Supersoidity of alpha cluster structure in nuclei

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田中一先生 (1924-2021)（北大名誉教授　湯川研）

1.　日本のクラスター研究の父　クラスター研究の開拓者

図１　1987年11月6日北大核理論研究室でのセミナーの後，田中教授室で．田中一先生（左）と筆者（大久保茂男）．

1987 北大

図２　2005 年基研シンポジウムでご一緒した田中一先生（３列目中央）と筆者（大久保茂男）（３列目右端）．前列には今はなき林忠四郎（1920－2010），南部陽一郎（1921－2015），益川敏英（1940-2021）の諸先生．

図３　2005 年基研シンポジウム懇親会の田中一先生．左より堀内昶，国広悌二，早川尚男．田中一，大久保茂男

2005　京都大学 基研シンポジウム「学問の系譜」ーアインシュタインから湯川・朝永まで

1. 「クラスター構造研究のfp殻への展開」 素粒子論研究
2. 玉垣先生追悼文　原子核研究 2016年
3. 学問の系譜 「クラスター模型の展開」 2005　素粒子論研究

David M Brink博士(1930-2021)　追憶

2006イタリア
Cortina 国際
会議

1985 Brink 邸
Oxford , Northmoor Road
大久保　Brink夫人　Brink

1993 Brink 邸
Oxford　Minister Road

- 共著論文
- S. Ohkubo and D. M. Brink Phys. Rev. 36, 966 (1987)

  Internal and barrier wave interpretation of the oscillations of the fusion excitation functions
- S. Ohkubo and D. M. Brink Phys. Rev. 36, 1375 (1987)
- Origin of the oscillations in the 12C+12C excitation function in terma of internal and barrier waves

What is a cluster?

- 1988 5th Cluster conference, Kyoto
- H. special session What is a cluster?
- いまやこのような問いを発する人はいない？
- 有馬朗人 A. Arima **[Shell model もクラスターをある確率で含む 8Be]**
- R. R. Betts 英語辞書には「a group of the same or similar elements gathered or occurring closely together]
  → What is not a cluster?
- 池田清美 K. Ikeda「spatially localized subsystem composed of strongly correlated nucleons」.

New picture of alpha clusters

- Geometrical configuration with superfluidity
  ーー＞ supersolidity
- Duality　粒子性と波動性　gemetrical structure & condndensate
- SSB (Spontaneous symmetry breaking)　of
  rotational invariance due to geometrical configuration and
  Global phase symmetry in gauge space

organization of this talk

1. Introduction
2. Duality of Brink alpha cluster model:
   crystallinity and condensation
3. Alpha cluster model with order parameter based on effective field theory
4. Result: Supersolidity of alpha cluster structure
5. Summary

Introduction: purpose

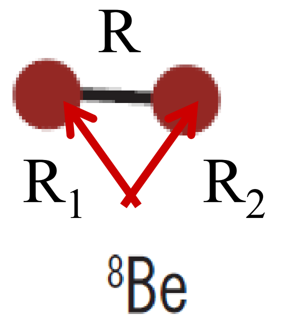
A supersolid is a **solid** that exhibits the property of **superfluidity**.

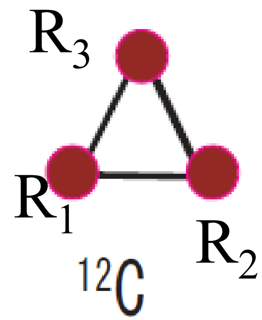
- ***α cluster structure*** has the **simultaneous** properties of **crystallinity** and **superfluidity**. That is,

the **α cluster structure is a stable supersolid**

II. Duality of Brink alpha cluster model: crystallinity and condensation

II. THE DUALITY OF THE α CLUSTER STRUCTURE: CRYSTALLINITY AND CONDENSATION





Brink wave function

$$\Phi^B_{n\alpha}(\boldsymbol{R}_1,\cdots,\boldsymbol{R}_n) = \frac{1}{\sqrt{(4n)!}}\det[\phi_{0s}(\boldsymbol{r}_1-\boldsymbol{R}_1)\chi_{\tau_1,\sigma_1}]\cdots\phi_{0s}(\boldsymbol{r}_{4n}-\boldsymbol{R}_n)\chi_{\tau_{4n},\sigma_{4n}}, \quad (1)$$

where $\boldsymbol{R}_i$ is a parameter that specifies the center of the i-th α cluster

$$\phi_{0s}(\boldsymbol{r}-\boldsymbol{R}) = \left(\frac{1}{\pi b^2}\right)^{3/4}\exp\left[-\frac{(\boldsymbol{r}-\boldsymbol{R})^2}{2b^2}\right], \quad (2)$$

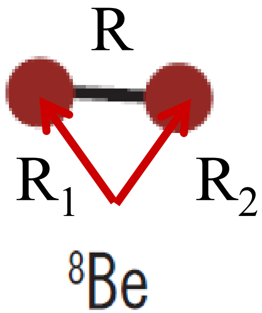
$$\Phi^B_{n\alpha}(\boldsymbol{R}_1,\cdots,\boldsymbol{R}_n) = \mathscr{A}\left[\prod_{i=1}^{n}\exp\left\{-2\frac{(\boldsymbol{X}_i-\boldsymbol{R}_i)^2}{b^2}\right\}\phi(\alpha_i)\right], \quad (3)$$

generator coordinate wave function $\Psi^{GCM}_{n\alpha}$ based on the geometrical configuration of the Brink wave function is given by

$$\Psi^{GCM}_{n\alpha} = \int d^3\boldsymbol{R}_1\cdots d^3\boldsymbol{R}_n f(\boldsymbol{R}_1,\cdots,\boldsymbol{R}_n)\Phi^B_{n\alpha}(\boldsymbol{R}_1,\cdots,\boldsymbol{R}_n). \quad (4)$$

cluster structure of ^{8}Be

$$\boldsymbol{R}_1 = \boldsymbol{R}_G + \frac{1}{2}\boldsymbol{R}, \quad \boldsymbol{R}_2 = \boldsymbol{R}_G - \frac{1}{2}\boldsymbol{R}. \tag{5}$$



