



On the competition graphs of d -partial orders



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ABSTRACT

In this paper, we study the competition graphs of d -partial orders and obtain their characterization which extends results given by Cho and Kim (2005). We also show that any graph can be made into the competition graph of a d -partial order for some positive integer d as long as adding isolated vertices is allowed. We then study graphs whose partial order competition dimensions are at most three, where the partial order competition dimension of a graph G is the smallest nonnegative integer d such that G together with some isolated vertices is the competition graph of a d -partial order.

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1. Introduction

The *competition graph* of a digraph D , which is denoted by $C(D)$, has the same vertex set as D and has an edge xy between two distinct vertices x and y if for some vertex $z \in V$, the arcs (x, z) and (y, z) are in D . The competition graph has been extensively studied over the last 40 years (see the survey articles by Kim [2] and Lundgren [6]).

Let d be a positive integer. For $\mathbf{x} = (x_1, x_2, \dots, x_d)$, $\mathbf{y} = (y_1, y_2, \dots, y_d) \in \mathbb{R}^d$, we write $\mathbf{x} < \mathbf{y}$ if $x_i < y_i$ for each $i = 1, \dots, d$. For a finite subset S of \mathbb{R}^d , let D_S be the digraph defined by $V(D_S) = S$ and $A(D_S) = \{(\mathbf{x}, \mathbf{v}) \mid \mathbf{v}, \mathbf{x} \in S, \mathbf{v} < \mathbf{x}\}$. A digraph D is called a d -partial order if there exists a finite subset S of \mathbb{R}^d such that D is isomorphic to the digraph D_S . By convention, the zero-dimensional Euclidean space \mathbb{R}^0 consists of a single point 0. In this context, we define a digraph with exactly one vertex as a 0-partial order. A 2-partial order is also called *doubly partial order* (see Fig. 1 for an example).

Cho and Kim [1] studied the competition graphs of doubly partial orders and showed that the competition graphs of these digraphs are interval graphs and any interval graph can be made into the competition graph of a doubly partial order by adding sufficiently many isolated vertices.

Theorem 1.1 ([1]). *The competition graph of a doubly partial order is an interval graph.*

Theorem 1.2 ([1]). *Every interval graph can be made into the competition graph of a doubly partial order by adding sufficiently many isolated vertices.*

Several variants of competition graphs of doubly partial orders also have been studied (see [3–5,7–9]).

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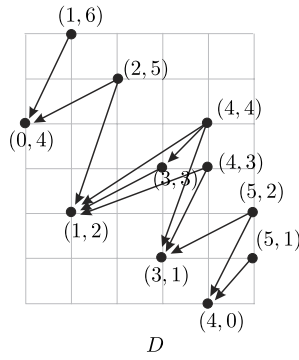


Fig. 1. A doubly partial order D .

In this paper, we study the competition graphs of d -partial orders. We obtain their characterization which nicely extends results given by Cho and Kim [1]. We also show that any graph can be made into the competition graph of a d -partial order for some positive integer d as long as adding isolated vertices is allowed. We then introduce the notion of the partial order competition dimension of a graph. Especially, we study graphs whose partial order competition dimensions are at most three.

2. The competition graphs of d -partial orders

In this section, we use the following notation. We use a bold faced letter to represent a point in \mathbb{R}^d ($d \geq 2$). For $\mathbf{x} \in \mathbb{R}^d$, let x_i denote the i th component of \mathbf{x} for each $i = 1, \dots, d$. Let $\mathbf{e}_i \in \mathbb{R}^d$ be the standard unit vector whose i th component is 1, i.e., $\mathbf{e}_1 := (1, 0, \dots, 0), \dots, \mathbf{e}_d := (0, \dots, 0, 1)$. Let $\mathbf{1}$ be the all-one vector $(1, \dots, 1)$ in \mathbb{R}^d . Note that, for $\mathbf{x} \in \mathbb{R}^d$, the standard inner product of \mathbf{x} and $\mathbf{1}$ is

$$\mathbf{x} \cdot \mathbf{1} = \sum_{i=1}^d x_i.$$

For $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^d$, let $\text{Conv}(\mathbf{v}_1, \dots, \mathbf{v}_n)$ denote the convex hull of $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^d$, i.e.,

$$\text{Conv}(\mathbf{v}_1, \dots, \mathbf{v}_n) := \left\{ \sum_{i=1}^n \lambda_i \mathbf{v}_i \mid \sum_{i=1}^n \lambda_i = 1, \lambda_i \geq 0, 1 \leq i \leq n \right\}.$$

