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The competition graphs of oriented complete bipartite graphs

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ABSTRACT

In this paper, we study the competition graphs of oriented complete bipartite graphs. We characterize graphs that can be represented as the competition graphs of oriented complete bipartite graphs. We also present the graphs having the maximum number of edges and the graphs having the minimum number of edges among such graphs.

1. Introduction

The *competition graph* $C(D)$ of a digraph D is the (simple undirected) graph G defined by $V(G) = V(D)$ and $E(G) = \{uv \mid u, v \in V(D), u \neq v, N_D^+(u) \cap N_D^+(v) \neq \emptyset\}$, where $N_D^+(x)$ denotes the set of out-neighbors of a vertex x in D. We denote the set of in-neighbors of a vertex x in a digraph D by $N_D^-(x)$ and denote the set of neighbors of a vertex x in a graph G by $N_G(x)$. Competition graphs arose in connection with an application in ecology (see [2]) and also have applications in coding, radio transmission, and modeling of complex economic systems. Early literature of the study on competition graphs is summarized in the survey papers by Kim [6] and Lundgren [10].

For a digraph D, the *underlying graph* of D is the graph G such that $V(G) = V(D)$ and $E(G) = \{uv \mid (u, v) \in A(D)\}$. An *orientation* of a graph G is a digraph having no directed 2-cycles, no loops, and no multiple arcs whose underlying graph is G. An *oriented graph* is a graph with an orientation. A *tournament* is an oriented complete graph. The competition graphs of tournaments have been actively studied (see [1,3,5], and [4] for papers related to this topic).

It seems to be a natural shift to take a look at the competition graphs of orientations of complete bipartite graphs. First, we can observe that the competition graph of an orientation of a complete bipartite graph is a disconnected graph as follows.

Proposition 1.1. *Let D be an orientation of a complete bipartite graph $K_{m,n}$ with bipartition (U, V), where $|U| = m$ and $|V| = n$. Then, the competition graph of D has no edges between the vertices in U and the vertices in V.*

Proof. Since D is an orientation of $K_{m,n}$, $N_D^+(x) \cup N_D^-(x) = N_{K_{m,n}}(x)$ holds for any vertex x in D. Take any vertex u in U and any vertex v in V. Since $N_{K_{m,n}}(u) \subseteq V$, $N_{K_{m,n}}(v) \subseteq U$, and $U \cap V = \emptyset$, we have $N_{K_{m,n}}(u) \cap N_{K_{m,n}}(v) = \emptyset$. Therefore, $N_D^+(u) \cap N_D^+(v) = \emptyset$. Thus, there is no edge between u and v in the competition graph of D. Hence the proposition holds. □

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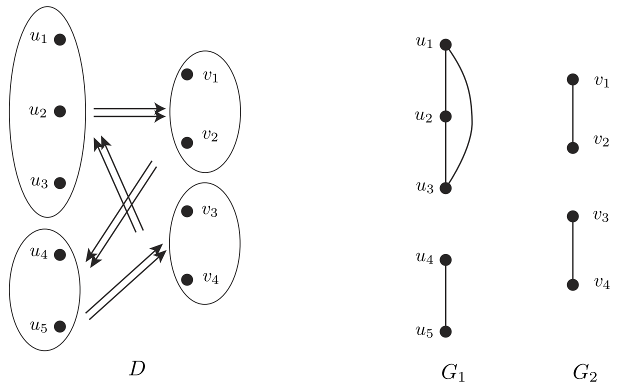

Fig. 1. An orientation D of the complete bipartite graph $K_{5,4}$ and its competition graph $G_1 \cup G_2$. (A pair of parallel arrows means that there is an arc from each vertex in the ellipse from which the arc initiates to each vertex in the ellipse to which the arc terminates.)

Based on Proposition 1.1, we introduce the notion of competition-realizable pairs.

Definition 1. Let G_1 and G_2 be graphs with m vertices and n vertices, respectively. The pair (G_1, G_2) is said to be *competition-realizable through* $K_{m,n}$ (in this paper, we only consider orientations of $K_{m,n}$ and therefore we omit "through $K_{m,n}$") if the disjoint union of G_1 and G_2 is the competition graph of an orientation of the complete bipartite graph $K_{m,n}$ with bipartition $(V(G_1), V(G_2))$.

Let us see an example. Let G_1 be the graph defined by $V(G_1) = \{u_1, u_2, u_3, u_4, u_5\}$ and $E(G_1) = \{u_1u_2, u_1u_3, u_2u_3, u_4u_5\}$, and let G_2 be the graph defined by $V(G_2) = \{v_1, v_2, v_3, v_4\}$ and $E(G_2) = \{v_1v_2, v_3v_4\}$. Then the pair $(G_1, G_2) \cong (K_3 \cup K_2, K_2 \cup K_2)$ is competition-realizable through $K_{5,4}$ (see Fig. 1).

In this paper, we study the competition graphs of oriented complete bipartite graphs by using the notion of competition-realizable pairs. We characterize graphs that can be represented as the competition graphs of oriented complete bipartite graphs. We also present the graphs having the maximum number of edges and the graphs having the minimum number of edges among such graphs.

2. A characterization of competition-realizable pairs in terms of edge clique covers

In this section, we present a theorem which characterizes a competition-realizable pair (G_1, G_2) in terms of edge clique covers of the graphs G_1 and G_2 without mentioning oriented complete bipartite graphs.

By a *family*, we mean a multiset of subsets of a set. A *clique* of a graph G is a set of vertices of G in which any two vertices are adjacent in G. We also consider an empty set $\emptyset$ as a clique. An *edge clique cover* of a graph G is a family $\mathcal{F}$ of cliques of G such that, for any two adjacent vertices of G, there is a clique in $\mathcal{F}$ containing both of them. For a graph G, we denote by $\theta_E(G)$ the minimum size of an edge clique cover of G.

The *intersection graph* $\Omega(\mathcal{F})$ of a family $\mathcal{F}$ is the graph whose vertex set is $\mathcal{F}$ and in which two sets X and Y in $\mathcal{F}$ are adjacent if and only if $X \cap Y \neq \emptyset$. Recall that a *graph isomorphism* from G_1 to G_2 is a bijection φ from $V(G_1)$ to $V(G_2)$ such that $xy \in E(G_1)$ if and only if $\varphi(x)\varphi(y) \in E(G_2)$. For a family $\mathcal{F}$ of subsets of a set V, the *dual family* of $\mathcal{F}$ is the family $\mathcal{F}^*$ defined by $\mathcal{F}^* = \{V \setminus S \mid S \in \mathcal{F}\}$.

