

Quantum Reality and Measurement: A Quantum Logical Approach

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Abstract

The recently established universal uncertainty principle revealed that two nowhere commuting observables can be measured simultaneously in some state, whereas they have no joint probability distribution in any state. Thus, one measuring apparatus can simultaneously measure two observables that have no simultaneous reality. In order to reconcile this discrepancy, an approach based on quantum logic is proposed to establish the relation between quantum reality and measurement. We provide a language speaking of values of observables independent of measurement based on quantum logic and we construct in this language the state-dependent notions of joint determinateness, value identity, and simultaneous measurability. This naturally provides a contextual interpretation, in which we can safely claim such a statement that one measuring apparatus measures one observable in one context and simultaneously it measures another nowhere commuting observable in another incompatible context.

KEYWORDS: quantum logic, quantum set theory, quantum measurement, joint determinateness, simultaneous measurability, contextual interpretation

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1 Introduction

In quantum mechanics, the value of any individual observable is postulated to be measured precisely. The values of several commuting observables can be reduced in principle by the functions of the value of a single observable, so that they are simultaneously measurable, and their joint probability distribution is predicted by the Born statistical formula. However, there is a generic difficulty in considering the values of non-commuting observables, or in considering their simultaneous measurement.

It has been accepted that two observables are simultaneously measurable if and only if they commute. However, this formulation is based on the state-independent formulation in the sense that two observables are simultaneously measurable *in any state* if and only if they commute *on the whole state space*. The recently established universal uncertainty principle [14, 12, 13, 15] revealed that two nowhere commuting observables can be measured simultaneously in some state [18], whereas they have no joint probability distribution in any state. Thus, a single measuring apparatus can simultaneously measure two observables that have no simultaneous reality. In order to reconcile this discrepancy, we consider a fundamental question as to the relation between quantum reality and measurement.

Here, an approach is proposed to establish the notion of the value of an observable independent of the measurement but obtained by the measurement. Our approach is based on quantum logic or quantum set theory [20, 19] to deal with quantum reality. First, we provide a language for the values of observables based on quantum logic and we construct in this language a sentence that means that two observables have the same value in a given state [16, 17]. Then, a contextual interpretation can be naturally obtained and within that we can safely claim that one measuring apparatus can reproduce the value of an observable A before measurement as the meter value after measurement in one context, and simultaneously that the same apparatus can reproduce the value of another nowhere commuting observable B in another context. This actually happens in the EPR state and in this interpretation the simultaneous measurability of nowhere commuting observables does not conclude the simultaneous reality of the values of those observables, for which once a hidden variable theory was called for.

In Section 2 we introduce the observational propositions, a propositional language constructed from atomic sentences meaning that a certain observable has a certain value, and assign them the truth values and probabilities based on standard rules in quantum logic. In Section 3 we define a new observational proposition in the above language meaning that given observables are jointly determinate in a given state, and show some basic properties of this proposition. In Section 4 we define a new observational proposition in the above language meaning that given two observables have the same value in a given state, and show some basic properties of this proposition. In Section 5 we introduce a mathematical theory of quantum measurements based on the notions of measuring processes, instruments, and POVMs. In Section 6 we define the notion of a measurement of an observable in a given state based on the observational proposition introduced in Section 4 so that a measuring process is a measurement of an observable in a given state if and only if the measured observable before the measurement and the meter observable after the measurement has the same value in the given state, and show some basic properties of this proposition. In Section 7 we define the notion of a simultaneous measurability of given observables in a given state based on the notion of a measurement of an observable in a given state introduced in Section 6. We clarify the conceptual difference between simultaneous measurability and joint determinateness by the existence of \mathbf{R}^2 -valued POVMs with certain properties. We also show some cases in which nowhere commuting observables are simultaneously measurable. Section 8 is devoted to discussions and conclusions.

2 Logic of observational propositions

In this paper, we assume for simplicity that every Hilbert space \mathcal{H} is finite dimensional and describes a quantum system $\mathbf{S}(\mathcal{H})$ unless stated otherwise. The infinite dimensional case will be discussed elsewhere. The observables are defined as self-adjoint operators, the states are defined as density operators, and a vector state ψ is identified with the state $|\psi\rangle\langle\psi|$. We denote by $\mathcal{O}(\mathcal{H})$ the set of observables, by $\mathcal{S}(\mathcal{H})$ the space of density operators, and by $\mathcal{L}(\mathcal{H})$ the space of linear operators on \mathcal{H} . We denote by \mathbf{R} the set of real numbers.

We define *observational propositions* for \mathcal{H} by the following rules.

(R1) For any $X \in \mathcal{O}(\mathcal{H})$ and $x \in \mathbf{R}$, the expression $X = x$ is an observational proposition, called an *atomic observational proposition*.

(R2) If ϕ_1 and ϕ_2 are observational propositions, $\neg\phi_1$ and $\phi_1 \wedge \phi_2$ are also observational propositions.

Thus, every observational proposition is built up from atomic observational propositions by means of the connectives \neg and \wedge . We introduce the connective \vee by definition.

(D1) $\phi_1 \vee \phi_2 := \neg(\neg\phi_1 \wedge \neg\phi_2)$.

We will freely use parentheses to clarify the construction. For example, if $X_1, X_2, X_3 \in \mathcal{O}(\mathcal{H})$ and $x_1, x_2, x_3 \in \mathbf{R}$, then $(X_1 = x_1 \wedge X_2 = x_2) \vee X_3 = x_3$ is an observational proposition.

The set of linear subspaces of a Hilbert space \mathcal{H} is a complete complemented modular lattice with the orthogonal complementation $M \mapsto M^\perp$ [2]. The lattice operations satisfy $M_1 \wedge M_2 = M_1 \cap M_2$ and $M_1 \vee M_2 = M_1 + M_2$. An operator P is called a *projection* if $P = P^\dagger = P^2$. The projection with range M is denoted by $\mathcal{P}(M)$ and the range of an operator X is denoted by $\mathcal{R}(X)$. Then, we have $\mathcal{P}(\mathcal{R}(P)) = P$ and $\mathcal{R}(\mathcal{P}(M)) = M$ for all projections P and subspaces M , so that the projection operators and the linear subspaces are in one-to-one correspondence, and the lattice structure is naturally introduced in the projections. We call the lattice of projections on \mathcal{H} as the *quantum logic* of \mathcal{H} and denote it by $\mathcal{Q}(\mathcal{H})$. For any $X \in \mathcal{O}(\mathcal{H})$ and $x \in \mathbf{R}$, we define the *spectral projection* $E^X(x)$ by $E^X(x) = \mathcal{P}(\{\psi \in \mathcal{H} \mid X\psi = x\psi\})$.

