Workshop ``Hilbert's Sixth Problem''
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Completion of von Neumann's Axiomatization of Quantum Mechanics: From the Repeatability Hypothesis to Quantum Instruments

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Von Neumann's Axioms for Quantum Mechanics

- **Axiom 1 (States and observables).** *Every quantum system* S *is described by a Hilbert space* $\mathcal{H}$ *called the* **state space** *of* S. **States** *of* S *are represented by density operators on* $\mathcal{H}$ *and* **observables** *of* S *are represented by self-adjoint operators on* $\mathcal{H}$.

- **Axiom 2 (Born statistical formula).** *If an observable* A *is measured in a state* ρ*, the outcome obeys the probability distribution of* A *in* ρ *defined by*

$$\Pr\{A \in \Delta \| \rho\} = \mathrm{Tr}[E^A(\Delta)\rho],$$

  where $\Delta \in \mathcal{B}(\mathrm{R})$.

- **Axiom 3 (Time evolution).** *Suppose that a system* S *is an isolated system with the (time-independent) Hamiltonian* H *between time* t *and* $t + \tau$*. If the system* S *is in a state* $\rho(t)$ *at time* t *then* S *is in the state* $\rho(t + \tau)$ *at time* $t + \tau$ *satisfying*

$$\rho(t + \tau) = e^{-i\tau H/\hbar}\rho(t)e^{i\tau H/\hbar}.$$

Repeatability Hypothesis

- **Axiom R (Repeatability hypothesis).** ***If an observable is measured twice in succession in a system, then we get the same value each time.***

  J. von Neumann, Mathematische Grundlagen der Quantenmechanik (1932)

- **Axiom M (Measurement axiom). If an observable A is measured in a system S to obtain the outcome a, then the system S is left in an eigenstate of A belonging to a.**

- **Theorem (von Neumann, 1932). Axiom R is equivalent to Axiom M.**

- **Schrödinger's definition of measurement**

  The systematically arranged interaction of two systems (measured object and measuring instrument) is called a measurement on the first system, if a directly-sensible variable feature of the second (pointer position) is always reproduced within certain error limits when the process is immediately repeated.

  E. Schrödinger, Naturwissenshaften 23, 807, 823, 844 (1935).

Approximate Repeatability Hypothesis

- **Axiom AR (Approximate repeatability hypothesis).** ***If an observable is measured with error ε and immediately afterward measured without error, then the first outcome is reproduced within error ε.***

- **Definition: A state ρ is called an ε-approximate eigenstate belonging to a if**

$$\|A\sqrt{\rho} - a\sqrt{\rho}\|_{HS} \le \varepsilon.$$

- **Axiom AM (Approximate measurement axiom). If an observable A is measured in a system S with mean error ε to obtain the outcome a, then the system S is left in an ε-approximate eigenstate of A belonging to a.**

- **Theorem. Axiom AR is equivalent to Axiom AM.**

Uncertainty Principle: Heisenberg's Original Formulation

- **Definition. Two observables Q, P are called canonically conjugate if**

$$QP - PQ = i\hbar.$$

- **Theorem (under Axioms 1–3). The standard deviations $\sigma(Q)$, $\sigma(P)$ of canonically conjugate observables Q, P satisfy**

$$\sigma(Q)\sigma(P) \geq \frac{\hbar}{2}.$$

W. Heisenberg, *Z. Phys.* **43**,172 (1927); E. H. Kennard, **44**, 326 (1927).

- **Theorem (under Axioms 1–3 and Axiom AR) (Heisenberg's uncertainty principle). Canonicallyconjugate observables can be measured simultaneously only with mean errors $\varepsilon(Q)$, $\varepsilon(P)$ satisfying**

$$\varepsilon(Q)\varepsilon(P) \geq \frac{\hbar}{2}.$$

W. Heisenberg, *Z. Phys.* **43**,172 (1927).

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Proof

- **Let ρ be the state just after a simultaneous measurement of Q and P with the mean errors $\varepsilon(Q), \varepsilon(P)$ to lead the outcomes q, p. By Axiom AR, we have**

$$\varepsilon(Q) \geq \|Q\sqrt{\rho} - q\sqrt{\rho}\|_{HS},$$
$$\varepsilon(P) \geq \|P\sqrt{\rho} - p\sqrt{\rho}\|_{HS}..$$

