

Quantum Measurement Theory for Systems with Finite Dimensional State Spaces

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Abstract In this paper, we present a general theory of finite quantum measurements, for which we assume that the state space of the measured system is a finite dimensional Hilbert space and that the possible outcomes of a measurement is a finite set of the real numbers. We develop the theory in a deductive manner from the basic postulates for quantum mechanics and a few plausible axioms for general quantum measurements. We derive an axiomatic characterization of all the physically realizable finite quantum measurements. Mathematical tools necessary to describe measurement statistics such as POVMs and quantum instruments are not assumed at the outset, but we introduce them as natural consequences of our axioms. Our objective is to show that those mathematical tools can be naturally derived from obvious theoretical requirements.

1 Introduction

Theories in physics are closely related to mathematics. In most cases, their basic principles are given undisputed mathematical expressions. Hilbert's sixth problem explicitly asked to treat such mathematical formulations of physical theories in the rigorous axiomatic method that was successful in the investigations on the foundations of geometry [9]. One of the typical outcomes of this problem is an axiomatization of quantum mechanics due to von Neumann, given in his book "Mathematical Foundations of Quantum Mechanics" (English translation) [16] originally published in 1932.

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In contrast to the prevailing view that this has completed the axiomatization of non-relativistic quantum mechanics without any superselection rules, von Neumann's axioms need to be completed by revising his treatments on quantum measurements. It is true that until the 1970s there were no problems in non-relativistic quantum mechanics that cannot be solved because of the deficiency of von Neumann's axioms. However, in the debate occurred in the 1980s on the sensitivity limit to gravitational-wave detectors, it was revealed that von Neumann's axioms were not sufficient.

In quantum mechanics, the notion of measurement plays an indispensable role. Nevertheless, "theory of measurement" was left incomplete in von Neumann's axiomatization. Quantum mechanics as axiomatized by von Neumann was only justified in experiments in which the object is prepared in the known state, undergoes time evolution with the known Hamiltonian, and then is subject to a measurement of one of its observables. Von Neumann's quantum postulates successfully predict the probability distribution of the outcome of the measurement. However, they are not sufficient to predict the joint probability distribution of the outcomes of successive measurements in time.

Experimental technology was not precise enough to carry out successive measurements for comparing with the theory for long. However, the emerging laser technology in the 1960s enabled us to control quantum states and measuring interaction between the object and the apparatus quantum mechanically to make precise successive measurements.

For the statistics of successive measurements, we need the notion of state changes caused by measurements, often called quantum state reductions, which has been considered one of the most difficult notions in quantum mechanics. In order to deal with quantum systems to be measured sequentially, we need a general notion of quantum state reductions and mathematical methods to calculate them. For this purpose, we have to mathematically characterize all of the possible state changes caused by the most general type of measurements.

Von Neumann [16] introduced the repeatability hypothesis to determine the quantum state reduction, which implies that the state of the measured system changes to the eigenstate of the measured observable corresponding to the outcome of the measurements. This principle uniquely determines the state change for measurements of non-degenerate discrete observables, of which all the eigenspaces are one dimensional. For degenerate observables the eigenstate is not uniquely determined. Subsequently, Lüders [11] proposed the projection postulate, called the von Neumann-Lüders projection postulate, to determine the unique eigenstate. A mathematical problem was considered to extend this notion to measurements of observables with continuous spectra (or continuous observables for brevity). Nakamura and Umegaki [15] pointed out that the unique state change described by von Neumann's repeatability hypothesis for measurements of non-degenerate observables is an instance of Umegaki's conditional expectations on operator algebras [45], which is a projection of the full operator algebra onto the subalgebra generated by the measured observable, and they conjectured that von Neumann's repeatability hypothesis can be extended to continuous observables as a conditional expectation of the full

operator algebra onto the subalgebra generated by the measured observable. In fact, for general discrete observables the state change described by the von Neumann-Lüders projection postulate is a conditional expectation of the full operator algebra onto the commutant of the measured observable. However, Arveson [1] proved that the corresponding conditional expectation does not exist as long as the measured observable has a continuous spectral.

Based on the above results, Davies and Lewis [5] proposed to abandon the repeatability hypothesis as the first principle, and introduced a general mathematical framework, based on the mathematical notion called instruments, which generalizes the notion of conditional expectations [45], the notion of operations described for discrete observables by Schwinger [40,42,43] and introduced in the context of algebraic quantum field theory by Haag and Kastler [6], and the notion of effects introduced by Ludwig [12,13]. The Davies-Lewis (DL) instruments were considered as a general mathematical notion that may describe all the possible quantum measurements. However, the consistency problem for their proposal remained. It was not clear whether the class of DL instruments are too general to describe quantum measurements, or whether every DL instrument is consistent with quantum mechanics, or whether every DL instrument has a quantum mechanical model.

The project of gravitational wave detection was one of the earliest attempts to use the new technology for precision measurements. This is an experiment to transfer the effect of gravitational wave on the motion of a macroscopic object to be monitored in time. The effect of gravitational wave is so weak that the monitoring the macroscopic object, called the gravitational wave antenna, is considered as a series of successive measurements of a quantum observable.

In the early days, two types of methods, called the resonator type and the interferometer type, were proposed to detect gravitational waves. Braginsky, Caves and their collaborators [2,3] promoted the resonator type detector. They claimed that there is a sensitivity limit called the standard quantum limit (SQL) for gravitational wave detection due to Heisenberg's uncertainty principle. According to them, however, the SQL can be circumvented by resonator type detectors that monitors the harmonic oscillator position by the technique called the back-action evading measurement based on the theory of quantum non-demolition measurements. In contrast, they claimed that the SQL is an unescapable limit for the interferometer type detector that monitors the free-mass position.

In 1983 Yuen [47] questioned the derivation of the SQL for monitoring of the free-mass position and claimed that the SQL can be broken by a measurement that measures the free-mass position leaving the object in a contractive state that evolves to a near position eigenstate at the time of the next position measurement. However, his claim was not readily accepted, because it was questioned whether a measurement with a sufficient accuracy leaving the free-mass in a contractive state is physically realizable or not. At that time no general way was known at all to judge whether a newly proposed measurement is physically realizable or not.

In fact, Yuen [48] proposed a problem to find a mathematical characterization of all the physically realizable quantum measurements, and he conjectured that the DL instruments are too general. It was quickly realized that this problem was already

solved by the present author [20] showing that all the physically realizable quantum measurements are faithfully characterized by completely positive instruments, the notion which modifies the notion of the Davies-Lewis instrument by requiring complete positivity. Later, Yuen's idea of contractive state measurements was realized by a solvable model that breaks the SQL to the arbitrary accuracy [14,24,26].

This completed von Neumann's axiomatization of quantum mechanics, since it provided the most general measurement axiom, or the most general description of quantum state reductions, consistent with the other axioms.

In this paper, we present a general theory of finite quantum measurements to introduce a general theory of quantum measurements developed as outlined above in a form accessible without sophisticated mathematics for operators in infinite dimensional Hilbert spaces and probability theory for continuous random variables. Thus, we assume that the state space of the measured system is a finite dimensional Hilbert space and that the possible outcomes of a measurement is a finite set of the real numbers. We develop the theory in a deductive manner from the basic postulate for quantum mechanics and a few plausible axioms for general quantum measurements, and we derive an axiomatic characterization of all of the physically realizable finite quantum measurements. Mathematical tools necessary to describe measurement statistics, such as POVMs and quantum instruments, are not assumed at the outset, but we introduce them as natural consequences of our intuitive axioms. Our objective is to show that those mathematical tools can be naturally derived from obvious theoretical requirements. We define observables as physical quantities of the system. We adopt the conventional postulate that observables corresponds to self-adjoint operators on the state space. The notion of POVMs are introduced as statistical properties of measurements at the outset. A POVM is considered to be an observable, a physical quantity, or a physical property belonging to the measured system only if it is a projection valued measure, uniquely associated with a self-adjoint operator on the state space of the system through the spectral decomposition.