For each observational proposition ϕ , we assign its truth value $\llbracket \phi \rrbracket \in \mathcal{Q}(\mathcal{H})$ by the following rules.

(T1) $\llbracket X = x \rrbracket = E^X(x)$.

(T2) $\llbracket \neg\phi \rrbracket = \llbracket \phi \rrbracket^\perp$.

(T3) $\llbracket \phi_1 \wedge \phi_2 \rrbracket = \llbracket \phi_1 \rrbracket \wedge \llbracket \phi_2 \rrbracket$.

From (D1), (T2) and (T3), we have

(T4) $\llbracket \phi_1 \vee \phi_2 \rrbracket = \llbracket \phi_1 \rrbracket \vee \llbracket \phi_2 \rrbracket$.

We have $\llbracket (\phi_1 \wedge \phi_2) \wedge \phi_3 \rrbracket = \llbracket \phi_1 \wedge (\phi_2 \wedge \phi_3) \rrbracket$, so that we do not distinguish the observational propositions $(\phi_1 \wedge \phi_2) \wedge \phi_3$ and $\phi_1 \wedge (\phi_2 \wedge \phi_3)$ to denote them by $\phi_1 \wedge \phi_2 \wedge \phi_3$. Similar conventions are also applied to longer propositions and the connective \vee .

We define the *probability* $\Pr\{\phi \parallel \rho\}$ of an observational proposition ϕ in a state ρ by $\Pr\{\phi \parallel \rho\} = \text{Tr}[\llbracket \phi \rrbracket \rho]$. We say that *an observational proposition ϕ holds in a state ρ* if $\Pr\{\phi \parallel \rho\} = 1$.

Suppose that $X_1, \dots, X_n \in \mathcal{O}(\mathcal{H})$ are mutually commuting. Let $x_1, \dots, x_n \in \mathbf{R}$. We have

$$\begin{aligned} \llbracket X_1 = x_1 \wedge \dots \wedge X_n = x_n \rrbracket &= \llbracket X_1 = x_1 \rrbracket \wedge \dots \wedge \llbracket X_n = x_n \rrbracket \\ &= E^{X_1}(x_1) \wedge \dots \wedge E^{X_n}(x_n) \\ &= E^{X_1}(x_1) \dots E^{X_n}(x_n). \end{aligned}$$

Hence, we reproduce the *Born statistical formula*

$$\Pr\{X_1 = x_1 \wedge \dots \wedge X_n = x_n \parallel \rho\} = \text{Tr}[E^{X_1}(x_1) \dots E^{X_n}(x_n) \rho]. \quad (1)$$

For any polynomial $p(X_1, \dots, X_n)$ we also have

$$\text{Tr}[p(X_1, \dots, X_n) \rho]$$

$$= \sum_{(x_1, \dots, x_n) \in \mathbf{R}^n} p(x_1, \dots, x_n) \Pr\{X_1 = x_1 \wedge \dots \wedge X_n = x_n \|\rho\} \quad (2)$$

for any state ρ . From the above, our definition of the truth vales of observational propositions are consistent with the standard quantum mechanics.

3 Joint determinateness

For observational propositions ϕ_1, \dots, ϕ_n , we define the observational proposition $\bigvee_j \phi_j$ by $\bigvee_j \phi_j = \phi_1 \vee \dots \vee \phi_n$. We denote by $\text{Sp}(X)$ the set of eigenvalues of an observable X .

For any observables X_1, \dots, X_n we define the observational proposition $\text{com}(X_1, \dots, X_n)$ by

$$\text{com}(X_1, \dots, X_n) := \bigvee_{x_1 \in \text{Sp}(X_1), \dots, x_n \in \text{Sp}(X_n)} X_1 = x_1 \wedge \dots \wedge X_n = x_n. \quad (3)$$

We say that observables X_1, \dots, X_n are *jointly determinate* in a state ρ if the observational proposition $\text{com}(X_1, \dots, X_n)$ holds in ρ . In general, we say that observables X_1, \dots, X_n are *jointly determinate* in a state ρ with probability $\Pr\{\text{com}(X_1, \dots, X_n) \|\rho\}$.

Then, we have the following.

Theorem 1. *Observables X_1, \dots, X_n are jointly determinate in a vector state ψ if and only if the state ψ is a superposition of common eigenvectors of X_1, \dots, X_n .*

Proof. It is easy to see that $\psi \in \mathcal{R}(\llbracket X_1 = x_1 \wedge \dots \wedge X_n = x_n \rrbracket)$ if and only if ψ is a common eigenstate belonging to eigenvalues x_1, \dots, x_n for observables X_1, \dots, X_n , respectively, so that the assertion follows easily. \square

Two observables X and Y are said to be *nowhere commuting* if there is no common eigenstate. From Theorem 1, X and Y are nowhere commuting if and only if $\llbracket \text{com}(X, Y) \rrbracket = 0$.

A probability distribution $\mu(x_1, \dots, x_n)$ on \mathbf{R}^n , i.e., $\mu : \mathbf{R}^n \rightarrow [0, 1]$ and $\sum_{(x_1, \dots, x_n) \in \mathbf{R}^n} \mu(x_1, \dots, x_n) = 1$, is called a *joint probability distribution* of $X_1, \dots, X_n \in \mathcal{O}(\mathcal{H})$ in $\rho \in \mathcal{S}(\mathcal{H})$ if

$$\mu(x_1, \dots, x_n) = \Pr\{X_1 = x_1 \wedge \dots \wedge X_n = x_n \|\rho\}. \quad (4)$$

It is easy to see that a joint probability distribution μ of X_1, \dots, X_n in ρ is unique, if any.