On the other hand, we have

$$\|Q\sqrt{\rho} - q\sqrt{\rho}\|_{HS} \geq \sigma(Q),$$
$$\|P\sqrt{\rho} - p\sqrt{\rho}\|_{HS} \geq \sigma(P).$$

Thus, Heisenberg's uncertainty relation follows from

$$\varepsilon(Q)\varepsilon(P) \geq \sigma(Q)\sigma(P) \geq \frac{\hbar}{2}.$$

- **Conclusion: Quantum Mechanics from 1932 to 1960's was axiomatized by Axioms 1–3 and Axiom AR.**

Abandoning of RH in Modern Quantum Theory

- **1970: Davies-Lewis Thesis**

  One of the crucial notions is that of repeatability which we show is implicitly assumed in most of the axiomatic treatments of quantum mechanics, but whose abandonment leads to a much more flexible approach to measurement theory.

  E.B. Davies and J.T. Lewis, CMP 17, 239 (1970)

Devies-Lewis Instruments

- **Notations.**

  $\tau c(\mathcal{H})$**=the space of trace-class operators on** $\mathcal{H}$**.**

  $P(\tau c(\mathcal{H}))$**=the space of positive maps on** $\tau c(\mathcal{H})$**.**

- **Definition. An *instrument* for** $\mathcal{H}$ **is a** $P(\tau c(\mathcal{H}))$**-valued Borel measure on** R **such that** $\mathcal{I}(\mathrm{R})$ **is trace-preserving.**

Davies-Lewis Axiom

- **Notations.**

  $\mathrm{Pr}\{\mathrm{x} \in \Delta \| \rho\}$**=the probability of obtaining the outcome** x **in a Borel set** Δ**.**

  $\rho_{\{\mathrm{x}\in\Delta\}}$**=the state after the measurement for the ensemble given** $\mathrm{x} \in \Delta$ **for input state** ρ

- **Axiom DL (Davies-Lewis axiom). For any measuring apparatus there exists an instrument** $\mathcal{I}$ **such that its statistical properties are determined by**

$$
\begin{aligned}
\mathrm{Pr}\{\mathrm{x} \in \Delta \| \rho\} &= \mathrm{Tr}[\mathcal{I}(\Delta)\rho], \\
\rho_{\{\mathrm{x}\in\Delta\}} &= \frac{\mathcal{I}(\Delta)\rho}{\mathrm{Tr}[\mathcal{I}(\Delta)\rho]}.
\end{aligned}
$$

Yuen's Realization Problem

- ***What is the mathematical characterization of a general quantum measurement? . . . I believe the (Davies-Lewis) operational approach is too general—many measurements within this approach are not realizable in the above sense. . . . Since precision measurement is a perennial problem, the characterization and realization of general quantum measurements, besides being cornerstones in the structure of quantum physics, will also be of perennial physical relevance.***

  H. P. Yuen, Characterization and Realization of General Quantum Measurements, Proc. ISQM-TOKYO'86, 360 (1986).

Measuring Processes

- **Definition. Measuring Process :** $(\mathcal{K}, \xi, U, M) \Leftrightarrow$

$$
\begin{aligned}
\mathcal{K} &= \text{a Hilbert space, modeling the state space of the probe} \\
\xi &= \text{a unit vector on } \mathcal{K}\text{, modeling the initial state of the probe} \\
U &= \text{a unitary on } \mathcal{H} \otimes \mathcal{K}\text{, modeling the measuring interaction} \\
M &= \text{a self-adjoint operators on } \mathcal{K}\text{, modeling the meter observable}
\end{aligned}
$$

Measuring Processes Determine Completely Positive Instruments

- **Definition. An instrument $\mathcal{I}$ is called completely positive if $\mathcal{I}(\Delta)$ is completely positive for all Borel sets Δ.**

- **Theorem. Any measuring process $(\mathcal{K}, \xi, U, M)$ determines a completely positive instrument $\mathcal{I}$ by**

$$\mathcal{I}(\Delta) = \mathrm{Tr}_{\mathcal{K}}[U(\rho \otimes |\xi\rangle\langle\xi|)U^{\dagger}(I \otimes E^{M}(\Delta))],$$

which is consistent with DL axiom:

(i) Output Probabiliy:

$$\Pr\{\mathrm{x} \in \Delta \| \rho\} = \mathrm{Tr}[U(\rho \otimes |\xi\rangle\langle\xi|)U^{\dagger}(I \otimes E^{M}(\Delta))] = \mathrm{Tr}[\mathcal{I}(\Delta)\rho].$$