As a chapter of a Festschrift celebrating Andrei Khrennikov and the quantum-like revolution, the author hopes that this work would help the readers to extend the scope of quantum measurement theory based on the notion of quantum instruments beyond quantum physics.

2 Quantum Mechanics

In this paper, we consider quantum systems described by finite dimensional Hilbert spaces. Mathematically, a finite dimensional Hilbert space is defined as any finite dimensional linear space \mathcal{H} over the complex number field with inner product (ξ, η) defined for all $\xi, \eta \in \mathcal{H}$, which we assume linear in η and conjugate linear in ξ following the physics convention.

Let \mathcal{H} be a finite dimensional Hilbert space. A linear operator on \mathcal{H} is a linear mapping defined everywhere on \mathcal{H} with values in \mathcal{H} . Denote by $\mathcal{L}(\mathcal{H})$ the space

of linear operators on \mathcal{H} . The adjoint of a linear operator A on \mathcal{H} is a linear operator A^\dagger uniquely determined by the condition $(\xi, A^\dagger \eta) = (A\xi, \eta)$ for all $\xi, \eta \in \mathcal{H}$. A linear operator A is said to be self-adjoint if $A^\dagger = A$, or equivalently $(\xi, A\xi)$ is a real number for all $\xi \in \mathcal{H}$. A linear operator A is said to be positive, in symbols $A \geq 0$, if $(\xi, A\xi) \geq 0$ for all $\xi \in \mathcal{H}$. The trace of a linear operator A is defined by $\text{Tr}[A] = \sum_j (\phi_j, A\phi_j)$ for any orthonormal basis $\{\phi_j\}$.¹ A linear operator ρ is called a density operator if positive and of unit trace, i.e., $\rho \geq 0$ and $\text{Tr}[\rho] = 1$.

Axioms for quantum mechanics of finite level systems without any superselection rules are given as follows.

Axiom Q1 (Quantum systems, states, and observables). *Every quantum system \mathbf{S} is described by a finite dimensional Hilbert space \mathcal{H} called the state space of \mathbf{S} . States of \mathbf{S} are represented by density operators on \mathcal{H} and observables of \mathbf{S} are represented by self-adjoint operators on \mathcal{H} . Every density operator on \mathcal{H} corresponds to a state of \mathbf{S} , and every self-adjoint operator corresponds to an observable of \mathbf{S} .*

By Axiom Q1 we shall identify states with density operators, and observables with self-adjoint operators. The state of the form $\rho = |\psi\rangle\langle\psi|$ is called a pure state.² If \mathbf{S} is in the state $\rho = |\psi\rangle\langle\psi|$, \mathbf{S} is said to be in the (vector) state ψ . We denote by $\mathcal{S}(\mathcal{H})$ the space of states, or density operators, on \mathcal{H} and by $\mathcal{O}(\mathcal{H})$ the space of observables, or self-adjoint operators, on \mathcal{H} .

Axiom Q2 (Born statistical formula). *If an observable A is measured in a state ρ , the outcome obeys the probability distribution of A in ρ defined by*

$$\Pr\{\mathbf{x} = x | \rho\} = \text{Tr}[P^A(x)\rho] \quad (1)$$

where $x \in \mathbb{R}$, and $P^A(x)$ stands for the projection onto the subspace $\{\psi \in \mathcal{H} | A\psi = x\psi\}$.

The projection $P^A(x)$ is called the spectral projection of A for the real number x . The map $P^A : x \mapsto P^A(x)$ is called the spectral measure of A [7]. From Axiom Q2, the mean value $\langle A \rangle$ and the standard deviation $\sigma(A)$ are given by

$$\langle A \rangle = \text{Tr}[A\rho], \quad (2)$$

$$\sigma(A)^2 = \langle A^2 \rangle - \langle A \rangle^2. \quad (3)$$

The standard deviations $\sigma(A), \sigma(B)$ of observables A, B in a state ρ satisfy Robertson's inequality

¹ The trace is independent of the choice of the orthonormal basis $\{\phi_j\}$. To see this let $\{\xi_k\}$ be another orthonormal basis. Then, we have $\text{Tr}[A] = \sum_j (\phi_j, A\phi_j) = \sum_{j,k} (\phi_j, \xi_k)(\xi_k, A\phi_j) = \sum_{j,k} (A^\dagger \xi_k, \phi_j)(\phi_j, \xi_k) = \sum_k (A^\dagger \xi_k, \xi_k) = \sum_k (\xi_k, A\xi_k)$. Thus, the trace is independent of the choice of the orthonormal basis.

² For any $\xi, \eta \in \mathcal{H}$, the operator $|\xi\rangle\langle\eta|$ is defined by $(|\xi\rangle\langle\eta|)\psi = (\eta, \psi)\xi$ for all $\psi \in \mathcal{H}$.

$$\sigma(A)\sigma(B) \geq \frac{1}{2} |\langle [A, B] \rangle|. \quad (4)$$

Axiom Q3 (Time evolution). Suppose that a system \mathbf{S} is an isolated system with the (time-independent) Hamiltonian H from time t to $t + \tau$. The system \mathbf{S} is in a state $\rho(t)$ at time t if and only if \mathbf{S} is in the state $\rho(t + \tau)$ at time $t + \tau$ satisfying

$$\rho(t + \tau) = e^{-i\tau H/\hbar} \rho(t) e^{i\tau H/\hbar}, \quad (5)$$

where \hbar stands for the Planck constant divided by 2π .

Axiom Q4 (Composite systems). The state space of the composite system $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ of a system \mathbf{S}_1 with the state space \mathcal{H} and a system \mathbf{S}_2 with the state space \mathcal{K} is given by the tensor product $\mathcal{H} \otimes \mathcal{K}$. The observable A of \mathbf{S}_1 is identified with the observable $A \otimes I$ of \mathbf{S} and the observable B of \mathbf{S}_2 is identified with $I \otimes B$ of \mathbf{S} .

For any orthonormal bases $\{\xi_j\}$ of \mathcal{H} and $\{\eta_k\}$ of \mathcal{K} , the family of their tensor products $\{\xi_j \otimes \eta_k\}$ forms an orthonormal basis of $\mathcal{H} \otimes \mathcal{K}$. The tensor product of $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$ is the linear operator $A \otimes B \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ defined by $(A \otimes B)(\xi \otimes \eta) = A\xi \otimes B\eta$ for all $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$. Every $A \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ is of the form $A = \sum_{j,k} C_j \otimes D_k$ where $C_j \in \mathcal{L}(\mathcal{H})$ and $D_k \in \mathcal{L}(\mathcal{K})$. The partial traces $\text{Tr}_{\mathcal{K}}[A]$ over \mathcal{K} and $\text{Tr}_{\mathcal{H}}[A]$ over \mathcal{H} of $A \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ are defined by $\text{Tr}_{\mathcal{K}}[A] = \sum_{j,k} C_j \otimes \text{Tr}[D_k]$ and $\text{Tr}_{\mathcal{H}}[A] = \sum_{j,k} \text{Tr}[C_j] \otimes D_k$ if $A = \sum_{j,k} C_j \otimes D_k$. It follows from Axiom Q4 that if the system $\mathbf{S}_1 + \mathbf{S}_2$ is in a state ρ_{12} , then the system \mathbf{S}_1 is in the state $\rho_1 = \text{Tr}_{\mathcal{K}}[\rho_{12}]$ and \mathbf{S}_2 is in the state $\rho_2 = \text{Tr}_{\mathcal{H}}[\rho_{12}]$.

3 Statistical properties of measuring apparatuses

In this section, we discuss statistical properties of measuring apparatus. We introduce plausible axioms of statistical properties of measuring apparatus required for every apparatus to satisfy, and then we show that statistical properties of physically realizable measuring apparatus can be naturally described by POVMs and completely positive instruments.