We take $\boldsymbol{R}_G$=0 to remove the spurious center-of-mass motion and use the notation $\Phi^B_{2\alpha}(\boldsymbol{R})$ for $\Phi^B_{2\alpha}(\frac{1}{2}\boldsymbol{R}, -\frac{1}{2}\boldsymbol{R})$. Thus Eq. (4) is written as

$$f(\boldsymbol{R}) = \int_0^\infty d\mu_x \int_0^\infty d\mu_y \int_0^\infty d\mu_z \exp\left[-(\mu_x {R_x}^2 + \mu_y {R_y}^2 + \mu_z {R_z}^2)\right] g(\boldsymbol{\mu}), \tag{7}$$

(6)

where $\boldsymbol{\mu} = (\mu_x, \mu_y, \mu_z)$. Then Eq.(6) reads

$$\Psi^{GCM}_{2\alpha} = \int d^3\boldsymbol{\mu} g(\boldsymbol{\mu}) \left[\int d^3\boldsymbol{R} \exp\left\{-(\mu_x {R_x}^2 + \mu_y {R_y}^2 + \mu_z {R_z}^2)\right\} \Phi^B_{2\alpha}(\boldsymbol{R})\right]. \tag{8}$$

$$\Phi^{PCM}_{2\alpha}(\boldsymbol{\mu}) \equiv \int d^3\boldsymbol{R} \exp\Big[-(\mu_x {R_x}^2 + \mu_y {R_y}^2 + \mu_z {R_z}^2)\Big] \Phi^B_{2\alpha}(\boldsymbol{R}), \tag{9}$$

Nonlocaalized Clustere Model

$$\propto \mathscr{A}\left[\prod_{i=1}^{2} \exp\left\{-2\left(\frac{X^2_{ix}}{B^2_x} + \frac{X^2_{iy}}{B^2_y} + \frac{X^2_{iz}}{B^2_z}\right)\right\}\phi(\alpha_i)\right], \tag{10}$$

where

$$B_k = \sqrt{b^2 + {\mu_k}^{-1}} \quad (k = x, y, z). \tag{11}$$

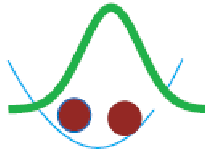
with $B_k = \sqrt{b^2 + \mu_k{}^{-1}}$ $(k = x, y, z)$, Eq.(8) reads

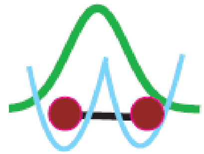
Coherent wave picture

$$\Psi^{GCM}_{2\alpha} = \int d^3\boldsymbol{\mu}\, g(\boldsymbol{\mu}) \Phi^{PCM}_{2\alpha}(\boldsymbol{\mu})$$

Crystallinity picture

$$\Psi^{GCM}_{2\alpha} = \int d^3\boldsymbol{R} f(\boldsymbol{R}) \Phi^{B}_{2\alpha}(\boldsymbol{R}).$$





Represented well by a single wave function

Duality

$\Phi^{B}_{2\alpha}(\boldsymbol{R})$. $\Phi^{PCM}_{2\alpha}(\boldsymbol{\mu})$.

The above discussion for the simplest two α cluster system can be generalized to the n-α cluster system. The Laplace transformation relation is generalized to

$$f(\boldsymbol{R}_1, \cdots, \boldsymbol{R}_n) = \int_0^\infty d\boldsymbol{\mu} \exp\left[-\sum_{i=1}^{n}(\mu_x R^2_{ix} + \mu_y R^2_{iy} + \mu_z R^2_{iz})\right] g(\boldsymbol{\mu}). \qquad (12)$$

$$\Phi_{n\alpha}^{PCM}(\boldsymbol{\mu}) = \int d^3\boldsymbol{R}_1 \cdots d^3\boldsymbol{R}_n \exp\Big[-\sum_{i=1}^{n}(\mu_x R_{ix}^2 + \mu_y R_{iy}^2 + \mu_z R_{iz}^2)\Big]\Phi_{n\alpha}^{B}(\boldsymbol{R}_1, \cdots, \boldsymbol{R}_n), \quad (13)$$

$$\propto \mathscr{A}\left[\prod_{i=1}^{n} \exp\left\{-2\left(\frac{X_{ix}^2}{B_x^2} + \frac{X_{iy}^2}{B_y^2} + \frac{X_{iz}^2}{B_z^2}\right)\right\}\phi(\alpha_i)\right].$$

Nonlocaalized Clustere Model

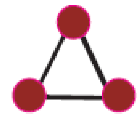
Similar to Eq.(11) one gets

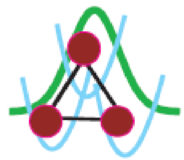
Coherent wave picture

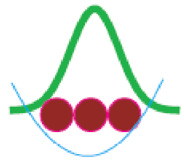
$$\Psi_{n\alpha}^{GCM} = \int d^3\boldsymbol{\mu}\, g(\boldsymbol{\mu})\Phi_{n\alpha}^{PCM}(\boldsymbol{\mu}).$$

Crystallinity picture

$$\Psi_{n\alpha}^{GCM} = \int d^3\boldsymbol{R} f(\boldsymbol{R})\Phi_{n\alpha}^{B}(\boldsymbol{R}). \quad (4)$$







Thus from Eq.(4) and Eq.(15) it is found generally hat the n-α cluster wave function in the geometrical luster model picture has the property of condensation. This shows generally that the GCM n-α cluster wave unction has the duality of crystallinity and condensation ndependent of the Hamiltonian used.

