Reforming Takeuti's Quantum Set Theory to Satisfy De Morgan's Laws

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Abstract. In 1981, Takeuti introduced set theory based on quantum logic by constructing a model analogous to Boolean-valued models for Boolean logic. He defined the quantum logical truth value for every sentence of set theory. He showed that equality axioms do not hold, while axioms of ZFC set theory hold if appropriately modified with the notion of commutators. Here, we consider the problem in Takeuti's quantum set theory that De Morgan's laws do not hold for bounded quantifiers. We construct a counter-example to De Morgan's laws for bounded quantifiers in Takeuti's quantum set theory. We redefine the truth value for the membership relation and bounded existential quantification to ensure that De Morgan's laws hold. Then, we show that the truth value of every theorem of ZFC set theory is lower bounded by the commutator of constants therein as quantum transfer principle.

Keywords: quantum logic, commutators, quantum set theory, De Morgan's laws, transfer principle

1 Introduction

Since quantum logic is an intrinsic logic governing observational propositions of quantum mechanics, it is an intriguing problem to develop mathematics based on quantum logic. In 1981, Takeuti [15] introduced quantum set theory for this purpose. As a start, he constructed a model of set theory based on quantum logic represented by the complete orthomodular lattice of projections on a Hilbert space, which is isomorphic to the lattice of closed subspaces in the Hilbert space. He defined the truth values for all sentences of set theory on the model assuming the Sasaki arrow for implication. In order to make quantum counter part of ZFC axioms, he introduced the notion of commutator in quantum logic, and he showed that the axioms of ZFC hold in quantum set theory if appropriately modified by commutators of elements of the model, while equality axioms do not generally hold in quantum set theory. He showed that the real numbers in the model correspond to the observables of the system to be described.

Following Takeuti's work, we explored the question how theorems of ZFC hold in quantum set theory [9]. We showed that every theorem of ZFC holds in quantum set theory with truth value greater than or equal to the commutator of elements of the model appearing therein. This result was extended to general complete orthomodular lattices and to a general class of operations for implication in Ref. [12]. Quantum set theory was effectively applied to quantum mechanics to extend the probabilistic interpretation from observational propositions to relations between observables [10, 11].

In this paper, we consider the problem in Takeuti's quantum set theory that De Morgan's laws do not hold for bounded quantifiers. Let $\mathcal{H}$ be a Hilbert space. The quantum logic $\mathcal{Q}$ on $\mathcal{H}$ is represented by the lattice of projections on $\mathcal{H}$, which is a complete orthomodular lattice, called the quantum logic on $\mathcal{H}$. The classical definition of implication, $P \to Q = P^\perp \vee Q$, does not work since the relation $P \to Q = 1$ and the order relation $P \leq Q$ are not equivalent, so that the implication in quantum logic is, according to the majority view [16], defined as the Sasaki arrow $\to$, a binary operation of $\mathcal{Q}$ defined by $P \to Q = P^\perp \vee (P \wedge Q)$.

Takeuti [15], applying the method of Boolean-valued models to quantum logic $\mathcal{Q}$, constructed the model $V^{(\mathcal{Q})}$ of quantum set theory. He defined the $\mathcal{Q}$-valued truth value $[\![\phi]\!]$ of a sentence ϕ in the language of set theory.

In particular, the truth values of bounded quantifications are directly defined as follows.

(1) $[\![(\forall x \in u)\,\phi(x)]\!] = \bigwedge_{u' \in \mathrm{dom}(u)}(u(u') \to [\![\phi(u')]\!])$.

(2) $[\![(\exists x \in u)\,\phi(x)]\!] = \bigvee_{u' \in \mathrm{dom}(u)}(u(u') \wedge [\![\phi(u')]\!])$.

Takeuti noted "In Boolean valued universes, $[\![(\forall x \in u)\phi(x)]\!] = [\![\forall x(x \in u \to \phi(x)]\!]$ and $[\![(\exists x \in u)\phi(x)]\!] = [\![\exists x(x \in u \wedge \phi(x)]\!]$. But this is not the case for $V^{(\mathcal{Q})}$" [15, p. 315]. However, it is problematic that he avoids the classical definition of implication $P \to Q = P^\perp \vee Q$ in the bounded universal quantification, whereas he still uses the classical definition of conjunction in the bounded existential quantification. Since the relation $P \wedge Q = (P \to Q^\perp)^\perp$ does not hold for the conjunction $\wedge$ and the Sasaki arrow $\to$, so that De Morgan's laws,

(3) $[\![\neg(\forall x \in u)\,\phi(x)]\!] = [\![(\exists x \in u)\,\neg\phi(x)]\!]$,

(4) $[\![\neg(\exists x \in u)\,\phi(x)]\!] = [\![(\forall x \in u)\,\neg\phi(x)]\!]$,

do not hold. In fact, if $\mathcal{Q}$ is not a Boolean algebra, we can construct a predicate $\phi(x)$ such that $[\![(\exists x \in u)\,\neg\phi(x)]\!] = 0$ but $[\![\neg(\forall x \in u)\,\phi(x)]\!] > 0$.

In this paper, we introduce a new binary operation $*$ by $P * Q = (P \to Q^\perp)^\perp$ and redefine the truth values of membership relation and bounded existential quantification as follows.

(5) $[\![u \in v]\!] = \bigvee_{v' \in \mathrm{dom}(v)}(v(v') * [\![v' = u]\!])$.

(6) $[\![(\exists x \in u)\,\phi(x)]\!] = \bigvee_{u' \in \mathrm{dom}(u)}(u(u') * [\![\phi(u')]\!])$.

Then, De Morgan's laws hold for bounded universal quantification and bounded existential quantification. Thus, for the language of quantum set theory we can assume only negation, conjunction, and bounded and unbounded universal quantification as primitive, while disjunction, bounded and unbounded existential quantification are considered to be introduced by definition.

The operator $*$ was found by Sasaki [14], and has been studied as the Sasaki projection in connection with residuation theory, whereas the operation $*$ has not been used for defining bounded quantifiers in quantum logic. Its intuitive meaning and significance will be discussed elsewhere.

We consider the commutator $\underline{\vee}(u_1, \ldots, u_n) \in \mathcal{Q}$ of elements $u_1, \ldots, u_n$ of $V^{(\mathcal{Q})}$ in order to explore how theorems of ZFC hold in the new interpretation for the model $V^{(\mathcal{Q})}$. Then the following quantum transfer principle holds: If a Δ_0-formula $\phi(x_1, \ldots, x_n)$ of the language $\mathcal{L}(\in)$ of set theory is provable in ZFC, for every $u_1, \ldots, u_n \in V^{(\mathcal{Q})}$ we have

$$\llbracket \phi(u_1, \ldots, u_n) \rrbracket \geq \underline{\vee}(u_1, \ldots, u_n).$$

This paper is organized as follows. Section 2 provides preliminaries on quantum logic, commutators, and conditionals. Section 3 introduces the model $V^{(\mathcal{Q})}$ and its interpretation that satisfies De Morgan's laws. We also discuss Takeuti's interpretation and construct the above mentioned counterexample. Section 4 derives the quantum transfer principle for the new interpretation.