3.1 Output probability distributions

Axiom M1 (Output probability distributions and quantum state reductions). An apparatus $\mathbf{A}(\mathbf{x})$ with output variable \mathbf{x} to measure a system \mathbf{S} determines the probability $\Pr\{\mathbf{x} = x | \rho\}$ of the outcome $\mathbf{x} = x$ of the measurement depending on the input state ρ (the state of \mathbf{S} just before the measurement), and determines the output

state $\rho_{\{\mathbf{x}=x\}}$ (the state of \mathbf{S} just after the measurement) depending on the input state ρ and the outcome $\mathbf{x} = x$ of the measurement.

The variable representing the outcome of the apparatus is called the **output variable**. Let \mathbf{S} be a quantum system, to be referred to the *object*, described by a Hilbert space \mathcal{H} of state vectors. Let $\mathbf{A}(\mathbf{x})$ be a *measuring apparatus* with an output variable \mathbf{x} to measure the object \mathbf{S} . We assume that \mathbf{x} takes values in the real line \mathbb{R} . For any real number $x \in \mathbb{R}$, we shall denote by “ $\mathbf{x} = x$ ” the probabilistic event that the output variable of apparatus $\mathbf{A}(\mathbf{x})$ takes the value x . By Axiom M1, the probability distribution of the output variable \mathbf{x} is determined by the input state ρ . Denote it by $\Pr\{\mathbf{x} = x \|\rho\}$, and call it as the output distribution of $\mathbf{A}(\mathbf{x})$ in the input state ρ . If the state ρ is a vector state $\rho = |\psi\rangle\langle\psi|$, we also write $\Pr\{\mathbf{x} = x \|\psi\} = \Pr\{\mathbf{x} = x \|\rho\}$. For any subset $\Delta \subseteq \mathbb{R}$, we define $\Pr\{\mathbf{x} \in \Delta \|\rho\} = \sum_{x \in \Delta} \Pr\{\mathbf{x} = x \|\rho\}$. We suppose that the output distribution satisfies the following conditions.

- (i) **Positivity:** $\Pr\{\mathbf{x} = x \|\rho\} \geq 0$ for every $x \in \mathbb{R}$.
- (ii) **Unity:** $\sum_{x \in \mathbb{R}} \Pr\{\mathbf{x} = x \|\rho\} = 1$.
- (iii) **Finiteness:** There exists a finite subset of $S \subseteq \mathbb{R}$ such that if $x \notin S$ then $\Pr\{\mathbf{x} = x \|\rho\} = 0$ for any state ρ .

The output probability distribution should satisfy the following postulate.

Axiom M2' (Mixing law of output probability). For any apparatus $\mathbf{A}(\mathbf{x})$, the function $\rho \mapsto \Pr\{\mathbf{x} = x \|\rho\}$ is an affine function of states ρ for any real number x .

This means that we have

$$\Pr\{\mathbf{x} = x \| p\rho_1 + (1-p)\rho_2\} = p\Pr\{\mathbf{x} = x \|\rho_1\} + (1-p)\Pr\{\mathbf{x} = x \|\rho_2\},$$

where ρ_1 and ρ_2 are density operators and $0 < p < 1$.

The above axiom is justified by the following interpretation of the mixture of states: The system \mathbf{S} is in the state $p\rho_1 + (1-p)\rho_2$ if it is in state ρ_1 with probability p and in state ρ_2 with probability $1-p$.

3.2 Probability operator-valued measures

In order to characterize output probability distributions, we introduce a mathematical definition. A mapping $\Pi : x \mapsto \Pi(x)$ of \mathbb{R} into the space $\mathcal{L}(\mathcal{H})$ of linear operators on \mathcal{H} is called a *probability operator-valued measure (POVM)*, also known as a *positive operator-valued measure* or a *probability operator measure (POM)* [8], if the following conditions are satisfied:

- (i) **Positivity:** $\Pi(x) \geq 0$ for all $x \in \mathbb{R}$.
- (ii) **Unity:** $\sum_{x \in \mathbb{R}} \Pi(x) = 1$.
- (iii) **Finiteness:** There exists a finite subset of $S \subseteq \mathbb{R}$ such that if $x \notin S$ then $\Pi(x) = 0$.

One of important consequences from the above postulate is the following characterization of output probability distributions [18].

Theorem 1 *Axiom M2' (the mixing law of output probability) holds if and only if for any apparatus $\mathbf{A}(\mathbf{x})$, there uniquely exists a POVM Π satisfying*

$$\Pr\{\mathbf{x} = x | \rho\} = \text{Tr}[\Pi(x)\rho] \quad (6)$$

for any real number x and density operator ρ .

The POVM Π defined by Eq. (6) is called the *POVM of $\mathbf{A}(\mathbf{x})$* .

3.3 The Born statistical formula

Let A be an observable of system \mathbf{S} . According to Axiom Q2 (Born statistical formula), we say that apparatus $\mathbf{A}(\mathbf{x})$ satisfies the *Born statistical formula (BSF)* for observable A on input state ρ , if we have

$$\Pr\{\mathbf{x} = x | \rho\} = \text{Tr}[P^A(x)\rho] \quad (7)$$

for every real number x , where $P^A(x)$ is the spectral projection of A for x . From Eqs. (6) and (7), apparatus $\mathbf{A}(\mathbf{x})$ satisfies the Born statistical formula on every input state if and only if the POVM Π of $\mathbf{A}(\mathbf{x})$ is the spectral measure P^A . In this case, the apparatus $\mathbf{A}(\mathbf{x})$ is called an *A-measuring apparatus*. Naturally, we assume that *for every observable A of \mathbf{S} there is at least one A-measuring apparatus*.

The Born statistical formula is a necessary condition for the state-dependent accuracy of measurement, or for the accurate measurements of an observable A in a single state ρ . For a necessary and sufficient condition for the state-dependent accuracy of measurement, we refer the reader to [37–39]. According to those studies of the state-dependent accuracy of measurement, an apparatus accurately measures an observable A in every state ρ if and only if the Born statistical formula holds for every state ρ . Thus, the state-dependent accuracy of measurement required for every state is consistent with the conventional state-independent accuracy of measurement.

3.4 Quantum state reductions

According to Axiom M1, depending on the input state ρ , any apparatus $\mathbf{A}(\mathbf{x})$ determines the state $\rho_{\{\mathbf{x}=x\}}$ just after the measurement for any possible outcome $\mathbf{x} = x$ with $\Pr\{\mathbf{x} = x | \rho\} > 0$. The operational meaning of the state $\rho_{\{\mathbf{x}=x\}}$ is given as follows. If a measurement using the apparatus $\mathbf{A}(\mathbf{x})$ on input state ρ is immediately followed by a measurement using another apparatus $\mathbf{A}(\mathbf{y})$ with output variable \mathbf{y} , we shall denote by $\Pr\{\mathbf{y} = y | \mathbf{x} = x | \rho\}$ the conditional probability of the outcome

$\mathbf{y} = y$ of the measurement using $\mathbf{A}(\mathbf{y})$ given the outcome $\mathbf{x} = x$ of the measurement using $\mathbf{A}(\mathbf{x})$. Then, under the condition $\mathbf{x} = x$ the state just before the measurement using $\mathbf{A}(\mathbf{y})$ is the state $\rho_{\{\mathbf{x}=x\}}$ so that we naturally have

$$\Pr\{\mathbf{y} = y | \mathbf{x} = x | \rho\} = \Pr\{\mathbf{y} = y | \rho_{\{\mathbf{x}=x\}}\}. \quad (8)$$

If $\Pr\{\mathbf{x} = x | \rho\} = 0$, the state $\rho_{\{\mathbf{x}=x\}}$ is taken to be indefinite. The state $\rho_{\{\mathbf{x}=x\}}$ is called the *output state of the apparatus $\mathbf{A}(\mathbf{x})$ given the outcome $\mathbf{x} = x$ on input state ρ* .

Two apparatuses are called *statistically equivalent* if they have the same output probabilities and the same output states for any outcomes and any input states.