Theorem 2.1. *Let G_1 and G_2 be graphs. Then, (G_1, G_2) is a competition-realizable pair if and only if there exist edge clique covers $\mathcal{F}_1$ and $\mathcal{F}_2$ of G_1 and G_2, respectively, such that*

(i) *there exist graph isomorphisms $\varphi_1 : G_2 \to \Omega(\mathcal{F}_1^*)$ and $\varphi_2 : G_1 \to \Omega(\mathcal{F}_2^*)$, where $\mathcal{F}_i^* := \{V(G_i) \setminus S \mid S \in \mathcal{F}_i\}$ for $i = 1, 2$;*
(ii) *for $u \in V(G_1)$ and $v \in V(G_2)$, $u \in \varphi_1(v)$ if and only if $v \notin \varphi_2(u)$.*

Before proving the theorem, let us see an example. Let

$$\mathcal{F}_1 = \{\{u_1, u_2, u_3\}, \{u_1, u_2, u_3\}, \{u_4, u_5\}, \{u_4, u_5\}\},$$
$$\mathcal{F}_2 = \{\{v_1, v_2\}, \{v_1, v_2\}, \{v_1, v_2\}, \{v_3, v_4\}, \{v_3, v_4\}\}.$$

Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are edge clique covers of $G_1 := K_3 \cup K_2$ and $G_2 := K_2 \cup K_2$, respectively, and we have $\mathcal{F}_1^* = \mathcal{F}_1$ and $\mathcal{F}_2^* = (\mathcal{F}_2 - \{\{v_1, v_2\}\}) \cup \{\{v_3, v_4\}\}$. We define maps φ_1 from $V(G_2)$ to $\mathcal{F}_1^*$ and φ_2 from $V(G_1)$ to $\mathcal{F}_2^*$ by $\varphi_1(v_1) = \varphi_1(v_2) = \{u_4, u_5\}$; $\varphi_1(v_3) = \varphi_1(v_4) = \{u_1, u_2, u_3\}$; $\varphi_2(u_1) = \varphi_2(u_2) = \varphi_2(u_3) = \{v_1, v_2\}$; $\varphi_2(u_4) = \varphi_2(u_5) = \{v_3, v_4\}$. It is easy to check that $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy conditions (i) and (ii) of Theorem 2.1.

Lemma 2.2. *Let G_1 and G_2 be graphs. Let D be an orientation of a complete bipartite graph with bipartition $(V(G_1), V(G_2))$ such that the competition graph of D is $G_1 \cup G_2$. Then, the family $\{N_D^-(v) \mid v \in V(G_2)\}$ is an edge clique cover of G_1, and the family $\{N_D^-(u) \mid u \in V(G_1)\}$ is an edge clique cover of G_2.*

Proof. By the definition of the competition graph of a digraph, $N_D^-(x)$ is a clique of the competition graph of D. Since D has no arcs between two vertices in $V(G_1)$, the family $\{N_D^-(v) \mid v \in V(G_2)\}$ forms an edge clique cover of G_1. Similarly, it holds that the family $\{N_D^-(v) \mid v \in V(G_1)\}$ forms an edge clique cover of G_2. □

Lemma 2.3. *If (G_1, G_2) is a competition-realizable pair, then $\theta_E(G_1) \leq |V(G_2)|$ and $\theta_E(G_2) \leq |V(G_1)|$.*

Proof. The lemma follows from Lemma 2.2. □

Proof of Theorem 2.1. First, we show the "only if" part. Let (G_1, G_2) be a competition-realizable pair. Then there exists an orientation D of the complete bipartite graph $K_{m,n}$ where $m = |V(G_1)|$ and $n = |V(G_2)|$. Let $\mathcal{F}_1$ be the family $\{N_D^-(v) \mid v \in V(G_2)\}$, and let $\mathcal{F}_2$ be the family $\{N_D^-(u) \mid u \in V(G_1)\}$. By Lemma 2.2, $\mathcal{F}_1$ and $\mathcal{F}_2$ are edge clique covers of G_1 and G_2, respectively. Since D is an orientation of a complete bipartite graph, $N_D^-(v) \cup N_D^+(v) = V(G_1)$ for any $v \in V(G_2)$ and that $N_D^-(u) \cup N_D^+(u) = V(G_2)$ for any $u \in V(G_1)$. Since D is loopless, $N_D^-(x) \cap N_D^+(x) = \emptyset$ for any $x \in V(D) = V(G_1) \cup V(G_2)$. Therefore, $\mathcal{F}_1^* = \{N_D^+(v) \mid v \in V(G_2)\}$ and $\mathcal{F}_2^* = \{N_D^+(u) \mid u \in V(G_1)\}$. We define a map φ_1 from $V(G_2)$ to $\mathcal{F}_1^*$ by $\varphi_1(v) = N_D^+(v)$ and a map φ_2 from $V(G_1)$ to $\mathcal{F}_2^*$ by $\varphi_2(u) = N_D^+(u)$. It is easy to check that φ_i is well-defined and bijective. To show that φ_i $(i = 1, 2)$ preserves the adjacency, take two vertices x and x' in G_{3-i}. Then, x and x' are adjacent in $C(D)$ if and only if $N_D^+(x) \cap N_D^+(x') \neq \emptyset$. Therefore φ_i is an isomorphism from G_{3-i} to the intersection graph of $\mathcal{F}_i^*$ and thus the condition (i) holds. Furthermore, $x \in N_D^+(y)$ if and only if $y \in N_D^-(x)$ for any vertices x and y of D. Therefore it follows from the definitions of $\varphi_1(v)$ and $\varphi_2(u)$ that $u \in \varphi_1(v)$ if and only if $v \notin \varphi_2(u)$ for $u \in V(G_1)$ and $v \in V(G_2)$. Thus the condition (ii) holds.

