# Quantum Set Theories Satisfying Both the Transfer Principle and De Morgan's Laws *

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In quantum logic introduced by Birkhoff and von Neumann, De Morgan's Laws play an important role in the projection-valued truth value assignment for observational propositions in quantum mechanics. Takeuti's quantum set theory (QST) extends this assignment to all the set-theoretical statements on the universe of quantum sets, in which the reals bijectively correspond to the quantum observables. The Transfer Principle for Takeuti's QST shows that the projection-valued truth value of every bounded formula provable in ZFC (the Zermelo-Faenkel set theory with the axiom of choice) is lower bounded by the commutator, the projection-valued truth value for the commutativity, of quantum sets appearing in the formula as constants. However, Takeuti's QST has a problem that De Morgan's Laws between universal and existential bounded quantifications do not hold. In this study, we solve this problem by reforming Takeuti's QST with a new truth value assignment for bounded quantifications to satisfy both the Transfer Principle and De Morgan's Laws. We further study QSTs with the most general class of truth value assignments. For QSTs with polynomially definable logical operations, we determine exactly 36 QSTs that satisfy the Transfer Principle, and exactly 6 QSTs that satisfy both the Transfer Principle and De Morgan's Laws including the above reform of Takeuti's QST.

## 1 Introduction

As quantum logic introduced by Birkhoff and von Neumann [2] is an intrinsic logic governing observational propositions of quantum mechanics, it is an intriguing program to develop mathematics based on quantum logic. However, the introduction of basic notions of sets and numbers in quantum logic was not realized before Takeuti [10] introduced quantum set theory for this purpose. As a start, Takeuti constructed the universe $V^{(\mathscr{Q})}$ of quantum sets based on quantum logic $\mathscr{Q}$ represented by the projection lattice of a Hilbert space $\mathscr{H}$, and to every formula $\phi(x_1,\dots,x_n)$ in set theory he assigned the $\mathscr{Q}$-valued truth value $[\![\phi(u_1,\dots,u_n)]\!]$ for quantum sets $u_1,\dots,u_n \in V^{(\mathscr{Q})}$ to satisfy $\phi(x_1,\dots,x_n)$, in a manner analogous to Boolean-valued models of set theory introduced by Scott–Solovay and Vopěnka to reformulate Paul Cohen's forcing method for the independence proof of the continuum hypothesis [1]. For the well-known arbitrariness of the implication in quantum logic, he adopted the Sasaki arrow $\to$ defined by $P \to Q = P^{\perp} \vee (P \wedge Q)$ for implication as the most favorable choice in the majority view [11].

He introduced the commutator $\underline{\vee}(u_1,\dots,u_n)$ of elements $u_1,\dots,u_n$ of the universe $V^{(\mathscr{Q})}$, the projection-valued truth value for the commutativity, and he showed that the axioms of ZFC hold in the universe $V^{(\mathscr{Q})}$ if appropriately modified by the commutator. Based on his pioneering work on Boolean-valued analysis [9], he showed that the real numbers in the universe $V^{(\mathscr{Q})}$ correspond bijectively to the self-adjoint

*Based on the preprint entitled "Quantum Set Theory: Transfer Principle and De Morgan's Laws" by the same author, available at https://arxiv.org/abs/2002.06692.

†Supported by JSPS KAKENHI No. 17K19970 and the IRI-NU collaboration.

Submitted to:
QPL 2020

operators on the underlying Hilbert space $\mathscr{H}$, suggesting rich applications to quantum physics and analysis. Following Takeuti's work, we explored the question as to how the theorems of ZFC hold in the universe $V^{(\mathscr{Q})}$. In [5] we established the following Transfer Principle for Takeuti's QST.

**Transfer Principle.** *Every $\Delta_0$-formula*[1] $\phi(x_1,\dots,x_n)$ *with free variables $x_1,\dots,x_n$ in the language of set theory provable in ZFC satisfies the relation*

$$[\![\phi(u_1,\dots,u_n)]\!] \geq \underline{\vee}(u_1,\dots,u_n)$$

*for any elements $u_1,\dots,u_n$ in the universe $V^{(\mathscr{Q})}$.*

Since $\mathscr{Q}$ includes many complete Boolean subalgebras $\mathscr{B}$, the universe $V^{(\mathscr{Q})}$ naturally includes many Boolean-valued models $V^{(\mathscr{B})}$ as submodels, in which the ZFC set theory and hence classical mathematics founded on the ZFC set theory hold. The Transfer Principle allows us to recover classical mathematics to the extent precisely indicated by the commutators. Specifically, if $\underline{\vee}(u_1,...,u_n) = 1$, there exists a complete Boolean subalgebra $\mathscr{B}$ of $\mathscr{Q}$ such that $u_1,...,u_n \in V^{(\mathscr{B})}$ and that $[\![\phi(u_1,...,u_n)]\!] = 1$ for any $\Delta_0$-formula $\phi(x_1,...,x_n)$ provable in ZFC.