2 Preliminaries

2.1 Quantum logic

Let $\mathcal{H}$ be a Hilbert space. For any subset $S \subseteq \mathcal{H}$, we denote by $S^{\perp}$ the orthogonal complement of S. Then, $S^{\perp\perp}$ is the closed linear span of S. Let $\mathcal{C}(\mathcal{H})$ be the set of all closed linear subspaces in $\mathcal{H}$. With the set inclusion ordering, the set $\mathcal{C}(\mathcal{H})$ is a complete lattice. The operation $M \mapsto M^{\perp}$ is an orthocomplementation on the lattice $\mathcal{C}(\mathcal{H})$, with which $\mathcal{C}(\mathcal{H})$ is a complete orthomodular lattices [7, p. 65], *i.e.*, the orthocomplementation satisfies

   (C1) if $P \leq Q$ then $Q^{\perp} \leq P^{\perp}$,
   (C2) $P^{\perp\perp} = P$,
   (C3) $P \vee P^{\perp} = 1$ and $P \wedge P^{\perp} = 0$,
where $0 = \bigwedge \mathcal{Q}$ and $1 = \bigvee \mathcal{Q}$, and the *orthomodular law*
   (OM) if $P \leq Q$ then $P \vee (P^{\perp} \wedge Q) = Q$.

We refer the reader to Kalmbach [7] for a standard textbook on orthomodular lattices.

Denote by $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$ and $\mathcal{Q}(\mathcal{H})$ the set of projections on $\mathcal{H}$. We define the *operator ordering* on $\mathcal{B}(\mathcal{H})$ by $A \leq B$ iff $(\psi, A\psi) \leq (\psi, B\psi)$ for all $\psi \in \mathcal{H}$. For any $A \in \mathcal{B}(\mathcal{H})$, denote by $\mathcal{R}(A) \in \mathcal{C}(\mathcal{H})$ the closure of the range of A, *i.e.*, $\mathcal{R}(A) = (A\mathcal{H})^{\perp\perp}$. For any $M \in \mathcal{C}(\mathcal{H})$, denote by $\mathcal{P}(M) \in \mathcal{Q}(\mathcal{H})$ the projection of $\mathcal{H}$ onto M. Then, $\mathcal{R}\mathcal{P}(M) = M$ for all $M \in \mathcal{C}(\mathcal{H})$ and $\mathcal{P}\mathcal{R}(P) = P$ for all $P \in \mathcal{Q}(\mathcal{H})$, and we have $P \leq Q$ if and only if $\mathcal{R}(P) \subseteq \mathcal{R}(Q)$ for all $P, Q \in \mathcal{Q}(\mathcal{H})$, so that $\mathcal{Q}(\mathcal{H})$ with the operator ordering is also a complete orthomodular lattice isomorphic to $\mathcal{C}(\mathcal{H})$. We consider $\mathcal{Q}(\mathcal{H})$ as the *standard quantum logic* of $\mathcal{H}$, or the logic of observational propositions in quantum mechanics for the system described by $\mathcal{H}$ [2,6]. The lattice operations

are characterized by $P \wedge Q = \text{weak-lim}_{n \to \infty}(PQ)^n$, $P^\perp = 1 - P$ for all $P, Q \in \mathcal{Q}(\mathcal{H})$.

A non-empty subset of $\mathcal{Q}(\mathcal{H})$ is called a *subalgebra* iff it is closed under $\wedge$, $\vee$, and $\perp$. A subalgebra $\mathcal{A}$ of $\mathcal{Q}(\mathcal{H})$ is said to be *complete* iff it has the supremum and the infimum in $\mathcal{Q}(\mathcal{H})$ of an arbitrary subset of $\mathcal{A}$.

Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$. We denote by $\mathcal{A}'$ the *commutant of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$*, *i.e.*,

$$\mathcal{A}' = \{A \in \mathcal{B}(\mathcal{H}) \mid AB = BA \text{ for any } B \in \mathcal{A}\}.$$

A self-adjoint subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$ is called a *von Neumann algebra* on $\mathcal{H}$ iff $\mathcal{M}'' = \mathcal{M}$. For any self-adjoint subset $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, $\mathcal{A}''$ is the von Neumann algebra generated by $\mathcal{A}$. We denote by $\mathcal{P}(\mathcal{M})$ the set of projections in a von Neumann algebra $\mathcal{M}$.

We say that P and Q in $\mathcal{Q}(\mathcal{H})$ *commute*, in symbols $P \,\substack{|\\\circ}\, Q$, iff $P = (P \wedge Q) \vee (P \wedge Q^\perp)$. For any $P, Q \in \mathcal{Q}(\mathcal{H})$, we have $P \,\substack{|\\\circ}\, Q$ iff $[P, Q] = 0$, where $[P, Q] = PQ - QP$.

For any subset $\mathcal{A} \subseteq \mathcal{Q}(\mathcal{H})$, the *commutant* $\mathcal{A}^!$ of $\mathcal{A}$ in $\mathcal{Q}(\mathcal{H})$ [7, p. 23] is defined by

$$\mathcal{A}^! = \{P \in \mathcal{Q}(\mathcal{H}) \mid P \,\substack{|\\\circ}\, Q \text{ for all } Q \in \mathcal{A}\}. \tag{1}$$

Then, $\mathcal{A}^!$ is a complete subalgebra of $\mathcal{Q}(\mathcal{H})$. A *sublogic* of $\mathcal{Q}(\mathcal{H})$ is a subset $\mathcal{A}$ of $\mathcal{Q}(\mathcal{H})$ satisfying $\mathcal{A} = \mathcal{A}^{!!}$. Any sublogic of $\mathcal{Q}(\mathcal{H})$ will be called a *logic* on $\mathcal{H}$. For any subset $\mathcal{A}$ of a logic $\mathcal{Q}$, the smallest logic including $\mathcal{A}$ is $\mathcal{A}^{!!}$ called the *logic generated by $\mathcal{A}$*. Then, a subset $\mathcal{Q} \subseteq \mathcal{Q}(\mathcal{H})$ is a logic on $\mathcal{H}$ if and only if $\mathcal{Q} = \mathcal{P}(\mathcal{M})$ for some von Neumann algebra $\mathcal{M}$ on $\mathcal{H}$ [9, Proposition 2.1]. A logic $\mathcal{Q}$ on $\mathcal{H}$ is a Boolean algebra if and only if $P \,\substack{|\\\circ}\, Q$ for all $P, Q \in \mathcal{Q}$ [7, pp. 24–25]

The center of a logic $\mathcal{Q}$, denoted by $Z(\mathcal{Q})$, is the set of elements of $\mathcal{Q}$ commute with every element of $\mathcal{Q}$, *i.e.*, $Z(\mathcal{Q}) = \mathcal{Q}^! \cap \mathcal{Q}$. Then, it is easy to see that a subset $\mathcal{A}$ is a Boolean sublogic, or equivalently a distributive sublogic, if and only if $\mathcal{A} = \mathcal{A}^{!!} \subseteq \mathcal{A}^!$. The center of $\mathcal{A}^{!!}$ is given by $Z(\mathcal{A}^{!!}) = \mathcal{A}^! \cap \mathcal{A}^{!!}$.

