the same thing, namely two instances of the one categorical concept.
- What is then the meaning of duality in categ. logic? My idea is that duality itself is a categorical form of logic.

Categorical Universal Logic

A topos **E** gives rise to

- $\mathrm{Sub}_{\mathbf{E}}(\text{-}) : \mathbf{E}^{\mathrm{op}} \to$ **HeytAlg** (and the fibration $\int \mathrm{Sub}_{\mathbf{E}} \to \mathbf{E}$)
- Lawvere-Pitts hyperdoctrines abstract such structures, providing semantics for both intuitionistic FOL and HOL.
  - Hyperdoctrines are of the form $P : \mathbf{C}^{\mathrm{op}} \to$ **HeytAlg** (Pitts' tripos version rather than Lawvere's original one).
  - Topoi are categories whose subobject functors form higher-order hyperdoctrines, or equivalently, fibrations.

A †-kernel cat **H** for CQL by Heunen-Jacobs (partially) yields

- $\mathrm{KSub}_{\mathbf{H}}(\text{-}) : \mathbf{H}^{\mathrm{op}} \to$ **OrthModLat**.

Both are of the form

- $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ for a monad T.

Categorical Universal Logic studies such monad-relativized hyperdoctrines, aiming at a universal conception of logic.

Categorical Logic of Duality

Duality induced by a schizophrenic object Ω:

- $\mathrm{Hom}_{\mathbf{D}}(\text{-}, \Omega) \dashv \mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega) : \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$.
  - Such duality theories have been developed by Johnstone, Porst-Tholen, etc. **C** and **D** have faithful functors into **Set**.
- $\mathbf{D} := \mathbf{Alg}(T)$ as in many cases. $\mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ looks like T-relativized hyperdoctrines.
  - I developed a duality theory specialized in such situation.
- $\mathrm{Hom}_{\mathbf{Top}}(\text{-}, \mathbf{2}) : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Frm}$ is a geometric hyperdoctrine.
- $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Alg}(T)$ is a (HO) T-hyperdoctrine.
  - In the Heyting case, these give rise to sheaf topoi via the tripos-topos construction by Hyland-Johnstone-Pitts.

We investigate into categorical logic of dual adjunctions. Duality hyperodctrines always have obj. classifiers as schizo. obj. Ω.

Finitary and Infinitary Stone Dualities

Finitary Stone-type dualitlies:

- involve finitary operations and compact specs.
- includes Stone duality for Boolean algebras.
- usually needs a form of AC (ontologically demanding).

Infinitary Stone-type dualities:

- involve infinitary operations and non-compact specs.
- includes duality b/w frames and topo. spaces; duality b/w continuous lat. and convex structures (essentially HMS).
- usually avoids AC (epistemologically more certain).

In both contexts, we focus on adjunctions rather than equiv.

Outline

Outline

Joyal Lem Revisited

Universal categorical semantics could be either semantics of proofs (CCC, $*$-aut. cat., etc.) or semantics of provability or deducibility (topos, reg. cat., etc.), which is proof-irrelevant.

- Joyal Lem tells us the collapsing of semantics of proofs for classical logic (some attempted to overcome it, though):
  - any CCC with a dualizing object (i.e., DNE) is actually a Boolean algebra up to equivalence.
- Joyal Lem extends to a bit more general, monoidal setting (relevant to No-Deleting Thm. in AC's $\dagger$-compact cats.):
  - any $*$-autonomous cat. $(\mathbf{C}, \otimes, I, \bot)$ with I being terminal and $\bot \otimes \bot \simeq \bot \times \bot$ is actually an ordered monoid up to equiv.
- Substruct. logics with (strong) weakening but without (full) contraction may involve difficulty in semantics of proofs.

Due to such Joyal-type paradoxes, in general, universal semantics can only aim at semantics of deducibility.