2.1. The regular $(d-1)$ -dimensional simplex $\Delta^{d-1}(\mathbf{p})$

Let \mathcal{H}^d be the hyperplane in \mathbb{R}^d defined by the equation $\mathbf{x} \cdot \mathbf{1} = 0$, and let \mathcal{H}_+^d be the open half space in \mathbb{R}^d defined by the inequality $\mathbf{x} \cdot \mathbf{1} > 0$, i.e.,

$$\mathcal{H}^d := \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{x} \cdot \mathbf{1} = 0\}, \quad \mathcal{H}_+^d := \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{x} \cdot \mathbf{1} > 0\}.$$

We fix a point \mathbf{p} in \mathcal{H}_+^d . Let $\Delta^{d-1}(\mathbf{p})$ be the intersection of the hyperplane \mathcal{H}^d and the closed cone

$$\{\mathbf{x} \in \mathbb{R}^d \mid x_i \leq p_i \ (i = 1, \dots, d)\}.$$

Lemma 2.1. For $\mathbf{p} \in \mathcal{H}_+^d$, the set $\Delta^{d-1}(\mathbf{p})$ is the convex hull $\text{Conv}(\mathbf{v}_1, \dots, \mathbf{v}_d)$ of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_d$ defined by

$$\mathbf{v}_i = \mathbf{p} - (\mathbf{p} \cdot \mathbf{1})\mathbf{e}_i \quad (i = 1, \dots, d).$$

Moreover, $\Delta^{d-1}(\mathbf{p})$ is a regular $(d-1)$ -simplex. (See Fig. 2.)

Proof. Let $\mathbf{v}_1, \dots, \mathbf{v}_d \in \mathbb{R}^d$ be the intersections of the hyperplane \mathcal{H}^d and the lines going through \mathbf{p} with directional vectors $\mathbf{e}_1, \dots, \mathbf{e}_d$, respectively. Then $\mathbf{v}_i = \mathbf{p} - (\mathbf{p} \cdot \mathbf{1})\mathbf{e}_i$ for $i = 1, \dots, d$. By definition, we have $\Delta^{d-1}(\mathbf{p}) = \text{Conv}(\mathbf{v}_1, \dots, \mathbf{v}_d)$. Since $\mathbf{v}_1, \dots, \mathbf{v}_d$ are linearly independent, the set $\Delta^{d-1}(\mathbf{p})$ is a $(d-1)$ -simplex. Moreover, since the length of each edge of $\Delta^{d-1}(\mathbf{p})$ is equal to $\sqrt{2}(\mathbf{p} \cdot \mathbf{1})$, the simplex $\Delta^{d-1}(\mathbf{p})$ is regular. \square

Note that the distance between \mathbf{p} and each vertex of $\Delta^{d-1}(\mathbf{p})$ is equal to $\mathbf{p} \cdot \mathbf{1}$. Moreover, the directional vector for the line passing through the vertices \mathbf{v}_i and \mathbf{v}_j is $\mathbf{e}_j - \mathbf{e}_i$ for distinct i, j in $\{1, \dots, d\}$. The center of $\Delta^{d-1}(\mathbf{p})$ is $\frac{1}{d} \sum_{i=1}^d \mathbf{v}_i = \mathbf{p} - \frac{1}{d}(\mathbf{p} \cdot \mathbf{1})\mathbf{1}$. Therefore, the directional vector from this center to the point \mathbf{p} is parallel to the all-one vector $\mathbf{1}$, and the distance between this center and the point \mathbf{p} is $\frac{1}{\sqrt{d}}(\mathbf{p} \cdot \mathbf{1})$ which is $\frac{1}{\sqrt{2d}}$ times the edge length of $\Delta^{d-1}(\mathbf{p})$.

We say that two geometric figures in \mathbb{R}^d are *homothetic* if they are related by a geometric contraction or expansion. From the above observation, we can conclude the following:

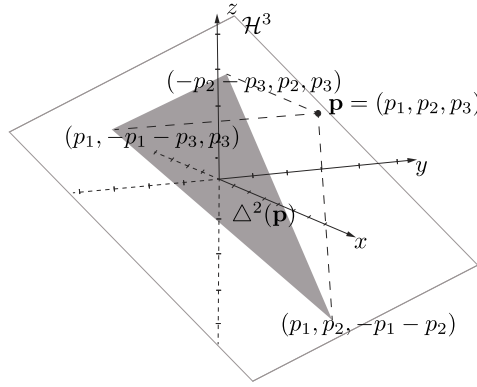


Fig. 2. A point $\mathbf{p} \in \mathcal{H}_+^3$ and the triangle $\Delta^2(\mathbf{p})$.