2.2 Commutators

Marsden [8] introduced the commutator $\perp\!\!\!\perp(P, Q)$ of two elements P and Q of an orthomodular lattice $\mathcal{Q}$ by

$$\perp\!\!\!\perp(P, Q) = (P \wedge Q) \vee (P \wedge Q^\perp) \vee (P^\perp \wedge Q) \vee (P^\perp \wedge Q^\perp). \tag{2}$$

Bruns and Kalmbach [3] generalized this notion to finite subsets of $\mathcal{Q}$ by

$$\perp\!\!\!\perp(\mathcal{F}) = \bigvee_{\theta:\mathcal{F} \to \{\text{id}, \perp\}} \bigwedge_{P \in \mathcal{F}} P^{\theta(P)} \tag{3}$$

for any finite subsets $\mathcal{F}$ of $\mathcal{Q}$, where $\{\text{id}, \perp\}$ stands for the set consisting of the identity operation id and the orthocomplementation $\perp$. Generalizing this notion to arbitrary subsets $\mathcal{A}$ of $\mathcal{Q}(\mathcal{H})$, Takeuti [15] defined $\perp\!\!\!\perp(\mathcal{A})$ by

$$\perp\!\!\!\perp(\mathcal{A}) = \bigvee\{E \in \mathcal{A}^! \mid P_1 \wedge E \,\substack{|\\\circ}\, P_2 \wedge E \text{ for all } P_1, P_2 \in \mathcal{A}\}, \tag{4}$$

for any subset $\mathcal{A}$ of $\mathcal{Q}(\mathcal{H})$. By Takeuti's definition it is not clear whether the commutator $\underset{\,}{\perp\!\!\!\perp}(\mathcal{A})$ is determined inside the logic $\mathcal{A}^{!!}$ generated by $\mathcal{A}$ or not, unlike the definition of $\underset{\,}{\perp\!\!\!\perp}(\mathcal{F})$ for finite subsets $\mathcal{F}$. To resolve this problem, it was shown in Ref. [10, Theorem 2.5] that $\underset{\,}{\perp\!\!\!\perp}(\mathcal{A}) \in \mathcal{A}^! \cap \mathcal{A}^{!!}$, and we obtain the relation

$$\underset{\,}{\perp\!\!\!\perp}(\mathcal{A}) = \bigvee\{E \in \mathcal{A}^! \cap \mathcal{A}^{!!} \mid P_1 \wedge E \,\underset{\,}{\downarrow}\, P_2 \wedge E \text{ for all } P_1, P_2 \in \mathcal{A}\}, \qquad (5)$$

as an alternative definition for $\underset{\,}{\perp\!\!\!\perp}(\mathcal{A})$.

We have the following characterizations of commutators [9, Theorems 2.5, 2.6, Proposition 2.2]: For any $\mathcal{A} \subseteq \mathcal{Q}(\mathcal{H})$, we have the following relations.

$$\underset{\,}{\perp\!\!\!\perp}(\mathcal{A}) = \mathcal{P}\{\psi \in \mathcal{H} \mid [P_1, P_2]P_3\psi = 0 \text{ for all } P_1, P_2, P_3 \in \mathcal{A}^{!!}\}. \qquad (6)$$

$$\underset{\,}{\perp\!\!\!\perp}(\mathcal{A}) = \mathcal{P}\{\psi \in \mathcal{H} \mid [A, B]\psi = 0 \text{ for all } A, B \in \mathcal{A}''\}. \qquad (7)$$

We refer the reader to Pulmannová [13] and Chevalier [4] for further results about commutators in orthomodular lattices.

2.3   Conditionals

In classical logic, the conditional operation $\rightarrow$ is defined by negation $\perp$ and disjunction $\vee$ as $P \rightarrow Q = P^\perp \vee Q$. In quantum logic there is well-known arbitrariness in choosing a binary operation for conditional. Following Hardegree [5], we define a *quantum material conditional* on a logic $\mathcal{Q}$ as a binary operation $\rightarrow$ on $\mathcal{Q}$ definable by an ortholattice polynomial $p(x, y)$ as $P \rightarrow Q = p(P, Q)$ for all $P, Q \in \mathcal{Q}$ satisfying the following "minimum implicative conditions":

(E) $P \rightarrow Q = 1$ if and only if $P \leq Q$.
(MP) (*modus ponens*) $P \wedge (P \rightarrow Q) \leq Q$.
(MT) (*modus tollens*) $Q^\perp \wedge (P \rightarrow Q) \leq P^\perp$.

Hardegree [5] showed that there are exactly three polynomially definable material conditionals:

(S) (Sasaki conditional) $P \rightarrow_S Q := P^\perp \vee (P \wedge Q)$,
(C) (Contrapositive Sasaki conditional) $P \rightarrow_C Q := (P \vee Q)^\perp \vee Q$,
(R) (Relevance conditional) $P \rightarrow_R Q := (P \wedge Q) \vee (P^\perp \wedge Q) \vee (P^\perp \wedge Q^\perp)$.

Following Takeuti [15] we adopt the Sasaki arrow, the most favorable according to the majority view [16], as the conditional $\rightarrow$ for a logic $\mathcal{Q}$, *i.e.*, $P \rightarrow Q = P^\perp \vee (P \wedge Q)$. The logical equivalence $\leftrightarrow$ is defined by

$$P \leftrightarrow Q = (P \rightarrow Q) \wedge (Q \rightarrow P). \qquad (8)$$

In Boolean logic, implication and conjunction are associated by the relation $P \wedge Q = (P \rightarrow Q^\perp)^\perp$, and this relation plays an essential role in the duality between bounded universal quantification $(\forall x \in A)\phi(x)$ and bounded existential

quantification $(\exists x \in A)\phi(x)$. In order to keep the above duality in quantum set theory, we introduce the binary operation $*$ dual to $\rightarrow$ by

$$P * Q = (P \rightarrow Q^{\perp})^{\perp}. \tag{9}$$

We have the following relations

$$P \rightarrow Q = (P \wedge Q) \vee (P^{\perp} \wedge Q) \vee (P^{\perp} \wedge Q^{\perp}) \vee (P^{\perp} \wedge \perp\!\!\!\perp (P,Q)^{\perp}). \tag{10}$$

$$P * Q = (P \wedge Q) \vee (P \wedge \perp\!\!\!\perp (P,Q)^{\perp}). \tag{11}$$

The following proposition is useful in later discussions [9, Proposition 2.4], [12, Proposition 3.1].

Proposition 1 *Let $\mathcal{Q}$ be a logic on $\mathcal{H}$. The following hold.*
(i) If $P_{\alpha} \in \mathcal{Q}$ and $P_{\alpha} \downarrow Q$ for all α, then $(\bigvee_{\alpha} P_{\alpha}) \downarrow Q$, $(\bigwedge_{\alpha} P_{\alpha}) \downarrow Q$, and $Q \wedge (\bigvee_{\alpha} P_{\alpha}) = \bigvee_{\alpha}(Q \wedge P_{\alpha})$.
(ii) If $P_1, P_2 \downarrow Q$, then $(P_1 \rightarrow P_2) \wedge Q = [(P_1 \wedge Q) \rightarrow (P_2 \wedge Q)] \wedge Q$.
(iii) If $P_1, P_2 \downarrow Q$, then $(P_1 * P_2) \wedge Q = [(P_1 \wedge Q) * (P_2 \wedge Q)] \wedge Q$.

3   Quantum set theory

We denote by V the universe of the Zermelo-Fraenkel set theory with the axiom of choice (ZFC). Let $\mathcal{L}(\in)$ be the language of first-order theory with equality augmented by a connective $\rightarrow$, a binary relation symbol $\in$, bounded quantifier symbols $\forall x \in y$, $\exists x \in y$, and no constant symbols. For any class U, the language $\mathcal{L}(\in, U)$ is the one obtained by adding a name for each element of U. We take the symbols $\neg$, $\wedge$, $\rightarrow$, $\forall x \in y$, and $\forall x$ as primitive, and the symbols $\vee$, $\exists x \in y$, and $\exists x$ as derived symbols by defining:

  (i) $\phi \vee \psi = \neg(\neg\phi \wedge \neg\psi)$,
  (ii) $\exists x \in y \, \phi(x) = \neg(\forall x \in y \, \neg\phi(x))$,
 (iii) $\exists x \phi(x) = \neg(\forall x \, \neg\phi(x))$.