3.5 Joint output probability distributions

If a measurement using apparatus $\mathbf{A}(\mathbf{x})$ on input state ρ is immediately followed by a measurement using apparatus $\mathbf{A}(\mathbf{y})$, then from Eq. (8) the joint probability distribution $\Pr\{\mathbf{x} = x, \mathbf{y} = y | \rho\}$ of the output variables \mathbf{x} and \mathbf{y} is given by

$$\Pr\{\mathbf{x} = x, \mathbf{y} = y | \rho\} = \Pr\{\mathbf{y} = y | \rho_{\{\mathbf{x}=x\}}\} \Pr\{\mathbf{x} = x | \rho\}. \quad (9)$$

Thus, the joint probability distribution of outputs of successive measurements depends only on the input state of the first measurement. We shall call the above joint probability distribution the *joint output probability distribution of $\mathbf{A}(\mathbf{x})$ followed by $\mathbf{A}(\mathbf{y})$* . The joint output probability distributions should satisfy the following axiom:

Axiom M2 (Mixing law of joint output probability). *For any apparatuses $\mathbf{A}(\mathbf{x})$ and $\mathbf{A}(\mathbf{y})$, the function $\rho \mapsto \Pr\{\mathbf{x} = x, \mathbf{y} = y | \rho\}$ is an affine function of density operators ρ for any real numbers x, y .*

Since the joint probability $\Pr\{\mathbf{x} = x, \mathbf{y} = y | \rho\}$ depends on the initial input state ρ , the above axiom is also justified by the following interpretation of the mixture of states: The system \mathbf{S} is in the state $p\rho_1 + (1 - p)\rho_2$ if it is in state ρ_1 with probability p and in state ρ_2 with probability $1 - p$.

By summing up y in Eq. (9) and using the unity relation $\sum_{y \in \mathbb{R}} \Pr\{\mathbf{y} = y | \rho_{\{\mathbf{x}=x\}}\} = 1$, we have

$$\sum_{y \in \mathbb{R}} \Pr\{\mathbf{x} = x, \mathbf{y} = y | \rho\} = \Pr\{\mathbf{x} = x | \rho\} \quad (10)$$

for any x and ρ . Thus, we conclude that *Axiom M2 (the mixing law of joint output probability) implies Axiom M2' (the mixing law of output probability)*.

3.6 Instruments

Davies and Lewis [4,5] introduced the following mathematical notion for unified description of statistical properties of measurements. A mapping $\mathcal{I} : x \mapsto \mathcal{I}(x)$ of \mathbb{R} into the space $\mathcal{L}(\mathcal{L}(\mathcal{H}))$ of linear transformations on $\mathcal{L}(\mathcal{H})$ is called a *Davies-Lewis (DL) instrument*, if the following conditions are satisfied.

(i) **Positivity:** $\mathcal{I}(x)$ maps positive operators in $\mathcal{L}(\mathcal{H})$ to positive operators in $\mathcal{L}(\mathcal{H})$ for any $x \in \mathbb{R}$.

(ii) **Unity:** $\sum_{x \in \mathbb{R}} \mathcal{I}(x)$ is trace-preserving.

(iii) **Finiteness:** There exists a finite subset $S \subset \mathbb{R}$ such that $\mathcal{I}(x) = 0$ for all $x \in S$.

For any apparatus $\mathbf{A}(\mathbf{x})$, we define the mapping $\mathcal{I}(x) : \rho \mapsto \mathcal{I}(x)\rho$ by

$$\mathcal{I}(x)\rho = \Pr\{\mathbf{x} = x | \rho\} \rho_{\{\mathbf{x}=x\}}, \quad (11)$$

where $\rho \in \mathcal{S}(\mathcal{H})$ and $x \in \mathbb{R}$. The mapping $\mathcal{I}(x)$ transforms any density operator ρ to a positive operator with the trace equal to $\Pr\{\mathbf{x} = x | \rho\}$. It follows from Axiom M2 (the mixing law of joint output probability) that $\mathcal{I}(x)$ is an affine mapping and can be extended to a linear transformation on the space $\mathcal{L}(\mathcal{H})$ of linear operators on \mathcal{H} [31,34]. Then it is easy to see that the mapping $\mathcal{I}(x) : \rho \mapsto \mathcal{I}(x)\rho$ satisfies the Davies and Lewis definition of instruments. Conversely, if any apparatus $\mathbf{A}(\mathbf{x})$ has a Davies-Lewis instrument \mathcal{I} satisfying Eq. (11), then Axiom M2 (the mixing law of joint output probability) holds. Thus, we have [34]

Theorem 2 *Axiom M2 (the mixing law of joint output probability) holds if and only if for any apparatus $\mathbf{A}(\mathbf{x})$ there uniquely exists a DL instrument \mathcal{I} satisfying Eq. (11) for any real number x and density operator ρ .*

The mapping $\mathcal{I}(x)$ defined by Eq. (11) for the apparatus $\mathbf{A}(\mathbf{x})$ is called the *operation of $\mathbf{A}(\mathbf{x})$ given the outcome $\mathbf{x} = x$* . The mapping \mathcal{I} is called the *instrument of $\mathbf{A}(\mathbf{x})$* . Then, the output probability and the output state can be expressed by

$$\Pr\{\mathbf{x} = x | \rho\} = \text{Tr}[\mathcal{I}(x)\rho], \quad (12)$$

$$\rho_{\{\mathbf{x}=x\}} = \frac{\mathcal{I}(x)\rho}{\text{Tr}[\mathcal{I}(x)\rho]} \quad (13)$$

where the second relation assumes $\Pr\{\mathbf{x} = x | \rho\} > 0$. Thus, if $\mathcal{I}_{\mathbf{x}}$ and $\mathcal{I}_{\mathbf{y}}$ are the instruments of $\mathbf{A}(\mathbf{x})$ and $\mathbf{A}(\mathbf{y})$, respectively, then the joint output probability distribution can be expressed by

$$\Pr\{\mathbf{x} = x, \mathbf{y} = y | \rho\} = \text{Tr}[\mathcal{I}_{\mathbf{y}}(y)\mathcal{I}_{\mathbf{x}}(x)\rho] \quad (14)$$

for any state ρ and any real numbers x, y .

Both the output probability distribution and the output states are determined by the instrument. Thus, two apparatuses are statistically equivalent if and only if they have the same instrument.

For any linear transformation T on the space $\mathcal{L}(\mathcal{H})$ of linear operators on \mathcal{H} , the *dual* of T is defined to be the linear transformation T^* on $\mathcal{L}(\mathcal{H})$ satisfying

$$\text{Tr}[A(T\rho)] = \text{Tr}[(T^*A)\rho] \quad (15)$$

for any $A, \rho \in \mathcal{L}(\mathcal{H})$. The dual of the operation $\mathcal{J}(x)$ is called the *dual operation* $\mathcal{J}(x)^*$ given $\mathbf{x} = x$; by Eq. (15) it is defined by the relation

$$\text{Tr}[A\mathcal{J}(x)\rho] = \text{Tr}\{[\mathcal{J}(x)^*A]\rho\} \quad (16)$$

for any $A, \rho \in \mathcal{L}(\mathcal{H})$.

The operator $\mathcal{J}(x)^*I$ obtained by applying the dual operation $\mathcal{J}(x)^*$ to the identity operator I is called the *effect of the operation* $\mathcal{J}(x)$. By Eq. (12) and Eq. (15) we have

$$\Pr\{\mathbf{x} = x | \rho\} = \text{Tr}[(\mathcal{J}(x)^*I)\rho]. \quad (17)$$

Since ρ is arbitrary, comparing with Eq. (6), we have

$$\Pi(x) = \mathcal{J}(x)^*I \quad (18)$$

for any real number x . Thus, the POVM of $\mathbf{A}(\mathbf{x})$ is determined by the effects of the instrument \mathcal{J} .