(ii) Posterior State:

$$\rho_{\{\mathrm{x}\in\Delta\}} = \frac{\mathrm{Tr}_{\mathcal{K}}[U(\rho \otimes |\xi\rangle\langle\xi|)U^{\dagger}(I \otimes E^{M}(\Delta))]}{\mathrm{Tr}[U(\rho \otimes |\xi\rangle\langle\xi|)U^{\dagger}(I \otimes E^{M}(\Delta))]} = \frac{\mathcal{I}(\Delta)\rho}{\mathrm{Tr}[\mathcal{I}(\Delta)\rho]}.$$

MO, JMP 25, 79 (1984).

Realization Theorem

- **Theorem (Realization theorem). For every completely positive instrument there is a measuring process $(\mathcal{K}, \xi, U, M)$ determining $\mathcal{I}$. Thus, physically realizable quantum measurements are characterized by the mathematical notion of completely positive instruments.**

  MO, JMP 25, 79 (1984).

Quantum Measurement Theory

- **Axiom M1 (Output Distribution, Quantum State Reduction). Every apparatus $\mathrm{A}(\mathrm{x})$ determines:**

  (i) Output Probability: $\rho \mapsto \Pr\{\mathrm{x} \in \Delta \| \rho\}$

  (ii) Posterior State: $(\rho, x) \mapsto \rho_{\{\mathrm{x}\in\Delta\}}$

- **Axiom M2 (Composition Law).**

  Every pair $\mathrm{A}(\mathrm{x})$ and $\mathrm{A}(\mathrm{y})$ have their composition $\mathrm{A}(\mathrm{x}, \mathrm{y})$:

  (i) $\Pr\{(\mathrm{x}, \mathrm{y}) \in \Delta \times \Gamma \| \rho\} = \Pr\{\mathrm{y} \in \Gamma \| \rho_{\{\mathrm{x}\in\Delta\}}\} \Pr\{\mathrm{x} \in \Delta \| \rho\}$,

  (ii) $\rho_{\{(\mathrm{x},\mathrm{y})\in\Delta\times\Gamma\}} = (\rho_{\{\mathrm{x}\in\Delta\}})_{\{\mathrm{y}\in\Gamma\}}$

- **Axiom M3 (Mixing Law).**

  Output probability is an affin function of input state, i.e.,

  $\Pr\{\mathrm{x} \in \Delta \| p\rho + (1-p)\rho'\} = p \Pr\{\mathrm{x} \in \Delta \| \rho\} + (1-p) \Pr\{\mathrm{x} \in \Delta \| \rho'\}$

- **Axiom M4 (Trivial Extendability).**

  Every apparatus $\mathrm{A}(\mathrm{x})$ has its trivial extension $\mathrm{A}(\mathrm{x} \otimes \boldsymbol{I})$:

  (i) $\Pr\{\mathrm{x} \otimes \boldsymbol{I} \in \boldsymbol{\Delta} \| \boldsymbol{\rho} \otimes \boldsymbol{\rho}'\} = \Pr\{\mathrm{x} \in \boldsymbol{\Delta} \| \boldsymbol{\rho}\}$

  (ii) $(\boldsymbol{\rho} \otimes \boldsymbol{\rho}')_{\{\mathrm{x} \otimes \boldsymbol{I} \in \boldsymbol{\Delta}\}} = \boldsymbol{\rho}_{\{\mathrm{x} \in \boldsymbol{\Delta}\}} \otimes \boldsymbol{\rho}'$

General Measurement Axiom

- **Theorem. Axioms M1–M3 are equivalent to Axiom DL. Axioms M1-M4 and Realization Theorem are equivalent to the following axiom:**

- **Axiom GM (General measurement axiom). For any measuring apparatus there exists a completely positive instrument $\mathcal{I}$ such that its statistical properties are determined by**

$$
\begin{aligned}
\Pr\{\mathrm{x} \in \Delta \| \rho\} &= \mathrm{Tr}[\mathcal{I}(\Delta)\rho], \\
\rho_{\{\mathrm{x}\in\Delta\}} &= \frac{\mathcal{I}(\Delta)\rho}{\mathrm{Tr}[\mathcal{I}(\Delta)\rho]},
\end{aligned}
$$

  and every completely positive instrument arises from a measuring apparatus in this way.