To each statement ϕ of $\mathcal{L}(\in, U)$, the satisfaction relation $\langle U, \in \rangle \models \phi$ is defined by the following recursive rules:

  1. $\langle U, \in \rangle \models u \in v$ iff $u \in v$.
  2. $\langle U, \in \rangle \models u = v$ iff $u = v$.
  3. $\langle U, \in \rangle \models \neg\phi$ iff $\langle U, \in \rangle \models \phi$ does not hold.
  4. $\langle U, \in \rangle \models \phi_1 \wedge \phi_2$ iff $\langle U, \in \rangle \models \phi_1$ and $\langle U, \in \rangle \models \phi_2$.
  5. $\langle U, \in \rangle \models \phi_1 \rightarrow \phi_2$ iff   if $\langle U, \in \rangle \models \phi_1$ then $\langle U, \in \rangle \models \phi_2$.
  6. $\langle U, \in \rangle \models (\forall x \in u)\,\phi(x)$ iff $\langle U, \in \rangle \models \phi(u')$ for all $u' \in u$.
  7. $\langle U, \in \rangle \models (\forall x)\,\phi(x)$ iff $\langle U, \in \rangle \models \phi(u)$ for all $u \in U$.

Our assumption that V satisfies ZFC means that if $\phi(x_1, \ldots, x_n)$ is provable in ZFC, *i.e.*, ZFC $\vdash \phi(x_1, \ldots, x_n)$, then $\langle V, \in \rangle \models \phi(u_1, \ldots, u_n)$ for any formula $\phi(x_1, \ldots, x_n)$ of $\mathcal{L}(\in)$ and all $u_1, \ldots, u_n \in V$.

In what follows let $\mathcal{Q}$ be a logic on $\mathcal{H}$. For each ordinal α, let

$$V_\alpha^{(\mathcal{Q})} = \{u \mid u : \operatorname{dom}(u) \to \mathcal{Q} \text{ and } (\exists \beta < \alpha) \operatorname{dom}(u) \subseteq V_\beta^{(\mathcal{Q})}\}. \qquad (12)$$

The *$\mathcal{Q}$-valued universe* $V^{(\mathcal{Q})}$ is defined by

$$V^{(\mathcal{Q})} = \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(\mathcal{Q})}, \qquad (13)$$

where On is the class of all ordinals. For every $u \in V^{(\mathcal{Q})}$, the rank of u, denoted by $\operatorname{rank}(u)$, is defined as the least α such that $u \in V_{\alpha+1}^{(\mathcal{Q})}$. It is easy to see that if $u \in \operatorname{dom}(v)$ then $\operatorname{rank}(u) < \operatorname{rank}(v)$.

An induction on rank argument leads to the following [1].

Theorem 2 (Induction Principle for $V^{(\mathcal{Q})}$). *For any predicate $\phi(x)$,*

$$\forall u \in V^{(\mathcal{Q})}[\forall u' \in \operatorname{dom}(u)\phi(u') \to \phi(u)] \to \forall u \in V^{(\mathcal{Q})}\phi(u)$$

For any $u, v \in V^{(\mathcal{Q})}$, the $\mathcal{Q}$-valued truth values $[\![u = v]\!]_\mathcal{Q}$ and $[\![u \in v]\!]_\mathcal{Q}$ of atomic formulas $u = v$ and $u \in v$ are assigned by the following rules recursive in rank.

(iv) $[\![u = v]\!]_\mathcal{Q} = \bigwedge_{u' \in \operatorname{dom}(u)}(u(u') \to [\![u' \in v]\!]_\mathcal{Q}) \wedge \bigwedge_{v' \in \operatorname{dom}(v)}(v(v') \to [\![v' \in u]\!]_\mathcal{Q})$.

(v) $[\![u \in v]\!]_\mathcal{Q} = \bigvee_{v' \in \operatorname{dom}(v)}(v(v') * [\![u = v']\!]_\mathcal{Q})$.

To each statement ϕ of $\mathcal{L}(\in, V^{(\mathcal{Q})})$ we assign the $\mathcal{Q}$-valued truth value $[\![\phi]\!]_\mathcal{Q}$ by the following rules.

(vi) $[\![\neg\phi]\!]_\mathcal{Q} = [\![\phi]\!]_\mathcal{Q}^\perp$.

(vii) $[\![\phi_1 \wedge \phi_2]\!]_\mathcal{Q} = [\![\phi_1]\!]_\mathcal{Q} \wedge [\![\phi_2]\!]_\mathcal{Q}$.

(viii) $[\![\phi_1 \to \phi_2]\!]_\mathcal{Q} = [\![\phi_1]\!]_\mathcal{Q} \to [\![\phi_2]\!]_\mathcal{Q}$.

(ix) $[\![(\forall x \in u)\, \phi(x)]\!]_\mathcal{Q} = \bigwedge_{u' \in \operatorname{dom}(u)}(u(u') \to [\![\phi(u')]\!]_\mathcal{Q})$.

(x) $[\![(\forall x)\, \phi(x)]\!]_\mathcal{Q} = \bigwedge_{u \in V^{(\mathcal{Q})}}[\![\phi(u)]\!]_\mathcal{Q}$.

By the definitions of derived logical symbols, (i)–(iii), we have the following relations.

(xi) $[\![\phi_1 \vee \phi_2]\!]_\mathcal{Q} = [\![\phi_1]\!]_\mathcal{Q} \vee [\![\phi_2]\!]_\mathcal{Q}$.

(xii) $[\![(\exists x \in u)\, \phi(x)]\!]_\mathcal{Q} = \bigvee_{u' \in \operatorname{dom}(u)}(u(u') * [\![\phi(u')]\!]_\mathcal{Q})$.

(xiii) $[\![(\exists x)\, \phi(x)]\!]_\mathcal{Q} = \bigvee_{u \in V^{(\mathcal{Q})}}[\![\phi(u)]\!]_\mathcal{Q}$.

To see (xii), we have

$$\begin{aligned}
[\![(\exists x \in u)\, \phi(x)]\!]_{\mathcal{Q}} &= [\![\neg(\forall x \in u\, \neg\phi(x))]\!]_{\mathcal{Q}} \\
&= \big(\bigwedge_{u' \in \mathrm{dom}(u)} u(u') \to [\![\neg\phi(u')]\!]_{\mathcal{Q}} \big)^{\perp} \\
&= \big(\bigwedge_{u' \in \mathrm{dom}(u)} u(u') \to [\![\phi(u')]\!]_{\mathcal{Q}}^{\perp} \big)^{\perp} \\
&= \bigvee_{u' \in \mathrm{dom}(u)} (u(u') \to [\![\phi(u')]\!]_{\mathcal{Q}}^{\perp})^{\perp} \\
&= \bigvee_{u' \in \mathrm{dom}(u)} (u(u') * [\![\phi(u')]\!]_{\mathcal{Q}}).
\end{aligned}$$

Note that according to the above, we have the following relations

(xiv) $[\![u = v]\!]_{\mathcal{Q}} = [\![\forall x \in u(x \in v) \wedge \forall x \in v(x \in u)]\!]_{\mathcal{Q}}$,
(xv) $[\![u \in v]\!]_{\mathcal{Q}} = [\![\exists x \in v(x = u)]\!]_{\mathcal{Q}}$.