Proposition 2.2. If $\mathbf{p}, \mathbf{q} \in \mathcal{H}_+^d$, then $\Delta^{d-1}(\mathbf{p})$ and $\Delta^{d-1}(\mathbf{q})$ are homothetic.

2.2. A bijection from \mathcal{H}_+^d to the set of certain regular $(d-1)$ -simplices

Lemma 2.3. The vertices of $\Delta^{d-1}(\mathbf{1})$ may be labeled as $\mathbf{w}_1, \dots, \mathbf{w}_d$ so that $\mathbf{w}_j - \mathbf{w}_i$ is a positive scalar multiple of $\mathbf{e}_i - \mathbf{e}_j$ for any distinct i, j in $\{1, \dots, d\}$.

Proof. By Lemma 2.1, the vertices of $\Delta^{d-1}(\mathbf{1})$ are $\mathbf{1} - d\mathbf{e}_i$ for $i = 1, \dots, d$. We denote $\mathbf{1} - d\mathbf{e}_i$ by \mathbf{w}_i to obtain the desired labeling. \square

Lemma 2.4. Let d be an integer with $d \geq 2$. Suppose that Λ is a regular $(d-1)$ -simplex contained in the hyperplane \mathcal{H}^d homothetic to $\Delta^{d-1}(\mathbf{1})$. Then, there exists $\mathbf{p} \in \mathcal{H}_+^d$ such that $\Lambda = \Delta^{d-1}(\mathbf{p})$.

Proof. Since Λ is homothetic to $\Delta^{d-1}(\mathbf{1})$, there exist $\mathbf{v}_1, \dots, \mathbf{v}_d \in \mathcal{H}^d$ which are linearly independent such that $\Lambda = \text{Conv}(\mathbf{v}_1, \dots, \mathbf{v}_d)$ and $\mathbf{v}_j - \mathbf{v}_i$ is a positive scalar multiple of $\mathbf{e}_i - \mathbf{e}_j$ for any distinct i and j in $\{1, \dots, d\}$ by Lemma 2.3. Moreover, $\mathbf{v}_i \cdot \mathbf{e}_i < \mathbf{v}_j \cdot \mathbf{e}_i$ for any distinct i and j in $\{1, \dots, d\}$. Let $\mathbf{p} \in \mathbb{R}^d$ be a vector defined by $p_i = \max\{\mathbf{v}_k \cdot \mathbf{e}_i \mid 1 \leq k \leq d\}$ ($i = 1, \dots, d$). Then, $\Lambda = \Delta^{d-1}(\mathbf{p})$. Since $\mathbf{v}_1 \in \mathcal{H}^d$, we have $\mathbf{v}_1 \cdot \mathbf{1} = 0$. Since $p_i \geq \mathbf{v}_1 \cdot \mathbf{e}_i$ for any $i = 1, \dots, d$, we have $\mathbf{p} \cdot \mathbf{1} \geq \mathbf{v}_1 \cdot \mathbf{1} = 0$. If $\mathbf{p} \cdot \mathbf{1} = 0$, then we obtain $\mathbf{p} = \mathbf{v}_1$ and $p_1 = \mathbf{v}_1 \cdot \mathbf{e}_1 < \mathbf{v}_2 \cdot \mathbf{e}_1$, which is a contradiction to the definition of p_1 . Therefore $\mathbf{p} \cdot \mathbf{1} > 0$, i.e., $\mathbf{p} \in \mathcal{H}_+^d$. Thus the lemma holds. \square

Let d be an integer with $d \geq 2$. Let \mathcal{F}_*^{d-1} be the set of the regular $(d-1)$ -simplices in the hyperplane \mathcal{H}^d which are homothetic to $\Delta^{d-1}(\mathbf{1})$. Let f_* be a map from \mathcal{H}_+^d to \mathcal{F}_*^{d-1} defined by $f_*(\mathbf{p}) = \Delta^{d-1}(\mathbf{p})$. By Lemma 2.1 and Proposition 2.2, $\Delta^{d-1}(\mathbf{p}) \in \mathcal{F}_*^{d-1}$ and therefore the map f_* is well-defined.

Proposition 2.5. For each integer $d \geq 2$, the map $f_* : \mathcal{H}_+^d \rightarrow \mathcal{F}_*^{d-1}$ is a bijection.

