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Soundness and completeness of quantum root-mean-square errors

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Classical Root-Mean-Square Error

- **Definition. If a quantity A is indirectly measured by measuring another quantity M with the joint probability distribution $\mu(a,m)$ of A and M, the root-mean-square error $\varepsilon_G(\mu)$ of the measurement of A is defined by**

$$\varepsilon_G(\mu) = \left(\sum_{a,m}(m-a)^2\mu(a,m)\right)^{1/2}.$$

- **Theorem. $\varepsilon_G(\mu) = 0$ if and only if $\mu\{M = A\} = 1$, where**

$$\mu\{M = A\} = \sum_{(a,m):m=a} \mu(a,m).$$

Universal Models for Quantum Measurements

- **Definition.** $\mathrm{M} = (\mathcal{K}, \boldsymbol{\xi}, \boldsymbol{U}, \boldsymbol{M})$: **Measuring Process for the system described by** $\mathcal{H} \Leftrightarrow$

$$
\begin{aligned}
\mathcal{K} &= \text{a Hilbert space, modeling the state space of the probe}\\
\boldsymbol{\xi} &= \text{a unit vector on } \mathcal{K}\text{, modeling the initial state of the probe}\\
\boldsymbol{U} &= \text{a unitary on } \mathcal{H} \otimes \mathcal{K}\text{, modeling the measuring interaction}\\
\boldsymbol{M} &= \text{a self-adjoint operator on } \mathcal{K}\text{, modeling the meter observable}
\end{aligned}
$$

- **For any** $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{O}(\mathcal{H})$**, the measuring process** M **from time 0 to** $\boldsymbol{\tau}$ **determines**

$$
\boldsymbol{A}(0) = \boldsymbol{A} \otimes \boldsymbol{I}, \quad \boldsymbol{B}(0) = \boldsymbol{B} \otimes \boldsymbol{I}, \quad \boldsymbol{M}(0) = \boldsymbol{I} \otimes \boldsymbol{M},
$$
$$
\boldsymbol{A}(\boldsymbol{\tau}) = \boldsymbol{U}^{\dagger}\boldsymbol{A}(0)\boldsymbol{U}, \quad \boldsymbol{B}(\boldsymbol{\tau}) = \boldsymbol{U}^{\dagger}\boldsymbol{B}(0)\boldsymbol{U}, \quad \boldsymbol{M}(\boldsymbol{\tau}) = \boldsymbol{U}^{\dagger}\boldsymbol{M}(0)\boldsymbol{U}.
$$

