Wittgenstein's Conception of Space and the Modernist Transformation of Geometry via Duality

Yoshihiro Maruyama
Quantum Group, Dept. of Computer Science, University of Oxford
maruyama@cs.ox.ac.uk

Abstract

Wittgenstein's disagreement with the set-theoretical view of mathematics led him to the idea that space is not an extensional collection of points, but the intensional realisation of a law. Brouwer's theory of the continuum is arguably based upon the point-free conception of space qua law; this would exhibit yet another case of Brouwer's influence on Wittgenstein's philosophy (in particular of space). I consider the conception of space qua law to represent epistemology of space, and the conception of space qua points to represent ontology of space. From this perspective, modern geometry, such as Topos Theory, Algebraic and Non-Commutative Geometry, and Formal Topology, some of which are conceptual ramifications of Brouwer's intuitionism, yields rich instances of duality between epistemology and ontology of space. Links with Husserl, Whitehead, Cassirer, Granger, Lawvere, and Japanese philosophers are briefly touched upon as well.

1 Introduction

Since ancient Greek philosophy, there have been a vast number of debates on whether or not the concept of a point precedes the concept of the space continuum. On the one hand, one may conceive of points as primary entities, and of the continuum as secondary ones to be understood as the collection of points; on the other, the whole space continuum may come first, and then the concept of a point is derived as a cut of it. This is more or less analogous to the well-known dichotomy between Newton's absolute space and Leibniz's relational space.

Wittgenstein gives a fresh look at the issue of the relationships between space and points:

> What makes it apparent that space is not a collection of points, but the realization of a law? (Wittgenstein 1975, p.216)

Wittgenstein's intensional view on space is a compelling consequence of his persistent disagreement with the set-theoretical extensional view of mathematics:

> Mathematics is ridden through and through with the pernicious idioms of set theory. One example of this is the way people speak of a line as composed of points. A line is a law and isn't composed of anything at all. (Wittgenstein 1974, p.211)

In the present article, I attempt to examine and articulate Wittgenstein's conception of space (in his intermediate philosophy) in relation to Brouwer's theory of the continuum and its mathematical descendants in a broad sense in modern geometry.

Wittgenstein's intensional conception of space is closely related, in its core idea, with "the modernist transformation of mathematics" (Gray 2008), especially the resulting revolutionary change of the concept of space in geometry: space is not a collection of points any more, but a sort of abstract algebraic structure, such as a C^*-algebra, topos, locale, formal space, or scheme (see, e.g., Cartier 2001). One of the first steps was taken by Brouwer in his intuitionistic conception of the continuum in terms of his peculiar (but indeed natural) notions of spreads and choice sequences, which, as explained below, may be seen as based upon the concept of a law rather than that of a point.

Although the enterprise of intuitionism did not succeed so much in convincing mathematicians of its significance, nevertheless, similar ideas on space have won widespread acceptance with the crucial help of duality theory, which elegantly enables us to derive points from an abstract algebraic structure as mentioned above, thereby establishing a tight (or functorial) link between point-set and point-free concepts of space; we could even say that duality justifies to regard an algebra as space, since it tells us that certain algebras carry the same amount of information as space itself. In this article, I aim at explicating the philosophical significance of such advances in modern geometry.

It is widely believed that Brouwer's intuitionism influenced Wittgenstein. Although the main point of this article is not on historical discussion, nevertheless, it could still be said that Wittgenstein's philosophy of space in particular was affected by Brouwer to some degree; his *Philosophical Remarks* and *Philosophical Grammar* contain a number of relevant descriptions, and even refers to Brouwer explicitly at several places.

The rest of the article is organised as follows. In Section 2, I discuss in more detail the relationships between Wittgenstein's and Brouwer's concepts of space. Section 3 is devoted to the explication of how modern geometry with duality conceptually involves Wittgenstein's philosophy of space. In Section 4, I conclude the article together with brief remarks on future work.

2 Wittgenstein's and Brouwer's Point-Free Conceptions of Space

Brouwer's influence on Wittgenstein is almost unquestionable, as for instance Rodych (2011) says, "There is little doubt that Wittgenstein was invigorated by L.E.J. Brouwer's March 10, 1928 Vienna lecture." At the same time, however, connections between their philosophies of space in particular have remained untouched and to be investigated; the present article embarks upon this project, taking a first step towards a full-fledged account of the relationships.

First of all, what Wittgenstein calls a law should be clarified. Just after the first sentence quoted above, he says, "In order to represent space we need–so it appears to me–something like an expansible sign" (Wittgenstein 1975, p.216); here, the concept of a sign already suggests relevance to algebra. What precisely is a sign, then? As he proceeds in the same page, it is "a sign that makes allowance for an interpolation, similar to the decimal system"; he then adds, "The sign must have the multiplicity and properties of space."