Let $\mathcal{J}_{\mathbf{x}}$ and $\mathcal{J}_{\mathbf{y}}$ be the instruments of $\mathbf{A}(\mathbf{x})$ and $\mathbf{A}(\mathbf{y})$, respectively, and let $\Pi_{\mathbf{y}}$ be the POVM of $\mathbf{A}(\mathbf{y})$. Then, we have

$$\begin{aligned} \text{Tr}[\mathcal{J}_{\mathbf{y}}(y)\mathcal{J}_{\mathbf{x}}(x)\rho] &= \text{Tr}\{[\mathcal{J}_{\mathbf{y}}(y)^*I][\mathcal{J}_{\mathbf{x}}(x)\rho]\} \\ &= \text{Tr}\{[\Pi_{\mathbf{y}}(y)][\mathcal{J}_{\mathbf{x}}(x)\rho]\} \\ &= \text{Tr}\{[\mathcal{J}(x)^*\Pi_{\mathbf{y}}(y)]\rho\} \end{aligned} \quad (19)$$

Thus, the joint output probability distribution can be expressed by

$$\Pr\{\mathbf{x} = x, \mathbf{y} = y | \rho\} = \text{Tr}\{[\mathcal{J}(x)^*\Pi_{\mathbf{y}}(y)]\rho\} \quad (20)$$

for any $x, y \in \mathbb{R}$.

3.7 Selective quantum state reduction

For any subset Δ of \mathbb{R} , the outcome event “ $\mathbf{x} \in \Delta$ ” means that the outcome of the measurement is an element of Δ . The probability of outcome event $\mathbf{x} \in \Delta$ is given by

$$\Pr\{\mathbf{x} \in \Delta | \rho\} = \sum_{x \in \Delta} \Pr\{\mathbf{x} = x | \rho\}. \quad (21)$$

If the input state is ρ , the state just after the measurement given the outcome event $\mathbf{x} \in \Delta$ is denoted by $\rho_{\{\mathbf{x} \in \Delta\}}$. This state is determined as follows. Let A be an observ-

able of the object \mathbf{S} . Suppose that the observer measures the object \mathbf{S} in the state $\rho_{\{\mathbf{x} \in \Delta\}}$ using another apparatus $\mathbf{A}(\mathbf{y})$ with the POVM $\Pi_{\mathbf{y}} = P^A$. Then we have

$$\begin{aligned} \Pr\{\mathbf{x} \in \Delta, \mathbf{y} = y | \rho\} &= \Pr\{\mathbf{y} = y | \rho_{\{\mathbf{x} \in \Delta\}}\} \Pr\{\mathbf{x} \in \Delta | \rho\} \\ &= \text{Tr}[P^A(y) \Pr\{\mathbf{x} \in \Delta | \rho\} \rho_{\{\mathbf{x} \in \Delta\}}] \end{aligned} \quad (22)$$

On the other hand, by Eq. (9) we have

$$\begin{aligned} \Pr\{\mathbf{x} \in \Delta, \mathbf{y} = y | \rho\} &= \sum_{x \in \Delta} \Pr\{\mathbf{x} = x, \mathbf{y} = y | \rho\} \\ &= \sum_{x \in \Delta} \Pr\{\mathbf{y} = y | \rho_{\{\mathbf{x} = x\}}\} \Pr\{\mathbf{x} = x | \rho\} \\ &= \text{Tr}[P^A(y) \sum_{x \in \Delta} \Pr\{\mathbf{x} = x | \rho\} \rho_{\{\mathbf{x} = x\}}]. \end{aligned} \quad (23)$$

Since A is an arbitrary observable, by comparing Eq. (22) and Eq. (23), we have

$$\Pr\{\mathbf{x} \in \Delta | \rho\} \rho_{\{\mathbf{x} \in \Delta\}} = \sum_{x \in \Delta} \Pr\{\mathbf{x} = x | \rho\} \rho_{\{\mathbf{x} = x\}}. \quad (24)$$

For any subset Δ of \mathbb{R} , we write $\mathcal{J}(\Delta) = \sum_{x \in \Delta} \mathcal{J}(x)$ and $\Pi(\Delta) = \sum_{x \in \Delta} \Pi(x)$. Let \mathcal{J} be the instrument of an apparatus $\mathbf{A}(\mathbf{x})$. For any state ρ we have

$$\mathcal{J}(\Delta)\rho = \Pr\{\mathbf{x} \in \Delta | \rho\} \rho_{\{\mathbf{x} \in \Delta\}}. \quad (25)$$

$\mathcal{J}(\Delta)$ is called *the operation given the outcome event $\mathbf{x} \in \Delta$ of the apparatus $\mathbf{A}(\mathbf{x})$* .

The state change from the state ρ to the state $\rho_{\{\mathbf{x} = x\}}$ is called an (*individual*) *quantum state reduction*. The state change from the state ρ to the state $\rho_{\{\mathbf{x} \in \Delta\}}$ is called a *selective quantum state reduction*. On the other hand, the state change $\rho \mapsto \rho_{\{\mathbf{x} \in \mathbb{R}\}}$ is called a *non-selective quantum state reduction*. For the instrument \mathcal{J} of the apparatus $\mathbf{A}(\mathbf{x})$, the operation $T = \mathcal{J}(\mathbb{R})$ is called the *non-selective operation* of $\mathbf{A}(\mathbf{x})$, and $T^* = \mathcal{J}(\mathbb{R})^*$ is called the *non-selective dual operation* of $\mathbf{A}(\mathbf{x})$. In general a linear transformation T on $\mathcal{L}(\mathcal{H})$ is called a *positive map* if $T\rho \geq 0$ for all $\rho \geq 0$. For any DL instrument \mathcal{J} , $\mathcal{J}(\Delta)$ is a positive map. The non-selective operation T is trace-preserving, i.e.,

$$\text{Tr}[T\rho] = \text{Tr}[\rho], \quad (26)$$

for any $\rho \in \mathcal{L}(\mathcal{H})$, while the non-selective dual operation T^* is unit-preserving,

$$T^*I = I. \quad (27)$$

3.8 Repeatability Hypothesis

In the early days of quantum mechanics, only a restricted class of measurements was seriously studied. The following axiom was broadly accepted in the 1930s.

(M) Measurement axiom. If an observable A is measured in a system \mathbf{S} to obtain the outcome a , then the system \mathbf{S} is left in an eigenstate of A for the eigenvalue a .

Von Neumann [16] showed that this assumption is equivalent to the following assumption called the *repeatability hypothesis* [16, p. 335], posed with a clear operational condition generalizing a feature of the Compton-Simons experiment [16, pp. 212–214].

(R) Repeatability hypothesis. *If an observable A is measured twice in succession in a system \mathbf{S} , then we get the same value each time.*

It can be seen from the following definition of measurement due to Schrödinger given in his famous “cat paradox” paper [41] that von Neumann’s repeatability hypothesis was broadly accepted in the 1930s.

The systematically arranged interaction of two systems (measured object and measuring instrument) is called a measurement on the first system, if a directly-sensible variable feature of the second (pointer position) is always reproduced within certain error limits when the process is immediately repeated (on the same object, which in the meantime must not be exposed to any additional influences) [41].

The repeatability hypothesis uniquely determines the state after the measurement if the measured observable A is non-degenerate, i.e., every eigensubspace is one-dimensional. Let $A = \sum_n a_n |\phi_n\rangle\langle\phi_n|$ be a non-degenerate observable, where $|\phi_n\rangle\langle\phi_n|$ stands for the projection onto the subspace spanned by the eigenvector ϕ_n for the eigenvalue a_n . Then the measuring apparatus satisfies the repeatability hypothesis if and only if the corresponding instrument is of the form:

$$\mathcal{I}(x)\rho = |\phi_n\rangle\langle\phi_n| \rho |\phi_n\rangle\langle\phi_n| \quad (28)$$

for any $\rho \in \mathcal{S}(\mathcal{H})$ if $x = a_n$; and $\mathcal{I}(x) = 0$ otherwise.

If the measured observable is degenerate, the repeatability hypothesis does not determine the unique eigenstate as the state after the measurement. Lüders [11] proposed the projection postulate to determine the eigenstate uniquely.