Now we show the "if" part. Suppose that there exist edge clique covers $\mathcal{F}_1$ and $\mathcal{F}_2$ of G_1 and G_2, respectively, satisfying the conditions (i) and (ii). We define an orientation D of $K_{m,n}$, where $m = |V(G_1)|$ and $n = |V(G_2)|$, by

$$\begin{aligned} A(D) = & \bigcup_{u \in V(G_1)} \left(\{(u, v) \mid v \in \varphi_2(u)\} \cup \{(v, u) \mid v \in V(G_2) \setminus \varphi_2(u)\}\right) \\ & \cup \bigcup_{v \in V(G_2)} \left(\{(v, u) \mid u \in \varphi_1(v)\} \cup \{(u, v) \mid u \in V(G_1) \setminus \varphi_1(v)\}\right). \end{aligned}$$

To show that D is an orientation of $K_{m,n}$, take an arc (x, y) in D. Without loss of generality, we may assume that $x \in V(G_1)$ and $y \in V(G_2)$. Then $y \in \varphi_2(x)$ or $x \in V(G_1) \setminus \varphi_1(y)$ by the definition of $A(D)$. By the condition (ii), $y \in \varphi_2(x)$ and $x \in V(G_1) \setminus \varphi_1(y)$. Thus $(y, x) \notin A(D)$. To show that every edge of $K_{m,n}$ is oriented by $A(D)$, take a vertex $x \in V(G_1)$ and a vertex $y \in V(G_2)$. Then $x \in V(G_1) \setminus \varphi_1(y)$ or $x \in \varphi_1(y)$. If $x \in V(G_1) \setminus \varphi_1(y)$, then $(x, y) \in A(D)$. If $x \in \varphi_1(y)$, then $(y, x) \in A(D)$. Thus D is an orientation of $K_{m,n}$. We now show that the competition graph of D is $G_1 \cup G_2$. Take any edge xy of $G_1 \cup G_2$. Without loss of generality, we may assume that xy is an edge of G_2. Then $\varphi_1(x) \cap \varphi_1(y) \neq \emptyset$ by the definition of an intersection graph. Therefore, there exists a vertex $z \in \varphi_1(x) \cap \varphi_1(y)$ and, by the definition of $A(D)$, $(x, z) \in A(D)$ and $(y, z) \in A(D)$. Thus xy is an edge in the competition graph of D. To show that any edge in the competition graph of D is an edge in $G_1 \cup G_2$, take two vertices x and y which are adjacent in $C(D)$. Then there is a vertex w such that $(x, w) \in A(D)$ and $(y, w) \in A(D)$. Without loss of generality, we may assume that $x \in V(G_2)$. By Proposition 1.1, $y \in V(G_2)$. By the definition of $A(D)$, we have $w \in \varphi_1(x)$. Since $w \in V(G_1)$ and $(y, w) \in A(D)$, we have $w \in \varphi_1(y)$. Therefore $w \in \varphi_1(x) \cap \varphi_1(y)$, which implies that x and y belong to $V(G_2) \setminus \varphi_2(w)$, by the condition (ii). Thus x and y are adjacent in G_2.

Hence the theorem holds. □

3. Competition-realizable pairs and the independence numbers of graphs

Recall that the *independence number* of a graph G is the maximum number of vertices in G that are pairwise nonadjacent, and is denoted by $\alpha(G)$.

Lemma 3.1. *Let (G_1, G_2) be a competition-realizable pair. If $\alpha(G_2) \geq 3$, then G_1 is a complete graph.*

Proof. Suppose that G_1 is not a complete graph. Then, we have $\alpha(G_1) \geq 2$. Let $\{u_1, u_2\}$ and $\{v_1, v_2, v_3\}$ be independent sets of G_1 and G_2, respectively. Since (G_1, G_2) is a competition-realizable pair, there exists an orientation D of a complete bipartite graph such that $C(D) = G_1 \cup G_2$. Note that the subgraph of $C(D)$ induced by $\{u_1, u_2\} \cup \{v_1, v_2, v_3\}$ has no edges and also note that the subgraph of the underlying graph of D induced by $\{u_1, u_2\} \cup \{v_1, v_2, v_3\}$ is the complete bipartite graph $K_{2,3}$. We can check that the competition graph of any orientation of $K_{2,3}$ has at least one edge, which is a contradiction. Thus the lemma holds. □

Note that, if (G_1, G_2) is a competition-realizable pair, then (G_2, G_1) is also a competition-realizable pair.

Theorem 3.2. *Let (G_1, G_2) be a competition-realizable pair. Then, one of the following holds:*

(i) $\alpha(G_1) = 1$ *or* $\alpha(G_2) = 1$;
(ii) $\alpha(G_1) = \alpha(G_2) = 2$.

Proof. Suppose that (i) does not hold, i.e., $\alpha(G_1) \geq 2$ and $\alpha(G_2) \geq 2$. Then, neither G_1 nor G_2 is a complete graph. By Lemma 3.1, we have $\alpha(G_1) \leq 2$ and $\alpha(G_2) \leq 2$. Thus (ii) holds. Hence the theorem holds. □

Based on Theorem 3.2, we consider the following two cases to characterize competition-realizable pairs: (i) G_1 or G_2 is a complete graph; (ii) $\alpha(G_1) = \alpha(G_2) = 2$.

In Case (i), we may assume without loss of generality that G_2 is a complete graph. Moreover, we divide Case (i) into the following two cases:

(i-a) G_1 has an isolated vertex (and G_2 is a complete graph);
(i-b) G_1 has no isolated vertices (and G_2 is a complete graph).

In the next section, we investigate competition-realizable pairs for these three cases.

4. Structural characterizations of competition-realizable pairs (G_1, G_2)

4.1. The case where G_1 has an isolated vertex and $\alpha(G_2) = 1$

When one of G_1 and G_2 is a complete graph and the other graph has an isolated vertex, the necessary condition given in Lemma 2.3 is also a sufficient condition for a pair of graphs being competition-realizable.

Theorem 4.1. *Let G be a graph and let n be a positive integer. Suppose that G has at least one isolated vertex. Then, (G, K_n) is a competition-realizable pair if and only if $\theta_E(G) \le n$.*

Proof. The "only if" part follows from Lemma 2.3. We show the "if" part. Suppose that $\theta_E(G) \le n$. Let S_0 be the set of isolated vertices in G. Since G has at least one isolated vertex, $S_0 \ne \emptyset$. Since $\theta_E(G - S_0) = \theta_E(G) \le n$, there exists an edge clique cover $\{S_1, \ldots, S_n\}$ of $G - S_0$. Let D be a digraph defined by $V(D) = V(G) \cup V(K_n) = V(G) \cup \{v_1, \ldots, v_n\}$ and

$$A(D) = \left(\bigcup_{j=1}^{n} \{(u, v_j) \mid u \in S_j\} \right) \cup \left(\bigcup_{j=1}^{n} \{(v_j, u) \mid u \in V(G) - S_j\} \right).$$

Then the underlying graph of D is the complete bipartite graph $K_{m,n}$, where $m = |V(G)|$. Since $S_0 \ne \emptyset$ and $S_0 \subseteq V(G) - S_j$ for each $j \in \{1, \ldots, n\}$, each vertex in S_0 is an out-neighbor of v_j for each $j \in \{1, \ldots, n\}$. Therefore, the vertices $v_1, \ldots, v_n$ form a clique in the competition graph of D. Thus, the competition graph of D is $G \cup K_n$. Hence, (G, K_n) is a competition-realizable pair. □

Corollary 4.2. *For any graph G, there exists a positive integer n such that the pair $(G \cup K_1, K_n)$ is competition-realizable.*

Proof. Let n be a positive integer such that $n \ge \theta_E(G)$. Note that $\theta_E(G \cup K_1) = \theta_E(G)$. By Theorem 4.1, the pair $(G \cup K_1, K_n)$ is competition-realizable. □

4.2. The case where G_1 has no isolated vertices and $\alpha(G_2) = 1$

In this subsection, we consider competition-realizable pairs (G_1, G_2) where G_1 has no isolated vertices and $\alpha(G_2) = 1$. We present a characterization for a pair (G, K_n) being competition-realizable.