Proof. By Lemma 2.4, the map f_* is surjective. Suppose that $\Delta^{d-1}(\mathbf{p}) = \Delta^{d-1}(\mathbf{q})$. Since the centers of $\Delta^{d-1}(\mathbf{p})$ and $\Delta^{d-1}(\mathbf{q})$ are the same, we have $\mathbf{p} - \frac{1}{d}(\mathbf{p} \cdot \mathbf{1})\mathbf{1} = \mathbf{q} - \frac{1}{d}(\mathbf{q} \cdot \mathbf{1})\mathbf{1}$. Since the lengths of edges of $\Delta^{d-1}(\mathbf{p})$ and $\Delta^{d-1}(\mathbf{q})$ are the same, we have $\sqrt{2}(\mathbf{p} \cdot \mathbf{1}) = \sqrt{2}(\mathbf{q} \cdot \mathbf{1})$. Therefore, we have $\mathbf{p} = \mathbf{q}$. Thus the map f_* is injective. Hence the map f_* is a bijection. \square

Let \mathcal{F}^{d-1} be the set of the interiors of the regular $(d-1)$ -simplices in the hyperplane \mathcal{H}^d which are homothetic to $\Delta^{d-1}(\mathbf{1})$. Then there is a clear bijection $\varphi : \mathcal{F}_*^{d-1} \rightarrow \mathcal{F}^{d-1}$ such that for each element in \mathcal{F}_*^{d-1} , its φ -value is its interior. Therefore we obtain the following corollary.

Corollary 2.6. For each integer $d \geq 2$, the map $\varphi \circ f_* : \mathcal{H}_+^d \rightarrow \mathcal{F}^{d-1}$ is a bijection.

2.3. A characterization of the competition graphs of d -partial orders

Let $A^{d-1}(\mathbf{p})$ be the interior of the regular simplex $\Delta^{d-1}(\mathbf{p})$, i.e., $A^{d-1}(\mathbf{p}) := \text{int}(\Delta^{d-1}(\mathbf{p}))$. Then

$$A^{d-1}(\mathbf{p}) = \{\mathbf{x} \in \mathcal{H}^d \mid \mathbf{x} < \mathbf{p}\}.$$

Proposition 2.7. For $\mathbf{p}, \mathbf{q} \in \mathcal{H}_+^d$, $\Delta^{d-1}(\mathbf{p})$ is contained in $A^{d-1}(\mathbf{q})$ if and only if $\mathbf{p} < \mathbf{q}$.

Proof. Suppose that $\mathbf{p} < \mathbf{q}$. Take a point \mathbf{a} in $\Delta^{d-1}(\mathbf{p})$. Then $a_k \leq p_k$ for each $k = 1, \dots, d$. By the assumption that $\mathbf{p} < \mathbf{q}$, $a_k < q_k$ for each $k = 1, \dots, d$, that is, $\mathbf{a} < \mathbf{q}$. Thus $\Delta^{d-1}(\mathbf{p})$ is contained in $A^{d-1}(\mathbf{q})$.

Suppose that $\Delta^{d-1}(\mathbf{p})$ is contained in $A^{d-1}(\mathbf{q})$. Then, by Lemma 2.1, $\mathbf{p} - (\mathbf{p} \cdot \mathbf{1})\mathbf{e}_i$ ($i = 1, \dots, d$) are points in $A^{d-1}(\mathbf{q})$. By the definition of $A^{d-1}(\mathbf{q})$, we have $\mathbf{p} - (\mathbf{p} \cdot \mathbf{1})\mathbf{e}_i < \mathbf{q}$ ($i = 1, \dots, d$), which implies $\mathbf{p} < \mathbf{q}$. \square

Lemma 2.8. Let d be a positive integer and let D be a d -partial order. Then, two vertices \mathbf{v} and \mathbf{w} of D are adjacent in the competition graph of D if and only if there exists a vertex \mathbf{a} in D such that $\Delta^{d-1}(\mathbf{a}) \subseteq A^{d-1}(\mathbf{v}) \cap A^{d-1}(\mathbf{w})$.

Proof. By definition, two vertices \mathbf{v} and \mathbf{w} are adjacent in the competition graph of D if and only if there exists a vertex \mathbf{a} in D such that $\mathbf{a} < \mathbf{v}$ and $\mathbf{a} < \mathbf{w}$. By Proposition 2.7, $\mathbf{a} < \mathbf{v}$ and $\mathbf{a} < \mathbf{w}$ hold if and only if $\Delta^{d-1}(\mathbf{a}) \subseteq A^{d-1}(\mathbf{v})$ and $\Delta^{d-1}(\mathbf{a}) \subseteq A^{d-1}(\mathbf{w})$, that is, $\Delta^{d-1}(\mathbf{a}) \subseteq A^{d-1}(\mathbf{v}) \cap A^{d-1}(\mathbf{w})$. Thus the lemma holds. \square

The following result extends Theorems 1.1 and 1.2.