The notion of joint probability distributions is inherently a state-dependent notion. It is well-known that the existence of the joint probability distribution in any state is equivalent to the commutativity of observables under consideration [7]. Since the joint determinateness is naturally considered to be the state-dependent notion of commutativity, it is naturally expected that the joint determinateness is equivalent to the state-dependent existence of the joint probability distribution, as shown below.

Theorem 2. *Observables X_1, \dots, X_n are jointly determinate in a state ρ if and only if there exists a joint probability distribution of X_1, \dots, X_n in ρ . In this case, for any polynomial $p(X_1, \dots, X_n)$ of observables X_1, \dots, X_n , we have*

$$\begin{aligned} & \text{Tr}[p(X_1, \dots, X_n)\rho] \\ &= \sum_{(x_1, \dots, x_n) \in \mathbf{R}^n} p(x_1, \dots, x_n) \Pr\{X_1 = x_1 \wedge \dots \wedge X_n = x_n \|\rho\}. \end{aligned} \quad (5)$$

Proof. The assertion follows from an argument similar to the proof of Theorem 3.3 of Gudder [7]. \square

Let $\phi(X_1, \dots, X_n)$ be an observational proposition that includes symbols for and only for observables X_1, \dots, X_n . Then, $\phi(X_1, \dots, X_n)$ is said to be *contextually well-formed* in a state ρ if X_1, \dots, X_n are jointly determinate in ρ . The following is an easy consequence from the transfer principle in quantum set theory [19].

Theorem 3. *Let $\phi(X_1, \dots, X_n)$ be an observational proposition that includes symbols for and only for observables X_1, \dots, X_n . Suppose that $\phi(X_1, \dots, X_n)$ is a tautology in classical logic. Then, we have*

$$\llbracket \text{com}(X_1, \dots, X_n) \rrbracket \leq \llbracket \phi(X_1, \dots, X_n) \rrbracket. \quad (6)$$

In particular, if $\phi(X_1, \dots, X_n)$ is contextually well-formed in a state ρ , then $\phi(X_1, \dots, X_n)$ holds in ρ .

4 Value identity of observables

For any observables X, Y , we define the observational proposition $X = Y$ by

$$X = Y := \bigvee_{x \in \text{Sp}(X)} X = x \wedge Y = x. \quad (7)$$

We say that observables X and Y *have the same value in a state ρ* if $X = Y$ holds in ρ . In this case, we shall write $X =_\rho Y$. In general, we say that observables X and Y *have the same value in a state ρ with probability $\Pr\{X = Y\|\rho\}$* .

Theorem 4. *For any $X, Y \in \mathcal{O}(\mathcal{H})$ and $\rho \in \mathcal{S}(\mathcal{H})$, we have $X =_\rho Y$ if and only if there exists a joint probability distribution μ of X, Y in ρ such that*

$$\sum_{x \in \mathbf{R}} \mu(x, x) = 1. \quad (8)$$

Proof. Suppose $X =_\rho Y$. We have $\Pr\{\text{com}(X, Y)\|\rho\} = 1$, so that from Theorem 2, X and Y have the joint probability distribution μ , which satisfies Eq. (8). Conversely, if we have the joint probability distribution μ satisfying Eq. (8), we have

$$\begin{aligned} \Pr\{X = Y\|\rho\} &= \text{Tr} \left[\left(\bigvee_{x \in \text{Sp}(X)} E^X(x) \wedge E^Y(x) \right) \rho \right] \\ &= \sum_{x \in \mathbf{R}} \text{Tr}[(E^X(x) \wedge E^Y(x))\rho] = \sum_{x \in \mathbf{R}} \mu(x, x) = 1. \end{aligned}$$

Thus, we have $X =_\rho Y$. \square

In order to consider the state-dependent notion of measurement of observables, the notion of quantum perfect correlation, or quantum identical correlation, between two observables has been previously introduced in Ref. [16, 17] as follows. We say that two observables $X, Y \in \mathcal{O}(\mathcal{H})$ are *identically correlated* in $\rho \in \mathcal{S}(\mathcal{H})$ if $\text{Tr}[E^X(x)E^Y(y)\rho] = 0$

for any $x, y \in \mathbf{R}$ with $x \neq y$. We say that two observables $X, Y \in \mathcal{O}(\mathcal{H})$ are *identically distributed* in a state $\rho \in \mathcal{S}(\mathcal{H})$ if $\text{Tr}[E^X(x)\rho] = \text{Tr}[E^Y(x)\rho]$ for any $x \in \mathbf{R}$. The *cyclic subspace* of \mathcal{H} spanned by an observable X and a state ρ is the linear subspace $\mathcal{C}(X, \rho)$ defined by

$$\mathcal{C}(X, \rho) = \{p(X)\psi \mid p(X) \text{ is a polynomial in } X \text{ and } \psi \in \mathcal{R}(\rho)\}^{\perp\perp}, \quad (9)$$

or equivalently $\mathcal{C}(X, \rho)$ is the linear subspace spanned by $X^n\psi$ for all $n = 0, 1, \dots$ and all $\psi \in \mathcal{R}(\rho)$. Then we obtain the following theorem.

Theorem 5. *For any two observables $X, Y \in \mathcal{O}(\mathcal{H})$ and any state $\rho \in \mathcal{S}(\mathcal{H})$, the following conditions are equivalent.*

- (i) X and Y have the same value in ρ .
- (ii) X and Y are identically correlated in ρ .
- (iii) X and Y are identically distributed in all $\psi \in \mathcal{C}(X, \rho)$.
- (iv) $f(X)\rho = f(Y)\rho$ for any function f .
- (v) $X = Y$ on $\mathcal{C}(X, \rho)$.

Proof. The equivalence between (i) and (ii) follows from Theorem 4 above and Theorem 5.3 in Ref. [17]. The rest of the assertions follow from Theorems 3.1, 3.2, and 3.4 in Ref. [17]. \square

The following theorem follows from Theorem 5 and Theorem 4.4 in Ref. [17].

Theorem 6. *The relation $=_\rho$ is an equivalence relation on $\mathcal{O}(\mathcal{H})$. In particular, it is transitive, i.e., if $X =_\rho Y$ and $Y =_\rho Z$, then $X =_\rho Z$ for all $X, Y, Z \in \mathcal{O}(\mathcal{H})$.*

The following theorem follows from Theorem 5 and Theorems 5.8 and 6.3 in Ref. [19].