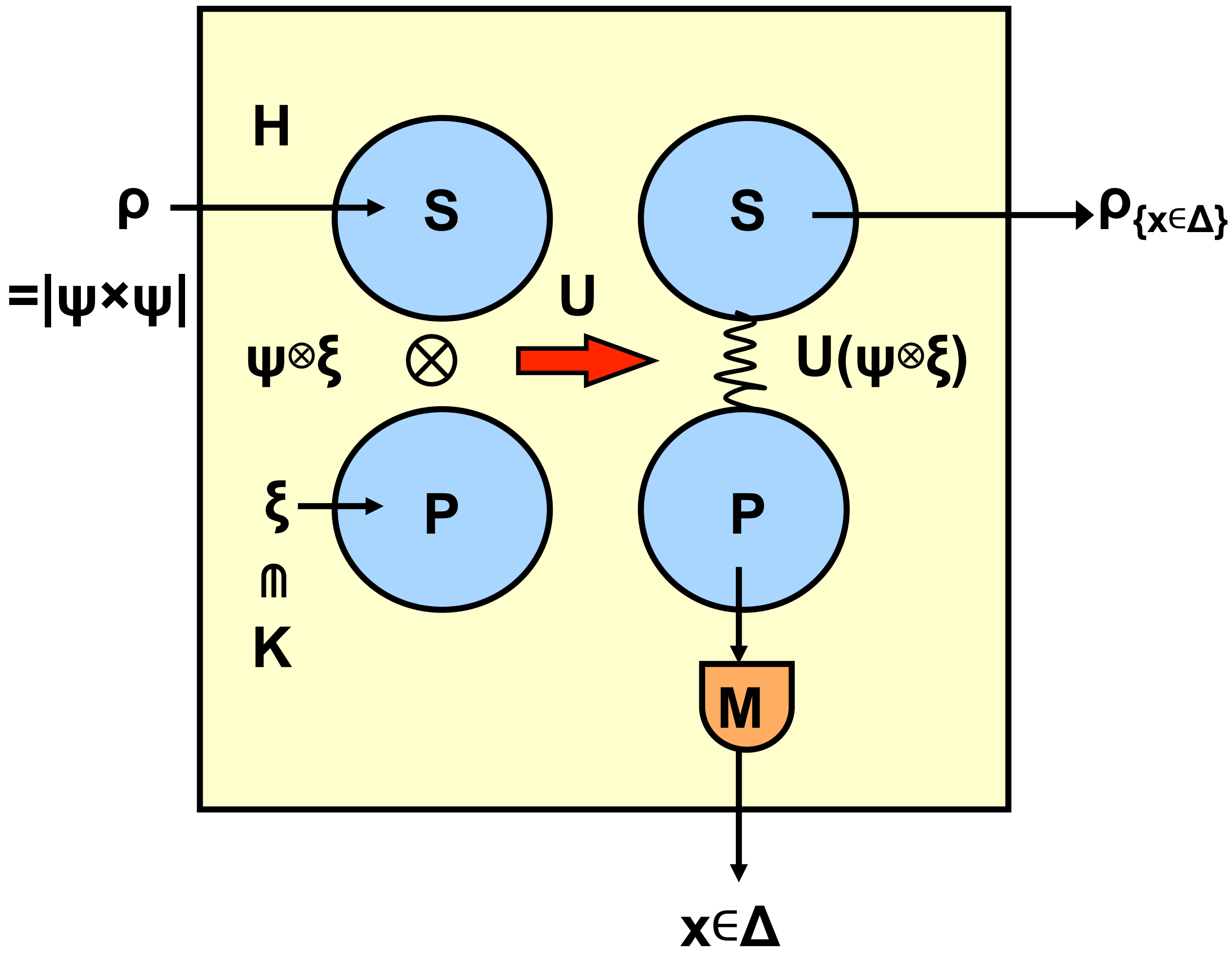

Figure 1: **Measuring Preocess**

- **When is the measurement $\mathrm{M} = (\mathcal{K}, \boldsymbol{\xi}, U, M)$ of A in ψ considered accurate? —The joint probability distribution μ of $M(\tau)$, $A(0)$ exists and satisfies $\mu\{M(\tau) = A(0)\} = 1$.**

State-Dependent Commutativity

- **Definition:** X **and** Y **commute in** Ψ **(**$X \leftrightarrow_{\Psi} Y$**) iff** $[f(X), g(Y)]\Psi = 0$ **for all polynomials** $f(X), g(Y)$**.**

- **Definition: A joint probability distribution (JPD) of observables** X, Y **in state** Ψ **is a probability distribution** $\mu_{\Psi}^{X,Y}(x, y)$ **on** R^2 **satisfying**

$$\langle \Psi | f(X, Y) | \Psi \rangle = \sum_{x,y} f(x, y)\, \mu_{\Psi}^{X,Y}(x, y)$$

for any polynomial $f(X, Y)$**.**

- **Theorem: There exists a JPD of** A**,** B **in** Ψ**.**

$\Leftrightarrow A \leftrightarrow_{\Psi} B$

$\Leftrightarrow$ Ψ **is a superposition of common eigenstates of** A **and** B**, i.e.,**

$$\Psi = \sum_{x,y} c_{x,y} \,| A = x, B = y \rangle \,.$$

Weak Joint Distribution

- **Definition: The weak joint distribution (WJD) $\nu_{\Psi}^{A,B}(x, y)$ of observables A, B in state Ψ is defined by**

$$\nu_{\Psi}^{A,B}(x, y) = \left\langle \Psi | P^{A}(x) P^{B}(y) | \Psi \right\rangle .$$

- **Remark: The WJD can be measured by weak measurement and post-selection, i.e.,**

$$\nu_{\Psi}^{A,B}(x, y) = \left\langle P^{B}(y) \right\rangle_{w, A=x, \Psi} \mu_{\Psi}^{A}(x).$$

Quantum Perfect Correlation

- **Definition:** A **and** B **are perfectly correlated in** Ψ **(**$A =_\Psi B$**)** $\Leftrightarrow A \leftrightarrow_\Psi B$ **and the JPD** $\mu_\Psi^{A,B}$ **satisfies**

$$\sum_x \mu_\Psi^{A,B}(x,x) = 1.$$

- **Theorem (MO 2005):**

$$\begin{aligned} A =_\Psi B \quad &\Leftrightarrow \quad \nu_\Psi^{A,B}(x,y) = 0 \quad \text{if } x \neq y \\ &\Leftrightarrow \quad \left\langle P^B(y) \right\rangle_{w,A=x,\Psi} = \delta_{x,y}. \end{aligned}$$