Theorem 2.9. Let G be a graph and let d be an integer with $d \geq 2$. Then, G is the competition graph of a d -partial order if and only if there exists a family \mathcal{F} of the interiors of regular $(d-1)$ -simplices in \mathbb{R}^d which are contained in the hyperplane \mathcal{H}^d and homothetic to $A^{d-1}(\mathbf{1})$ and there exists a one-to-one correspondence between the vertex set of G and \mathcal{F} such that

(\star) two vertices v and w are adjacent in G if and only if two elements in \mathcal{F} corresponding to v and w have the intersection containing the closure of another element in \mathcal{F} .

Proof. First we show the “only if” part. Let D be a d -partial order and let G be the competition graph of D . Without loss of generality, we may assume that $V(D) \subseteq \mathcal{H}_+^d$ by translating each of the vertices of D in the same direction and by the same amount since the competition graph of D is determined only by the adjacency among vertices of D . Consequently $A^{d-1}(\mathbf{v}) \neq \emptyset$ for each vertex \mathbf{v} of D . Let $\mathcal{F} = \{A^{d-1}(\mathbf{v}) \mid \mathbf{v} \in V(D)\}$ and let $f : V(G) \rightarrow \mathcal{F}$ be the map defined by $f(\mathbf{v}) = A^{d-1}(\mathbf{v})$ for $\mathbf{v} \in V(D)$. Note that $\mathcal{F} \subseteq \mathcal{F}^{d-1}$. Since the map $f : V(G) \rightarrow \mathcal{F}$ is a restriction of the map $\varphi \circ f_* : V(G) \rightarrow \mathcal{F}^{d-1}$, it follows from Corollary 2.6 that f is a bijection. By Lemma 2.8, the condition (\star) holds.

Second, we show the “if” part. Suppose that there exist a family $\mathcal{F} \subseteq \mathcal{F}^{d-1}$ and a bijection $f : V(G) \rightarrow \mathcal{F}$ such that the condition (\star) holds. By Corollary 2.6, each element in \mathcal{F} can be represented as $A^{d-1}(\mathbf{p})$ for some $\mathbf{p} \in \mathcal{H}_+^d$. Let D be a digraph with vertex set $V(D) = \{\mathbf{p} \in \mathbb{R}^d \mid A^{d-1}(\mathbf{p}) \in \mathcal{F}\}$ and arc set $A(D) = \{(\mathbf{p}, \mathbf{q}) \mid \mathbf{p}, \mathbf{q} \in V(D), \mathbf{p} \neq \mathbf{q}, \Delta^{d-1}(\mathbf{q}) \subseteq A^{d-1}(\mathbf{p})\}$. By Proposition 2.7, $(\mathbf{p}, \mathbf{q}) \in A(D)$ if and only if $\mathbf{q} < \mathbf{p}$, so D is a d -partial order. Now, take two vertices v and w in G . Then, by the hypothesis and above argument, v and w correspond to some points \mathbf{p} and \mathbf{q} in \mathbb{R}^d , respectively, so that v and w are adjacent if and only if both $A^{d-1}(\mathbf{p})$ and $A^{d-1}(\mathbf{q})$ contain the closure of an element in \mathcal{F} , that is, $\Delta^{d-1}(\mathbf{r})$ for some $\mathbf{r} \in \mathbb{R}^d$. By the definition of D , $(\mathbf{p}, \mathbf{r}) \in A(D)$ and $(\mathbf{q}, \mathbf{r}) \in A(D)$. Consequently, v and w are adjacent if and only if the corresponding vertices \mathbf{p} and \mathbf{q} have a common out-neighbor in D . Hence G is the competition graph of the d -partial order D . \square

2.4. Intersection graphs and the competition graphs of d -partial orders

Theorem 2.10. If G is the intersection graph of a finite family of homothetic open regular $(d-1)$ -simplices, then G together with sufficiently many new isolated vertices is the competition graph of a d -partial order.