  MO, Ann. Physics 311, 350 (2004).

- **Conclusion: Quantum mechanics after 1984 is axiomatized by Axioms 1– 3 and Axiom GM.**

Controversy over the Limit for Gravitational Wave Detection

- **In 1980, Braginsky, Thorne, Caves and collaborators derived the standard quantum limit (SQL) to interferometer type gravitational wave detectors from Heisenberg's uncertainty principle, promoting resonator type approach.**

  V. B. Braginsky, K. S. Thorne et al., *Science* **209**, 547 (1980); C. M. Caves, K. S. Thorne et al., *Rev. Mod. Phys.* **52**, 341 (1980)

- **In 1983, Yuen claimed that such a limit can be broken by a contractive state measurement.**

  H. P. Yuen, *Phys. Rev. Lett.* **51**, 719 (1983).

- **In 1985, Caves criticised Yuen's argument with a new proof of SQL and questioned the realizability of the contractive state measurement.**

  C. M. Caves, *Phys. Rev. Lett.* **54**, 2465 (1985).

- **In 1988, M.O. showed that a contractive state measurement can be realized and indeed the SQL is broken to settle the debate.**

  1. **Caves proof used the approximate repeatability hypothesis.**
  2. **A solvable measuring process realizing Yuen's idea is constructed to beat the SQL.**

  M. O., *Phys. Rev. Lett.* **60**, 385 (1988).

- **In 1992, Thorne and coworkers obtained the NSF support for LIGO (Laser Interferometer Gravitational-Wave Observatory) project.**

- **In 2016, LIGO succeeded in detecting the gravitational wave.**

NATURE VOL. 331 18 FEBRUARY 1988 NEWS AND VIEWS 559

Beating the quantum limits (cont'd)

Heisenberg's Uncertainty Principle is for many an irksome constraint on the freedom to make measurements accurately. Can the constraint be overturned?

Ozawa's calculation will undoubtedly lift the spirits of those involved with the design of gravitational wave detectors; it will be Interesting to see where this leads.

[J. Maddox, Nature 331 (1988), 559]

What Ozawa has now done is to pull apart Cave's definition of "resolution",

ascendant. But now Masanao Ozawa of Nagoya University has put a cat among the pigeons by specifying a quantum system in which, he says, it is possible to do better than SQL (*Phys. Rev. Lett.* **54**,

ventional position on the SQL. One of the virtues of Ozawa's case is that the quantities arising in his calculations are indeed precisely defined and are related directly to quantities that can be measured. On the

Naturally, Ozawa's conclusion is that such a state of affairs is indeed attainable. The crux of his argument is the construction of a solvable model to represent the interaction between the measuring equipment and the free particle whose position is to be measured, which has the virtue that (regarded as a quantum mechanical hamiltonian) it can be solved exactly. In

Universal Uncertainty Relation

- **2003: Universal uncertainty relations is proved under Axioms 1-3 and Axiom GM:**

$$\varepsilon(A)\varepsilon(B) + \varepsilon(A)\sigma(B) + \sigma(A)\varepsilon(B) \geq C_{AB}.$$

where

$$C_{AB} = \frac{1}{2}|\mathrm{Tr}([A, B]\rho)|.$$

- **If $\varepsilon(B) = 0$ then**

$$\varepsilon(A)\sigma(B) \geq C_{AB}.$$

MO, PRA **67**, 042105 (2003); IJQI **1**, 569 (2003)

Experimental demonstration in neutron spin measurements

- **The apparatus make a projective measurement of $\sigma_\phi = \sigma_x \cos\phi + \sigma_y \sin\phi$ for the neutron spin.**
- **The apparatus is described by measurement operators $E^\phi(+1) = (1+\sigma_\phi)/2$ and $E^\phi(-1) = (1-\sigma_\phi)/2$ with $O_A = \sum_{x=\pm1} xE^\phi(x)$.**
- **For $A = \sigma_x$, $B = \sigma_y$, and $\rho = |\sigma_z = +1\rangle\langle\sigma_z = +1|$, we have**

$$\sigma(A) = \sigma(B) = 1, \quad \epsilon(A) = 2\sin\frac{\phi}{2}, \quad \eta(B) = \sqrt{2}\cos\phi.$$