We also have the following relations satisfying De Morgan's laws:

(xvi) $[\![\neg(\phi_1 \wedge \phi_2)]\!]_{\mathcal{Q}} = [\![\neg\phi_1 \vee \neg\phi_2]\!]_{\mathcal{Q}}$,
(xvii) $[\![\neg(\phi_1 \vee \phi_2)]\!]_{\mathcal{Q}} = [\![\neg\phi_1 \wedge \neg\phi_2]\!]_{\mathcal{Q}}$,
(xviii) $[\![\neg(\forall x \in u\, \phi(x))]\!]_{\mathcal{Q}} = [\![\exists x \in u\, (\neg\phi(x))]\!]_{\mathcal{Q}}$,
(xix) $[\![\neg(\exists x \in u\, \phi(x))]\!]_{\mathcal{Q}} = [\![\forall x \in u\, (\neg\phi(x))]\!]_{\mathcal{Q}}$,
(xx) $[\![\neg(\forall x\, \phi(x))]\!]_{\mathcal{Q}} = [\![\exists x\, (\neg\phi(x))]\!]_{\mathcal{Q}}$,
(xxi) $[\![\neg(\exists x\, \phi(x))]\!]_{\mathcal{Q}} = [\![\forall x\, (\neg\phi(x))]\!]_{\mathcal{Q}}$.

A formula in $\mathcal{L}(\in)$ is called a Δ_0-*formula* iff it has no unbounded quantifiers $\forall x$ nor $\exists x$. The following theorem holds.

Theorem 3 (Δ_0-Absoluteness Principle). *For any Δ_0-formula $\phi(x_1,\ldots,x_n)$ of $\mathcal{L}(\in)$ and $u_1,\ldots,u_n \in V^{(\mathcal{Q})}$, we have*

$$[\![\phi(u_1,\ldots,u_n)]\!]_{\mathcal{Q}} = [\![\phi(u_1,\ldots,u_n)]\!]_{\mathcal{Q}(\mathcal{H})}.$$

Proof. The assertion is proved by the induction on the complexity of formulas and the rank of elements of $V^{(\mathcal{Q})}$. Let $u, v \in V^{(\mathcal{Q})}$. By induction hypothesis, for any $u' \in \mathrm{dom}(u)$ and $v' \in \mathrm{dom}(v)$ we have $[\![u' \in w]\!]_{\mathcal{Q}} = [\![u' \in w]\!]_{\mathcal{Q}(\mathcal{H})}$, $[\![v' \in w]\!]_{\mathcal{Q}} = [\![v' \in w]\!]_{\mathcal{Q}(\mathcal{H})}$, and $[\![w = v']\!]_{\mathcal{Q}} = [\![w = v']\!]_{\mathcal{Q}(\mathcal{H})}$ for all $w \in V^{(\mathcal{Q})}$. Thus,

$$\begin{aligned}
[\![u = v]\!]_{\mathcal{Q}} &= \bigwedge_{u' \in \mathrm{dom}(u)} (u(u') \to [\![u' \in v]\!]_{\mathcal{Q}}) \wedge \bigwedge_{v' \in \mathrm{dom}(v)} (v(v') \to [\![v' \in u]\!]_{\mathcal{Q}}) \\
&= \bigwedge_{u' \in \mathrm{dom}(u)} (u(u') \to [\![u' \in v]\!]_{\mathcal{Q}(\mathcal{H})}) \wedge \bigwedge_{v' \in \mathrm{dom}(v)} (v(v') \to [\![v' \in u]\!]_{\mathcal{Q}(\mathcal{H})}) \\
&= [\![u = v]\!]_{\mathcal{Q}(\mathcal{H})},
\end{aligned}$$

and we also have

$$[\![u \in v]\!]_{\mathcal{Q}} = \bigvee_{v' \in \mathrm{dom}(v)} (v(v') * [\![u = v']\!]_{\mathcal{Q}})$$

$$= \bigvee_{v' \in \mathrm{dom}(v)} (v(v') * [\![u = v']\!]_{\mathcal{Q}(\mathcal{H})})$$

$$= [\![u \in v]\!]_{\mathcal{Q}(\mathcal{H})}.$$

Thus, the assertion holds for atomic formulas. Any induction step adding a logical symbol works easily, even when bounded quantifiers are concerned, since the ranges of the supremum and the infimum are common for evaluating $[\![\cdots]\!]_{\mathcal{Q}}$ and $[\![\cdots]\!]_{\mathcal{Q}(\mathcal{H})}$.

Henceforth, for any Δ_0-formula $\phi(x_1, \ldots, x_n)$ and $u_1, \ldots, u_n \in V^{(\mathcal{Q})}$, we abbreviate $[\![\phi(u_1, \ldots, u_n)]\!] = [\![\phi(u_1, \ldots, u_n)]\!]_{\mathcal{Q}}$, which is the common $\mathcal{Q}(\mathcal{H})$-valued truth value for $u_1, \ldots, u_n \in V^{(\mathcal{Q})}$.

The universe V can be embedded in $V^{(\mathcal{Q})}$ by the following operation $\vee : v \mapsto \check{v}$ defined by the $\in$-recursion: for each $v \in V$, $\check{v} = \{\check{u} | \ u \in v\} \times \{1\}$. Then we have the following.

Theorem 4 (Δ_0-Elementary Equivalence Principle). *Let $\phi(x_1, \ldots, x_n)$ be a Δ_0-formula of $\mathcal{L}(\in)$. For any $u_1, \ldots, u_n \in V$, we have*

$$\langle V, \in \rangle \models \phi(u_1, \ldots, u_n) \quad \textit{if and only if} \quad [\![\phi(\check{u}_1, \ldots, \check{u}_n)]\!] = 1.$$

Proof. Let $\mathbf{2}$ be the sublogic such that $\mathbf{2} = \{0, 1\}$. Then, by induction it is easy to see that $\langle V, \in \rangle \models \phi(u_1, \ldots, u_n)$ if and only if $[\![\phi(\check{u}_1, \ldots, \check{u}_n)]\!]_{\mathbf{2}} = 1$ for any $\phi(x_1, \ldots, x_n)$ in $\mathcal{L}(\in)$, and this is equivalent to $[\![\phi(\check{u}_1, \ldots, \check{u}_n)]\!] = 1$ for any Δ_0-formula $\phi(x_1, \ldots, x_n)$ by the Δ_0-absoluteness principle.

Instead of (v) and (xii), Takeuti [15] defined the truth values of membership relation and existential quantification as follows.

(v') $[\![u \in v]\!] = \bigvee_{v' \in \mathrm{dom}(v)} (v(v') \wedge [\![u = v']\!])$.
(xii') $[\![(\exists x \in u)\, \phi(x)]\!] = \bigvee_{u' \in \mathrm{dom}(u)} (u(u') \wedge [\![\phi(u')]\!])$.

In this case, De Morgan's laws do not hold in general as follows.

Suppose that $\mathcal{Q}$ is not a Boolean algebra. Then, there exists a pair $P_0, Q_0 \in \mathcal{Q}$ such that P_0 does not commute with Q_0, so that $\perp\!\!\!\perp (P_0, Q_0)^{\perp} > 0$. Let $E = \perp\!\!\!\perp (P_0, Q_0)^{\perp}$, $P = P_0 \wedge E$, and $Q = Q_0 \wedge E$. If $P = 0$ then $P_0 = P_0 \wedge \perp\!\!\!\perp (P_0, Q_0)$ so that $P_0 \, \substack{|\\ \circ} \, Q_0$, a contradiction. Thus, $P \neq 0$. We also have that $P \wedge Q = P_0 \wedge Q_0 \wedge \perp\!\!\!\perp (P_0, Q_0)^{\perp} = 0$, so that $P \wedge Q = 0$. Let $u = \{\langle \check{0}, P \rangle\}$ and $v = \{\langle \check{0}, Q \rangle\}$. Consider the formula $\phi(x) = \neg(x \in v)$. Then, we can show

$$[\![\neg(\forall x \in u)\phi(x)]\!] > [\![(\exists x \in u)\neg\phi(x)]\!] = 0. \tag{14}$$