Proof. Let $\mathcal{A} = \{A_1, \dots, A_n\}$ be a finite family of homothetic open regular $(d-1)$ -simplices, and let G be the intersection graph of \mathcal{A} with bijection $\phi : \mathcal{A} \rightarrow V(G)$. For each distinct pair of i and j in $\{1, \dots, n\}$ such that $A_i \cap A_j \neq \emptyset$, let B_{ij} be an open regular $(d-1)$ -simplex homothetic A_1 such that the closure of B_{ij} is contained in $A_i \cap A_j$. We can take such B_{ij} so that $B_{ij} \cap B_{i'j'} = \emptyset$ for distinct pairs $\{i, j\}$ and $\{i', j'\}$. Let $\mathcal{B} = \{B_{ij} \mid i, j \in \{1, \dots, n\}, i \neq j, A_i \cap A_j \neq \emptyset\}$. Then the family $\mathcal{F} := \mathcal{A} \cup \mathcal{B}$ and a map $f : \mathcal{F} \rightarrow V(G) \cup \{z_1, \dots, z_{|\mathcal{B}|}\}$ such that $f|_{\mathcal{A}} = \phi$ and $f(\mathcal{B}) = \{z_1, \dots, z_{|\mathcal{B}|}\}$ satisfy the condition (\star) in Theorem 2.9. Thus $G \cup \{z_1, \dots, z_{|\mathcal{B}|}\}$ is the competition graph of a d -partial order. \square

Lemma 2.11. If G is the intersection graph of a finite family \mathcal{F} of homothetic closed regular $(d-1)$ -simplices, then G is the intersection graph of a finite family of homothetic open regular $(d-1)$ -simplices.

Proof. Let $\mathcal{D} = \{\Delta_1, \dots, \Delta_n\}$ be a finite family of homothetic closed regular $(d-1)$ -simplices, and let G be the intersection graph of \mathcal{D} with bijection $\phi : \mathcal{D} \rightarrow V(G)$. Let

$$\varepsilon = \frac{1}{3} \min\{d(\Delta_i, \Delta_j) \mid i, j \in \{1, \dots, n\}, \Delta_i \cap \Delta_j = \emptyset\}$$

where $d(\Delta_i, \Delta_j) = \inf\{d(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in \Delta_i, \mathbf{y} \in \Delta_j\}$.

We make each simplex in \mathcal{D} $(1 + \varepsilon)$ times bigger while the center of each simplex is fixed. Then we take the interiors of these closed simplices. By the choice of ε , the graph G is the intersection graph of the family of newly obtained open simplices. Hence the lemma holds. \square

Theorem 2.12. If G is the intersection graph of a finite family \mathcal{F} of homothetic closed regular $(d-1)$ -simplices, then G together with sufficiently many new isolated vertices is the competition graph of a d -partial order.

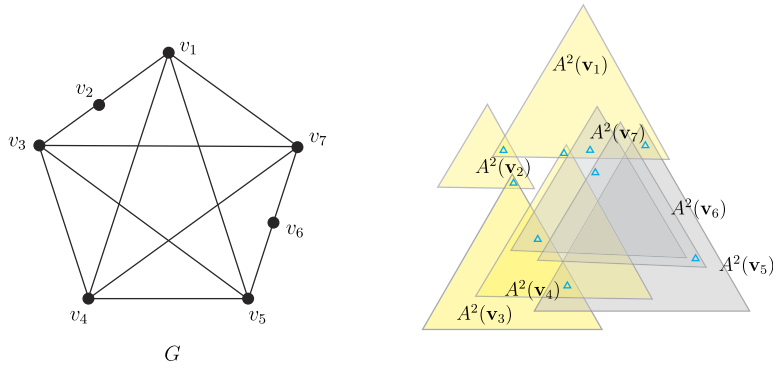


Fig. 3. A subdivision G of K_5 and a family of homothetic equilateral triangles making G together with 9 isolated vertices into the competition graph of a 3-partial order.

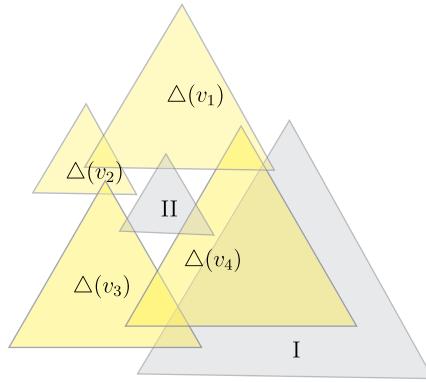


Fig. 4. An assignment of homothetic equilateral triangles to vertices $v_1, v_2, v_3, v_4, v_5, v_7$ of G given in Fig. 3.

Proof. The theorem follows from [Lemma 2.11](#) and [Theorem 2.10](#). \square

Remark 2.13. In the case where $d = 2$, [Theorem 2.12](#) is the same as [Theorem 1.2](#). Due to [Theorem 1.1](#), the converse of [Theorem 2.12](#) is true for $d = 2$. In fact, we can show that the converse of [Theorem 2.10](#) is also true for $d = 2$.