Then for $0 \le \phi \le \pi/2$ we have the violation

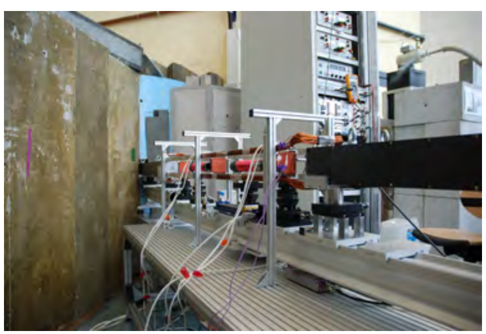
$$\epsilon(A)\eta(B) < 1.$$

- **The experiment was carried out at the research reactor facility TRIGA Mark II of the Vienna University of Technology (TU Vienna) by Yuji Hasegawa's group.**

Atom institute at TU Vienna

Research Reactor in Atom Institute , TU Vienna

Polarimeter Beamline



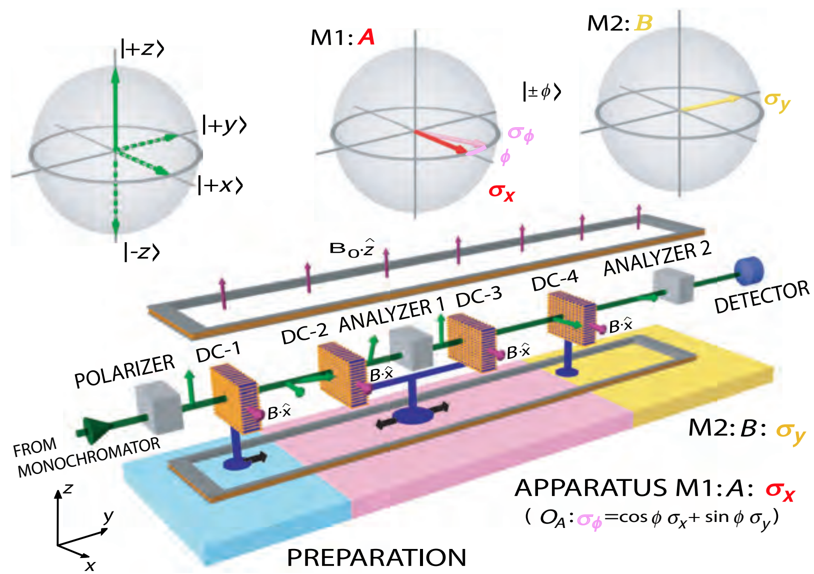
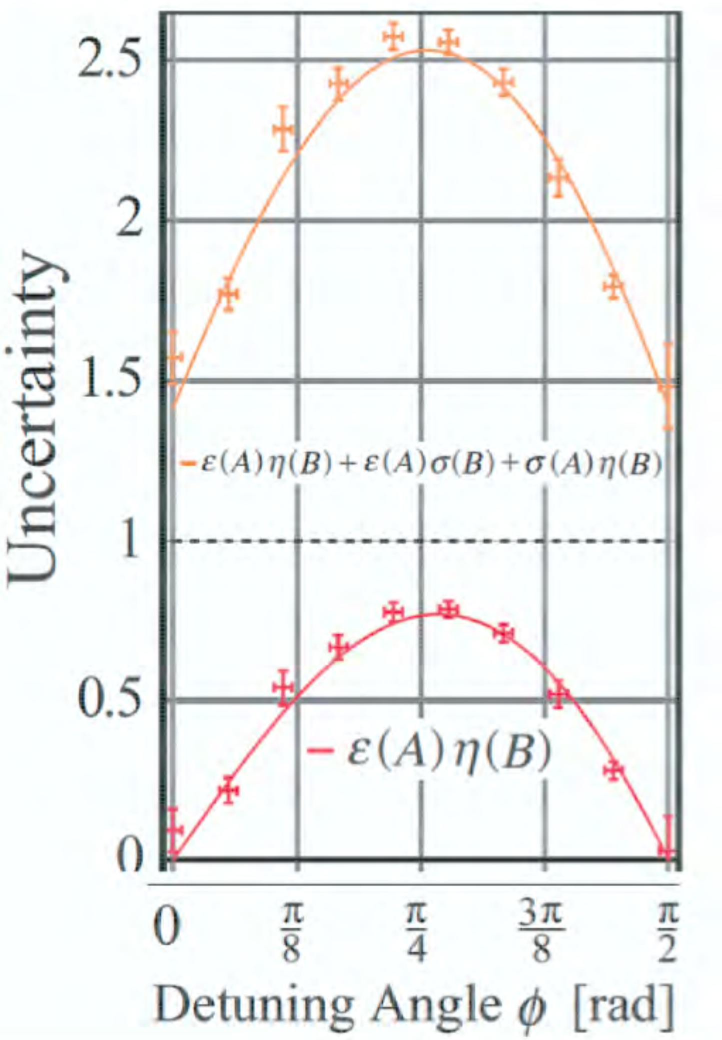