Monad as Propositional Logic

From the viewpoint of algebraic logic:

- Logic is the free algebra over propositional variables.
- Monads give a generic concept of free algebras.
  - Since monads on **C** represent **C**-structured algebras, we focus on monads on **Set**, i.e., alg with no additional struct.

Algebras of most logical systems can be expressed as algebras of monads on **Set**, including:

- **BoolAlg**, **HeytAlg**, **ResAlg**, **OrthModLat**, **Frm**, ...
  - Infinitary ones are included; e.g., geometric logic.

We see **ContLat**, which is **Alg**(Filt), as the propositional (point-free) logic of convex sets. Adj. exist b/w **ContLat** and convex structures. Recall frames give the logic of open sets.

T-hyperdoctrines for a monad T

A T-hyperdoctrine is an **Alg**(T)-valued presheaf (fibred T-alg.)

$$P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$$

where **C** has at least products (**C** may be monoidal, e.g., in the case of tensored quantum logic; it has a monoidal type struct.).

- It has $\forall$ iff for any proj. π in **C**, $P(\pi)$ has a right adjoint $\forall_\pi$.
  - The entailment or order relations of T-algs. are assumed.
- It has $\exists$ iff for any proj. π in **C**, $P(\pi)$ has a left adjoint $\exists_\pi$.

We assume the corresponding Beck-Chevalley conditions, but not Frobenius Reciprocity in the view of quantum logic, which restricts context formuli in a seq.-style formulation to the effect of “visibility" in Sambin’s terms; Frob. is caused by non-visibility.

Are Propositions Types?

The hyperdoctrine (or fibred alg.) perspective leads us to reexamination of the idea of Propositions as Types.

- Primarily, types and propositions are not presupposed to be the same in hyperdoc. (cf. Aczel's logic-enriched TT).
    - **C**: type structure. **V**: proposition structure.
- In the case of intuitionistic logic, the type structure and the proposition structure are in harmony.
    - The logic of **C** is coherent with the logic of **V**.
- *In general*, however, propositions are not types. Rather, they *turn out* to be the same in certain logical systems.

Yet, in a duality hyperdoctrine, the proposition structure **V** is reflected onto the type structures **C** via adjunction. Another relevant issue: two different conceptions of logic.

Lindenbaum Fibred Algebra

We can accommodate both cartesian logic and monoidal logic.

- For a cartesian logic, we interpret a sequent $\Gamma \Rightarrow \varphi$ as $\bigwedge [\![\Gamma]\!] \leq [\![\varphi]\!]$.
- For a monoidal logic, we interpret a sequent $\Gamma \Rightarrow \varphi$ as $\bigotimes [\![\Gamma]\!] \leq [\![\varphi]\!]$.

From both kinds of systems, we can construct Lindenbaum fibred algebras in a standard way:

- **C** is cat of lists $[x_1, ..., x_n]$ of (typed) variables and of terms. P maps $[x_1, ..., x_n]$ to alg of formuli with variables $[x_1, ..., x_n]$. Arrow part is defined by substitution: $P(t)(\varphi(x)) = \varphi[t/x]$.

Lindenbaum fibred algebras (or classifying, syntactic hypdoc.) are used to prove completeness.

More Logical Structures

A T-hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ gives rise to a fibred category $\int P$ via the Grothendieck construction.

- $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ has comprehenion $\{\text{-}\}$ iff the truth functor $\top : \mathbf{C} \to \int P$ has a right adjoint $\{\text{-}\} : \int P \to \mathbf{C}$.
  - Each fibre $P(C)$ is supposed to have the truth const. $\top_{P(C)}$.
- It has $=$ iff for any diagonal δ, $P(\delta)$ has a left adjoint.
  - Most such ideas come from Lawvere.

Higher-order logic can be treated as follows.

- A T-hyperdoctrine P is higher order iff the base cat. $\mathbf{C}$ is a CCC, and it has an object classifier (or truth value object), i.e., $\exists \Omega \in \mathbf{C} \;\; P \simeq \mathrm{Hom}_{\mathbf{D}}(\text{-}, \Omega)$.