Theorem 7. *Let $X_1, \dots, X_n \in \mathcal{O}(\mathcal{H})$. We have*

$$\llbracket X_1 = X_2 \wedge \dots \wedge X_{n-1} = X_n \rrbracket \leq \llbracket \text{com}(X_1, \dots, X_n) \rrbracket. \quad (10)$$

In particular, for any $\rho \in \mathcal{S}(\mathcal{H})$, if we have $X_1 =_\rho X_2, \dots, X_{n-1} =_\rho X_n$, then X_1, \dots, X_n are jointly determinate in ρ .

5 Measuring processes

A *measuring process* for \mathcal{H} is defined to be a quadruple $(\mathcal{K}, \sigma, U, M)$ consisting of a Hilbert space \mathcal{K} , a state σ on \mathcal{K} , a unitary operator U on $\mathcal{H} \otimes \mathcal{K}$, and an observable M on \mathcal{K} [11]. A measuring process $\mathbf{M}(\mathbf{x}) = (\mathcal{K}, \sigma, U, M)$ with *output variable* \mathbf{x} describes a measurement carried out by an interaction, called the *measuring interaction*, between the measured system $\mathbf{S} = \mathbf{S}(\mathcal{H})$ described by \mathcal{H} and the *probe* system $\mathbf{P} = \mathbf{S}(\mathcal{K})$ described by \mathcal{K} that is prepared in the state σ just before the measuring interaction. The unitary operator U describes the time evolution during the measuring interaction. The outcome of this measurement is obtained by measuring the observable M , called the *meter observable*, in the probe at the time just after the measuring interaction. Thus, the *output distribution*

$\Pr\{\mathbf{x} = x\|\rho\}$, the probability distribution of the output variable \mathbf{x} of this measurement on input state $\rho \in \mathcal{S}(\mathcal{H})$, is naturally defined by

$$\Pr\{\mathbf{x} = x\|\rho\} = \Pr\{U^\dagger(I \otimes M)U = x\|\rho \otimes \sigma\}. \quad (11)$$

Then, we have

$$\begin{aligned} \Pr\{\mathbf{x} = x\|\rho\} &= \Pr\{I \otimes M = x\|U(\rho \otimes \sigma)U^\dagger\} \\ &= \text{Tr}[(I \otimes E^M(x))U(\rho \otimes \sigma)U^\dagger]. \end{aligned} \quad (12)$$

Moreover, the *quantum state reduction*, the state change from the input state ρ to the state $\rho_{\{\mathbf{x}=x\}}$ just after the measurement under the condition $\mathbf{x} = x$, is given by

$$\rho \mapsto \rho_{\{\mathbf{x}=x\}} = \frac{\text{Tr}_{\mathcal{K}}[(I \otimes E^M(x))U(\rho \otimes \sigma)U^\dagger]}{\text{Tr}[(I \otimes E^M(x))U(\rho \otimes \sigma)U^\dagger]}, \quad (13)$$

where $\text{Tr}_{\mathcal{K}}$ stands for the partial trace over \mathcal{K} , provided that $\Pr\{\mathbf{x} = x\|\rho\} > 0$.

An *instrument* is a mapping \mathcal{I} from \mathbf{R} to the space $\text{CP}(\mathcal{H})$ of completely positive maps on the space $\mathcal{L}(\mathcal{H})$; for general theory of instruments see Ref. [11, 5]. Let $\mathbf{M}(\mathbf{x}) = (\mathcal{K}, \sigma, U, M)$ be a measuring process with output variable \mathbf{x} . The *instrument* of $\mathbf{M}(\mathbf{x})$ is defined by

$$\mathcal{I}(x)\rho = \text{Tr}_{\mathcal{K}}[(I \otimes E^M(x))U(\rho \otimes \sigma)U^\dagger]. \quad (14)$$

The POVM of $\mathbf{M}(\mathbf{x})$ is defined by

$$\Pi(x) = \text{Tr}_{\mathcal{K}}[U^\dagger(I \otimes E^M(x))U(I \otimes \sigma)]. \quad (15)$$

They satisfy the relations [11]

$$\Pr\{\mathbf{x} = x\|\rho\} = \text{Tr}[\mathcal{I}(x)\rho] = \text{Tr}[\Pi(x)\rho], \quad (16)$$

$$\rho_{\{\mathbf{x}=x\}} = \frac{\mathcal{I}(x)\rho}{\text{Tr}[\mathcal{I}(x)\rho]}. \quad (17)$$

The dual map T^* of $T \in \text{CP}(\mathcal{H})$ is defined by $\text{Tr}[(T^*a)\rho] = \text{Tr}[a(T\rho)]$. The POVM Π and the instrument \mathcal{I} are related by the relation [5]

$$\Pi(x) = \mathcal{I}(x)^*1, \quad (18)$$

where $\mathcal{I}(x)^*$ is the dual map of $\mathcal{I}(x)$. Conversely, it has been proved in Ref. [11] that for every POVM Π there exists an instrument \mathcal{I} satisfying Eq. (18), and for every instrument \mathcal{I} there is a measuring process $(\mathcal{K}, \sigma, U, M)$ satisfying Eq. (14). For further information on the theory of quantum measurements based on the notions of measuring processes, instruments, and POVMs, we refer the reader to Refs. [5, 4, 11, 15].

6 Measurements of observables

A measuring process $\mathbf{M}(\mathbf{x}) = (\mathcal{K}, \sigma, U, M)$ with output variable \mathbf{x} is said to be a *measurement of an observable A in a state ρ* , or said to *measure A in ρ* , if $A \otimes I$ and $U^\dagger(I \otimes M)U$ has the same value in the state $\rho \otimes \sigma$. In the measuring process $\mathbf{M}(\mathbf{x})$, the observer actually measures the meter observable just after the measuring interaction, but reports the value of this observable as the value of the observable A just before the measuring interaction; or in other words the observer actually measures $U^\dagger(I \otimes M)U$ in the state $\rho \otimes \sigma$, but reports the outcome to be the value of the observable $A \otimes I$ in the state $\rho \otimes \sigma$. Thus, this procedure is justified if and only if $A \otimes I$ and $U^\dagger(I \otimes M)U$ has the same value in the state $\rho \otimes \sigma$.

