A new classification of semilinear damped wave equations by lifespan estimates [a]

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★ I.V.P. with Small Data and Critical Exponent

For scalar unknowns $u = u(x,t),\ x \in \mathbf{R}^n,\ t \in [0,\infty)$

Classical Damped Wave : $u_{tt} - \Delta u + u_t = |u|^p$,

or Heat : $-\Delta u + u_t = |u|^p$

$\Longrightarrow$ Fujita exp. $p_F(n) := 1 + \frac{2}{n}$

Wave : $u_{tt} - \Delta u = |u|^p\ (n \geq 2)$

$\Longrightarrow$ Strauss exp. $p_S(n) := \frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}$

[Note] $p_F(n) < p_S(n)\ (n \geq 2)$

★ I.V.P. for Semilinear Damped Wave Equations

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\mu}{(1+t)^{\beta}} u_t = |u|^p \\ \qquad \text{in} \quad \mathbf{R}^n \times (0,\infty), \\ u(x,0) = \varepsilon f(x),\ u_t(x,0) = \varepsilon g(x) \end{cases} \tag{1}$$

$\varepsilon > 0$: small, $\mu > 0$, $\beta \in \mathbf{R}$.

$(f,g) \in H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$: compact supp.

$T(\varepsilon)$: maximal existence time of energy solution of (1) for $\forall (f,g) \not\equiv (0,0)$ (fixed).

★ Wirth's Classification of Linear Equations

$$u_{tt} - \Delta u + \frac{\mu}{(1+t)^{\beta}} u_t = 0$$

$\beta < -1$	overdamping
$-1 \leq \beta < 1$	effective damping
$\beta = 1$	scaling invariant $0 < \mu < 1 \Rightarrow$ non-effective damping
$1 < \beta$	scattering damping

by Mochizuki('76), Matsumura('77) $\sim$ Wirth('07)

★ Overdamping & Effective Damping ($\beta < 1$)

$$u_{tt} - \Delta u + \frac{\mu}{(1+t)^{\beta}} u_t = |u|^p, \quad p_F(n) := 1 + \frac{2}{n}$$

$T(\varepsilon)$	$1 < p < p_F(n)$	$p = p_F(n)$	$p_F(n) < p$ $n \geq 3 \Rightarrow \leq p_F(n-2)$
$\beta < -1$	$= \infty$	$= \infty$	$= \infty$
$\beta = -1$	$\sim \exp\left(C\varepsilon^{-\frac{2(p-1)}{2-n(p-1)}}\right)$	$\sim \exp\left(\exp(C\varepsilon^{-(p-1)})\right)$	$= \infty$
$-1 < \beta < 1$	$\sim C\varepsilon^{-\frac{2(p-1)}{(1+\beta)\{2-n(p-1)\}}}$	$\sim \exp\left(C\varepsilon^{-(p-1)}\right)$	$= \infty$

by Li&Zhou('95), Li('96), Todorova&Yordanov('01), Zhang('01), Nishihara('03), Nishihara('11),

Lin&Nishihara&Zhai('12), Ikeda&Wakasugi('15) Ikeda&Ogawa('16), Lai&Zhou('19),

Fujiwara&Ikeda&Wakasugi('19), Ikeda&Inui('19), Ikeda&Wakasugi('20)

★ Scattering Damping ($\beta > 1$); No.1

$$u_{tt} - \Delta u + \frac{\mu}{(1+t)^\beta} u_t = |u|^p, \quad p_S(n) := \frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}$$

Conjecture 1 When $n \geq 3$, or $n = 2$ and $p > 2$,

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{2p(p-1)}{\gamma(p,n)}} & \text{for } p < p_S(n), \\ \exp(C\varepsilon^{-p(p-1)}) & \text{for } p = p_S(n), \end{cases}$$

$$T(\varepsilon) = \infty \qquad \text{for} \quad p > p_S(n),$$

where $\gamma(p,n) := 2 + (n+1)p - (n-1)p^2$.