Theorem 4.3. *Let G be a graph and let n be a positive integer. Suppose that G has no isolated vertices. Then, (G, K_n) is a competition-realizable pair if and only if there exists an edge clique cover $\mathcal{F}$ of G of size at most n such that*

$$|S \cup S'| \le |V(G)| - 1$$

holds for any two cliques S and S' in $\mathcal{F}$.

Proof. Let $G_1 := G$ and $G_2 := K_n$. To show the "only if" part, suppose that $(G_1, G_2) = (G, K_n)$ is a competition-realizable pair. Then there exist edge clique covers $\mathcal{F}_1$ and $\mathcal{F}_2$ of G_1 and G_2, respectively, and graph isomorphisms $\varphi_1 : G_2 \to \Omega(\mathcal{F}_1^*)$ and $\varphi_2 : G_1 \to \Omega(\mathcal{F}_2^*)$ satisfying the conditions (i) and (ii) of Theorem 2.1. Then $\Omega(\mathcal{F}_1^*) \cong K_n$ and therefore $|\mathcal{F}_1^*| = n$, or equivalently $|\mathcal{F}_1| = n$. We show that $\mathcal{F}_1$ is a desired edge clique cover of G_1. Take any two elements S and S' in $\mathcal{F}_1$. Then $\varphi_1(v) = V(G_1) \setminus S$ and $\varphi_1(v') = V(G_1) \setminus S'$ for some vertices v and v' in G_2. Since the intersection graph of $\mathcal{F}_1^*$ is isomorphic to the complete graph K_n, we have $\varphi_1(v) \cap \varphi_1(v') \ne \emptyset$. Since

$$\varphi_1(v) \cap \varphi_1(v') = (V(G_1) \setminus S) \cap (V(G_1) \setminus S') = V(G_1) \setminus (S \cup S'),$$

we have $V(G_1) \setminus (S \cup S') \ne \emptyset$. Thus $|S \cup S'| \le |V(G_1)| - 1$.

To show the "if" part, suppose that there exists an edge clique cover $\mathcal{F}_1$ of G_1 of size at most n such that $|S \cup S'| \le |V(G_1)| - 1$ for any cliques S and S' in $\mathcal{F}_1$. If the size of $\mathcal{F}_1$ is less than n, then we add empty sets to the family $\mathcal{F}_1$ to make the size of $\mathcal{F}_1$ is equal to n. We label the vertices of G_2 as $v_1, \ldots, v_n$ and label the elements of $\mathcal{F}_1$ as $S^{(1)}_{v_1}, \ldots, S^{(1)}_{v_n}$. For $u \in V(G_1)$, let

$$S^{(2)}_u := \{v_i \in V(G_2) \mid u \in V(G_1) \setminus S^{(1)}_{v_i}\},$$

and let $\mathcal{F}_2$ be the family $\{S^{(2)}_u \mid u \in V(G_1)\}$. We now show that $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy the conditions (i) and (ii) of Theorem 2.1. To show that $\mathcal{F}_2$ is an edge clique cover of $G_2 = K_n$, take two distinct vertices v_i and v_j of G_2. Then there exists a vertex

$u \notin S_{v_i}^{(1)} \cup S_{v_j}^{(1)}$ by the hypothesis, and therefore $u \in V(G_1) \setminus S_{v_i}^{(1)}$ and $u \in V(G_1) \setminus S_{v_j}^{(1)}$. Thus $S_u^{(2)}$ contains v_i and v_j, which implies that $\mathcal{F}_2$ is an edge clique cover of G_2. Let $\mathcal{F}_i^*$ denote the family $\{V(G_i) \setminus S \mid S \in \mathcal{F}_i\}$ for $i = 1, 2$. We define maps $\varphi_1 : V(G_2) \to \mathcal{F}_1^*$ and $\varphi_2 : V(G_1) \to \mathcal{F}_2^*$ by

$$\varphi_1(v_i) := V(G_1) \setminus S_{v_i}^{(1)}, \qquad \varphi_2(u) := V(G_2) \setminus S_u^{(2)}.$$

We show that φ_1 and φ_2 satisfy the conditions (i) and (ii) of Theorem 2.1. For $u \in V(G_1)$ and $v_i \in V(G_2)$, it follows from the definition of $S_u^{(2)}$ that $v_i \in S_u^{(2)}$ if and only if $u \in V(G_1) \setminus S_{v_i}^{(1)}$, or equivalently $u \notin \varphi_1(v_i)$ if and only if $v_i \in \varphi_2(u)$. Thus φ_1 and φ_2 satisfy the condition (ii) of Theorem 2.1. We now show that φ_1 and φ_2 satisfy the condition (i) of Theorem 2.1. For two vertices u and u' of G_1, we can easily check that the following statements are equivalent:

- u and u' are adjacent in G_1;
- $S_{v_i}^{(1)}$ contains both u and u' for some $v_i \in V(G_1)$;
- $v_i \in V(G_2) \setminus S_u^{(2)}$ and $v_i \in V(G_2) \setminus S_{u'}^{(2)}$ for some $v_i \in V(G_1)$;
- $v_i \in (V(G_2) \setminus S_u^{(2)}) \cap (V(G_2) \setminus S_{u'}^{(2)}) = \varphi_2(u) \cap \varphi_2(u')$ for some $v_i \in V(G_1)$;
- $\varphi_2(u) \cap \varphi_2(u') \neq \emptyset$.

Therefore φ_2 is an isomorphism from G_1 to the intersection graph of $\mathcal{F}_2^*$. For two vertices v_i and v_j of G_2, since v_i and v_j are adjacent in G_2 and $\mathcal{F}_2$ is an edge clique cover of G_2, there exists $u \in V(G_1)$ such that $S_u^{(2)}$ contains both v_i and v_j, i.e., $u \in (V(G_1) \setminus S_{v_i}^{(1)}) \cap (V(G_1) \setminus S_{v_j}^{(1)}) = \varphi_1(v_i) \cap \varphi_1(v_j)$ for some $u \in V(G_1)$. Therefore φ_1 is an isomorphism from G_2 to the intersection graph of $\mathcal{F}_1^*$. Thus φ_1 and φ_2 satisfy the condition (i) of Theorem 2.1. Hence, by Theorem 2.1, $(G_1, G_2) = (G, K_n)$ is a competition-realizable pair. □

By using this theorem, we can obtain several results.