III. Alpha cluster model with order parameter based on effective field theory

PHYSICAL REVIEW C **94**, 014314 (2016)

Traditional cluster models for ^{12}C without order parameter

1. GCM : Uegaki et al
2. RGM: Kamimura et al
3. OCM: Kurokawa et al
4. Boson model: Matsumura et al
5. Local potential model :Buck et al

Particle picture

No concept of Vacuum

A theory with order parameter

6. Effective field theory :

superfluid cluster model (SCM)

14

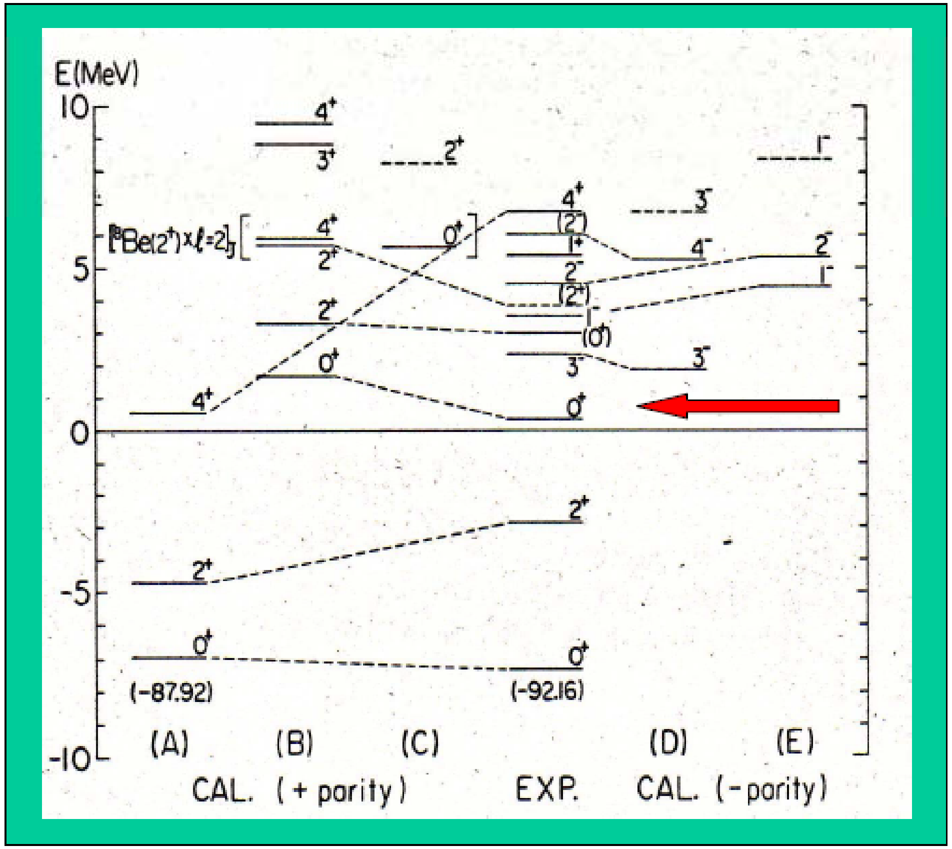
Uegaki et al **1977**

^{12}C **gas-like alpha-cluster states** ”new phase”

1977

Uegaki 3 alpha cluster model

:Energy level ^{12}C **(PTP 57,1262(1977)** GCM



Pointed out for the first time the existence of 3 alpha gas state

3 α Hoyle state

Uegaki et al

gas state of α particles

PTP 57,1262(1977)

Tohsaki et al

condensate PRL **87**, 192501(2001)

α condensate

No order parameter

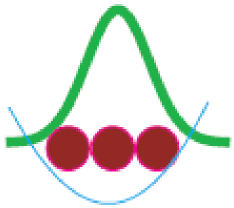
PTEP Prog. Theor. Exp. Phys. 2020, 041D01 (10 pages)
DOI: 10.1093/ptep/ptaa043

Letter

Supersolidity of the α cluster structure in the nucleus ^{12}C

S. Ohkubo[1,*], J. Takahashi[2], and Y. Yamanaka[2]

alpha field and the model



$$H = \int d^3x \left[\psi^\dagger_\alpha(x) \left(-\frac{\hbar^2}{2m}\nabla^2 - \mu + V_{\rm ext}(x) \right) \psi_\alpha(x) \right] + \frac{1}{2}\int\int d^3x d^3x' \, \psi^\dagger_\alpha(x)\psi^\dagger_\alpha(x')V(|x-x'|)\psi_\alpha(x')\psi_\alpha(x)$$

(b)
^{12}C

$$V(r) = V^{Nucl}(r) + V^{Coulomb}(r)$$

The Heisenberg equation

$$i\hbar\frac{\partial}{\partial t}\psi_\alpha(x) = \left(-\frac{2m}{\hbar^2}\nabla^2 - \mu + V_{\rm ext}(x)\right)\psi_\alpha(x) + \int d^3x'\,\psi^\dagger_\alpha(x')V(|x-x'|)\psi_\alpha(x')\psi_\alpha(x)$$

canonical commutation relation for $t=t'$

$$\left[\psi_\alpha(x,t)\,,\,\psi^\dagger_\alpha(x',t)\right] = \delta(x-x')$$

For stationary system (independent of t)

$$\psi_\alpha(x) = \xi(x) + \varphi_\alpha(x)$$

Here $\xi(x)$ condensate

$\varphi_\alpha(x)$ operator for excitation field

$$\left[\varphi_\alpha(x,t)\,,\,\varphi^\dagger_\alpha(x',t)\right] = \delta(x-x')$$

Goldstone theorem (Ward Takahashi identity) is respected.

From the condition $H_1 = 0$ $< \varphi_\alpha >= 0$

the Gross-Pitaevski equation is derived as follows:

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(x) + \int d^3x' \, |\xi(x')|^2 V(|x - x'|)\right) \xi(x) = \mu\xi(x)$$

If we take the unperturbed Hamiltonian $H_0 = H_2$ in the interaction picture, the equation of motion for the field operator is given in the matrix form as follows:

$$i\hbar\frac{\partial}{\partial t}\Phi_\alpha(x) = (\mathcal{T}\Phi_\alpha)(x)$$

where

$$\Phi_\alpha(x) = \begin{pmatrix} \varphi_\alpha(x) \\ \varphi_\alpha^\dagger(x) \end{pmatrix}$$

$$(\mathcal{T}\Phi_\alpha)(x) = \int dy \, T(x, y)\Phi_\alpha(y)$$

2x2 matrix is given by

$$T_{11}(x,y) = \left\{ -\frac{\hbar^2}{2m}\nabla^2 - \mu + V_{\rm ext}(x) + \int d^3x' \, |\xi(x')|^2 \, V(|x-x'|) \right\} \delta(x-y)$$
$$+ \xi^*(y) V(|x-y|)\xi(x)$$
$$T_{12}(x,y) = V(|x-y|)\xi(x)\xi(y)$$
$$T_{21}(x,y) = -\xi^*(x)\xi^*(y)V(|x-y|)$$
$$T_{22}(x,y) = -\left\{ -\frac{\hbar^2}{2m}\nabla^2 - \mu + V_{\rm ext}(x) + \int d^3x' \, |\xi(x')|^2 \, V(|x-x'|) \right\} \delta(x-y)$$
$$- \xi^*(x)V(|x-y|)\xi(y)$$