- **Remark: Perfect correlation is experimentally accessible.**

- **Theorem (MO 2005): The relation** $=_\Psi$ **is an equivalence relation among observables. In particular, if** $A =_\Psi B$ **and** $B =_\Psi C$**, then** $A =_\Psi C$**.**

State-Dependent Accurate Measurements

- **Definition (MO 2005): A measuring process $\mathrm{M} = (\mathcal{K}, \xi, U, M)$ *accurately* measures A in $\psi \Leftrightarrow$**

$$A(0) =_{\psi\otimes\xi} M(\tau).$$

- **Remark: The above condition is operationally accessible, since it is equivalent to**

$$\sum_{x\neq y} |\nu_{\psi\otimes\xi}^{A(0),M(\tau)}(x,y)| = 0.$$

- **Theorem A measuring process $\mathrm{M} = (\mathcal{K}, \xi, U, M)$ *accurately* measures A in $\psi \Leftrightarrow$**

$$\psi \otimes \xi = \sum_x c_x \, |A(0) = x, M(\tau) = x\rangle,$$

where $|c_x|^2 = |\langle\psi|A = x\rangle|^2$.

Noise Operator based Quantum Root-Mean-Square Error

- **Definition. For any measuring process $\mathrm{M} = (\mathcal{K}, \xi, U, M)$ the noise operator $N(A)$ for measuring A is defined by**

$$N(A) = M(\tau) - A(0).$$

- **The NO based QRMSE for measuring A in Ψ is defined by**

$$\varepsilon_{NO}(A, \psi) = \langle N(A)^2 \rangle^{1/2} = \|[M(\tau) - A(0)]\psi \otimes \xi\|.$$

- **Theorem (Lund-Wiseman 2010) The NO based QRMSE $\varepsilon_{NO}(A, \psi)$ can be measured by weak measurement and post-selection**

$$\varepsilon_{NO}(A, \psi)^2 = \sum_{a,m} (m - a)^2 \mathrm{Re}\, \nu_{\psi\otimes\xi}^{\mathrm{M}(\tau),\mathrm{A}(0)}(\mathrm{m}, \mathrm{a}).$$

Requirements for Quantum RMS Errors

(i) Device-independent definability: The error measure should be definable by the POVM of the measuring process, the observable to be measured, and the state of the object.

(ii) Correspondence principle: The error measure should be identical with the classical rms error if the joint probability distribution of $M(\tau)$ and $A(0)$ exists.

(iii) Soundness: The error measure should take the value zero for any accurate measurements.

(iv) Completeness: The error measure should take the value zero if and only if the measurement is accurate.

Device-Independent Definability

- **The NO based QRMSE ε_{NO}, satisfies the device-independent definability.**
- **The POVM of** M**:** $\Pi(x) = \left\langle \xi | E^{M(\tau)}(x) | \xi \right\rangle$
- **Moment operator of POVM** Π**:** $m^{(n)}(\Pi) = \sum_{x \in \mathrm{R}} x^n \Pi(x)$
- $\varepsilon_{NO}(A, \psi)$ **satisfies**

$$\varepsilon_{NO}(A, \psi)^2 = \mathrm{Re} \left\langle \psi | \mathrm{m}^{(2)}(\Pi) - 2\mathrm{m}(\Pi)\mathrm{A} + \mathrm{A}^2 | \psi \right\rangle .$$

Correspondence Principle

- **The NO based QRMSE ε_{NO} satisfies the correspondence principle.**

- **If $M(\tau)$ and $A(0)$ commute in $\psi \otimes \xi$, there exists the joint probability distribution $\mu(a, m) = \mu_{\psi\otimes\xi}^{A(0),M(\tau)}(a, m)$ and we have**

$$\varepsilon_{NO}(A, \psi)^2 = \sum_{a,m}(m - a)^2\mu(a, m).$$

- **Note that any error notions having been proposed based on the distance between the probability distributions do not satisfy the Correspondence Principle.**

Soundness

- **The NO based QRMSE ε_{NO} satisfies the soundness condition.**
- **If $A(0) =_{\Psi\otimes\sigma} M(\tau)$ then**

$$\mu_{\psi\otimes\xi}^{M(\tau),A(0)}(m,a) = 0 \quad \textbf{for } m \neq a,$$

and hence

$$\varepsilon_{NO}(A,\psi)^2 = \sum_{a,m}(m-a)^2\mu_{\psi}^{M(\tau),A(0)}(m,a) = 0.$$