Proposition 4.4. *Let m and n be integers such that $m \geq 6$ and $n \geq 6$. Then, the pair (K_m, K_n) is competition-realizable.*

Proof. Without loss of generality, we may assume that $m \geq n$. Let $V(K_m) = \{u_1, \ldots, u_m\}$. Let

$$\begin{aligned} R_1 &= \{u_1, u_4, u_6\}, & R_2 &= \{u_2, u_4, u_5\}, & R_3 &= \{u_3, u_5, u_6\}, \\ R_4 &= \{u_2, u_3, u_4\}, & R_5 &= \{u_1, u_3, u_5\}, & R_6 &= \{u_1, u_2, u_6\}. \end{aligned}$$

Note that $\{R_1, \ldots, R_6\}$ is an edge clique cover of K_6. Let $S_i := R_i \cup \{u_j \mid 7 \leq j \leq m\}$ for $i = 1, \ldots, 6$ and let $\mathcal{F} := \{S_i \mid 1 \leq i \leq 6\}$. Then the family $\mathcal{F}$ is an edge clique cover of K_m. Moreover, for any i and j with $1 \leq i, j \leq 6$, $|S_i \cup S_j| = |R_i \cup R_j| + (m-6) \leq 5 + (m-6) = m-1$. By Theorem 4.3, the pair (K_m, K_n) is competition-realizable. □

Proposition 4.5. *Let n be a positive integer. Then, the pair (K_n, K_n) is competition-realizable if and only if $n = 1$ or $n \geq 6$.*

Proof. By Proposition 4.4, the pair (K_n, K_n) is competition-realizable for $n \geq 6$. Moreover, (K_1, K_1) is competition-realizable. Thus the "if" part holds.

To show the "only if" part, let (K_n, K_n) be a competition-realizable pair for a positive integer n. Assume that $n \neq 1$, i.e., $n \geq 2$. By Theorem 4.3, there exists an edge clique cover $\mathcal{F}$ consisting at most n cliques of K_n such that the union of any two cliques in $\mathcal{F}$ is not the whole vertex set of K_n. If $\mathcal{F}$ has an element S of size $n-1$, then the union of S and any clique in $\mathcal{F}$ containing the vertex that is not contained in S is the whole vertex set of K_n. Therefore, $|S| \leq n-2$ for any $S \in \mathcal{F}$. Since any element of $\mathcal{F}$ has size at least two, we have $n \geq 4$. Suppose that $n = 4$. Then $|S| \leq 2$ for any $S \in \mathcal{F}$. This implies that $\mathcal{F}$ must contain at least $\binom{4}{2} = 6$ cliques, which is a contradiction to $|\mathcal{F}| \leq 4$. Therefore $n \geq 5$. Suppose that $n = 5$. Then $\mathcal{F}$ has at most 5 cliques and the maximum size of a clique in $\mathcal{F}$ is at most 3. Since K_5 has 10 edges, $\mathcal{F}$ contains at least three cliques of size 3. Let S_1 and S_2 be cliques of size 3 in $\mathcal{F}$. Since $|S_1 \cup S_2| \leq 4$ by the choice of $\mathcal{F}$, we have $|S_1 \cap S_2| \geq 2$, which implies that the triangles induced by S_1 and S_2 share an edge. Therefore a subset of $\mathcal{F}$ consisting of three cliques of size 3 covers at most seven edges of K_5, which implies that $\mathcal{F}$ must contain at least four cliques of size 3. However, it is impossible that four triangles of K_5 mutually share an edge. Thus $n \geq 6$. Hence the proposition holds. □

We denote by K_n^m the complete multipartite graph on m partite sets in which each partite set has n vertices.

Proposition 4.6. *Let m and n be positive integers such that $2 \leq m < n$. Then, the pair (K_2^m, K_n) is competition-realizable.*

Proof. Let $\{x_l, y_l\}$ be the lth partite set of K_2^m for each $l \in [m] := \{1, \ldots, m\}$. Let $S_0 := \{x_1, x_2, \ldots, x_m\}$ and $S_i := (\{y_1, y_2, \ldots, y_m\} \setminus \{y_i\}) \cup \{x_i\}$ for $i \in [m]$, and let $\mathcal{F} = \{S_0, S_1, \ldots, S_m\}$. Since no two vertices in S_l belong to the same partite set, S_l forms a clique in K_2^m for each $l \in \{0\} \cup [m]$. Now we take an edge e of K_2^m. Then, $e = x_i x_j$, $e = x_i y_j$, or $e = y_i y_j$ for some $i, j \in [m]$ with $i \neq j$. If $e = x_i x_j$, then e is covered by the set S_0. If $e = x_i y_j$, then e is covered by S_i. If $e = y_i y_j$, then e is covered by S_l for some $l \in [m] \setminus \{i, j\}$. Thus $\mathcal{F}$ is an edge clique cover of K_2^m of size at most n (by the hypothesis that $n \geq m+1$). Since S_i has exactly one of $x_1, \ldots, x_m$ and $m-1$ of $y_1, \ldots, y_m$ for $i \in [m]$ and $S_0 = \{x_1, \ldots, x_m\}$, we have $|S_i \cap S_j| \geq 1$ for any $i, j \in \{0\} \cup [m]$. Thus $|S_i \cup S_j| \leq 2m-1 = |V(K_2^m)| - 1$ for any $i, j \in \{0\} \cup [m]$. Hence (K_2^m, K_n) is competition-realizable by Theorem 4.3. □

By Proposition 4.6, we know that the pair (K_n^3, K_4) is competition-realizable if $n = 2$. For $n \geq 3$, the following holds. Let $L(n)$ denote the largest size of a family of mutually orthogonal Latin squares of order n. Park *et. al* [11] showed that, if m and n are positive integers such that $3 \leq m \leq L(n) + 2$, then there is a minimum edge clique cover of K_n^m of size n^2 in which each clique in the cover is a complete graph of size m (see also [7] and [9]).

Proposition 4.7. *Let m, n, and t be positive integers such that $3 \leq m \leq L(n)+2$. Then, the pair (K_n^m, K_t) is competition-realizable if and only if $n^2 \leq t$.*

Proof. Suppose that $n^2 \leq t$. Since K_n^m has an edge clique cover $\mathcal{F}$ of size n^2 (at most t) in which each clique has size m, the union of any two cliques in $\mathcal{F}$ has size at most $2m \leq 3m - 1 \leq nm - 1 = |V(K_n^m)| - 1$. Thus, by Theorem 4.3, (K_n^m, K_t) is competition-realizable.