[Note] $\gamma(p,n) = 0 \iff p = p_S(n), \ p_S(2) > 2$

★ Scattering Damping ($\beta > 1$); No.2

$$u_{tt} - \Delta u + \frac{\mu}{(1+t)^\beta} u_t = |u|^p, \quad u(x,0) = \varepsilon f(x),\ u_t(x,0) = \varepsilon g(x)$$

Conjecture 2 When $n = 2$ and $p \leq 2 (< p_S(2))$,

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{p-1}{3-p}} & \text{for } p < 2 \\ Ca(\varepsilon) & \text{for } p = 2 \end{cases} \quad \text{if} \int_{\mathbf{R}^2} g(x)dx \neq 0,$$

where $a = a(\varepsilon)$ satisfies $a^2\varepsilon^2 \log(1+a) = 1$. While Conjecture 1 still holds if $\int_{\mathbf{R}^2} g(x)dx = 0$.

[Note] $\frac{p-1}{3-p} < \frac{2p(p-1)}{\gamma(p,2)} \iff 1 < p < 2,\ \gamma(2,2) = 4,$ so that $T(\varepsilon)$ with $\int_{\mathbf{R}^2} g(x)dx \neq 0 < T(\varepsilon)$ with $= 0$.

★ Scattering Damping ($\beta > 1$); No.3

$$u_{tt} - \Delta u + \frac{\mu}{(1+t)^\beta} u_t = |u|^p, \quad u(x,0) = \varepsilon f(x),\ u_t(x,0) = \varepsilon g(x)$$

Conjecture 3 When $n = 1$,

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{p-1}{2}} & \text{if } \int_{\mathbf{R}} g(x)dx \neq 0, \\ C\varepsilon^{-p\frac{p-1}{p+1}} & \text{if } \int_{\mathbf{R}} g(x)dx = 0. \end{cases}$$

[Note] $\frac{p-1}{2} < p\frac{p-1}{p+1} \Longleftrightarrow p > 1$,

so that $T(\varepsilon)$ with $\int_{\mathbf{R}} g(x)dx \neq 0 < T(\varepsilon)$ with $= 0$.

Conjecture 1-3 are partially verified by Lai&T('19),

Wakasa&Yordaov('19), Liu&Wang('20).

★ Main Question

In view of the critical exponent and the lifespan estimate of

$$u_{tt} - \Delta u + \frac{\mu}{(1+t)^{\beta}} u_t = |u|^p,$$

- Effective damping without threshold $(-1 < \beta < 1)$
  $\sim -\Delta u + \dfrac{\mu}{(1+t)^{\beta}} u_t = |u|^p.$
- Scattering damping $(\beta > 1)$
  $\sim u_{tt} - \Delta u = |u|^p.$

So, how about the scale-invariant damping $(\beta = 1)$?

★ Scale-Invariant Damping; No.1

$$u_{tt} - \Delta u + \frac{\mu}{1+t} u_t = |u|^p, \quad u(x,0) = \varepsilon f(x), \ u_t(x,0) = \varepsilon g(x)$$

Conjecture 4 The critical exponent is

$$\begin{cases} p_F(n) & \text{for } \mu > \mu_0(n) : \text{heat-like}, \\ p_F(n) = p_S(n+\mu) & \text{for } \mu = \mu_0(n) : \text{intermediate}, \\ p_S(n+\mu) & \text{for } \mu < \mu_0(n) : \text{wave-like}, \end{cases}$$

where $\mu_0(n) := \dfrac{n^2+n+2}{n+2}$.

[Note] $p_F(n) < p_S(n)$ and $p_S(n+\mu) \searrow (\mu \nearrow)$.