Figure 3: Illustration of the experimental setup for demonstration of the universally valid uncertainty relation for error and disturbance in neutron spin-measurements.

Comparison Result



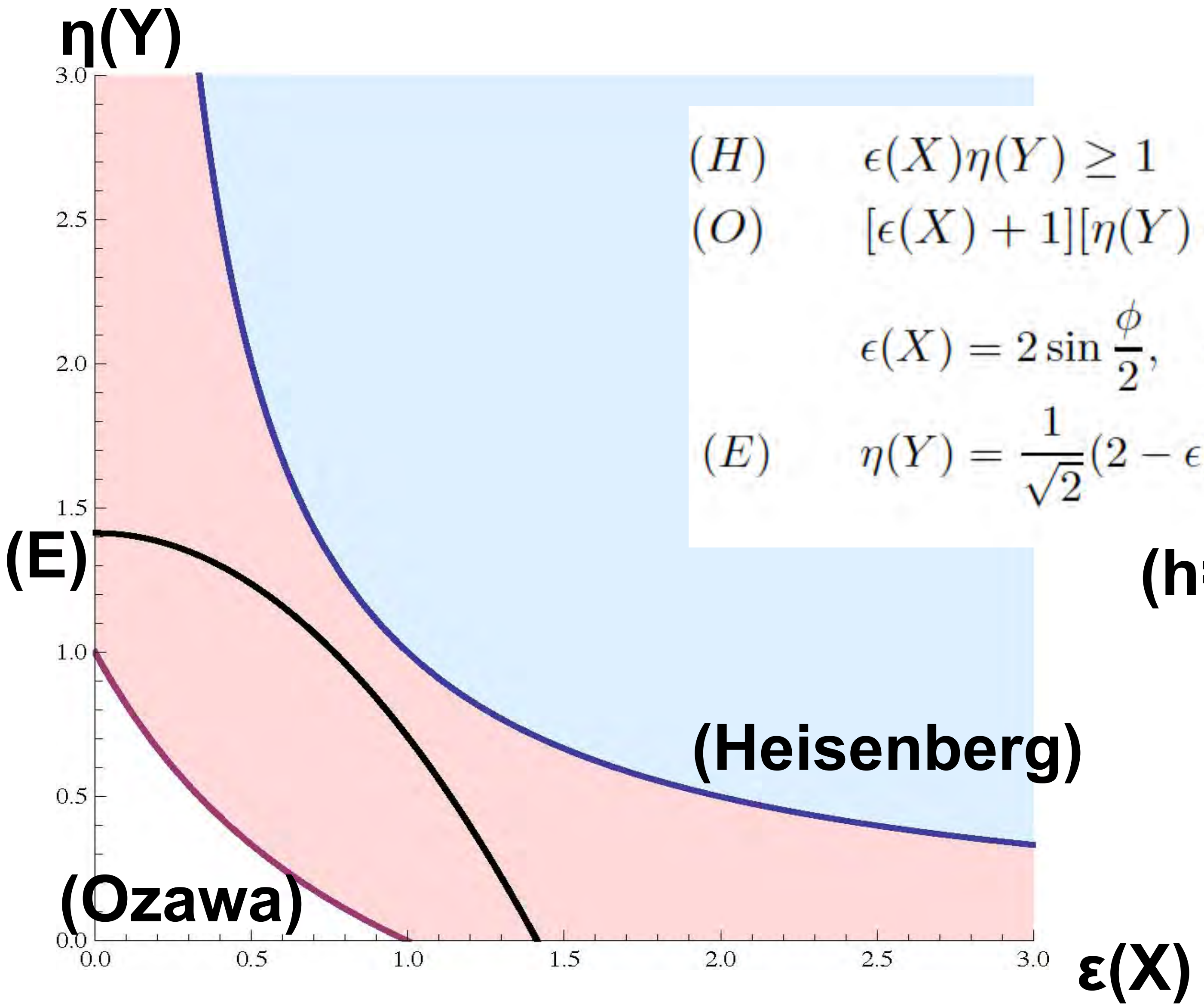
- Ozawa's relation holds:

$$\Delta X \Delta Y + \Delta X \sigma(Y) + \sigma(X) \Delta Y \geq \left(\frac{h}{4\pi}\right)^2$$