In fact, we have

$$
\begin{aligned}
[\![(\exists x \in u)\neg\phi(x)]\!] &= \bigvee_{u' \in \mathrm{dom}(u)} (u(u') \wedge [\![\neg\phi(u')]\!]) \\
&= u(\breve{0}) \wedge [\![\neg\neg(\breve{0} \in v)]\!] \\
&= u(\breve{0}) \wedge [\![\breve{0} \in v]\!] \\
&= u(\breve{0}) \wedge \bigvee_{v' \in \mathrm{dom}(v)} (v(v') \wedge [\![\breve{0} = v']\!]) \\
&= u(\breve{0}) \wedge (v(\breve{0}) \wedge [\![\breve{0} = \breve{0}]\!]) \\
&= u(\breve{0}) \wedge v(\breve{0}) \\
&= P \wedge Q \\
&= 0.
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
[\![\neg(\forall x \in u)\phi(x)]\!] &= [\![(\forall x \in u)\phi(x)]\!]^{\perp} \\
&= \Big(\bigwedge_{u' \in \mathrm{dom}(u)} (u(u') \to [\![\phi(u')]\!]) \Big)^{\perp} \\
&= (u(\breve{0}) \to [\![\phi(\breve{0})]\!])^{\perp} \\
&= u(\breve{0}) * [\![\phi(\breve{0})]\!]^{\perp} \\
&= u(\breve{0}) * [\![\neg(\breve{0} \in v)]\!]^{\perp} \\
&= u(\breve{0}) * [\![\breve{0} \in v]\!] \\
&= u(\breve{0}) * \bigvee_{v' \in \mathrm{dom}(v)} (v(v') \wedge [\![\breve{0} = v']\!]) \\
&= u(\breve{0}) * (v(\breve{0}) \wedge [\![\breve{0} = \breve{0}]\!]) \\
&= u(\breve{0}) * v(\breve{0}) \\
&= P * Q \\
&= (P \wedge Q) \vee (P \wedge \perp\!\!\!\perp(P,Q)^{\perp}) \\
&= P.
\end{aligned}
$$

Since $P \neq 0$, Eq. (14) follows.

Thus, if $\mathcal{Q}$ is not a Boolean algebra, there exists a predicate $\phi(x)$ such that $[\![(\exists x \in u)\,\neg\phi(x)]\!] = 0$ but $[\![\neg(\forall x \in u)\,\phi(x)]\!] > 0$.

4   Transfer principle

In this section, we investigate the transfer principle that gives any Δ_0-formula provable in ZFC a lower bound for its truth value, which is determined by the degree of the commutativity of the elements of $V^{(\mathcal{Q})}$ appearing in the formula as constants. The results in this section was obtained in Ref. [9] for Takeuti's

original formulation. Here, we extends the argument in a self-contained manner to the present formulation, in which De Morgan's laws hold for bounded quantifiers.

For $u \in V^{(\mathcal{Q})}$, we define the *support* of u, denoted by $L(u)$, by transfinite recursion on the rank of u by the relation

$$L(u) = \bigcup_{x \in \mathrm{dom}(u)} L(x) \cup \{u(x) \mid x \in \mathrm{dom}(u)\} \cup \{0\}. \tag{15}$$

For $\mathcal{A} \subseteq V^{(\mathcal{Q})}$ we write $L(\mathcal{A}) = \bigcup_{u \in \mathcal{A}} L(u)$ and for $u_1, \ldots, u_n \in V^{(\mathcal{Q})}$ we write $L(u_1, \ldots, u_n) = L(\{u_1, \ldots, u_n\})$. Then, we obtain the following characterization of subuniverses of $V^{(\mathcal{Q}(\mathcal{H}))}$.

Proposition 5 *Let $\mathcal{Q}$ be a logic on $\mathcal{H}$ and α an ordinal. For any $u \in V^{(\mathcal{Q}(\mathcal{H}))}$, we have $u \in V_\alpha^{(\mathcal{Q})}$ if and only if $u \in V_\alpha^{(\mathcal{Q}(\mathcal{H}))}$ and $L(u) \subseteq \mathcal{Q}$. In particular, $u \in V^{(\mathcal{Q})}$ if and only if $u \in V^{(\mathcal{Q}(\mathcal{H}))}$ and $L(u) \subseteq \mathcal{Q}$. Moreover, $\mathrm{rank}(u)$ is the least α such that $u \in V_{\alpha+1}^{(\mathcal{Q}(\mathcal{H}))}$ for any $u \in V^{(\mathcal{Q})}$.*

Proof. Immediate from transfinite induction on α.

Let $\mathcal{A} \subseteq V^{(\mathcal{Q})}$. The *commutator of $\mathcal{A}$*, denoted by $\underline{\vee}(\mathcal{A})$, is defined by

$$\underline{\vee}(\mathcal{A}) = \underline{\perp}(L(\mathcal{A})). \tag{16}$$

For any $u_1, \ldots, u_n \in V^{(\mathcal{Q})}$, we write $\underline{\vee}(u_1, \ldots, u_n) = \underline{\vee}(\{u_1, \ldots, u_n\})$.

Let $u \in V^{(\mathcal{Q})}$ and $p \in \mathcal{Q}$. The *restriction $u|_p$ of u to p* is defined by the following transfinite recursion:

$$u|_p = \{\langle x|_p, u(x) \wedge p \rangle \mid x \in \mathrm{dom}(u)\} \cup \{\langle u, 0 \rangle\}.$$

The last term $\{\langle u, 0 \rangle\}$ has no essential role but ensures the well-definedness of the function $u|_p : \mathrm{dom}(u|_p) \to \mathcal{Q}$.

Proposition 6 *For any $\mathcal{A} \subseteq V^{(\mathcal{Q})}$ and $p \in \mathcal{Q}$, we have*

$$L(\{u|_p \mid u \in \mathcal{A}\}) = L(\mathcal{A}) \wedge p. \tag{17}$$

Proof. By induction, it is easy to see the relation $L(u|_p) = L(u) \wedge p$, so that the assertion follows easily.

Let $\mathcal{A} \subseteq V^{(\mathcal{Q})}$. The *logic generated by $\mathcal{A}$*, denoted by $\mathcal{Q}(\mathcal{A})$, is defined by

$$\mathcal{Q}(\mathcal{A}) = L(\mathcal{A})^{!!}. \tag{18}$$

For $u_1, \ldots, u_n \in V^{(\mathcal{Q})}$, we write $\mathcal{Q}(u_1, \ldots, u_n) = \mathcal{Q}(\{u_1, \ldots, u_n\})$.

Proposition 7 *For any Δ_0-formula $\phi(x_1, \ldots, x_n)$ in $\mathcal{L}(\in)$ and $u_1, \cdots, u_n \in V^{(\mathcal{Q})}$, we have $[\![\phi(u_1, \ldots, u_n)]\!] \in \mathcal{Q}(u_1, \ldots, u_n)$.*

Proof. Let $\mathcal{A} = \{u_1, \ldots, u_n\}$. Since $L(\mathcal{A}) \subseteq \mathcal{Q}(\mathcal{A})$, it follows from Proposition 5 that $u_1, \ldots, u_n \in V^{(\mathcal{Q}(\mathcal{A}))}$. By the Δ_0-absoluteness principle, we have $[\![\phi(u_1, \ldots, u_n)]\!] = [\![\phi(u_1, \ldots, u_n)]\!]_{\mathcal{Q}(\mathcal{A})} \in \mathcal{Q}(\mathcal{A})$.