A measuring process $\mathbf{M}(\mathbf{x}) = (\mathcal{K}, \sigma, U, M)$ is said to satisfy the *Born statistical formula* (BSF) for $A \in \mathcal{O}(\mathcal{H})$ in $\rho \in \mathcal{S}(\mathcal{H})$ if it satisfies $\Pr\{\mathbf{x} = x | \rho\} = \text{Tr}[E^A(x)\rho]$ for all $x \in \mathbf{R}$. A POVM Π is said to be *identically correlated* with an observable A in ρ if $\text{Tr}[\Pi(x)E^A(y)\rho] = 0$ for any $x, y \in \mathbf{R}$ with $x \neq y$.

The following theorem characterizes measurements of an observable in a given state.

Theorem 8. *Let $\mathbf{M}(\mathbf{x}) = (\mathcal{K}, \sigma, U, M)$ be a measuring process for \mathcal{H} . Let Π be the POVM of $\mathbf{M}(\mathbf{x})$. For any observable A and any state ρ , the following conditions are all equivalent.*

- (i) $\mathbf{M}(\mathbf{x})$ is a measurement of A in ρ .
- (ii) $\mathbf{M}(\mathbf{x})$ satisfies the BSF for A in any $\psi \in \mathcal{C}(A, \rho)$.
- (iii) Π is identically correlated with A in ρ .

Proof. The assertion follows from Theorem 8.2 of Ref. [17]. \square

Theorem 9. *A measuring process $\mathbf{M}(\mathbf{x}) = (\mathcal{K}, \sigma, U, M)$ for \mathcal{H} is a measurement of an observable $A \in \mathcal{O}(\mathcal{H})$ in any state if and only if its POVM Π coincides with the spectral measure of A , i.e., $\Pi = E^A$.*

Proof. The assertion follows immediately from the fact that Π is identically correlated with A in any state if and only if $\Pi(x) = E^A(x)$ for all $x \in \mathbf{R}$. \square

7 Simultaneous measurability

For any measuring process $\mathbf{M}(\mathbf{x}) = (\mathcal{K}, \sigma, U, M)$ for \mathcal{H} with output variable \mathbf{x} and a real function f , the measuring process $\mathbf{M}(f(\mathbf{x}))$ with output variable $f(\mathbf{x})$ is defined by $\mathbf{M}(f(\mathbf{x})) = (\mathcal{K}, \sigma, U, f(M))$. Observables A_1, \dots, A_n are said to be *simultaneously measurable* in a state $\rho \in \mathcal{S}(\mathcal{H})$ by $\mathbf{M}(\mathbf{x})$ if there are real functions f_1, \dots, f_n such that $\mathbf{M}(f_j(\mathbf{x}))$ measures A_j in ρ for $j = 1, \dots, n$. Observables A_1, \dots, A_n are said to be *simultaneously measurable* in ρ if there is a measuring process $\mathbf{M}(\mathbf{x})$ such that A_1, \dots, A_n are simultaneously measurable in ρ by $\mathbf{M}(\mathbf{x})$.

The cyclic subspace $\mathcal{C}(A, B, \rho)$ generated by A, B, ρ is defined by

$$\begin{aligned} \mathcal{C}(A, B, \rho) &= \{p(A, B)\psi \mid p(A, B) \text{ is a polynomial in } A, B \text{ and } \psi \in \mathcal{R}(\rho)\}^{\perp\perp}. \end{aligned} \quad (19)$$

The simultaneous measurability and the commutativity are not equivalent notion under the state-dependent formulation, as the following theorem clarifies.

Theorem 10. (i) Two observables $A, B \in \mathcal{O}(\mathcal{H})$ are jointly determinate in a state $\rho \in \mathcal{S}(\mathcal{H})$ if and only if there is an \mathbf{R}^2 -valued POVM Π such that

$$\sum_y \Pi(x, y) = E^A(x) \quad \text{on } \mathcal{C}(A, B, \rho), \quad (20)$$

$$\sum_x \Pi(x, y) = E^B(y) \quad \text{on } \mathcal{C}(A, B, \rho). \quad (21)$$

(ii) Two observables $A, B \in \mathcal{O}(\mathcal{H})$ are simultaneously measurable in a state $\rho \in \mathcal{S}(\mathcal{H})$ if and only if there is an \mathbf{R}^2 -valued POVM Π such that

$$\sum_y \Pi(x, y) = E^A(x) \quad \text{on } \mathcal{C}(A, \rho), \quad (22)$$

$$\sum_x \Pi(x, y) = E^B(y) \quad \text{on } \mathcal{C}(B, \rho). \quad (23)$$

(iii) Two observables on \mathcal{H} are simultaneously measurable in a state $\rho \in \mathcal{S}(\mathcal{H})$ if they are jointly determinate in ρ .

Proof. Suppose that there is an \mathbf{R}^2 -valued POVM Π satisfying Eqs. (20) and (21). Let $P = \mathcal{P}(\mathcal{C}(A, B, \rho))$. Let Π' be such that $\Pi'(x, y) = P\Pi(x, y)P$. Then, marginals of Π' are projection-valued measures $E^A(x)P$ and $E^B(y)P$, so that from a well-know theorem, e.g., Theorem 3.2.1 of Ref. [4], the marginals commute and Π' is the product of the marginals. Thus, we have

$$\begin{aligned} P\Pi(x, y)P &= \Pi'(x, y) = E^A(x)PE^B(y)P = (E^A(x)P \wedge E^B(y)P) \\ &= (E^A(x) \wedge E^B(y))P \end{aligned}$$