★ Scale-Invariant Damping; No.2

Conjecture 4 is partially verified by Waksasugi('14) with

- partial heat-like blow-up for $\mu \geq 1$ & $1 < p < p_F(n)$.
- partial wave-like blow-up (super-Fujita exp.) for $0 < \mu < 1$ & $1 < p < p_F(n + \mu - 1)$.

and by D'Abbicco('15) with partial heat-like existence for

$$p > p_F(n) \;\&\; \begin{cases} n = 1 \;\&\; \mu \geq 5/3 > \mu_0(1) = 4/3, \\ n = 2 \;\&\; \mu \geq 3 > \mu_0(2) = 2, \\ n \geq 3 \;\&\; \mu \geq n + 2 \; {\scriptstyle (p \leq p_F(n-2))}. \end{cases}$$

★ Scale-Invariant Damping; No.3

Global existence part of Conjecture 4 also is partially verified by

- Inui & Mizutani('21) with

$$3 \leq n \leq 5,\ \mu > 0,\ p = 1 + \frac{4}{n-2},$$

- Lai & Zhou ('21) with

$$n = 3,\ \frac{3}{2} \leq \mu < 2,\ p_S(n+\mu) < p \leq 2$$

for the radial solution. (cf. $\mu_0(3) = \dfrac{14}{5}$).

★ Scale-Invariant Damping; No.4

$$u_{tt} - \Delta u + \frac{\mu}{1+t} u_t = |u|^p, \; u(x,0) = \varepsilon f(x), \; u_t(x,0) = \varepsilon g(x).$$

Liouville trans.

Set $v(x,t) := (1+t)^{\frac{\mu}{2}} u(x,t)$. Then we have

$$\begin{cases} v_{tt} - \Delta v + \dfrac{\mu(2-\mu)}{4(1+t)^2} v = \dfrac{|v|^p}{(1+t)^{\frac{\mu}{2}(p-1)}} \\ \quad \text{in} \quad \mathbf{R}^n \times (0,\infty), \\ v(x,0) = \varepsilon f(x), \; v_t(x,0) = \varepsilon\{\mu f(x)/2 + g(x)\}. \end{cases}$$

$\Longrightarrow$ Conjecture 4 is also partially verified in $\mu = 2$ by

D'Abbicco&Lucente&Reissig('15), D'Abbicco&Lucente('15).

★ Lifespan Estimates for Scale-Invariant Damping

$$u_{tt} - \Delta u + \frac{\mu}{1+t} u_t = |u|^p,\ u(x,0) = \varepsilon f(x),\ u_t(x,0) = \varepsilon g(x).$$

- By Lai&T&Wakasa('17) partially and Tu&Lin(arXiv) finally,

$$T(\varepsilon) \le C\varepsilon^{-\frac{2p(p-1)}{\gamma(p,n+\mu)}} \quad \text{for } p < p_S(n+\mu).$$

- By Ikeda&Sobajima('19),

$$T(\varepsilon) \le \exp\left(C\varepsilon^{-p(p-1)}\right) \quad \text{for } p = p_S(n+\mu).$$

[Rem] In view of $\mu = 0$, the estimates above may be optimal when $\mu < \mu_0$ (wave-like) except for $n = 1$, or $n = 2$ and $p \le 2$.

★ Scale-Invariant Damping in the Special Case

$$u_{tt} - \Delta u + \frac{2}{1+t} u_t = |u|^p, \ u(x,0) = \varepsilon f(x), \ u_t(x,0) = \varepsilon g(x).$$

- By Wakasa('16), in case of $n = 1$,

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{p-1}{3-p}} & \text{for } p < p_F(1) = 3, \\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{for } p = p_F(1) = 3. \end{cases}$$

- By Kato&Sakuraba('19), in case of $n = 3$,

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{2p(p-1)}{\gamma(p,5)}} & \text{for } p < p_S(5), \\ \exp\left(C\varepsilon^{-p(p-1)}\right) & \text{for } p = p_S(5). \end{cases}$$

[Note] $\mu_0(1) = 4/3 < 2$ (heat-like) and $\mu_0(3) > 2$ (wave-like).