To solve the equation we expand the field operator in the complete set

$$\varphi_\alpha(x) = \sum_n a_n(t) w_n(x)$$

The complete set wave functions satisfy the following completeness condition

$$\sum_n w_n(x) w_n(x') = \delta(x-x')$$

From the canonical commutation relations

$$[a_n(t), a^\dagger_{n'}(t)] = \delta_{nn'}$$

The eigenvalue equation of Bogoliubov-de-Gennes is given by

$$(\mathcal{T}Y_n)(x) = \varepsilon_n Y_n(x)\,, \qquad Y_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}$$

From the symmetry property of the eigenvalue equation, the following function

$$Z_n(x) = \sigma_1 Y^*_n(x)$$

σ_1: Pauli matrix

is also an eigenfunction with $-\varepsilon_n$

For the zero mode with $\varepsilon_n = 0$

we can define the function $Y_{-1}(x)$

$$(\mathcal{T}Y_0)(x) = IY_{-1}(x) \qquad (\,I = \text{constant})$$

The completeness condition is given by

$$Y_0(x)Y_{-1}^\dagger(x') + Y_{-1}(x)Y_0^\dagger(x') + \sum_{n\neq 0}\{Y_n(x)Y_n^\dagger(x') - Z_n(x)Z_n^\dagger(x')\} = \sigma_3\delta(x-x')$$

By using the complete set $\{Y_0, Y_{-1}, Y_n, Z_n\}$ the wave function Φ_α is expanded as follows;

$$\Phi_\alpha(x) = \begin{pmatrix} \varphi_\alpha(x) \\ \varphi_\alpha^\dagger(x) \end{pmatrix}$$

$$\Phi_\alpha(x) = -iq(t)Y_0(x) + p(t)Y_{-1}(x) + \sum_{n\neq 0}\{b_n(t)Y_n(x) + b_n^\dagger(t)Z_n(x)\}$$

for $n=0$ we used $q(t)$ and $p(t)$ instead of b_0 and b_0^+

From the canonical commutation relation for $\varphi_\alpha(x)$

the operators $\{q(t), p(t), b_n(t), b_n^\dagger(t)\}$ satisfy

$$[q(t), p(t)] = i\ , \quad [b_n(t), b_{n'}^\dagger(t)] = \delta_{nn'}$$

By putting $\Phi_\alpha(x)$ into the Hamiltonian,

we obtain the diagonalized Hamiltonian H_0 as follows:

$$H_0 = \frac{I}{2}p^2(t) + \sum_{n\neq 0} \varepsilon_n b_n^\dagger(t) b_n$$

Goldstone theorem is respected

The vacuum for the operator b_n is defined by $b_n|0\rangle = 0$

Bogoliubov-de-Genne mode

$$|0\rangle = |\Psi\rangle \otimes |0\rangle_b \,,\ \Psi(q) = \langle q|\Psi\rangle$$

Now the non-perturbative field operator, the vacuum, non-perturbative hamiltonian and the interaction potential hamiltonian are given.

NG mode sector is modified to include higher power terms of NG quantum coordinate *p, q*

Zero mode equation (NB mode)

$$\hat{H}_u^{QP} |\Psi_\nu >= E_\nu |\Psi >$$ V : (n, ℓ, m)

Modes due to SSB

▪ α –α nuclear interaction

Ali-Bodmer potential： determined from α—α scattering to fit the s-wave phase shift fit potential

$$V^{nucl}(r) = V_r \exp[-\mu_r{}^2 r^2] + V_a \exp[-\mu_a{}^2 r^2]$$

$$V_r = 500 MeV,\ \mu_r = 0.7\ fm^{-1},\ V_a = -130 MeV,\ \mu_a = 0.474\ fm^{-1}$$

(610 MeV for ^{12}C and 591 MeV for ^{40}Ca)

▪ α –α Coulomb interaction：

α—α folding potential

$$V^{\mathrm{Coul}}_{\alpha-\alpha}(r) = (4e^2/r)\mathrm{erf}(\sqrt{3}r/2b)$$

size parameter of the α particle b is 1.44 fm

▪ Number of α clusters: N_0

$$\int d^3x\ |\xi(r)|^2 = N_0\,.$$

▪ External field potential： harmonic oscillator

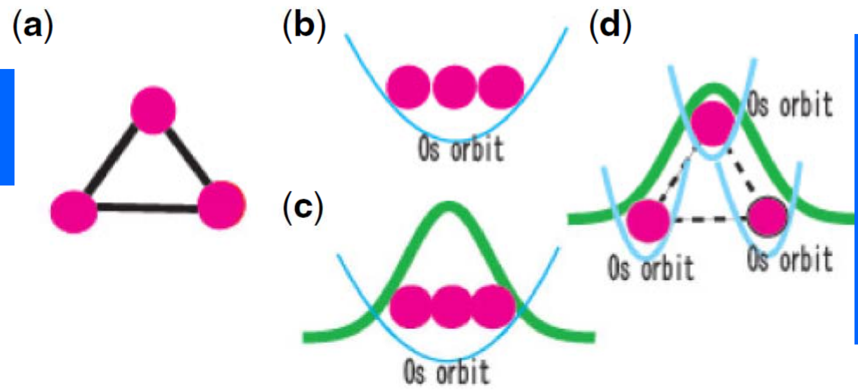
$$V_{\mathrm{ex}}(r) = m\Omega^2 r^2/2$$

(4.093 MeV for ^{12}C and 2.97 MeV for ^{40}Ca)