Let $\mu(x, y) = \text{Tr}[\Pi(x, y)\rho]$. Since Π is a POVM, μ is a probability distribution. Since $P\rho = \rho P = \rho$, we have $\mu(x, y) = \text{Tr}[(E^A(x) \wedge E^B(y))\rho]$, so that A and B has the joint probability distribution in ρ and they are jointly determinate. Conversely, suppose that A and B are jointly determinate in a state ρ . Then, $\mathcal{R}\rho \subseteq \mathcal{R}(\llbracket \text{com}(A, B) \rrbracket)$ and $\mathcal{R}(\llbracket \text{com}(A, B) \rrbracket)$ is both A -invariant and B -invariant, so that we have $\mathcal{C}(A, B, \rho) \subseteq \mathcal{R}(\llbracket \text{com}(A, B) \rrbracket)$. Then, we have $AB = BA$ on $\mathcal{C}(A, B, \rho)$ and $\Pi'(x, y) = (E^A(x) \wedge E^B(y))P$ can be extended to a POVM Π satisfying Eqs. (20) and (21). Thus, statement (i) follows. Suppose that $A, B \in \mathcal{O}(\mathcal{H})$ are simultaneously measurable in $\rho \in \mathcal{S}(\mathcal{H})$. Then, we have a measuring process $\mathbf{M}(\mathbf{x}) = (\mathcal{K}, \sigma, U, M)$ and real functions f, g such that $\mathbf{M}(f(\mathbf{x}))$ measures A in ρ and $\mathbf{M}(g(\mathbf{x}))$ measures B in ρ . Let $\Pi_0(x)$ be the POVM of $\mathbf{M}(\mathbf{x})$. Let $\Pi(x, y) = \sum_{x': (x, y) = (f(x'), g(x'))} \Pi_0(x')$. Then, it is easy to see that Π satisfies Eqs. (22) and (23). Conversely, suppose that there is an \mathbf{R}^2 -valued POVM Π satisfying Eqs. (22) and (23). Then, from the realization theorem of instruments and POVMs, Theorem 5.1 of Ref. [11], we have a measuring process $\mathbf{M}(\mathbf{x}) = (\mathcal{K}, \sigma, U, M)$ and real functions f, g such that

$$\Pi(x, y) = \text{Tr}_{\mathcal{K}}[U^\dagger(I \otimes \sum_{x': (x, y) = (f(x'), g(x'))} E^M(x'))U(I \otimes \sigma)]. \quad (24)$$

Then, we have

$$\sum_y \Pi(x, y) = \text{Tr}_{\mathcal{K}}[U^\dagger(I \otimes E^{f(M)}(x))U(I \otimes \sigma)], \quad (25)$$

so that from Eq. (22) we have $\mathbf{M}(f(\mathbf{x}))$ measures A in ρ . Similarly, we can show that $\mathbf{M}(g(\mathbf{x}))$ measures B in ρ . Thus, statement (ii) follows. Now, statement (iii) follows (i) and (ii) \square

The conventional relation between simultaneous measurability and commutativity, or joint determinateness, in the state-independent formulation is recovered in our quantum logical approach as follows.

Theorem 11. *For any observables $A, B \in \mathcal{O}(\mathcal{H})$, the following conditions are all equivalent.*

- (i) A and B are jointly determinate in any state $\rho \in \mathcal{S}(\mathcal{H})$.
- (ii) A and B are simultaneously measurable in any state $\rho \in \mathcal{S}(\mathcal{H})$.
- (iii) A and B commute on \mathcal{H} .

Proof. From Theorem 10 conditions (i) and (ii) are both equivalent to that there is an \mathbf{R}^2 -valued POVM Π such that their marginals are equal to E^A and E^B . This condition is well-known to be equivalent to condition (iii); see for example Theorem 3.2.1 of Ref. [4]. \square

An observable is said to be *non-degenerate* if every eigenvalue has one-dimensional eigenspace. In the case where $\dim(\mathcal{H}) = 2$, every observable is non-degenerate or scalar.

Theorem 12. *Suppose that $\dim(\mathcal{H}) = 2$. For any non-degenerate observables $A, B \in \mathcal{O}(\mathcal{H})$ and any state $\rho \in \mathcal{S}(\mathcal{H})$ that is not an eigenstate of A or B , the following conditions are all equivalent.*

- (i) A and B are jointly determinate in ρ .
- (ii) A and B are simultaneously measurable in ρ .
- (iii) A and B commute on \mathcal{H} .

Proof. In this case, we have $\mathcal{C}(A, \rho) = \mathcal{C}(B, \rho) = \mathcal{C}(A, B, \rho) = \mathcal{H}$, and hence the assertion follows easily. \square

The following theorems show that we can simultaneously measure two nowhere commuting observables.

Theorem 13. *In any Hilbert space, every pair of observables are simultaneously measurable in any eigenstate of either observable.*

Proof. Let $A, B \in \mathcal{O}(\mathcal{H})$. Suppose that we have $A\psi = a\psi$ and $\|\psi\| = 1$ with $\psi \in \mathcal{H}$ and $a \in \mathbf{R}$. Then, $\Pi(x, y) = \delta_{x,a}|\psi\rangle\langle\psi|E^B(y)$ satisfies Eqs. (22) and (23), so that A and B are simultaneously measurable in ψ . \square

From Theorems 12 and 13, we can characterize all pairs of simultaneously measurable observables in the case where $\dim(\mathcal{H}) = 2$, as follows. If A and B commute, then they are simultaneously measurable in every state. If A and B do not commute, then they are non-degenerate, and hence simultaneously measurable if and only if the state is an eigenstate of A or B .

Theorem 14. *In any Hilbert space with dimension more than 3, there are nowhere commuting observables that are simultaneously measurable in a state that is not an eigenstate of either observable.*

Proof. Let \mathcal{H} be a Hilbert space with $\dim(\mathcal{H}) = n > 3$. First we suppose $n = 4$. In this case we can assume without any loss of generality that \mathcal{H} is a Hilbert space of a pair of spin 1/2 particles, i.e., $\mathcal{H} = \mathbf{C}^2 \otimes \mathbf{C}^2$. Obviously, the observables $A = \sigma_z \otimes I$ and $B = \sigma_x \otimes I$ on \mathcal{H} are nowhere commuting. Let

$$\psi = \frac{1}{\sqrt{2}}(|\sigma_x = +1\rangle|\sigma_x = +1\rangle + |\sigma_x = -1\rangle|\sigma_x = -1\rangle). \quad (26)$$