Topoi are cats. whose $\mathrm{Sub}(\text{-})$ form higher-order hyperdoctrines. Duality hyperdoctrines always have object classifiers.

Axiomatisability of Fibred Logical Algebras

Is it possible under some conditions to (finitely) axiomatise logic of $\mathcal{V}$ and logic of fibred algebras $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{V}$? An example:

- Suppose L is semi-primal with a lattice reduct and $\mathcal{V} = \mathbb{ISP}(L)$,
- We can then axiomatise $\mathbb{ISP}(L)$ as L-valued logic, and fibred $\mathbb{ISP}(L)$-algebras as L-valued predicate logic.

We can even modalise this by using $\mathbb{ISP}_{\mathbb{M}}(L)$, a universal-algebraic way to modalise algebras in $\mathbb{ISP}(L)$.

Outline

Universal Tripos Theorem

A dual adj. $\mathrm{Hom}_{\mathbf{Alg}}(\text{-},\Omega) \dashv \mathrm{Hom}_{\mathbf{C}}(\text{-},\Omega) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ does not necessarily induce a T-hyperdoctrine.

- There are criteria for knowing if a duality T-hyperdoctrine has logical struct. ($\forall$, $\exists$, $=$, $\{\text{-}\}$, etc.); we ass. Ω is compl.
  - Using them, we can give models of convex geometric logic as $\mathrm{Hom}(\text{-},\mathbf{2})_{\mathbf{Top}} : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Frm}$ is a geometric hyperdoc.

Compatibility b/w **C** and logical structures is a major problem. We can show **Set** is universally compatible with logical struct.:

Theorem

Dual adjunction

$$\mathrm{Hom}_{\mathbf{Alg}}(\text{-},\Omega) \dashv \mathrm{Hom}_{\mathbf{Set}}(\text{-},\Omega) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Alg}(T)$$

always exists and gives rise to higher-order T-hyperdoctrines.

Categorical Quantum Logic

A problem of Heunen-Jacobs' $\mathrm{KSub}_{\mathbf{H}}(\text{-}) : \mathbf{H}^{\mathrm{op}} \to \mathbf{OrthModLat}$ for a $\dagger$-kernel cat. **H** is: it does not have universal quantifier $\forall$.

- Let H be a Hilbert space, and $\mathrm{Proj}(H)$ the lattice of projection operators on H.
- $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \mathrm{Proj}(H)) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{OrthModLat}$ has both quantifiers $\forall, \exists$, and even higher-order stcutures.

For a locale Ω, $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HeytAlg}$ gives the sheaf topos $\mathbf{Sh}(\Omega)$ (Heyting-val. universe of set theory) via HJP const.
It derives cat. of $(X \in \mathbf{C}, \mathrm{eq} \in P(X \times X))$; eq is sym. trans. in $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{V}$.

- What happens in the case of our quantum hyperdoctrine?
- $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \mathrm{Proj}(H)) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{OrthModLat}$ induces, via HJP const., a universe of $\mathrm{Proj}(H)$-valued sets, related to Takeuti-Ozawa's quantum-valued models of set theory.

Convexity Theory

Let $\mathcal{D}$: **Set** $\rightarrow$ **Set** be the distribution monad: $\mathcal{D}(X) :=$ the set of probability distributions on X with finite supports.

- Algebras of $\mathcal{D}$ can be described concretely as barycentric algebras, i.e., sets with convex combination operations.

A barycentric algebra is a set X with a ternary function

$$\langle \text{-}, \text{-}, \text{-} \rangle : [0,1] \times X \times X \rightarrow X$$

such that (i) $\langle r, x, x \rangle = x$; (ii) $\langle 0, x, y \rangle = y$;
(iii) $\langle r, x, y \rangle = \langle 1 - r, y, x \rangle$;
(iv) $\langle r, x, \langle s, y, z \rangle \rangle = \langle r + (1-r)s, \langle r/(r + (1-r)s), x, y \rangle, z \rangle$.