Ω： parameter

THSR, Nolocalized Cluster Model (NCM):
No order parameter

Geometrical picture



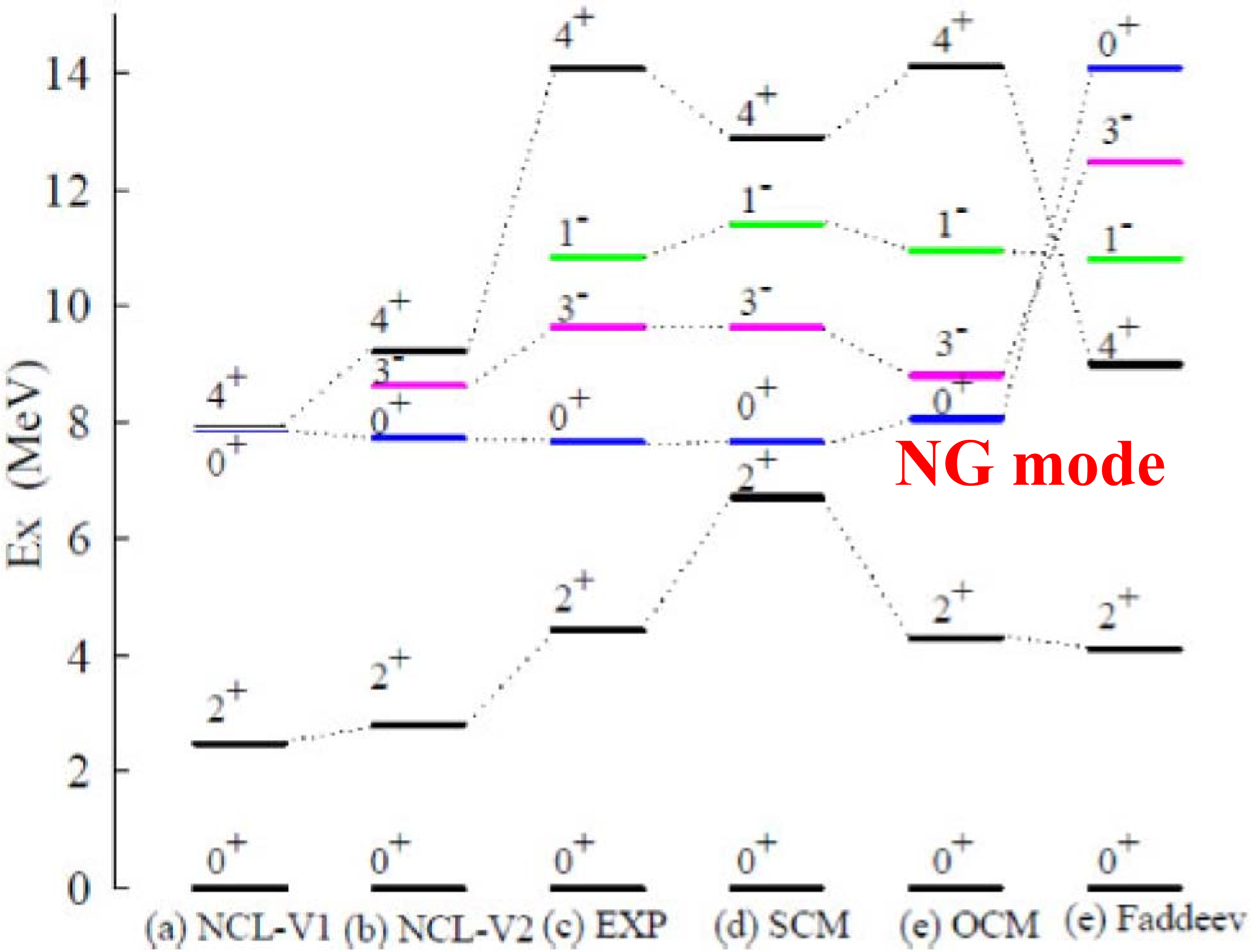
Supersolid picture with Duality
crystallinity & coherent wave

Superfluid cluster Model (SCM) :
with order parameter

Fig. 1. Illustrative pictures of the α cluster structure in ^{12}C. (a) Geometrical crystalline picture of the three α clusters. (b) Nonlocalized cluster picture of the three α clusters in the same 0s orbit of the potential. (c) Superfluid cluster model picture of the α clusters trapped in the potential with the associated coherent wave (broad curve). (d) Supersolid picture of the crystalline α clusters trapped in the distinct (due to the Pauli principle) 0s orbit of each potential associated with the coherent wave (broad curve).