Suppose that (K_n^m, K_t) is competition-realizable. Then, by Theorem 4.3, there exists an edge clique cover $\mathcal{F}$ of K_n^m with size at most t. Thus, $n^2 = \theta_E(K_n^m) \leq |\mathcal{F}| \leq t$. $\square$

We now consider (G, K_n) where G is a triangle-free graph having no isolated vertices.

Proposition 4.8. *Let G be a triangle-free graph having no isolated vertices. Then, the pair (G, K_n) is competition-realizable if and only if either $n = |E(G)| = 1$ or $n \geq |E(G)| \geq 3$ and $G \neq P_4$, where P_4 is the path on 4 vertices.*

Proof. Let $\mathcal{F} := \{\{u, v\} \mid uv \in E(G)\}$. Then the family $\mathcal{F}$ is an edge clique cover of G. If $n = |E(G)| = 1$, then $G \cong K_2$ since G has no isolated vertices and $(G, K_n) = (K_2, K_1)$ is clearly a competition-realizable pair. Suppose that $n \geq |E(G)| \geq 3$. Then $|V(G)| \geq 4$ since G is triangle-free. If $|V(G)| = 4$, then $G \cong K_{1,3}$ since $G \neq P_4$. Let $V(K_{1,3}) = \{x, y, z, w\}$, where x is the vertex of degree 3. Then $\mathcal{F} = \{\{x, y\}, \{x, z\}, \{x, w\}\}$ and $\mathcal{F}$ satisfies the condition of Theorem 4.3. If $|V(G)| \geq 5$, then $|\{u, v\} \cup \{x, y\}| \leq 4 \leq |V(G)| - 1$ for any two edges uv and xy of G and so (G, K_n) is competition-realizable by Theorem 4.3.

To show the converse, suppose that a pair (G, K_n) is competition-realizable. If $|E(G)| = 1$, then $G \cong K_2$ since G has no isolated vertices and, by Proposition 4.5, $n = 1$. Suppose that $|E(G)| = 2$. Then $G \cong K_{1,2}$ or $K_2 \cup K_2$ since G is triangle-free and has no isolated vertices. By Theorem 4.3, neither $(K_{1,2}, K_n)$ nor $(K_2 \cup K_2, K_n)$ is competition-realizable. Suppose that $|E(G)| \geq 3$. Then $n \geq \theta_E(G) = |E(G)| \geq 3$ by Lemma 2.3. By Theorem 4.3, the pair (P_4, K_n) is not competition-realizable. Therefore $G \neq P_4$. Hence the proposition holds. $\square$

4.3. The case where $\alpha(G_1) = \alpha(G_2) = 2$

Let C_n denote the cycle with n vertices. By checking all the orientations of the complete bipartite graph $K_{5,5}$, we can confirm that (C_5, C_5) is not a competition-realizable pair. Since $\alpha(C_5) = 2$, we can observe that the condition $\alpha(G_1) = \alpha(G_2) = 2$ does not guarantee the pair (G_1, G_2) to be competition-realizable. For a graph G, we denote the complement of G by $\overline{G}$. Note that $\overline{C_5} = C_5$. More generally, we can show that the pairs of the complements of odd cycles are such examples (see [8] for a related topic).

Proposition 4.9. *Let s and t be positive integers. Let $(G_1, G_2) = (\overline{C_{2s+3}}, \overline{C_{2t+3}})$ Then (G_1, G_2) is not competition-realizable and $\alpha(G_1) = \alpha(G_2) = 2$.*

We use the following lemma to prove this proposition.

Lemma 4.10. *Let (G_1, G_2) be a competition-realizable pair. If G_1 is not a complete graph, then the vertex set of G_2 can be partitioned into two cliques V_1 and V_2 of G_2.*

Proof. Since (G_1, G_2) is a competition-realizable pair, there exists an orientation D of the complete bipartite graph $K_{m,n}$ such that $C(D) = G_1 \cup G_2$, where $m = |V(G_1)|$ and $n = |V(G_2)|$. Since G_1 is not a complete graph, there are two vertices u_1 and u_2 in $V(G_1)$ which are not adjacent in G_1. Let $V_1 := N_D^+(u_1)$ and $V_2 := V(G_2) \setminus V_1$. Note that $V_1 \subseteq V(G_2)$ and $V_2 \subseteq V(G_2)$. Take any vertex v_1 in V_1. Since v_1 is an out-neighbor of u_1 and since v_1 cannot be a common out-neighbor of u_1 and u_2, we have $v_1 \notin N_D^+(u_2)$. Then $v_1 \in N_D^-(u_2)$. Therefore $(v_1, u_2) \in A(D)$. Since v_1 was arbitrarily taken from V_1, u_2 is a common out-neighbor of all the vertices in V_1 and thus V_1 is a clique in G_2. Take any vertex v_2 in V_2. Since v_2 is not an out-neighbor of u_1, v_2 is an in-neighbor of u_1. Therefore $(v_2, u_1) \in A(D)$. Since v_2 is arbitrarily taken from V_2, u_1 is a common out-neighbor of all the vertices in V_2 and thus V_2 is a clique in G_2. $\square$

Proof of Proposition 4.9. Let $(G_1, G_2) = (\overline{C_{2s+3}}, \overline{C_{2t+3}})$. It is easy to see that $\alpha(G_1) = \alpha(G_2) = 2$. Since the vertex set of an odd cycle of length at least 5 cannot be partitioned into two independent sets, the vertices of the complement $\overline{C_{2s+3}}$ of an odd cycle cannot be covered by two cliques of $\overline{C_{2s+3}}$. Thus it follows from Lemma 4.10 that (G_1, G_2) is not competition-realizable. $\square$

We can characterize the competition-realizable pairs (G_1, G_2) when both G_1 and G_2 are cycles.

Proposition 4.11. *Let m and n be integers greater than or equal to 3. Then, the pair (C_m, C_n) is competition-realizable if and only if $(m, n) = (4, 4)$.*

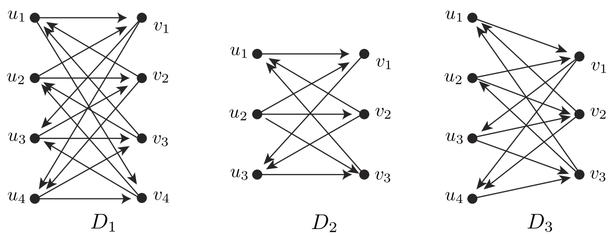

Fig. 2. The digraphs D_1, D_2, and D_3.