- Heisenberg's relation fails:

$$\Delta X \Delta Y < \left(\frac{h}{4\pi}\right)^2$$

Comparison between admissible regions



$$(H) \qquad \epsilon(X)\eta(Y) \geq 1$$

$$(O) \qquad [\epsilon(X)+1][\eta(Y)+1] \geq 2$$

$$\epsilon(X) = 2\sin\frac{\phi}{2}, \quad \eta(Y) = \sqrt{2}\cos\phi$$

$$(E) \qquad \eta(Y) = \frac{1}{\sqrt{2}}(2-\epsilon(X)^2)$$

nature physics

LETTERS

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Experimental demonstration of a universally valid error–disturbance uncertainty relation in spin measurements

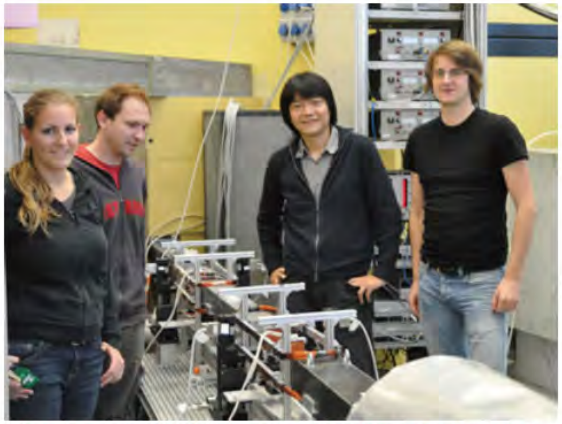
Jacqueline Erhart[1], Stephan Sponar[1], Georg Sulyok[1], Gerald Badurek[1], Masanao Ozawa[2] and Yuji Hasegawa[1]*

Physics World - the member magazine of the Institute of Physics

"This is certainly the first experiment to test Ozawa's formulation, so I think this should draw more attention to Ozawa's formulation, and how it is universally valid unlike a naive Heisenberg measurement-disturbance relation," said Howard Wiseman of Griffith University in Australia.

Neutrons revive Heisenberg's first take on uncertainty

Jan 20, 2012 10 comments



Yuji Hasegawa and colleagues disturbing a few neutrons

Media

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Common Interpretation of Heisenberg's Uncertainty Principle Is Proved False

A new experiment shows that measuring a quantum system does not necessarily introduce uncertainty

By Geoff Brumfiel | Tuesday, September 11, 2012 | 18 comments

BBC NEWS

SCIENCE & ENVIRONMENT

7 September 2012 Last updated at 16:24 GMT

Heisenberg uncertainty principle stressed in new test

By Jason Palmer
Science and technology reporter, BBC News

Pioneering experiments have cast doubt on a founding idea of the branch of physics called quantum mechanics.

Summary

- In 1932 von Neumann axiomatized quantum mechanics assuming the repeatability hypothesis and this is incomplete about what are quantum measurements.
- In 1927 Heisenberg derived the uncertainty relation for simultaneous measurements under this assumption and this is not universally valid.
- In 1970 Davies and Lewis proposed to abandon the repeatability hypothesis and introduced the mathematical notion of instruments.
- In 1980 Braginsky and Thorne claimed the standard quantum limit for gravitational wave detection based on Heisenberg's uncertainty principle or the repeatability hypothesis and this was proved wrong in 1988.
- In 1986 Yuen claimed that instruments are too general and proposed the problem of mathematical characterization of realizable measurements.
- In 1984 this problem was solved by the notion of completely positive instruments and von Neumann's axiomatization is completed.
- In 2003 a universally valid uncertainty relation was derived under this general axioms.
- In 2012 universally valid uncertainty relation was experimentally confirmed by neutron spin measurements.
- In 2016 LIGO project succeeded in detection of gravitational wave.