IV. Result
Supersolidity of alpha cluster structure

Results and comparison with other models

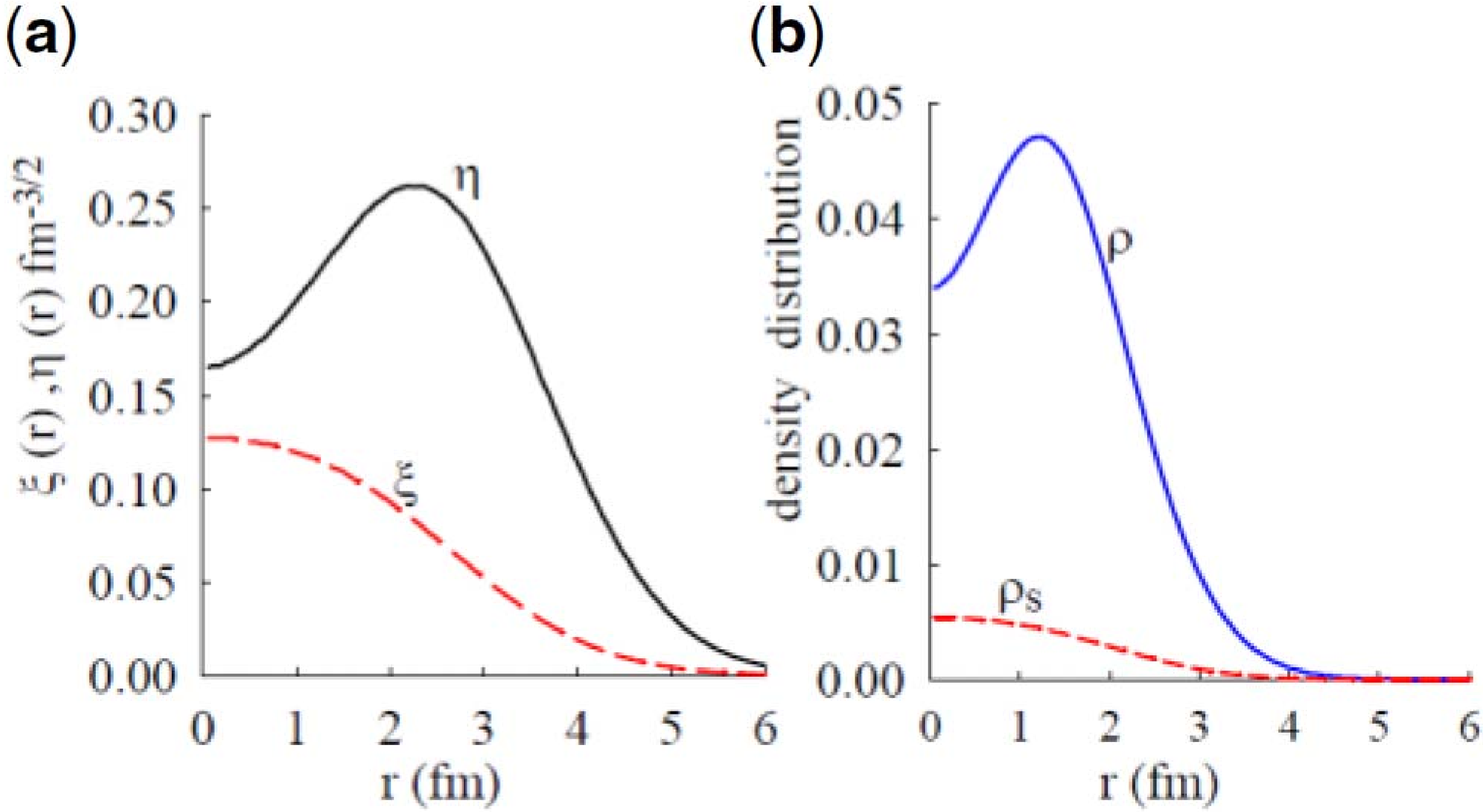


Particle picture

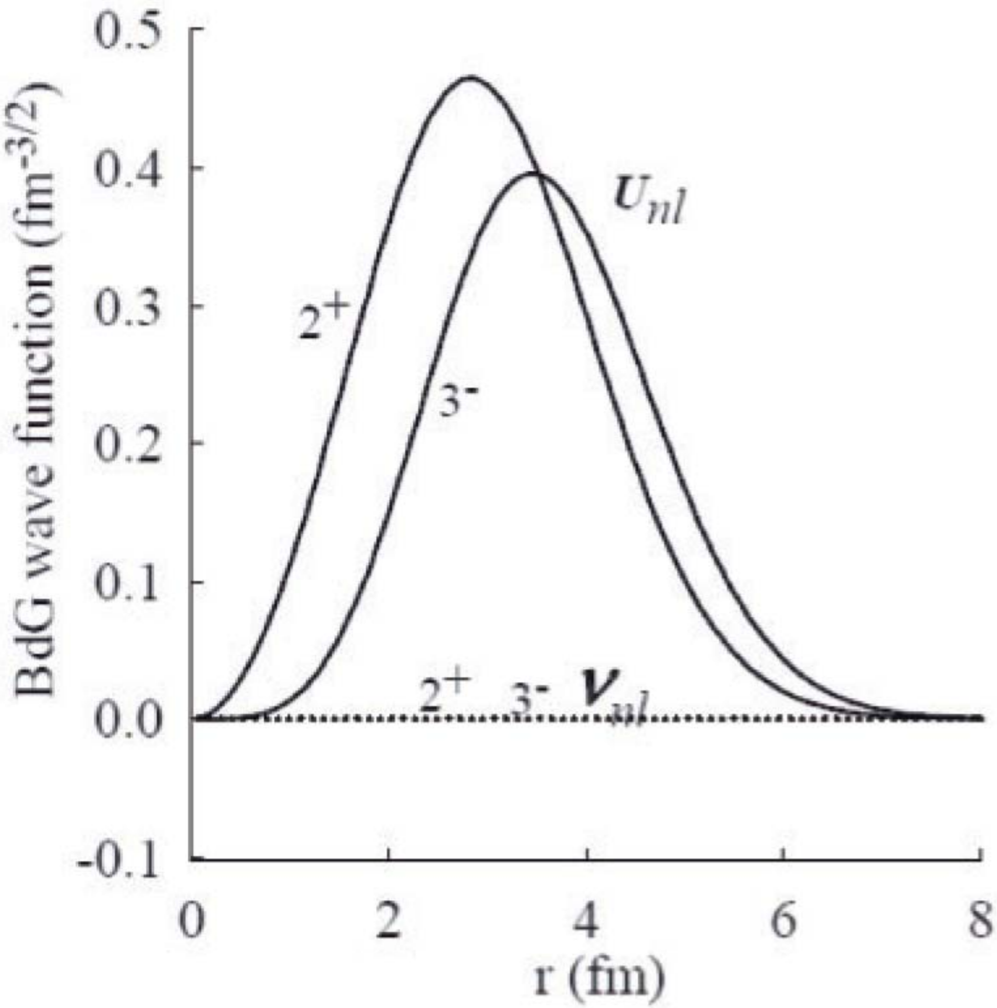
Superfluid picture

Particle picture

Density distributions of the ground state (^{12}C)