- A barycentric algebra $(X, \langle \text{-}, \text{-}, \text{-} \rangle)$ is idempotent iff for any $x, y \in X$, for any $r, s \in (0,1)$, $\langle r, x, y \rangle = \langle s, x, y \rangle$.

A convexity sp. is a set with a family of subsets closed under arbitrary intersections and directed unions.

Convex Geometric Logic

- Jacobs showed a dual adjunct. b/w **Alg**($\mathcal{D}$) and **ContLat**. **Alg**($\mathcal{D}$) is the cat. of barycentric algebras.
  - Jacobs left open the equivalence induced by this adj. We can show it is the dual equiv. b/w idem. **D**-algs. and **AlgLat**; in fact, it is essentially Hofmann-Mislove-Stralka Duality.
- We showed another dual adj. b/w **ConvSp** and **ContLat**, whose equiv. part is also essentially HMS Duality.
  - Convex structures are relevant to domain theory. As frames give logic of open sets, conti. lat. give logic of convex sets.
- Both adj. give fibred continuous lattices with quantifier $\forall$.
  - Convex geometric logic only allows $\forall$, as geometric logic only admits $\exists$.
  - Recall **Top**-**Frm** adj. gives a fibred **Frm**-algebra with $\exists$.

Outline

1. Monad-Relativized Hyperdoctrines
   - Joyal Lem: Proof vs. Provability
   - Monad as Propositional Logic
   - T-Hyperdoctrines for a Monad T
2. Categorical Logic of Duality
   - Duality Hyperdoctrines
   - Categorical Quantum Logic
   - Point-Free Geometric Logics
3. Lawvere-Tierney as Translation
   - Logical Translation in CUL
   - Gödel, Girard, Baaz, Gödel

Lawvere-Tierney Topology

A Lawvere-Tierney topology $j : \Omega \to \Omega$ in a topos $\mathbf{E}$ may be regarded as a natural transformation

$$j : \mathrm{Sub}_{\mathbf{E}}(\text{-}) \to \mathrm{Sub}_{\mathbf{E}}(\text{-})$$

such that j_C is a left-exact monad on $\mathrm{Sub}_{\mathbf{C}}(C)$ for every $C \in \mathbf{C}$.

Definition

A Lawvere-Tierney topology (or operator) on a T-hyperdoctrine $P : \mathbf{C} \to \mathbf{Alg}(T)$ is a natural transformation

$$j : P \to P$$

s.t. $j_C : P(C) \to P(C)$ is a left-exact monad on $P(C)$ for $\forall C \in \mathbf{C}$.

Co-topology (or co-operator) is defined in a similar way.

Gödel-Gentzen Translation

Consider the syntactic **HA**-hyperdoctrine P, and the double negation topology $d : P \to P$ defined by $d_C(\varphi) = \neg\neg\varphi$. The double negation topology induces Boolean

$$P_d : \mathbf{C}^{\mathrm{op}} \to \mathbf{BA}$$

defined as follows: $P_d(C) := \{d_C(\varphi) \mid \varphi \in P(C)\}$.

Proposition

*The **BA**-hyperdoctrine $P_d : \mathbf{C}^{\mathrm{op}} \to \mathbf{BA}$ has quantifiers $\forall, \exists$. The mapping $P \mapsto P_d$ preserves quantified hyperdoctrine structures.*

We can relativize this to monads T as follows.

Universal Translation Theorem

Fix a Lawvere-Tierney toplogy j on $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$. Suppose the image of $j_C : P(C) \to P(C)$ is an algebra of a monad S. Then, we define an $\mathbf{Alg}(S)$-valued presheaf

$$P_j : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(S)$$

as follows: $P_j(C) = \mathrm{Fix}(j_C) = \{j_C(\varphi) \mid \varphi \in P(C)\}$.