(P) The von Neumann-Lüders projection postulate. If a measurement of an observable A in a state ρ leads to the outcome $\mathbf{x} = x$, the state $\rho_{\{\mathbf{x}=x\}}$ just after the measurement is given by

$$\rho_{\{\mathbf{x}=x\}} = \frac{P^A(x)\rho P^A(x)}{\text{Tr}[P^A(x)\rho]}. \quad (29)$$

Thus, the projection postulate uniquely determines the instrument for measurement of A as

$$\mathcal{I}(x)\rho = P^A(x)\rho P^A(x) \quad (30)$$

for all $x \in \mathbb{R}$ and $\rho \in \mathcal{S}(\mathcal{H})$.

It is well known that the same observable can be measured with many different ways that do not satisfy the projection postulate. Thus, the von Neumann-Lüders projection postulate should not be taken as a universal postulate for quantum mechanics but should be taken as a defining condition for a class of measurements called projective measurements.

For any sequence of projective measurements, we can determine the joint probability distribution of the outcomes of measurements [46].

Theorem 3 (Wigner's formula) *Let A_1, \dots, A_n be observables with a discrete spectrum of a system \mathbf{S} in a state ρ at time 0. If one carries out projective measurements of observables A_1, \dots, A_n at times $(0 <) t_1 < \dots < t_n$ and otherwise leaves the system \mathbf{S} isolated with the Hamiltonian H , then the joint probability distribution of the outcomes $\mathbf{x}_1, \dots, \mathbf{x}_n$ of those measurements is given by*

$$\begin{aligned} \Pr\{\mathbf{x}_1 = x_1, \dots, \mathbf{x}_n = x_n \mid \rho\} = & \text{Tr}[E^{A_n}(\{x_n\}) \cdots U(t_2 - t_1) E^{A_1}(\{x_1\}) U(t_1) \rho \\ & \times U(t_1)^\dagger E^{A_1}(\{x_1\}) U(t_2 - t_1)^\dagger \cdots E^{A_n}(\{x_n\})], \end{aligned} \quad (31)$$

where $U(t) = e^{-iHt/\hbar}$.

3.9 Abandoning the Repeatability Hypothesis

The repeatability hypothesis applies only to a restricted class of measurements and does not generally characterize the state changes caused by quantum measurements. In fact, there exist commonly used measurements of discrete observables, such as photon counting, that do not satisfy the repeatability hypothesis [10]. Moreover, it has been shown that the repeatability hypothesis cannot be generalized to continuous observables in the standard formulation of quantum mechanics [20, 22, 25, 44]. In 1970, Davies and Lewis [5] proposed abandoning the repeatability hypothesis and introduced a new mathematical framework to treat all the physically realizable quantum measurements.

One of the crucial notions is that of repeatability which we show is implicitly assumed in most of the axiomatic treatments of quantum mechanics, but whose abandonment leads to a much more flexible approach to measurement theory [5, p. 239].

The proposal of Davies and Lewis [5] can be stated as follows.

(DL) The Davies-Lewis thesis. *For every measuring apparatus $\mathbf{A}(\mathbf{x})$ with output variable \mathbf{x} there exists a unique DL instrument \mathcal{I} satisfying*

$$\Pr\{\mathbf{x} = x \mid \rho\} = \text{Tr}[\mathcal{J}(x)\rho], \quad (32)$$

$$\rho \rightarrow \rho_{\{\mathbf{x}=x\}} = \frac{\mathcal{J}(x)\rho}{\text{Tr}[\mathcal{J}(x)\rho]}. \quad (33)$$

We have previously shown that under Axiom M1, the Davies-Lewis thesis is equivalent to Axiom M2 (the mixing law of joint output probability).

3.10 Complete positivity

Is every Davies-Lewis instrument physically relevant? We shall show that this is not the case. In physics various phenomena can be described by mathematical models, and even in a single physical theory a single phenomenon can be modeled by various different mathematical models. Nevertheless, we should have consistency relations among all the models describing a single physical phenomenon such as invariance under the change of coordinate systems. In quantum measurement theory, a single measuring apparatus can have different models even with a fixed coordinate system, according to the arbitrariness of the spacial boundary of the measured object. As described by Axiom Q4, the observable A in a system \mathbf{S} can have a different mathematical representative $A \otimes I$ in a larger system $\mathbf{S} + \mathbf{S}'$. Thus, an apparatus measuring the observable A is accompanied by another model describing it as an apparatus measuring the observable $A \otimes I$. Namely, any model of an apparatus measuring the system \mathbf{S} is always accompanied by the model for an apparatus measuring the system $\mathbf{S} + \mathbf{S}'$. It is interesting that this rather obvious fact leads to an important common property of measuring apparatuses.

Axiom M3 (Extendability). *For any apparatus $\mathbf{A}(\mathbf{x})$ measuring a system \mathbf{S} and any quantum system \mathbf{S}' not interacting with $\mathbf{A}(\mathbf{x})$ nor \mathbf{S} , there exists an apparatus $\mathbf{A}(\mathbf{x}')$ measuring system $\mathbf{S} + \mathbf{S}'$ with the following statistical properties:*

$$\Pr\{\mathbf{x}' = x \mid \rho \otimes \rho'\} = \Pr\{\mathbf{x} = x \mid \rho\}, \quad (34)$$

$$(\rho \otimes \rho')_{\{\mathbf{x}'=x\}} = \rho_{\{\mathbf{x}=x\}} \otimes \rho' \quad (35)$$

for any $x, x' \in \mathbb{R}$, any state ρ of \mathbf{S} , and any state ρ' of \mathbf{S}' .

The above postulate is justified as follows. Suppose that apparatus $\mathbf{A}(\mathbf{x})$ measures object \mathbf{S} . Let \mathbf{S}' be any system, described by a Hilbert space \mathcal{H}' , remote from \mathbf{S} and $\mathbf{A}(\mathbf{x})$. Naturally, the apparatus $\mathbf{A}(\mathbf{x})$ makes no measurement on \mathbf{S}' but then the same apparatus can be formally described as an apparatus $\mathbf{A}(\mathbf{x}')$ measuring the system $\mathbf{S} + \mathbf{S}'$ with the following statistical properties:

$$\Pr\{\mathbf{x}' = x \mid \rho \otimes \rho'\} = \Pr\{\mathbf{x} = x \mid \rho\}, \quad (36)$$

$$(\rho \otimes \rho')_{\{\mathbf{x}'=x\}} = \rho_{\{\mathbf{x}=x\}} \otimes \rho' \quad (37)$$

for any real number x , any state ρ of \mathbf{S} and any state ρ' of \mathbf{S}' . Let \mathcal{I}' be the instrument of the apparatus $\mathbf{A}(\mathbf{x}')$. Then, we have

$$\begin{aligned}\mathcal{I}'(x)(\rho \otimes \rho') &= \Pr\{\mathbf{x}' = x \mid \rho \otimes \rho'\}(\rho \otimes \rho')_{\{\mathbf{x}'=x\}} \\ &= \Pr\{\mathbf{x} = x \mid \rho\} \rho_{\{\mathbf{x}=x\}} \otimes \rho' \\ &= [\mathcal{I}(x)\rho] \otimes \rho'.\end{aligned}\tag{38}$$

It follows that the operation $\mathcal{I}'(x)$ of the extended apparatus $\mathbf{A}(\mathbf{x}')$ given $\mathbf{x}' = x$ is represented by $\mathcal{I}'(x) = \mathcal{I}(x) \otimes \text{id}$, where id stands for the identity map on $\mathcal{L}(\mathcal{H}')$.

Let \mathcal{H}' be a finite dimensional Hilbert space. Any linear transformation T on $\mathcal{L}(\mathcal{H})$ can be extended naturally to the linear transformation $T \otimes \text{id}_{\mathcal{H}'}$ on $\mathcal{L}(\mathcal{H} \otimes \mathcal{H}') = \mathcal{L}(\mathcal{H}) \otimes \mathcal{L}(\mathcal{H}')$ by

$$(T \otimes \text{id})(\sum_j \rho_j \otimes \rho'_j) = \sum_j T(\rho_j) \otimes \rho'_j\tag{39}$$

for any $\rho_j \in \mathcal{L}(\mathcal{H})$ and $\rho'_j \in \mathcal{L}(\mathcal{H}')$. Then, T is called *completely positive (CP)*, if $T \otimes \text{id}$ maps positive operators in $\mathcal{L}(\mathcal{H} \otimes \mathcal{H}')$ to positive operators in $\mathcal{L}(\mathcal{H} \otimes \mathcal{H}')$ for any \mathcal{H}' . A DL instrument \mathcal{I} is called a *completely positive (CP) instrument*, if the operation $\mathcal{I}(x)$ is CP for every $x \in \mathbb{R}$.