Proof. Let (C_m, C_n) be a competition-realizable pair. By Lemma 2.3, we have $m = n$. By Theorem 3.2, we have $m \le 5$. By Proposition 4.5, $(C_3, C_3) \cong (K_3, K_3)$ is not a competition-realizable pair. Since neither (C_3, C_3) nor (C_5, C_5) is a competition-realizable pair, we have $(m, n) = (4, 4)$. In fact, the pair (C_4, C_4) is competition-realizable. To show this, we consider a digraph D_1 defined by $V(D_1) = \{u_1, u_2, u_3, u_4\} \cup \{v_1, v_2, v_3, v_4\}$ and $A(D_1) = \{(u_1, v_1), (u_1, v_4), (u_2, v_1), (u_2, v_2), (u_3, v_2), (u_3, v_3), (u_4, v_3), (u_4, v_4), (v_2, u_1), (v_3, u_1), (v_3, u_2), (v_4, u_2), (v_1, u_3), (v_4, u_3), (v_1, u_4), (v_2, u_4)\}$ (see Fig. 2). Then, the underlying graph of D_1 is the complete bipartite graph $K_{4,4}$ and the competition graph of D is $C_4 \cup C_4$. Thus the pair (C_4, C_4) is competition-realizable. Hence the proposition holds. □

We can also characterize a competition-realizable pair (G_1, G_2) when both G_1 and G_2 are paths by using the following lemma.

Lemma 4.12. *Let (G_1, G_2) be a competition-realizable pair. If G_1 and G_2 are connected triangle-free graphs, then $|V(G_2)| - 1 \le |V(G_1)| \le |V(G_2)| + 1$.*

Proof. Since G_1 and G_2 are triangle-free, $\theta_E(G_1) = |E(G_1)|$ and $\theta_E(G_2) = |E(G_2)|$. Since G_1 and G_2 are connected, $|V(G_1)| - 1 \le |E(G_1)|$ and $|V(G_2)| - 1 \le |E(G_2)|$. By Lemma 2.3, $\theta_E(G_1) \le |V(G_2)|$ and $\theta_E(G_2) \le |V(G_1)|$. Therefore $|V(G_2)| - 1 \le |E(G_2)| = \theta_E(G_2) \le |V(G_1)| \le |E(G_1)| + 1 = \theta_E(G_1) + 1 \le |V(G_2)| + 1$. □

Let P_n denote the path with n vertices.

Proposition 4.13. *Let m and n be positive integers such that $m \ge n$. Then, the pair (P_m, P_n) is competition-realizable if and only if (m, n) is one of $(1, 1)$, $(2, 1)$, $(3, 3)$, and $(4, 3)$.*

Proof. First, we show the "if" part. Since $P_1 \cong K_1$ and $P_2 \cong K_2$, the pairs (P_1, P_1) and (P_2, P_1) are competition-realizable by Theorem 4.1. Consider the case where $(m, n) = (3, 3)$. Let D_2 be the digraph defined by $V(D_2) = \{u_1, u_2, u_3\} \cup \{v_1, v_2, v_3\}$ and $A(D_2) = \{(u_1, v_1), (u_2, v_1), (u_2, v_2), (u_2, v_3), (u_3, v_3), (v_1, u_3), (v_2, u_1), (v_2, u_3), (v_3, u_1)\}$ (see Fig. 2). Then the underlying graph of D_2 is the complete bipartite graph $K_{3,3}$ and the competition graph of D_2 is $P_3 \cup P_3$. Thus the pair (P_3, P_3) is competition-realizable. Consider the case where $(m, n) = (4, 3)$. Let D_3 be a digraph defined by $V(D_3) = \{u_1, u_2, u_3, u_4\} \cup \{v_1, v_2, v_3\}$ and $A(D_3) = \{(u_1, v_1), (u_2, v_1), (u_2, v_2), (u_3, v_2), (u_3, v_3), (u_4, v_3), (v_1, u_3), (v_1, u_4), (v_2, u_1), (v_2, u_4), (v_3, u_1), (v_3, u_2)\}$ (see Fig. 2). Then the underlying graph of D_3 is the complete bipartite graph $K_{4,3}$ and the competition graph of D_3 is $P_4 \cup P_3$. Thus the pair (P_4, P_3) is competition-realizable.

We now show the "only if" part. Assume that (P_m, P_n) is a competition-realizable pair. Note that, if $n \ge 5$, then $\alpha(P_n) = \lceil \frac{n}{2} \rceil \ge 3$. By Theorem 3.2, we have $1 \le n \le 4$. By Lemma 4.12, we have $m = n$ or $m = n + 1$. Therefore, $(m, n) \in \{(1, 1), (2, 1), (2, 2), (3, 2), (3, 3), (4, 3), (4, 4)\}$. If (G, P_2) is a competition-realizable pair, then G must contain an isolated vertex which is a common out-neighbor of the two vertices of P_2 in an orientation D of the complete bipartite graph $K_{m,2}$. Therefore neither (P_3, P_2) nor (P_2, P_2) is a competition-realizable pair. To show by contradiction that (P_4, P_4) is not competition-realizable, suppose that $P_4 \cup P_4$ is the competition graph of some orientation D of the complete bipartite graph $K_{4,4}$. Since P_4 is triangle-free, $|N_D^-(x)| \le 2$ for any vertex x of D. Then we have

$$16 = |A(D)| = \sum_{x \in V(D)} |N_D^-(x)| \le 16.$$

Therefore, $|N_D^-(x)| = 2$ holds for any vertex x of D. Moreover, $|N_D^+(x)| = 4 - |N_D^-(x)| = 2$ holds for any vertex x of D. Since there are only six edges in $P_4 \cup P_4$, there exists an edge e such that the endvertices x and y of e have two common out-neighbors in D. Then, at least one of x and y, say x, has degree 2 in $P_4 \cup P_4$. Therefore, the vertex x has an out-neighbor other than the two common out-neighbors of x and y in D. This implies that there is a vertex of out-degree at least 3 in D, which is a contradiction. Thus, (P_4, P_4) is not a competition-realizable pair. Hence $(m, n) \in \{(1, 1), (2, 1), (3, 3), (4, 3)\}$. □

5. Extremal competition-realizable pairs

In this section, we present competition-realizable pairs with the maximum number of edges and with the minimum number of edges.