To elucidate what he means, the following discussion on a coin-tossing game seems crucial:

> Imagine we are throwing a two-sided die, such as a coin. I now want to determine a point of the interval AB by continually tossing the coin, and always bisecting the side prescribed by the throw: say: heads means I bisect the right-hand interval, tails the left-hand one. (ibid., pp.218-219)

The point here is that a point is being derived from the coin-tossing game, a sort of law, which Wittgenstein thinks realises space. The process of tossing the coin, of course, does not terminate within finite time, so Wittgenstein remarks, "I have an unlimited process, whose results as such don't lead me to the goal, but whose unlimited possibility is itself the goal" (ibid., p.219). To put it differently, such a rule for determining a point only gives us the point in infinite time, but still we may regard a rule itself as a sort of point; this idea of identifying points with rules or functions is now prevailing in mainstream mathematics, such as Algebraic and Non-Commutative Geometry. It should be noted here that a shift of emphasis is lurking behind the scene, from static entities like points to dynamic processes like laws.

Those who are familiar with Brouwer's theory of the continuum would already have noticed that there is a close connection between Brouwer and Wittgenstein on the nature of space. The above illustration of tossing a coin almost defines the Cantor space in terms of contemporary mathematics: the Cantor space is the space of infinite sequences consisting of zeros and ones only, which in turn correspond to heads and tails of a coin in Wittgenstein's terms; actually, he himself discusses this correspondence (ibid., p.220).

Now let me quote a passage which, together with the quotations above, strikingly exhibits a remarkable link between Brouwer's and Wittgenstein's ideas of space (Brouwer 1918, p.1; translation by van Atten 2007):

> A spread is a law on the basis of which, if again and again an arbitrary complex of digits [a natural number] of the sequence ζ [the natural number sequence] is chosen, each of these choices either generates a definite symbol, or nothing, or brings about the inhibition of the process together with the definitive annihilation of its result; for every n, after every uninhibited sequence of $n-1$ choices, at least one complex of digits can be specified that, if chosen as n-th complex of digits, does not bring about the inhibition of the process. Every sequence of symbols generated from the spread in this manner (which therefore is generally not representable in finished form) is called an element of the spread.

For Brouwer, a law is a rule to make a sequence of countably many digits. The difference between Brouwer's and Wittgenstein's laws basically lies in which to use two digits only (the Cantor space in modern terms) or all natural numbers (the so-called Baire space). Although this in fact gives rise to a certain technical difference, however, there is no doubt that the underlying conceptual view of capturing the concept of space in terms of laws is fundamentally the same in their thoughts.

It may thus be concluded that Wittgenstein's and Brouwer's conceptions of space build upon the same core idea of regarding space as a law to form infinite digital sequences (important differences in the light of Wittgenstein's distinction between arithmetical and geometrical space shall be remarked in Section 4); interestingly, their philosophically motivated idea has now become a standard method, in Computer Science, to implement exact computation over continuous infinitary structures.

3 The Modernist Transformation of Geometry via Duality

Modern mathematics has encountered drastic changes in both conceptual and technical senses. Prominent among them are a shift of emphasis from space itself to the structure of functions on it. In Algebraic Geometry, for example, properties of algebraic varieties are often shown through analysis of their associated function algebras; this was already noticed by Riemann in his study of what we now call Riemann surfaces. Duality expressed in terms of category theory is lurking behind the efficacy of function algebras in mathematics; in the case of Riemann surfaces, there is category-theoretical duality between Riemann surfaces and rational function fields, which tells us that Riemann surfaces can be reconstructed from the purely algebraic information of rational function fields on them, and vice versa.

The modernist transformation of geometry has naturally led mathematicians to regard algebras themselves as spaces that do not presuppose the concept of points in the first place; points are entities derived as prime ideals of algebras under suitable conditions. In Non-Commutative Geometry, for example, spaces are indeed defined as certain algebras, and commutative spaces are equivalent, via Gelfand Duality, to locally compact Hausdorff spaces in a classical sense. The same duality phenomenon happens to exist in mathematical logic, and is pursued under the name of Stone Duality, which is duality between syntax and semantics, thus functioning as a strengthened version of logical completeness theorems. In this case, we can regard logical systems on the side of syntax as spaces of their models on the side of semantics; classical logic amounts to the Cantor space in this sense.

Philosophically phrasing, we have both ontology and epistemology of space. As in many philosophies of space, points could be seen as metaphysical ultimate constituents of space, therefore an ontological account of space may be given in terms of points. At the same time, however, we cannot really see points with no extension, and hence the point-based concept of space is not acceptable from an epistemological point of view. In a sense, we can actually recognise regions with some extensions, or properties of space, so that a region-based or property-based notion of space, which has been implemented in algebraic concepts of space, is suitable for an epistemological account of space.