Then, from the positivity of the operation $\mathcal{I}'(x)$, the complete positivity of the original operation $\mathcal{I}(x)$ follows. Thus, Axiom M3 (Extendability) is equivalent to the following Axiom M3' (Complete positivity).

Axiom M3' (Complete positivity). *The instrument of every apparatus should be a CP instrument.*

We have posed two plausible requirements for the measurement statistics to be satisfied by any apparatus, the mixing law of joint output probability distributions $\Pr\{\mathbf{x} = x, \mathbf{y} = y \mid \rho\}$ and the extendability of measurement statistics, as a set of necessary conditions for every apparatus to satisfy. Under these conditions, we have shown that every apparatus corresponds uniquely to a CP instrument that determines the output probability distributions and the quantum state reduction caused by the apparatus. Thus, the problem of determining physically possible output probability distributions and quantum state reductions is reduced to the problem as to which CP instrument corresponds to a physically realizable apparatus. This problem will be discussed in the next section and it will be shown that every CP instruments corresponds to a physically realizable apparatus.

Now we note that there indeed exists a DL instrument that is not a CP instrument. The transpose operation of matrices in a fix basis is a typical example of a positive linear map which is not CP [17]. Let T be a transpose operation on $\mathcal{L}(\mathcal{H})$ for \mathcal{H} , and let $\mu(x)$ be any probability distribution supported in a finite subset of \mathbb{R} , i.e., there exists a finite subset $S \subseteq \mathbb{R}$ such that $\mu(x) = 0$ if $x \notin S$. Then, the relation

$$\mathcal{I}(x)\rho = \mu(x)T(\rho)\tag{40}$$

for any subset Δ and any operator ρ defines a DL instrument. However, since T is not CP, the operation $\mathcal{I}(\Delta)$ is not CP, so the \mathcal{I} is not a CP instrument. The extendability postulate implies that there is no physically realizable apparatus corresponding to the above instrument.

4 Measuring processes

In this section, we discuss measuring processes. We introduce indirect measurement models, a class of universal models for measuring processes, carried out by physically realizable measuring apparatuses. We analyze them according to quantum mechanics, and show that every indirect measurement model uniquely determines the instrument of the apparatus that is a completely positive instrument, and conversely that every completely positive instrument is described by an indirect measurement model of an apparatus. Since any apparatus described by an indirect measurement model is considered physically realizable, in principle, we conclude that an apparatus is physically realizable if and only if its statistical properties are described by a CP instrument.

4.1 Indirect measurement models

Let $\mathbf{A}(\mathbf{x})$ be a measuring apparatus with the macroscopic output variable \mathbf{x} to measure the object \mathbf{S} . The measuring interaction turns on at time t , the *time of measurement*, and turns off at time $t + \Delta t$ between object \mathbf{S} and apparatus $\mathbf{A}(\mathbf{x})$. We assume that *the object and the apparatus do not interact each other before t nor after $t + \Delta t$ and that the composite system $\mathbf{S} + \mathbf{A}(\mathbf{x})$ is isolated in the time interval $(t, t + \Delta t)$* . The *probe* \mathbf{P} is defined to be the minimal part of apparatus $\mathbf{A}(\mathbf{x})$ such that the composite system $\mathbf{S} + \mathbf{P}$ is isolated in the time interval $(t, t + \Delta t)$. By minimality, we naturally assume that probe \mathbf{P} is a quantum system represented by a Hilbert space \mathcal{H} . Denote by U the unitary operator on $\mathcal{H} \otimes \mathcal{H}$ representing the time evolution of $\mathbf{S} + \mathbf{P}$ for the time interval $(t, t + \Delta t)$.

At the time of measurement the object is supposed to be in an arbitrary input state ρ and the probe is supposed to be prepared in a fixed state σ . Thus, the composite system $\mathbf{S} + \mathbf{P}$ is in the state $\rho \otimes \sigma$ at time t and in the state $U(\rho \otimes \sigma)U^\dagger$ at time $t + \Delta t$. Just after the measuring interaction, the object is separated from the apparatus, and the probe is subjected to a local interaction with the subsequent stages of the apparatus. The last process is assumed to measure an observable M , called the *meter observable*, of the probe, without further interacting with the object \mathbf{S} , and the output is represented by the value of the output variable \mathbf{x} .

Any physically realizable apparatus $\mathbf{A}(\mathbf{x})$ can be modeled as above by a quadruple $(\mathcal{H}, \sigma, U, M)$, called an *indirect measurement model*, of a Hilbert space \mathcal{H} , a density operator σ in \mathcal{H} , a unitary operator U on $\mathcal{H} \otimes \mathcal{H}$, and a self-adjoint

operator M on \mathcal{K} , where \mathcal{K} represents the state space of the probe, σ the preparation of the probe, U the interaction between the object and the probe, and M the probe observable to be detected. An indirect measurement model $(\mathcal{K}, \sigma, U, M)$ is called *pure*, if σ is a pure state; we shall write $(\mathcal{K}, \sigma, U, M) = (\mathcal{K}, \xi, U, M)$, if $\rho = |\xi\rangle\langle\xi|$.

4.2 Output probability distributions

Let $\mathbf{A}(\mathbf{x})$ be an apparatus with indirect measurement model $(\mathcal{K}, \sigma, U, M)$. Since the outcome of this measurement is obtained by the measurement of the probe observable M at time $t + \Delta t$, by the BSF for observable M on input state $U(\rho \otimes \sigma)U^\dagger$ the output probability distribution of $\mathbf{A}(\mathbf{x})$ is determined by

$$\Pr\{\mathbf{x} = x | \rho\} = \text{Tr}\{[I \otimes P^M(x)]U(\rho \otimes \sigma)U^\dagger\}. \quad (41)$$

By linearity of operators and the trace, it is easy to check that the output probability distribution of $\mathbf{A}(\mathbf{x})$ satisfies the mixing law of output probability. Thus, by Theorem 1 there exists the POVM Π of $\mathbf{A}(\mathbf{x})$. To determine Π , using the partial trace operation $\text{Tr}_{\mathcal{K}}$ over \mathcal{K} we rewrite Eq. (41) as

$$\Pr\{\mathbf{x} = x | \rho\} = \text{Tr}[\text{Tr}_{\mathcal{K}}\{U^\dagger[I \otimes P^M(x)]U(I \otimes \sigma)\}\rho]. \quad (42)$$

Since ρ is arbitrary, comparing Eqs. (6) and (42), POVM of $\mathbf{A}(\mathbf{x})$ is determined as

$$\Pi(x) = \text{Tr}_{\mathcal{K}}\{U^\dagger[I \otimes P^M(x)]U(I \otimes \sigma)\} \quad (43)$$

for any subset Δ .

4.3 Quantum state reductions

Since the composite system $\mathbf{S} + \mathbf{P}$ is in the state $U(\rho \otimes \sigma)U^\dagger$ at time $t + \Delta t$, from Axiom Q4 it is standard that the object state at the time $t + \Delta t$ is obtained by tracing out the probe part of that state. Thus, the nonselective state change is determined by

$$\rho \mapsto \rho' = \text{Tr}_{\mathcal{K}}[U(\rho \otimes \sigma)U^\dagger]. \quad (44)$$

In order to determine the quantum state reduction caused by apparatus $\mathbf{A}(\mathbf{x})$, suppose that at time $t + \Delta t$ the observer would locally measure an arbitrary observable B of the same object \mathbf{S} . Let $\mathbf{A}(\mathbf{y})$ be a B -measuring apparatus with output variable \mathbf{y} . Since both the M measurement on \mathbf{P} and the B measurement on \mathbf{S} at time $t + \Delta t$ are local, the joint probability distribution of their outcomes satisfies the joint prob-

ability formula for the simultaneous measurement of $I \otimes M$ and $B \otimes I$ in the state $U(\rho \otimes \sigma)U^\dagger$ [35].