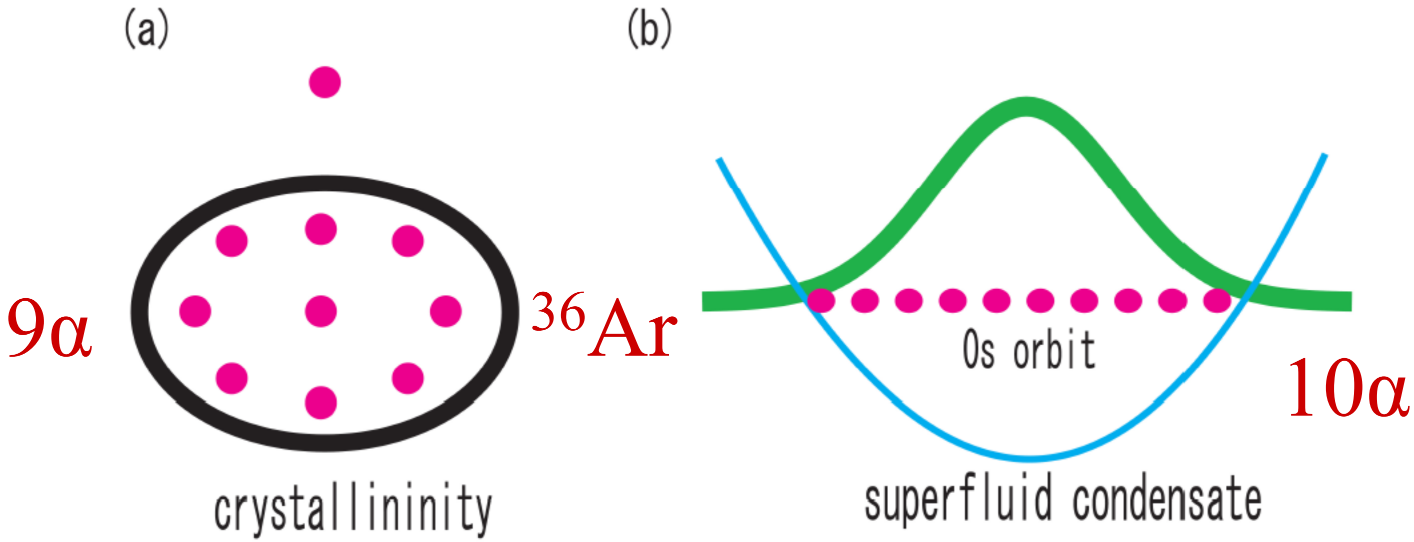


BdG Wave function （^{12}C)



^{40}Ca

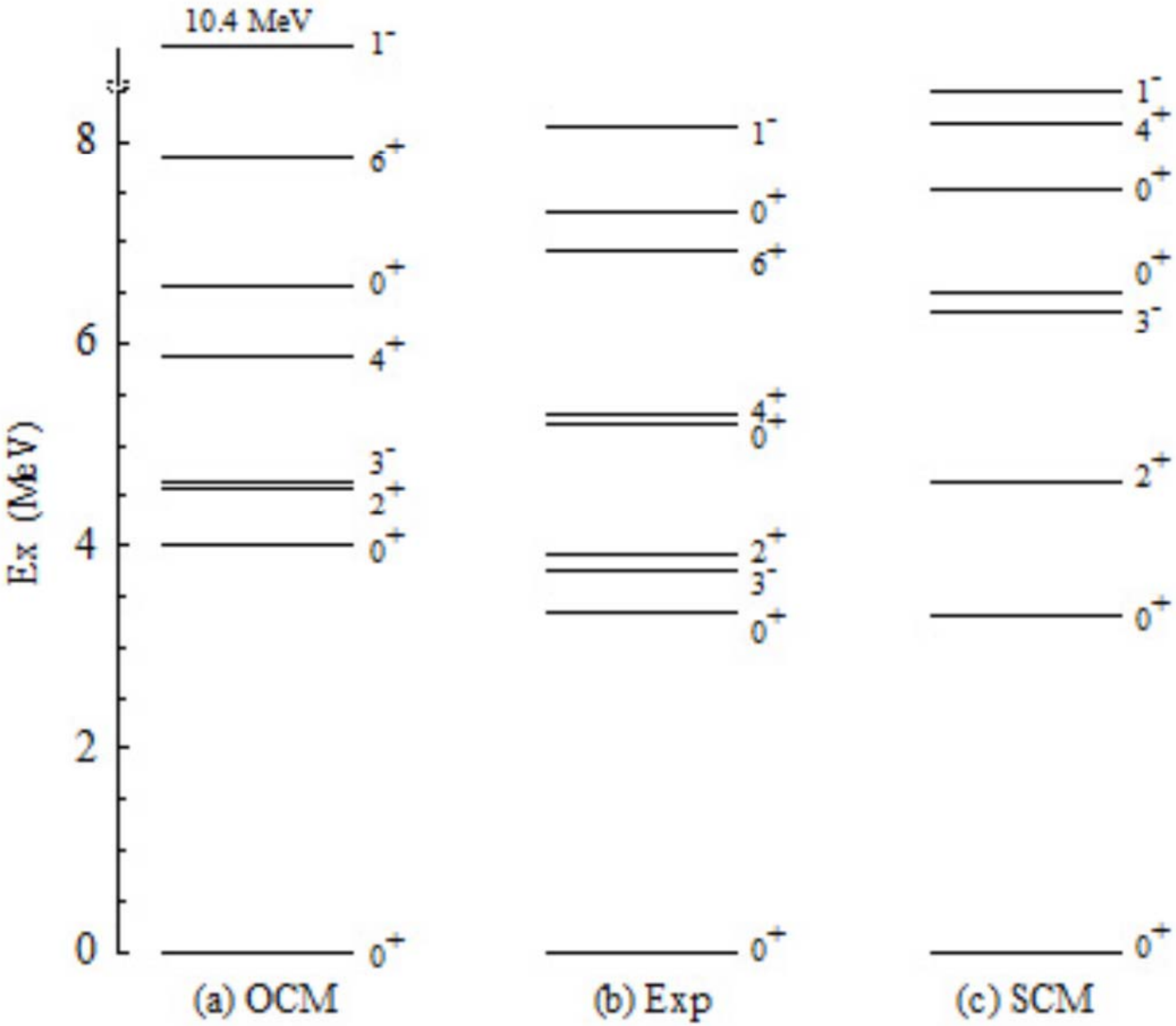
SO Phys. Rev. 106, 034324 (2022)



^{36}Ar + α model

Sakuda and Ohkubo, PTP Suppl. 132, 103 (1998)