- **The NO based QRMSE satisfies the device-independent-definability, the correspondence principle, and the soundness.**

Locally Uniform Quantum Root Mean Square Error

- **For any $t \in \mathrm{R}$, define**

$$\varepsilon_t(A, \psi) = \varepsilon_{NO}(A, e^{-itA}\psi).$$

- **The locally uniform rms error is defined by**

$$\overline{\varepsilon}(A, \psi) = \sup_{t \in \mathrm{R}} \varepsilon_t(A, \psi).$$

- **Theorem: (1) If $A(0)$ and $M(\tau)$ commute in $\psi \otimes \xi$, then**

$$\overline{\varepsilon}(A, \psi) = \varepsilon_{NO}(A, \psi).$$

(2) $\overline{\varepsilon}$ satisfies all the requirements (i)–(iv).

(3) $\varepsilon_{NO}(A, \psi) \leq \overline{\varepsilon}(A, \psi)$.

(4) If $A(0)^2 = M(\tau)^2 = I$, then $\overline{\varepsilon}(A, \psi) = \varepsilon_{NO}(A, \psi)$.

(5) The relation

$$\overline{\varepsilon}(Q, \psi)\overline{\varepsilon}(P, \psi) \geq \frac{\hbar}{2}$$

is violated.

Example

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\ |\psi\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

with $\Pi(y) = P^M(y)$. Then we have

$$\varepsilon_{NO}(A, \Pi, \psi) = 0,$$

but the measurement is not accurate, since A and Π are not identically distributed as $\left\langle \psi | P^A(2) | \psi \right\rangle = 1/2$ but $\langle \psi | \Pi(2) | \psi \rangle = 0$.

We have

$$\varepsilon_t(A, \Pi, \psi) = 2|\sin t|, \quad \text{and} \quad \overline{\varepsilon}(A, \Pi, \psi) = 2, \tag{1}$$

despite of the relation $\varepsilon_{NO}(A, \Pi, \psi) = 0$, the relation $\overline{\varepsilon}(A, \Pi, \psi) = 2$ correctly indicate that the above measurement of A is not accurate.

Simultaneous Measurements

- **Let $\mathrm{M} = (\mathcal{K}, \xi, U, M)$ be a measuring process. For any function f define $f(\mathrm{M}) = (\mathcal{K}, \xi, U, f(M))$**

- **Definition. A simultaneous measurement for A, B in Ψ is is defined as a pair of measuring processes $(f(\mathrm{M}), g(\mathrm{M}))$.**

- **Definition: A simultaneous measurement $(f(\mathrm{M}), g(\mathrm{M}))$ for A, B in ξ is *accurate* iff**

$$
\begin{aligned}
f(M(\tau)) &=_{\Psi\otimes\xi} A(0), \\
g(M(\tau)) &=_{\Psi\otimes\xi} B(0).
\end{aligned}
$$

Uncertainty Relations for Simultaneous Measurements

- **Definition.** **The error** $(\overline{\varepsilon}(A), \overline{\varepsilon}(B))$ **of a simultaneous measurement** $(f(\mathrm{M}), g(\mathrm{M}))$ **for** A, B **in** ξ **is defined by** $\overline{\varepsilon}(A) = \overline{\varepsilon}(A, \psi, f(\mathrm{M}))$ **and** $\overline{\varepsilon}(B) = \overline{\varepsilon}(B, \psi, g(\mathrm{M}))$.

- **Theorem.** $(f(\mathrm{M}), g(\mathrm{M}))$ **is accurate if and only if** $(\overline{\varepsilon}(A), \overline{\varepsilon}(B)) = (0, 0)$.

- **Theorem.** **Let** $C_{AB} = \frac{1}{2}| \langle\psi|[A, B]|\psi\rangle |$. **The following relations hold**

  (i) $\overline{\varepsilon}(A)\overline{\varepsilon}(B) + \sigma(B)\overline{\varepsilon}(A) + \sigma(A)\overline{\varepsilon}(B) \geq C_{AB}$.

  (ii) $$\sigma(B)^2\overline{\varepsilon}(A)^2 + \sigma(A)^2\overline{\varepsilon}(B)^2 + 2\overline{\varepsilon}(A)\overline{\varepsilon}(B)\sqrt{\sigma(A)^2\sigma(B)^2 - C_{AB}^2} \geq C_{AB}^2.$$