Let $C = I \otimes \sigma_x$. Let $\Pi(x, y) = E^{\sigma_z}(x) \otimes E^{\sigma_x}(y)$. Then $\Pi(x, y)$ is a \mathbf{R}^2 -valued POVM satisfying

$$\sum_y \Pi(x, y) = E^A(x), \quad (27)$$

$$\sum_x \Pi(x, y) = E^C(y). \quad (28)$$

From Eq. (26) we have $C =_\psi B$, so that $E^C(y) = E^B(y)$ on $\mathcal{C}(B, \psi)$. By the transitivity of the relation $=_\psi$, we have

$$\sum_x \Pi(x, y) = E^B(y) \quad (29)$$

on $\mathcal{C}(B, \psi)$. Thus, A and B are simultaneously measurable. In the general case, we can assume that the space $\mathbf{C}^2 \otimes \mathbf{C}^2$ is a subspace of \mathcal{H} . Then, it is easy to see that A and B are extended to two nowhere commuting observables on \mathcal{H} and that they are simultaneously measurable in the state ψ . \square

8 Discussions and conclusions

We have considered the language for propositional logic constructed from atomic formulas of the form $A = a$, where A denotes an observable of a quantum system described by a Hilbert space \mathcal{H} and a denotes a real number. Every sentence in this language is called an observational proposition for \mathcal{H} . To every observational proposition ϕ we have assigned the $\mathcal{Q}(\mathcal{H})$ -valued truth value $\llbracket \phi \rrbracket$, where $\mathcal{Q}(\mathcal{H})$ is the lattice of projections on \mathcal{H} . It is easy to check that this assignment of observational propositions to projections on \mathcal{H} , or equivalently to subspaces of \mathcal{H} , is equivalent to the assignment first proposed by Birkhoff and von Neumann [2]. Then, for any state ρ we have assigned to every observational proposition ϕ the probability $\Pr\{\phi \parallel \rho\}$, which is consistent with the Born statistical formula. We say that an observational proposition ϕ holds in a state ρ if $\Pr\{\phi \parallel \rho\} = 1$. In the case where ρ is a vector state $\rho = |\psi\rangle\langle\psi|$, $\Pr\{\phi \parallel \rho\}$ is the squared length of the vector $\llbracket \phi \rrbracket \psi$, and the observational proposition ϕ holds in the state ρ if and only $\psi \in \mathcal{R}(\llbracket \phi \rrbracket)$.

Our language of observational propositions obviously commits the notion of the values of observables. Our language gives a systematic way to determine which sentences on the values of observables are true, partially true, or false based on quantum logic. The notion of the values of observables has been known to involve a difficulty suggested by the Kochen-Specker theorem [9], if we treat this notion in classical logic. A key notion in our language that circumvents this difficulty is the notion of joint determinateness. This

notion determines the states in which given observables A_1, \dots, A_n have simultaneous values as follows. We have introduced an observational proposition $\text{com}(A_1, \dots, A_n)$ defined by Eq. (3), and we say that A_1, \dots, A_n are jointly determinate in state ρ if $\text{com}(A_1, \dots, A_n)$ holds in ρ . This condition for a vector state $\rho = |\psi\rangle\langle\psi|$ is equivalent to that ψ is a superposition of common eigenvectors of A_1, \dots, A_n . Since this proposition is a built-in proposition in the sense that it is constructed from atomic formulas, quantum logic can determine the limitation for the notion of values of observables in its own right.

If our language could be interpreted in classical logic, we would have sufficiently many two-valued truth-value assignments for all the sentences in the language, and each assignment would give any observable A only one real number a such that $A = a$ is true, so that A is assigned the value a ; and then in any state the probability of the observational proposition would be interpreted as the ignorance as to which truth-value is actually assigned.

What the Kochen-Specker theorem denies is the existence of such a two-valued assignment. Instead, quantum logic allows contextual two-valued assignments in the sense that if A_1, \dots, A_n are jointly determinate in a state ρ , then we have sufficiently many two-valued assignment for all sentences in the sublanguage constructed by atomic formulas of the form $A_1 = a_1, \dots$, or $A = a_n$, where a_1, \dots, a_n denote arbitrary real numbers; and then the probability of an observational proposition in this sublanguage in the state ρ allows ignorance interpretation. The observational propositions in this sublanguage are called contextually well-formed for A_1, \dots, A_n in ρ . Thus, in quantum logic the probability assignment is non-contextual but the two-valued assignment is contextual. This corresponds to the fact that the two-valued assignment commits a result of a single measurement, but the probability assignment commits only the statistics of results of many measurements.

One of our purpose of developing the language of observational propositions is to extract another built-in proposition $A = B$ defined by Eq. (7) meaning that the observable A and the observable B has the same value. We say that A and B has the same value in state ρ if $A = B$ holds in ρ . This condition for a vector state $\rho = |\psi\rangle\langle\psi|$ is equivalent to that ψ is a superposition of common eigenvectors of A and B belonging to the same eigenvalues; in general ρ is a mixture of such vector states. As naturally expected, if $A = B$ holds in ρ then A and B are jointly determinate in ρ . It is easy to see that if $A = B$ holds in ρ , we can simultaneously measure A and B in ρ , and each measurement gives the same measured value. Moreover, in this case, we can simultaneously measure A and any polynomial $f(A, B)$, so that if we obtain the measurement outcome $A = a$ then we obtain the measurement outcome $f(A, B) = f(a, a)$.

Now, we are in a position to discuss the relation between quantum reality and measurement. Every measurement is statistically equivalent to a model describing a physical interaction between the measured object and the probe followed by a subsequent measurement of the meter observable in the probe. In this model, the measurement of the observable $A(0)$ in the state ρ is replaced by the measurement of the meter observable $M(\Delta t)$ after the interaction.