Here note that, in the context of topology, regions (i.e., open sets) bijectively correspond to properties or predicates on space (i.e., Boolean-valued continuous functions). Wittgenstein (1974) says, “A line as a coloured length in visual space can be composed of shorter coloured lengths (but, of course, not of points)" (p.211). This gets quite closer to the idea of Locale Theory and Formal Topology, which replace topological spaces by the structures of (basic) open sets. Formal Topology by Sambin and Martin-Löf is a descendant of Brouwer's intuitionism, contriving a predicative framework for constructive topology. Locale Theory models intuitionistic propositional logic, and closely connected with Topos Theory by Grothendieck and Lawvere, which gives a category-theoretical concept of point-free spaces serving as models of intuitionistic predicate logic. Deligne's theorem in Topos Theory, which guarantees the existence of sufficiently many points of certain toposes, amounts to Gödel's completeness theorem.

Just as intuitionism is an epistemological enterprise, so Wittgenstein's intensional conception of space is arguably epistemological in its nature. In contrast, Cantor's or Hausdorff's set-theoretical extensional concept of space may be conceived of as ontological in the sense above. Duality theory establishes a categorical-theoretical equivalence (or adjunction) between point-free and point-set spaces, and may thus be interpreted as exposing a remarkable link between ontology and epistemology of space in a mathematically profound and effective fashion.

Furthermore, it should be noted that the idea of duality between the ontological and the epistemological makes sense in surprisingly diverse disciplines. In Quantum Physics, there is duality between quantum states and observables, which, in a sense, may be seen as ontological and epistemological respectively. In Computer Science, there is duality, due to Abramsky, between computer systems (or programs) and observable properties of them, which are ontological and epistemological respectively. Duality in logic was already mentioned above.

What matters here is that all this can indeed be understood as duality between set-theoretical extensional concepts of space and algebraic intensional concepts of space.

4 Concluding Remarks

Wittgenstein's conception of space has been articulated in connection with Brouwer's and other mathematical concepts of space in modern geometry. The philosophical significance of duality in mathematics and sciences has been elucidated in terms of duality between ontology and epistemology of space.

Several remarks are to be made here, including those on future directions. First and foremost, it should be emphasised that Wittgenstein's and Brouwer's conceptions of space are not claimed to be absolutely the same in any sense. I am aware of discussions as in Rodych (2011) regarding discrepancies between Brouwer and Wittgenstein on real numbers (and lawlike or lawless sequences).

I have deliberately avoided to talk about real numbers so far, not just for this reason but more crucially because Wittgenstein carefully distinguishes between real numbers and points (or positions), and between arithmetical space and geometrical visual space (even though they are often identified in set-theoretical modern mathematics).

In the light of this distinction between arithmetical space and geometric space, the claim here is concerning geometrical space, and indeed consistent with the orthodox view of discrepancies in terms of numbers or arithmetical space. And I believe Wittgenstein's true characteristic as opposed to Brouwer's lies in this conceptual articulation of the notion of space.

From a historical perspective, Brentano and Husserl, as well as Wittgenstein, rejected to see space as a collection of points (but in their case for phenomenological reasons); a link between Brouwer and Husserl is pursued in van Atten (2007). Merleau-Ponty argued for the phenomenological priority of anthropological space (i.e., lived actual space) over geometric space (i.e., posited ideal space).

Whitehead vigorously developed his process philosophy, putting strong emphasis on dynamic processes (like laws) rather than static entities (like points); in other words, he advocated the shift from Being to Becoming. He considered a point to be a bunch of shrinking regions (which converges to the point); it is exactly a prime filter in mathematical terms. Brentano, Husserl, and Whitehead may be seen as origins of what is now called Mereology.

Cassirer weaved his philosophy based upon the dichotomy between substances and functions, which roughly amounts to the dichotomy between objects and operations in the more recent case of Granger, a French analytic philosopher. In category-theoretical logic, Lawvere discussed duality between the conceptual and the formal.

Looking at Japanese philosophy, I would remark that Hajime Tanabe in the Kyoto School of Philosophy was directly influenced, as recently uncovered in Susumu Hayashi's philological research, by Brouwer's theory of the continuum, in contriving what he calls logic of species; he worked in the context of German idealism (the Marburg School of Neo-Kantianism, to be more specific), prominently seeing a parallelism between individuals and points, and between societies and regions (or laws).

Kitaro Nishida, the leader of the Kyoto School of Philosophy, asserts "it is not that experience exists because there is an individual, but that an individual exists because there is experience" (Nishida 1990; originally 1921; xv in the Introduction). Just as a point is a bunch of observable regions for Whitehead, so an individual is a bundle of actual experiences for Nishida.

Wataru Hiromatsu, a Tokyo philosopher of a later generation than the Kyoto School, similarly propounded the shift from things (or objects) to events, which is analogous to the shift from points to laws in philosophy of space by Wittgenstein and Brouwer, and more conspicuously to the aforementioned shift from substances to processes and functions in Whitehead and Cassirer respectively.

These philosophers (and some others) seem to share certain ideas in some way or other; however, a coherent and rigorous perspective on them is yet to be explored in future work.

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