Results and comparison with other model

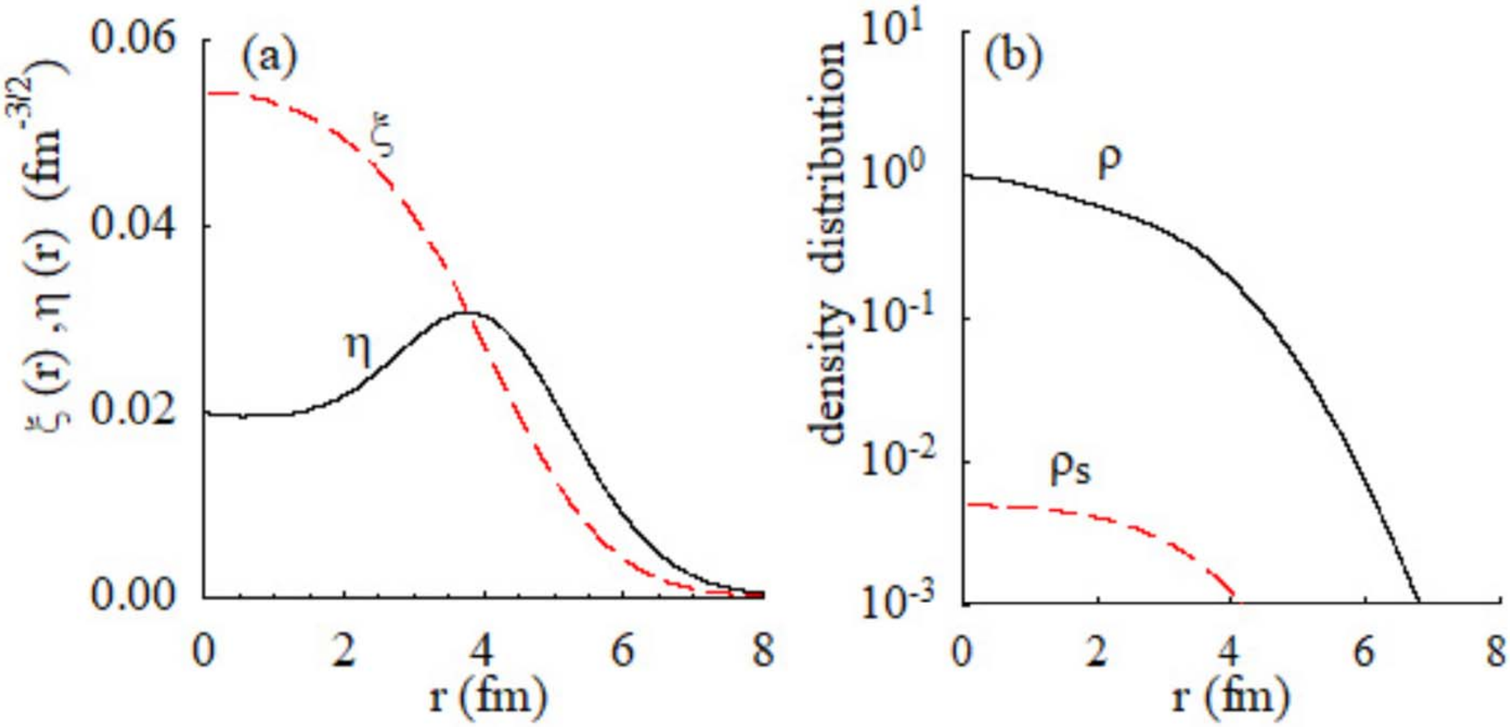


NG mode

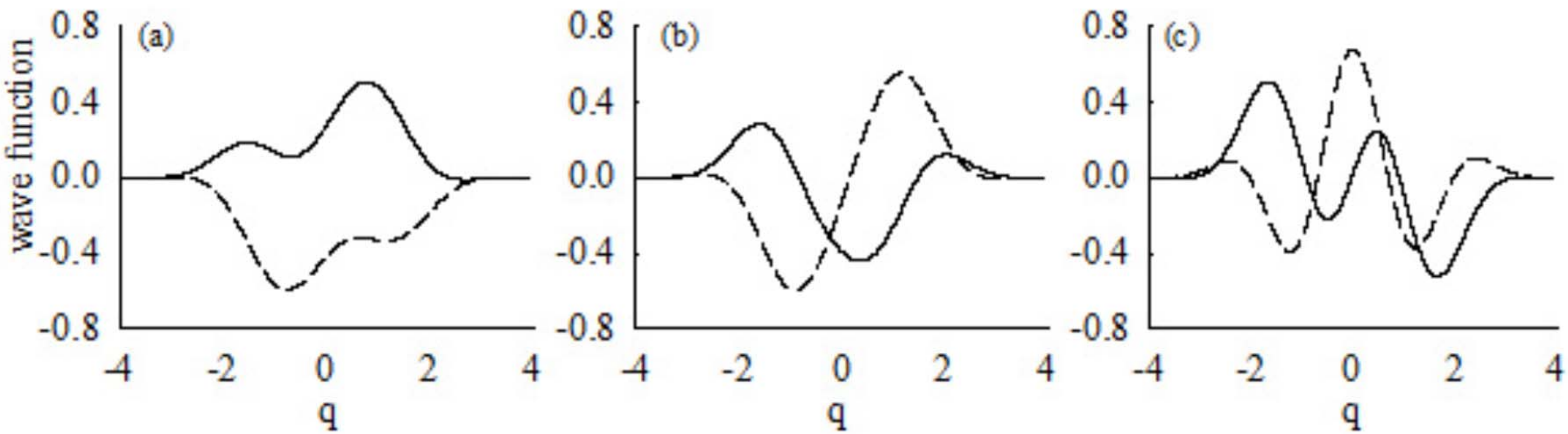
Particle picture

Superfluid picture

Density distributions of the ground state



NG mode wave functions



Summary

1. **Duality** of alpha cluster structure: **crystallinity** and **condensation:** **supersolidity**

2. Superfluid alpha cluster model (SCM) with order parameter treating Nambu-Goldstone mode rigorously is presented and applied to ^{12}C and ^{40}Ca :

   ^{12}C, ^{40}Ca :alpha cluster structure is understood by the superfluid cluster model

3. The emergence of the **mysterious** $\mathbf{0^+}$ state is understood to be a **manifestation of Nambu-Goldstone mode**

Ohkubo Phys. Rev. 106, 034324 (2022)
Ohkubo et al PTEP 041d01 (2021)