In the conventional approach, the measurement of $A(0)$ is considered to be correct if and only if $A(0)$ and $M(\Delta t)$ have the same probability distribution in the initial state $\rho \otimes \sigma$ for an arbitrary ρ and a fixed σ , or equivalently the POVM of the measurement

coincides with the spectral measure of the measured observable. However, then the question arises about the status of the measured value: how the measured value commits the reality of the measured system just before the measurement. A natural requirement is that the measured value obtained from the $M(\Delta t)$ measurement reproduce the value of $A(0)$, but this requirement has a difficulty due to the Kochen-Specker theorem, which prohibits a context-free assignment of the values of observables. Our approach circumvents this difficulty using quantum logic. In order to ensure that the given measuring process reproduces the value of measured observable before the interaction, we do not need to assume the context-free assignment. Instead, we can justify it by our well-founded statement that $A(0)$ and $M(\Delta t)$ have the same value in the state $\rho \otimes \sigma$. In that, we do not need a context-free value assignment, but it is only required that there should be a context in which the values of $A(0)$ and $M(\Delta t)$ are jointly assigned and they are the same. We call this value the outcome obtained by the measurement. Actually, this approach justifies the conventional definition of correct measurements of an observable as follows: the measuring process reproduces the probability distribution in any state ρ if and only if the measuring process also reproduces the value of the measured observable in any state ρ . Thus, we can speak of the value of observables in a context-free language based on quantum logic, and this language consistently implies the reality of the measurement outcome in the contextual language based on classical logic.

A major significance of our approach is that it gives us a criterion, which does not exist in the conventional approach, on what is the correct measurement of an observable in a given state. As mentioned above, in the conventional approach the correct measurement is characterized by the probability reproducibility in arbitrary states, but the probability reproducibility cannot be used as the criterion on the correct measurement in a given state, as is obvious even in the classical case. However, our new criterion applies to the measurement in a given state stating that the measurement of $A(0)$ in a given state ρ is correct if and only if $A(0)$ and $M(\Delta t)$ have the same value in the initial state $\rho \otimes \sigma$.

Another feature of our approach is to enable us to consider the state-dependent notion of simultaneous measurability. The state-independent notion of simultaneous measurability is known to be equivalent with the commutativity. However, the state-dependent notion of simultaneous measurability has not been given a right place in quantum mechanics, although a few has been considered as pathological exceptions of the uncertainty principle.

It has been claimed for long that if at time 0 the object is prepared in an eigenstate of $A(0)$ and the observer actually measures the value of another observable $B(0)$ at time 0, then the observer can know both the values of two observables $A(0)$ and $B(0)$, even though they are nowhere commuting. Heisenberg discussed this case in his book [8] published in 1930. His reluctance to accept this case as a simultaneous measurement is mainly due to the fact that this does not leave the system in a joint eigenstate of $A(0)$ and $B(0)$. However, the notions of state preparation and measurement should have been clearly distinguished. In fact, it is widely accepted nowadays that any observable can be measured correctly without leaving the object in an eigenstate of the measured observable; for instance, a projection E can be correctly measured in a state ψ with the outcome being 1 leaving the object in the state $M\psi/\|M\psi\|$, where the operator M depends on the apparatus and satisfies $E = M^\dagger M$ (see, for example, a widely accepted

text book by Nielsen and Chuang [10]).

On the other hand, one of the variations of the Einstein-Podolsky-Rosen (EPR) argument [6] runs as follows. In the EPR state of two particles, I and II, the momentum of particle I can be measured by directly and locally measuring the momentum of particle II taking into account the EPR correlation; this follows from the EPR original argument stating that the locality of measurement ensures that the predicted correlation determines the value of momentum of particle I. The locality of the momentum measurement of particle II also concludes that it does not disturb the particle I, and hence we can simultaneously measure the position of particle I by a direct measurement on particle I. Thus, the momentum and position of particle I are simultaneously measurable, so that both the measured values corresponds to elements of reality. However, quantum mechanics has no state to describe those results, and hence it should be incomplete.

In the conventional approach, we cannot discuss those cases in the light of general theory of quantum measurement, since those simultaneous measurements are state-dependent. In our approach, we have provided a general theory of state-dependent simultaneous measurements, and actually the above two cases are two special cases of simultaneous measurements characterized by our rigorous definition.

According to our theory, the simultaneous measurement of $A(0)$ and $B(0)$ by the meter $M(\Delta t)$ is defined by the following two conditions:

- (i) $A(0) = f(M(\Delta t))$ holds in the state $\rho \otimes \sigma$.
- (ii) $B(0) = g(M(\Delta t))$ holds in the state $\rho \otimes \sigma$.

From (i) we can conclude that $A(0)$ and $f(M(\Delta t))$ are jointly determinate in $\rho \otimes \sigma$. From (ii) the same is true for $B(0)$ and $g(M(\Delta t))$. However, this does not imply that $A(0)$ and $B(0)$ are jointly determinate in $\rho \otimes \sigma$. Thus, the simultaneous measurability of $A(0)$ and $B(0)$ does not ensure that the two outcomes from the simultaneous measurements has simultaneous reality. This is because of the contextuality of the two defining conditions of simultaneous measurement. Condition (i) ensures $A(0) = f(M(\Delta t))$ is contextually well-formed in $\rho \otimes \sigma$, and condition (ii) ensures the same is true for $B(0) = g(M(\Delta t))$. The statement “ $A(0) = f(M(\Delta t))$ and $B(0) = g(M(\Delta t))$ ” holds in $\rho \otimes \sigma$, but this is not contextually well-formed unless $A(0)$ and $B(0)$ are jointly determinate in $\rho \otimes \sigma$. Thus, if the apparatus has made a simultaneous measurements of nowhere commuting observables $A(0)$ and $B(0)$ and obtained the outcome $A(0) = a$ and $B(0) = b$, we can use the fact $A(0) = a$ in one of the context including $A(0)$ as the reality of the measured object subject to classical logic and the same is true for $B(0) = b$, but we have no right to use both $A(0) = a$ and $B(0) = b$ as elements of the unified reality.

In conclusion, quantum logic sheds a unique light on two facets of quantum mechanics. The quantum state is defined as a probability measure on the lattice of observational propositions. Then, the projection-valued truth-value assignment and the probability assignment, consistent with the Born probabilistic interpretation, are non-contextual, and explain our experience about the statistics of results of many measurements. This aspect of quantum mechanics has been emphasized as the statistical interpretation typically formulated by Ballentine [1]. On the other hand, the two-valued assignment to the observational propositions is contextual, and explains our experience about the results of single measurements together with theoretical predictions in a context of sublanguage of observational propositions subject to classical logic. This aspect of quantum mechanics

has explained how we should apply our intuition about physical reality stemmed from classical physics to quantum mechanical objects, and was typically stressed by Bohr's complementarity principle [3].

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