The following example shows that the converses of [Theorems 2.10](#) and [2.12](#) are not true for $d = 3$.

Example 2.14. Let G be a subdivision of K_5 given in [Fig. 3](#). Then, by [Theorem 2.9](#), the family of homothetic equilateral triangles given in the figure makes G together with 9 isolated vertices into the competition graph of a 3-partial order. However, G is not the intersection graph of any family of homothetic equilateral closed triangles. By [Lemma 2.11](#), G is not the intersection graph of any family of homothetic equilateral open triangles, either.

Proof. Suppose that there exists a family $\mathcal{F} := \{\Delta(v) \mid v \in V(G)\}$ of homothetic equilateral closed triangles such that G is the intersection graph of \mathcal{F} . Since $v_1v_2v_3v_4v_1$ is an induced cycle in G , the triangles $\Delta(v_1)$, $\Delta(v_2)$, $\Delta(v_3)$, and $\Delta(v_4)$ are uniquely located as in [Fig. 4](#) up to the sizes of triangles. Since the vertices v_1, v_3 , and v_4 are neighbors of both v_5 and v_7 in G whereas v_2 is not, and the vertices v_5 and v_7 are not adjacent in G , we may conclude that the locations of $\Delta(v_5)$ and $\Delta(v_7)$ should be those for the triangles I and II given in [Fig. 4](#). Since the triangle II cannot have intersections with $\Delta(v_2)$ and the triangle I, all of its sides are surrounded by $\Delta(v_1)$, $\Delta(v_2)$, $\Delta(v_3)$, and $\Delta(v_4)$. Now, since v_6 is adjacent to v_5 and v_7 , $\Delta(v_6)$ must have intersections with both $\Delta(v_5)$ and $\Delta(v_7)$. However, it cannot be done without having an intersection with one of $\Delta(v_1)$, $\Delta(v_2)$, $\Delta(v_3)$, and $\Delta(v_4)$, which is a contradiction to the fact that none of v_1, v_2, v_3 , and v_4 is adjacent to v_6 in G . \square

3. The partial order competition dimension of a graph

Proposition 3.1. Let d be a positive integer. If G is the competition graph of a d -partial order, then G is the competition graph of a $(d + 1)$ -partial order.

Proof. Let D be a d -partial order such that G is the competition graph of D . For each $\mathbf{v} \in V(D) \subseteq \mathbb{R}^d$, we define $\tilde{\mathbf{v}} \in \mathbb{R}^{d+1}$ by

$$\tilde{\mathbf{v}} = \left(v_1, \dots, v_d, \sum_{i=1}^d v_i \right).$$

Then $\{\tilde{\mathbf{v}} \mid \mathbf{v} \in V(D)\}$ defines a $(d+1)$ -partial order \tilde{D} . Since

$$\begin{aligned}\tilde{\mathbf{v}} < \tilde{\mathbf{w}} &\Leftrightarrow v_i < w_i \ (i = 1, \dots, d) \text{ and } \sum_{i=1}^d v_i < \sum_{i=1}^d w_i \\ &\Leftrightarrow v_i < w_i \ (i = 1, \dots, d) \\ &\Leftrightarrow \mathbf{v} < \mathbf{w},\end{aligned}$$

the $(d+1)$ -partial order \tilde{D} is the same digraph as the d -partial order D . Hence G is the competition graph of the $(d+1)$ -partial order \tilde{D} . \square

Proposition 3.2 can be shown by using Theorem 8 in [10]. We present a new proof from which **Proposition 3.5** also follows.

Proposition 3.2. *For any graph G , there exist positive integers d and k such that G together with k isolated vertices is the competition graph of a d -partial order.*

Proof. Let $n = |V(G)|$ and label the vertices of G as v_1, \dots, v_n . Fix four real numbers r_1, r_2, r_3 , and r_4 such that $r_1 < r_2 < r_3 < r_4$. We define a map $\phi : V(G) \rightarrow \mathbb{R}^n$ by

$$\phi(v_i)_j = \begin{cases} r_2 & \text{if } j = i; \\ r_4 & \text{if } j \neq i. \end{cases}$$

We define a map $\psi : E(G) \rightarrow \mathbb{R}^n$ by

$$\psi(e)_k = \begin{cases} r_1 & \text{if } v_k \in e; \\ r_3 & \text{if } v_k \notin e. \end{cases}$$