In [8], Takeuti's QST was extended to quantum logics $\mathscr{Q}$ represented by general complete orthomodular lattices. This generalization makes QST to fully include Boolean-valued models of set theory as the special case where $\mathscr{Q}$ is a Boolean algebra, and naturally incorporates powerful methods of forcing through Boolean-valued analysis [2]. Quantum set theory extended the Born rule for the probabilistic predictions of observational propositions to relations between observables, such as commutativity, equality, and order relations [6, 7]. The relations to paraconsistent set theory and topos quantum mechanics are also studied recently [3]. It was also applied to computer science [12].

In spite of the above successful development of the theory, one problem has eluded a solution. Takeuti's QST does not satisfy De Morgan's Laws for bounded quantifications. Takeuti noted "In Boolean-valued universes, $[\![(\forall x \in u)\phi(x)]\!] = [\![\forall x(x \in u \rightarrow \phi(x)]\!]$ and $[\![(\exists x \in u)\phi(x)]\!] = [\![\exists x(x \in u \wedge \phi(x)]\!]$ [hold]. But this is not the case for $V^{(\mathscr{Q})}$" [10, p. 315], and he defined the truth values of bounded quantifications using the Sasaki arrow $\rightarrow$ as follows.

(1) $[\![(\forall x \in u)\,\phi(x)]\!] = \bigwedge_{u' \in \mathrm{dom}(u)}(u(u') \rightarrow [\![\phi(u')]\!])$.

(2) $[\![(\exists x \in u)\,\phi(x)]\!] = \bigvee_{u' \in \mathrm{dom}(u)}(u(u') \wedge [\![\phi(u')]\!])$.

However, it is problematic that the classical implication $P \to Q = P^{\perp} \vee Q$ was avoided in the bounded universal quantification, and yet the classical conjunction $\wedge$ was used in the bounded existential quantification. Since the relation $P \wedge Q = (P \rightarrow Q^{\perp})^{\perp}$ does not hold for the classical conjunction $\wedge$ and the Sasaki arrow $\rightarrow$, De Morgan's Laws,

(3) $[\![\neg(\forall x \in u)\,\phi(x)]\!] = [\![(\exists x \in u)\,\neg\phi(x)]\!]$,

(4) $[\![\neg(\exists x \in u)\,\phi(x)]\!] = [\![(\forall x \in u)\,\neg\phi(x)]\!]$,

do not hold. In fact, there exist a predicate $\phi(x)$ and a quantum set $u \in V^{(\mathscr{Q})}$ such that $[\![(\exists x \in u)\,\neg\phi(x)]\!] = 0$ but $[\![\neg(\forall x \in u)\,\phi(x)]\!] > 0$, while the relation $[\![(\exists x \in u)\,\neg\phi(x)]\!] \leq [\![\neg(\forall x \in u)\,\phi(x)]\!]$ always holds.

In this study, we consider the binary operation $*$ given by $P * Q = (P \rightarrow Q^{\perp})^{\perp}$ and redefine the truth values of membership relation and bounded existential quantification as follows.

(5) $[\![u \in v]\!] = \bigvee_{v' \in \mathrm{dom}(v)}(v(v') * [\![v' = u]\!])$.

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[1] A $\Delta_0$-formula is a formula not including unbounded quantifiers $(\forall x)$ or $(\exists x)$ but bounded quantifiers $(\forall x \in u)$ or $(\exists x \in u)$.

[2] See [9, 4] and [1, Chapter 7] for Boolean valued analysis. In [4] we resolved Kaplansky's conjecture on the classification of type I AW*-algebras applying the cardinal collapsing forcing through the technique of Boolean-valued analysis.

(6) $[\![(\exists x \in u)\,\phi(x)]\!] = \bigvee_{u' \in \mathrm{dom}(u)} (u(u') * [\![\phi(u')]\!])$.

Then, De Morgan's Laws for the pair of universal–existential bounded quantifications as well as the Transfer Principle hold. Thus, for the language of quantum set theory we can assume only the negation, conjunction, and bounded and unbounded universal quantification as primitive, while disjunction, bounded and unbounded existential quantification are considered to be introduced by definition to avoid ramification and ambiguity in interpreting quantifications. The operation $*$ has been studied as the Sasaki projection in connection with residuation theory, whereas up to our knowledge this operation has not been used for defining bounded quantifiers in quantum logic.

We further consider a general class of binary operations $\to$ and $*$ for the implication and the conjunction, respectively, and we explore the consistency between the Transfer Principle and De Morgan's Laws. We show necessary and sufficient conditions for a pair $(\to, *)$ of operations to support the Transfer Principle, and to support both the Transfer Principle and De Morgan's Laws, respectively. For polynomially definable operations $\to$ and $*$, we determine all the 36 pairs $(\to, *)$ that admit the Transfer Principle, and we derive 6 out of 36 pairs that admit both the Transfer Principle and De Morgan's Laws, including the pair $(\to, *)$ of the Sasaki arrow $\to$ and the Sasaki projection $*$ and also the pair $(\to, *)$ of the classical implication $\to$ and the classical conjunction $*$ as previously mentioned in [8].

The future study of QST is expected to focus on those 6 QSTs. It is an interesting problem whether each of them has its own role in applications or only some of them do.

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