Proposition 8 *For any Δ_0-formula $\phi(x_1, \ldots, x_n)$ in $\mathcal{L}(\in)$ and $u_1, \ldots, u_n \in V^{(\mathcal{Q})}$, if $p \in L(u_1, \ldots, u_n)^!$, then $p \,\substack{\downarrow\\\circ}\, [\![\phi(u_1, \ldots, u_n)]\!]$ and $p \,\substack{\downarrow\\\circ}\, [\![\phi(u_1|_p, \ldots, u_n|_p)]\!]$.*

Proof. Let $u_1, \ldots, u_n \in V^{(\mathcal{Q})}$. If $p \in L(u_1, \ldots, u_n)^!$, then $p \in \mathcal{Q}(u_1, \ldots, u_n)^!$. From Proposition 7, $[\![\phi(u_1, \ldots, u_n)]\!] \in \mathcal{Q}(u_1, \ldots, u_n)$, so that $p \,\substack{\downarrow\\\circ}\, [\![\phi(u_1, \ldots, u_n)]\!]$. From Proposition 6, $L(u_1|_p, \ldots, u_n|_p) = L(u_1, \ldots, u_n) \wedge p$, and hence $p \in L(u_1|_p, \ldots, u_n|_p)^!$, so that $p \,\substack{\downarrow\\\circ}\, [\![\phi(u_1|_p, \ldots, u_n|_p)]\!]$.

We define the binary relation $x_1 \subseteq x_2$ by $\forall x \in x_1 (x \in x_2)$. Then, by definition for any $u, v \in V^{(\mathcal{Q})}$ we have

$$[\![u \subseteq v]\!] = \bigwedge_{u' \in \mathrm{dom}(u)} (u(u') \to [\![u' \in v]\!]), \tag{19}$$

and we have $[\![u = v]\!] = [\![u \subseteq v]\!] \wedge [\![v \subseteq u]\!]$.

Proposition 9 *For any $u, v \in V^{(\mathcal{Q})}$ and $p \in L(u, v)^!$, we have the following relations.*

(i) $[\![u|_p \in v|_p]\!] = [\![u \in v]\!] \wedge p$.

(ii) $[\![u|_p \subseteq v|_p]\!] \wedge p = [\![u \subseteq v]\!] \wedge p$.

(iii) $[\![u|_p = v|_p]\!] \wedge p = [\![u = v]\!] \wedge p$

Proof. We prove the relations by induction on the ranks of u, v. If $\mathrm{rank}(u) = \mathrm{rank}(v) = 0$, then $\mathrm{dom}(u) = \mathrm{dom}(v) = \emptyset$, so that the relations trivially hold. Let $u, v \in V^{(\mathcal{Q})}$ and $p \in L(u, v)^!$. To prove (i), let $v' \in \mathrm{dom}(v)$. Then, we have $p \,\substack{\downarrow\\\circ}\, v(v')$ by the assumption on p. By induction hypothesis, we have also $[\![u|_p = v'|_p]\!] \wedge p = [\![u = v']\!] \wedge p$. By Proposition 8, we have $p \,\substack{\downarrow\\\circ}\, [\![u = v']\!]$, so that $v(v'), [\![u = v']\!] \in \{p\}^!$. From Eq. (11) we have $(v(v') \wedge p) * [\![u|_p = v'|_p]\!] \leq p$, and hence we have

$$(v(v') \wedge p) * [\![u|_p = v'|_p]\!] = (v(v') \wedge p) * ([\![u|_p = v'|_p]\!] \wedge p)$$
$$= (v(v') \wedge p) * ([\![u = v']\!] \wedge p)$$
$$= (v(v') * [\![u = v']\!]) \wedge p.$$

Thus, we have

$$\llbracket u|_p \in v|_p \rrbracket = \bigvee_{v' \in \mathrm{dom}(v|_p)} (v|_p(v') * \llbracket u|_p = v' \rrbracket)$$

$$= \bigvee_{v' \in \mathrm{dom}(v)} (v|_p(v'|_p) * \llbracket u|_p = v'|_p \rrbracket)$$

$$= \bigvee_{v' \in \mathrm{dom}(v)} [(v(v') \wedge p) * \llbracket u|_p = v'|_p \rrbracket]$$

$$= \bigvee_{v' \in \mathrm{dom}(v)} [(v(v') * \llbracket u = v' \rrbracket) \wedge p]$$

$$= \left(\bigvee_{v' \in \mathrm{dom}(v)} (v(v') * \llbracket u = v' \rrbracket) \right) \wedge p,$$

where the last equality follows from Proposition 1 (i). Thus, by definition of $\llbracket u \in v \rrbracket$ we obtain the relation $\llbracket u|_p \in v|_p \rrbracket = \llbracket u \in v \rrbracket \wedge p$, and relation (i) has been proved. To prove (ii), let $u' \in \mathrm{dom}(u)$. Then, we have $\llbracket u'|_p \in v|_p \rrbracket = \llbracket u' \in v \rrbracket \wedge p$ by induction hypothesis. Thus, we have

$$\llbracket u|_p \subseteq v|_p \rrbracket = \bigwedge_{u' \in \mathrm{dom}(u|_p)} (u|_p(u') \to \llbracket u' \in v|_p \rrbracket)$$

$$= \bigwedge_{u' \in \mathrm{dom}(u)} (u|_p(u'|_p) \to \llbracket u'|_p \in v|_p \rrbracket)$$

$$= \bigwedge_{u' \in \mathrm{dom}(u)} [(u(u') \wedge p) \to (\llbracket u' \in v \rrbracket \wedge p)].$$

We have $p \downarrow u(u')$ by assumption on p, and $p \downarrow \llbracket u' \in v \rrbracket$ by Proposition 8, so that $p \downarrow u(u') \to \llbracket u' \in v \rrbracket$ and $p \downarrow (u(u') \wedge p) \to (\llbracket u' \in v \rrbracket \wedge p)$. Thus, by Proposition 1 we have

$$p \wedge \llbracket u|_p \subseteq v|_p \rrbracket = p \wedge \bigwedge_{u' \in \mathrm{dom}(u)} [(u(u') \wedge p) \to (\llbracket u' \in v \rrbracket \wedge p)]$$

$$= p \wedge \bigwedge_{u' \in \mathrm{dom}(u)} (u(u') \to \llbracket u' \in v \rrbracket)$$

$$= p \wedge \llbracket u \subseteq v \rrbracket.$$

Thus, we have proved relation (ii). Relation (iii) follows easily from relation (ii).

We have the following theorem.

Theorem 10 (Δ_0-Restriction Principle). *For any Δ_0-formula $\phi(x_1, \ldots, x_n)$ in $\mathcal{L}(\in)$ and $u_1, \ldots, u_n \in V^{(\mathcal{Q})}$, if $p \in L(u_1, \ldots, u_n)^!$, then $\llbracket \phi(u_1, \ldots, u_n) \rrbracket \wedge p = \llbracket \phi(u_1|_p, \ldots, u_n|_p) \rrbracket \wedge p$.*

Proof. We shall write $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{u}|_p = (u_1|_p, \ldots, u_n|_p)$. We prove the assertion by induction on the complexity of $\phi(x_1, \ldots, x_n)$. From Proposition 9, the assertion holds for atomic formulas. Thus, it suffices to consider the following induction steps: (i) $\phi \Rightarrow \neg\phi$, (ii) $\phi_1, \phi_2 \Rightarrow \phi_1 \wedge \phi_2$ (iii) $\phi_1, \phi_2 \Rightarrow \phi_1 \rightarrow \phi_2$, (iv) $\{\phi(x) \mid x \in \mathrm{dom}(u)\} \rightarrow \bigwedge_{x \in \mathrm{dom}(u)} \phi(x)$.