Theorem

Assume: for $\pi : C \times D \to D$, $\varphi \in P_j(C \times D)$, and $\psi \in P_j(D)$,

$$j_D(\forall_\pi(j_{C\times D}(\varphi))) \vdash \forall_\pi(j_{C\times D}(\varphi)).$$

If $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ has quantifiers $\forall, \exists$, then $P_j : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(S)$ has quantifiers $\forall, \exists$ as well (i.e., j pres. quantifier structures).

Gödel, Girard, Baaz, Gödel

Translation Theorem can be applied in the Heyting case w.r.t. double negation topology. The dual of the translation theorem holds as well, and can be applied in the following cases.

- Given a lat. L (e.g., $[0,1]$) with $\bot, \top$, define Baaz' $\Delta : L \to L$ by: $\Delta(x) = \top$ if $x = \top$ and $\Delta(x) = \bot$ if $x \neq \top$. Δ gives a co-top., transforming a fibred L-val. alg. into a Bool. one.
- $\Box$ in S4 modal logic is a co-monad, and the theorem applies, giving a fibred algebraic form of Gödel translation b/w S4 and intuitionistic logic.

! in linear logic gives a co-topology ! (in an extended sense) on a linear hypdoc. P. Then, ! tansforms P into an int. hypdoc. $P_!$.

Conclusions

- Fibred T-algebra or T-hyperdoctrine can give sound and complete semantics for most predicate logics (class., int., geom., modal, fuzzy, linear); any subst. logic is included.
- Duality gives a model of pred. logic (under certain cond.).
    - Quantum: a duality hypdoc. gives both quantifiers, and a universe of projection-valued sets.
    - Point-free: domain-convexity duality has quantifier, and reveals a new, convexity-th. aspect of HMS duality.
- Translation b/w various systems can be treated in a uniform manner, using Lawvere-Tierney top. on T-hypdoc.

Future Work

Coalgebraic predicate logic as the study of modal T-hypdoc.

- Quantifier structures of propositional duality (e.g., **Set**-**BA**) lift to those of modalized duality (e.g., **Coalg**(S)-**Alg**(T) duality) under certain strong conditions.

Tensored quantum logic.

- Tensor is incorporated into quantum logic as translation b/w different fibres; tensors of proections are projections.
- The hyperdoctrine perspective suggests the type structure of QM is monoidal and the propositional structure of QM is cartesian, reconciling traditional BvN QL and AC CQM.

Philosophy of Duality

Duality exists b/w the epistemic and the ontic. The shift of emphasis from the ontic to the epistemic has been crucial in modernisation of math, physics, phil., art, literary theory, ...

- Math: from varieties to function alg; from points to opens.
- Physics: from things to actions (Piet Hut at IAS). Mermin: "Correlations have physical reality; that which they correlate does not."
- Phil: from reality to process (Whitehead), to phenomena (Husserl), and to language (linguistic turn; Frege, Russell).
- Literary theory: from authors to texts (Russian formalism, new criticism, etc.). Derrida: "There is no outside-text".

Wittgenstein: "What makes it apparent that space is not a collection of points, but the realization of a law?". Brentano: "I regard it as absurd to interpret a continuum as a set of points".

Philosophy of Duality (cont.)

Thus, rich, philosophical and historical contexts are arguably lurking behind duality and point-free geometry.

- Lawvere also discusses “duality b/w the formal and the conceptual", or syntax-semantics duality.
- Duality may be regarded as a dichotomy b/w the epistemic and the ontic, or syntax (PTS) and semantics (MTS).
- Categorical logic, 3rd way in semantics, deconstructs the dichotomy in the sense that it integrates proof-theoretic and model-theoretic semantics into the one semantics.
- In categorical logic, duality is not a dichotomy any more, but semantics, source of meaning.

Still, duality or dichotomy is a vital way of thinking, telling us how to see the world, or the meaning of the world. Accordingly, the two perspectives on duality are not really inconsistent.