5.1. The maximum number of edges in a competition-realizable pair

Theorem 5.1. *Let m and n be positive integers. If $m \geq 6$ and $n \geq 6$, then the pair (K_m, K_n) has the maximum number of edges among all the competition realizable pairs (G_1, G_2) with $|V(G_1)| = m$ and $|V(G_2)| = n$.*

Proof. The theorem immediately follows from Proposition 4.4. □

Therefore we have an upper bound for the number of edges of the competition graph of an oriented bipartite graph.

Corollary 5.2. *Let m and n be positive integers. Let G be the competition graph of an orientation D of the complete bipartite graph $K_{m,n}$. If $m \geq 6$ and $n \geq 6$, then*

$$|E(G)| \leq \tfrac{1}{2}m(m-1) + \tfrac{1}{2}n(n-1).$$

Proof. The corollary follows from Theorem 5.1. □

5.2. The minimum number of edges in a competition-realizable pair

Lemma 5.3. *Let m and n be positive integers. Let G be the competition graph of an orientation D of the complete bipartite graph $K_{m,n}$. Then*

$$|E(G)| \geq \min\left\{ t(m,n), \frac{m(m-1)}{2}, \frac{n(n-1)}{2} \right\},$$

where $t(m,n)$ is the number of edges in $K_{\lfloor \frac{m}{2} \rfloor} \cup K_{\lceil \frac{m}{2} \rceil} \cup K_{\lfloor \frac{n}{2} \rfloor} \cup K_{\lceil \frac{n}{2} \rceil}$, i.e.,

$$t(m,n) = \frac{1}{8} \left\lfloor \frac{m}{2} \right\rfloor \left\lceil \frac{m}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \left(\left\lfloor \frac{m}{2} \right\rfloor - 1 \right) \left(\left\lceil \frac{m}{2} \right\rceil - 1 \right) \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \left(\left\lceil \frac{n}{2} \right\rceil - 1 \right).$$

Proof. Let D be any orientation of the complete bipartite graph $K_{m,n}$, and let G be the competition graph of D. Let (U, V) be the bipartition of $K_{m,n}$ where $|U| = m$ and $|V| = n$. If one of U and V is a clique of G, then $|E(G)| \geq \min\{\frac{m(m-1)}{2}, \frac{n(n-1)}{2}\}$ and therefore the lemma holds. Suppose that neither U nor V is a clique of G. Then there are four vertices $u_1, u_2 \in U$ and $v_1, v_2 \in V$ such that u_1 and u_2 are not adjacent in G and v_1 and v_2 are not adjacent in G. By Lemma 4.10, U can be partitioned into two cliques U_1 and U_2 of G, and V can be partitioned into two cliques V_1 and V_2 of G. Therefore,

$$\begin{aligned}
|E(G)| &\geq |E(G[U_1])| + |E(G[U_2])| + |E(G[V_1])| + |E(G[V_2])| \\
&= |E(K_{|U_1|})| + |E(K_{m-|U_1|})| + |E(K_{|V_1|})| + |E(K_{n-|V_1|})| \\
&\geq |E(K_{\lfloor \frac{m}{2} \rfloor})| + |E(K_{\lceil \frac{m}{2} \rceil})| + |E(K_{\lfloor \frac{n}{2} \rfloor})| + |E(K_{\lceil \frac{n}{2} \rceil})| \\
&= |E(K_{\lfloor \frac{m}{2} \rfloor} \cup K_{\lceil \frac{m}{2} \rceil} \cup K_{\lfloor \frac{n}{2} \rfloor} \cup K_{\lceil \frac{n}{2} \rceil})| \\
&= t(m,n).
\end{aligned}$$

Hence the lemma holds. □

Theorem 5.4. *Let m and n be positive integers with $m \geq n$. Then, either the pair $(K_{\lfloor \frac{m}{2} \rfloor} \cup K_{\lceil \frac{m}{2} \rceil}, K_{\lfloor \frac{n}{2} \rfloor} \cup K_{\lceil \frac{n}{2} \rceil})$ or the pair $(\overline{K_m}, K_n)$ has the minimum number of edges among all the competition-realizable pairs (G_1, G_2) with $|V(G_1)| = m$ and $|V(G_2)| = n$.*

Proof. We first show that the pair $(K_{\lfloor \frac{m}{2} \rfloor} \cup K_{\lceil \frac{m}{2} \rceil}, K_{\lfloor \frac{n}{2} \rfloor} \cup K_{\lceil \frac{n}{2} \rceil})$ is competition-realizable. Let U_1, U_2, V_1, and V_2 be disjoint sets with $|U_1| = \lfloor \frac{m}{2} \rfloor$, $|U_2| = \lceil \frac{m}{2} \rceil$, $|V_1| = \lfloor \frac{n}{2} \rfloor$, and $|V_2| = \lceil \frac{n}{2} \rceil$. Let D be the digraph defined by $V(D) = U_1 \cup U_2 \cup V_1 \cup V_2$ and $A(D) = \{(u, v) \mid u \in U_1, v \in V_1\} \cup \{(u, v) \mid u \in U_2, v \in V_2\} \cup \{(v, u) \mid u \in U_1, v \in V_2\} \cup \{(v, u) \mid u \in U_2, v \in V_1\}$. Then, the underlying graph of D is the complete bipartite graph $K_{m,n}$, and the competition graph of D is $(K_{\lfloor \frac{m}{2} \rfloor} \cup K_{\lceil \frac{m}{2} \rceil}) \cup (K_{\lfloor \frac{n}{2} \rfloor} \cup K_{\lceil \frac{n}{2} \rceil})$. Thus $(K_{\lfloor \frac{m}{2} \rfloor} \cup K_{\lceil \frac{m}{2} \rceil}, K_{\lfloor \frac{n}{2} \rfloor} \cup K_{\lceil \frac{n}{2} \rceil})$ is competition-realizable.

Second, we show that the pair $(\overline{K_m}, K_n)$ is competition-realizable. Let U and V be disjoint sets with $|U| = m$ and $|V| = n$. Let D be the digraph defined by $V(D) = U \cup V$ and $A(D) = \{(v, u) \mid u \in U, v \in V\}$. Then, the underlying graph of D is the complete bipartite graph $K_{m,n}$, and the competition graph of D is $\overline{K_m} \cup K_n$. Thus the pair $(\overline{K_m}, K_n)$ is competition-realizable.

Since $|E((K_{\lfloor \frac{m}{2} \rfloor} \cup K_{\lceil \frac{m}{2} \rceil}) \cup (K_{\lfloor \frac{n}{2} \rfloor} \cup K_{\lceil \frac{n}{2} \rceil}))| = t(m,n)$ and $|E(\overline{K_m} \cup K_n)| = \frac{1}{2}n(n-1)$, the theorem follows from Lemma 5.3. □

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