(i) If $a \mathbin{\underset{\circ}{\downharpoonleft}} p$, the relation

$$a^\perp \wedge p = (a \wedge p)^\perp \wedge p \tag{20}$$

follows easily. Let $p \in L(\mathbf{u})^!$. Suppose $[\![\phi(\mathbf{u})]\!] \wedge p = [\![\phi(\mathbf{u}|_p)]\!] \wedge p$. From Eq. (20) we have

$$\begin{aligned} [\![\phi(\mathbf{u})]\!]^\perp \wedge p &= ([\![\phi(\mathbf{u})]\!] \wedge p)^\perp \wedge p \\ &= ([\![\phi(\mathbf{u}|_p)]\!] \wedge p)^\perp \wedge p \\ &= [\![\phi(\mathbf{u}|_p)]\!]^\perp \wedge p, \end{aligned}$$

so that we have

$$[\![\neg\phi(\mathbf{u})]\!] \wedge p = [\![\neg\phi(\mathbf{u}|_p)]\!] \wedge p.$$

(ii) Let $p \in L(\mathbf{u})^!$. Suppose $[\![\phi_j(\mathbf{u})]\!] \wedge p = [\![\phi_j(\mathbf{u}|_p)]\!] \wedge p$ for $j = 1, 2$. Then, it follows easily from associativity of $\wedge$, we have

$$[\![\phi_1(\mathbf{u}) \wedge \phi_2(\mathbf{u})]\!] \wedge p = [\![\phi_1(\mathbf{u}|_p) \wedge \phi_2(\mathbf{u}|_p)]\!] \wedge p.$$

(iii) Recall the relation

$$(a \rightarrow b) \wedge p = [(a \wedge p) \rightarrow (b \wedge p)] \wedge p$$

for all $a, b \in \{p\}^!$ as shown in Proposition 1 (ii). Let $p \in L(\mathbf{u})^!$. Suppose $[\![\phi_j(\mathbf{u})]\!] \wedge p = [\![\phi_j(\mathbf{u}|_p)]\!] \wedge p$ for $j = 1, 2$. It follows from the above relation and the induction hypothesis that

$$\begin{aligned} [\![\phi_1(\mathbf{u}) \rightarrow \phi_2(\mathbf{u})]\!] \wedge p &= [([\![\phi_1(\mathbf{u})]\!] \wedge p) \rightarrow ([\![\phi_2(\mathbf{u})]\!] \wedge p)] \wedge p \\ &= [([\![\phi_1(\mathbf{u}|_p)]\!] \wedge p) \rightarrow ([\![\phi_2(\mathbf{u}|_p)]\!] \wedge p)] \wedge p \\ &= ([\![\phi_1(\mathbf{u}|_p)]\!] \rightarrow [\![\phi_2(\mathbf{u}|_p)]\!]) \wedge p, \end{aligned}$$

so that we have

$$[\![\phi_1(\mathbf{u}) \rightarrow \phi_2(\mathbf{u})]\!] \wedge p = [\![\phi_1(\mathbf{u}|_p) \rightarrow \phi_2(\mathbf{u}|_p)]\!] \wedge p.$$

(iv) Note that the relation

$$\left(\bigwedge_\alpha P_{1,\alpha} \rightarrow P_{2,\alpha}\right) \wedge Q = \left(\bigwedge_\alpha (P_{1,\alpha} \wedge Q) \rightarrow (P_{2,\alpha} \wedge Q)\right) \wedge Q$$

holds if $P_{j,\alpha} \mathbin{\underset{\circ}{\downharpoonleft}} Q$ for $j = 1, 2$, which follows from Proposition 1 (i) and (ii). Suppose $[\![\phi_j(u)]\!] \wedge p = [\![\phi_j(u|_p)]\!] \wedge p$ for $j = 1, 2$ for any $u \in V^{(\mathcal{Q})}$ and $p \in L(u)^!$.

Suppose $u \in V^{(\mathcal{Q})}$ and $p \in L(u)^!$. Let $u' \in \mathrm{dom}(u)$. Since $L(u') \subseteq L(u)$, we have $p \in L(u')^!$. It follows that

$$\llbracket \phi_j(u') \rrbracket \wedge p = \llbracket \phi_j(u'|_p) \rrbracket \wedge p \quad \text{and} \quad p \mathbin{\underset{\circ}{\downarrow}} \llbracket \phi(u') \rrbracket, \llbracket \phi(u'|_p) \rrbracket$$

for all $u' \in \mathrm{dom}(u)$. Thus, we have

$$
\begin{aligned}
\llbracket (\forall x \in u)\phi(x) \rrbracket \wedge p &= \left(\bigwedge_{u' \in \mathrm{dom}(u)} (u(u') \to \llbracket \phi(u') \rrbracket) \right) \wedge p \\
&= \bigwedge_{u' \in \mathrm{dom}(u)} [(u(u') \to \llbracket \phi(u') \rrbracket) \wedge p] \\
&= \bigwedge_{u' \in \mathrm{dom}(u)} \{[(u(u') \wedge p) \to (\llbracket \phi(u') \rrbracket \wedge p)] \wedge p\} \\
&= \bigwedge_{u' \in \mathrm{dom}(u|_p)} \{[u|_p(u') \wedge p \to (\llbracket \phi(u') \rrbracket \wedge p)] \wedge p\} \\
&= \bigwedge_{u' \in \mathrm{dom}(u|_p)} \{[u|_p(u') \to (\llbracket \phi(u') \rrbracket)] \wedge p\} \\
&= \left(\bigwedge_{u' \in \mathrm{dom}(u|_p)} (u|_p(u') \to \llbracket \phi(u') \rrbracket) \right) \wedge p.
\end{aligned}
$$

It follows that

$$\llbracket (\forall x \in u)\phi(x) \rrbracket \wedge p = \llbracket (\forall x \in u|_p)\phi(x) \rrbracket \wedge p.$$

Now, we obtain the following transfer principle for bounded theorems of ZFC in the new truth-value assignments for membership and existential quantifications to fully satisfy De Morgan's laws.

Theorem 11 (Δ_0-ZFC Transfer Principle). *For any Δ_0-formula $\phi(x_1, \ldots, x_n)$ of $\mathcal{L}(\in)$ and $u_1, \ldots, u_n \in V^{(\mathcal{Q})}$, if $\phi(x_1, \ldots, x_n)$ is provable in ZFC, then we have*

$$\llbracket \phi(u_1, \ldots, u_n) \rrbracket \geq \underline{\vee}(u_1, \ldots, u_n). \tag{21}$$

Proof. Let $p = \underline{\vee}(u_1, \ldots, u_n)$. Then, we have $a \wedge p \mathbin{\underset{\circ}{\downarrow}} b \wedge p$ for any $a, b \in L(u_1, \ldots, u_n)$, and hence there is a Boolean sublogic $\mathcal{B}$ such that $L(u_1, \ldots, u_n) \wedge p \subseteq \mathcal{B}$. From Proposition 6, we have $L(u_1|_p, \ldots, u_n|_p) \subseteq \mathcal{B}$. From Proposition 5, we have $u_1|_p, \ldots, u_n|_p \in V^{(\mathcal{B})}$. By the ZFC transfer principle of the Boolean-valued universe [1, Theorem 1.33], we have $\llbracket \phi(u_1|_p, \ldots, u_n|_p) \rrbracket_{\mathcal{B}} = 1$. By the Δ_0-absoluteness principle, we have $\llbracket \phi(u_1|_p, \ldots, u_n|_p) \rrbracket = 1$. From Proposition 10, we have $\llbracket \phi(u_1, \ldots, u_n) \rrbracket \wedge p = \llbracket \phi(u_1|_p, \ldots, u_n|_p) \rrbracket \wedge p = p$, and the assertion follows.

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