It follows that the joint output probability distribution of $\mathbf{A}(\mathbf{x})$ and $\mathbf{A}(\mathbf{y})$ is given by

$$\Pr\{\mathbf{x} = x, \mathbf{y} = y | \rho\} = \text{Tr}\{[P^B(y) \otimes P^M(x)]U(\rho \otimes \sigma)U^\dagger\}. \quad (45)$$

Thus, using the partial trace $\text{Tr}_{\mathcal{H}}$ we have

$$\Pr\{\mathbf{x} = x, \mathbf{y} = y | \rho\} = \text{Tr}[P^B(y)\text{Tr}_{\mathcal{H}}\{[I \otimes P^M(x)]U(\rho \otimes \sigma)U^\dagger\}]. \quad (46)$$

On the other hand, from Eq. (9) the same joint output probability distribution can be represented by

$$\begin{aligned} \Pr\{\mathbf{x} = x, \mathbf{y} = y | \rho\} &= \text{Tr}[P^B(y)\rho_{\{\mathbf{x}=x\}}]\Pr\{\mathbf{x} = x | \rho\} \\ &= \text{Tr}[P^B(y)\Pr\{\mathbf{x} = x | \rho\}\rho_{\{\mathbf{x}=x\}}]. \end{aligned} \quad (47)$$

Since B is chosen arbitrarily, comparing Eqs. (46) and (47), we have

$$\Pr\{\mathbf{x} = x | \rho\}\rho_{\{\mathbf{x}=x\}} = \text{Tr}_{\mathcal{H}}\{[I \otimes P^M(x)]U(\rho \otimes \sigma)U^\dagger\} \quad (48)$$

From Eq. (48), the state $\rho_{\{\mathbf{x}=x\}}$ is uniquely determined as

$$\rho_{\{\mathbf{x}=x\}} = \frac{\text{Tr}_{\mathcal{H}}\{[I \otimes P^M(x)]U(\rho \otimes \sigma)U^\dagger\}}{\text{Tr}\{[I \otimes P^M(x)]U(\rho \otimes \sigma)U^\dagger\}} \quad (49)$$

By Eq. (48) the instrument \mathcal{J} of the apparatus $\mathbf{A}(\mathbf{x})$ is determined by

$$\mathcal{J}(x)\rho = \text{Tr}_{\mathcal{H}}\{[I \otimes P^M(x)]U(\rho \otimes \sigma)U^\dagger\} \quad (50)$$

for any $x \in \mathbb{R}$ and any state ρ . From the above relation, it is easy to see that $\mathcal{J}(x)$ satisfies the complete positivity; as an alternative characterization, it is well-known that a linear transformation T on $\mathcal{L}(\mathcal{H})$ is completely positive if and only if

$$\sum_{ij}(\xi_i, T(\rho_i^\dagger \rho_j) \xi_j) \geq 0 \quad (51)$$

for any finite sequences $\xi_1, \dots, \xi_n \in \mathcal{H}$ and $\rho_1, \dots, \rho_n \in \mathcal{L}(\mathcal{H})$ [20]. In fact, we have

$$\begin{aligned} \sum_{ij}(\xi_i, \mathcal{J}(x)(\rho_i^\dagger \rho_j) \xi_j) &= \sum_{ij} \text{Tr}[\mathcal{J}(x)(\rho_i^\dagger \rho_j) |\xi_j\rangle \langle \xi_i|] \\ &= \sum_{ij} \text{Tr}\{[|\xi_j\rangle \langle \xi_i| \otimes P^M(x)]U(\rho_i^\dagger \rho_j \otimes \sigma)U^\dagger\} \\ &= \text{Tr}[X^\dagger X] \geq 0, \end{aligned}$$

where

$$X = \sum_j U(\rho_j \otimes \sqrt{\sigma})U^\dagger |\xi_j\rangle \langle \phi| \otimes P^M(x)$$

for an arbitrary unit vector ϕ .

Thus, we conclude that *the instrument of any apparatus with indirect measurement model $(\mathcal{K}, \sigma, U, M)$ is a CP instrument.*

The converse of this assertion was proven in [20], and hence we conclude this section by the following theorem.

Theorem 4 (Realization theorem) *The instrument of any apparatus with an indirect measurement model is a CP instrument, and conversely every CP instrument is obtained in this way with a pure indirect measurement model.*

According to the above theorem, we conclude that an apparatus satisfying Axiom M1 is physically realizable if and only if it satisfied Axiom M2 (the mixing law of joint output probability) and Axiom M3 (the extendability).

5 Conclusion

For quantum systems with finite dimensional state spaces, we can now complete von Neumann's axiomatization of quantum mechanics by augmenting it by the general measurement axiom that describes all the physically realizable measurement consistent with other axioms of quantum mechanics.

Axiom Q5 (General measurement axiom). *Every physically realizable apparatus $\mathbf{A}(\mathbf{x})$ for the system \mathbf{S} with the state space \mathcal{H} uniquely corresponds to a completely positive instrument \mathcal{I} for \mathcal{H} such that the statistical properties of $\mathbf{A}(\mathbf{x})$ are determined by*

$$\mathcal{I}(x)\rho = \Pr\{\mathbf{x} = x | \rho\} \rho_{\{\mathbf{x}=x\}}, \quad (52)$$

or equivalently

$$\Pr\{\mathbf{x} = x | \rho\} = \text{Tr}[\mathcal{I}(x)\rho], \quad (53)$$

$$\rho_{\{\mathbf{x}=x\}} = \frac{\mathcal{I}(x)\rho}{\text{Tr}[\mathcal{I}(x)\rho]}, \quad \text{if } \text{Tr}[\mathcal{I}(x)\rho] > 0 \quad (54)$$

for all $x \in \mathbb{R}$ and $\rho \in \mathcal{S}(\mathcal{H})$. Conversely, every completely positive instrument \mathcal{I} for \mathcal{H} has at least one physically realizable apparatus $\mathbf{A}(\mathbf{x})$ with the above statistical properties.

According to the General measurement axiom, we can generalize Wigner's formula to arbitrary sequence of measurements, and determine the joint probability distribution of the outcomes of any sequence of measurements using physically realizable apparatuses.

Theorem 5 (Generalized Wigner's formula) *Let $\mathcal{I}_1, \dots, \mathcal{I}_n$ be completely positive instruments for the system with the state space \mathcal{H} in a state ρ at time 0. If one carries out measurements described by $\mathcal{I}_1, \dots, \mathcal{I}_n$ at times $(0 <) t_1 < \dots < t_n$ and otherwise leaves the system \mathbf{S} isolated with the Hamiltonian H , then the joint probability distribution of the outcomes $\mathbf{x}_1, \dots, \mathbf{x}_n$ of those measurements is given by*

$$\begin{aligned} \Pr\{\mathbf{x}_1 = x_1, \mathbf{x}_2 = x_2, \dots, \mathbf{x}_n = x_n \mid \rho\} \\ = \text{Tr}[\mathcal{I}_n(x_n) \alpha(t_n - t_{n-1}) \cdots \mathcal{I}_2(x_2) \alpha(t_2 - t_1) \mathcal{I}_1(x_1) \alpha(t_1) \rho], \end{aligned} \quad (55)$$

for any $x_1, x_2, \dots, x_n \in \mathbb{R}$ and $\rho \in \mathcal{S}(\mathcal{H})$, where α is defined by $\alpha(t)\rho = e^{-iHt/\hbar} \rho e^{iHt/\hbar}$ for all $t \in \mathbb{R}$ and $\rho \in \mathcal{S}(\mathcal{H})$.

Foundations of quantum measurement theory based on the notion of completely positive instruments and indirect measurement models have been developed in [19–39].

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