★ Expectation (?) on Scale-Invariant Damping

$$u_{tt} - \Delta u + \frac{\mu}{1+t} u_t = |u|^p, \; u(x,0) = \varepsilon f(x), \; u_t(x,0) = \varepsilon g(x).$$

So, one may expect that

- For $\mu > \mu_0(n)$: heat-like, i.e. $p_S(n+\mu) < p_F(n)$,

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{p-1}{2-n(p-1)}} & \text{for } p < p_F(n), \\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{for } p = p_F(n). \end{cases}$$

- For $\mu < \mu_0(n)$: wave-like, i.e., $p_S(n+\mu) > p_F(n)$,

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{2p(p-1)}{\gamma(p,n+\mu)}} & \text{for } p < p_S(n+\mu), \\ \exp\left(C\varepsilon^{-p(p-1)}\right) & \text{for } p = p_S(n+\mu). \end{cases}$$

★ Lifespan Estimates in the Special Case; No.1

$$u_{tt} - \Delta u + \frac{2}{1+t} u_t = |u|^p, \ u(x,0) = \varepsilon f(x), \ u_t(x,0) = \varepsilon g(x).$$

The expectation is not always true as

Thm 1 [Kato&T.&Wakasa('19)] For $n = 1$ and $\mu = 2$, then

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{2p(p-1)}{\gamma(p,3)}} & \text{for } 1 < p < 2, \\ Cb(\varepsilon) & \text{for } p = 2, \\ C\varepsilon^{-\frac{p(p-1)}{3-p}} & \text{for } 2 < p < 3, \\ \exp\left(C\varepsilon^{-p(p-1)}\right) & \text{for } p = p_F(1) = 3, \end{cases}$$

where $b\varepsilon^2 \log(1+b) = 1$ provided $\int_{\mathbf{R}} \{f(x) + g(x)\} dx = 0$.

★ Lifespan Estimates in the Special Case; No.2

$$u_{tt}-\Delta u+\frac{2}{1+t}u_t=|u|^p,\ u(x,0)=\varepsilon f(x),\ u_t(x,0)=\varepsilon g(x).$$

As a conclusion, Wakasa's result on heat-like in $n=1$;

$$T(\varepsilon)\sim\begin{cases} C\varepsilon^{-\frac{p-1}{3-p}} & \text{for } p<p_F(1)=3,\\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{for } p=p_F(1)=3,\end{cases}$$

is valid only for $\int_{\mathbf{R}}\{f(x)+g(x)\}dx\neq 0$.

So, we may say that

$$u_{tt}-\Delta u+\frac{\mu}{1+t}u_t=|u|^p\sim v_{tt}-\Delta v=\frac{|v|^p}{(1+t)^{\frac{\mu}{2}(p-1)}}.$$

★ Lifespan Estimates in 2D Case; No.1

$$u_{tt} - \Delta u + \frac{2}{1+t} u_t = |u|^p, \quad u(x,0) = \varepsilon f(x), \quad u_t(x,0) = \varepsilon g(x).$$

[Note] $\mu_0(2) = 2$: intermediate.

Thm 2 [Imai&Kato&T.&Wakasa('20)] For $n = 2$ and $\mu = 2$,

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{p-1}{4-2p}} & \text{for } 1 < p < 2, \\ \exp\left(C\varepsilon^{-\frac{1}{2}}\right) & \text{for } p = p_S(4) = p_F(2) = 2 \end{cases}$$

provided $\int_{\mathbf{R}^2} \{f(x) + g(x)\} dx \neq 0$.

★ Lifespan Estimates in 2D Case; No.2

$$u_{tt} - \Delta u + \frac{2}{1+t} u_t = |u|^p, \ u(x,0) = \varepsilon f(x), \ u_t(x,0) = \varepsilon g(x).$$

[Note] $\mu_0(2) = 2$: intermediate.

Thm 3 [Imai&Kato&T.&Wakasa('20)] For $n = 2$ and $\mu = 2$,

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{2p(p-1)}{\gamma(p,4)}} & \text{for } 1 < p < 2, \\ \exp\left(C\varepsilon^{-\frac{2}{3}}\right) & \text{for } p = p_S(4) = p_F(2) = 2 \end{cases}$$

provided $\int_{\mathbf{R}^2} \{f(x) + g(x)\} dx = 0$.

★ Conclusion

Semilinear damped wave equation

$\sim$ Semilinear wave equation

at least

in the scale-invariant and scattering cases

according to

the lifespan estimates.