Let $V = \{\phi(v_i) \mid v_i \in V(G)\} \cup \{\psi(e) \mid e \in E(G)\} \subseteq \mathbb{R}^n$. Then V defines an n -partial order D . By definition, the in-neighborhood of the vertex $\psi(e) \in V$ is $\{\phi(v_i), \phi(v_j)\}$ for an edge $e = \{v_i, v_j\}$ of G and the in-neighborhood of the vertex $\phi(v) \in V$ is the empty set for a vertex v . Thus the competition graph of D is G together with isolated vertices as many as $|E(G)|$. Hence, by taking $d = n$ and $k = |E(G)|$, we complete the proof. \square

Now we may introduce the following notion. Recall that for a finite subset S of \mathbb{R}^d , D_S is the digraph defined by $V(D_S) = S$ and $A(D_S) = \{(\mathbf{x}, \mathbf{v}) \mid \mathbf{x} \in S, \mathbf{v} < \mathbf{x}\}$.

Definition 3.3. For a graph G , we define the *partial order competition dimension* $\dim_{\text{poc}}(G)$ of G as the smallest nonnegative integer d such that G together with k isolated vertices is the competition graph of D for some d -partial order D and some nonnegative integer k , i.e.,

$$\dim_{\text{poc}}(G) := \min\{d \in \mathbb{Z}_{\geq 0} \mid \exists k \in \mathbb{Z}_{\geq 0}, \exists S \subseteq \mathbb{R}^d, \text{ s.t. } G \cup I_k = C(D_S)\},$$

where $\mathbb{Z}_{\geq 0}$ is the set of nonnegative integers and I_k is a set of k isolated vertices.

Remark 3.4. Wu and Lu [10] introduced the notion of the *dimension- d poset competition number* of a graph G , denoted by $\mathcal{PK}^d(G)$, which is defined to be the smallest nonnegative integer p such that G together p additional isolated vertices is isomorphic to the competition graph of a poset of dimension at most d if such a poset exists, and to be ∞ otherwise. By using $\mathcal{PK}^d(G)$, the partial order competition dimension of G can be represented as $\dim_{\text{poc}}(G) = \min\{d \mid \mathcal{PK}^d(G) < \infty\}$. Therefore $\mathcal{PK}^d(G) < \infty$ implies that $\dim_{\text{poc}}(G) \leq d$. In this respect, **Proposition 3.2** and the “if” part of **Proposition 3.10** may follow from their result presenting the dimension- d poset competition numbers of a complete graph with or without isolated vertices, which are also shown to be trivially true in this paper.

Proposition 3.5. *For any graph G , we have $\dim_{\text{poc}}(G) \leq |V(G)|$.*

Proof. The proposition follows from the construction of a d -partial order in the proof of **Proposition 3.2**. \square

For a graph G , the partial order competition dimension of an induced subgraph of G is less than or equal to that of G . To show this, we need the following lemmas.

Lemma 3.6. *Let D be a digraph and let G be the competition graph of D . Let S be a set of vertices. The competition graph of $D[S]$ is a subgraph of $G[S]$, where $D[S]$ and $G[S]$ mean the subdigraph of D and the subgraph of G , respectively, induced by S .*

Proof. Let H be the competition graph of $D[S]$. Obviously, $V(H) = S$. Take an edge $\{u, v\}$ of H . By definition, there exists a vertex w in $D[S]$ such that (u, w) and (v, w) are arcs of $D[S]$. Consequently, (u, w) and (v, w) are arcs of D and so $\{u, v\}$ is an edge of G . Since $u, v \in S$, $\{u, v\}$ is an edge of $G[S]$. Hence H is a subgraph of $G[S]$. \square

Lemma 3.7. *Let D be a transitive acyclic digraph and let G be the competition graph of D . For any non-isolated vertex u of G , there exists an isolated vertex v of G such that (u, v) is an arc of D .*

Proof. Take a non-isolated vertex u of G . Since u has a neighbor w in G , u and w have a common out-neighbor in D . Take a longest directed path in D originating from u . We denote by v the terminal vertex of the directed path. Since D is acyclic, the out-degree of v in D is zero and so v is isolated in G . By the hypothesis that D is transitive, (u, v) is an arc of D . \square

Proposition 3.8. *Let G be a graph and let H be an induced subgraph of G . Then $\dim_{\text{poc}}(H) \leq \dim_{\text{poc}}(G)$.*

Proof. Let $d = \dim_{\text{poc}}(G)$. Then, there exists a d -partial order D whose competition graph is the disjoint union of G and a set J of isolated vertices. Let I be the set of isolated vertices in G . Let $S = V(H) \cup I \cup J \subseteq \mathbb{R}^d$. Then the digraph D_S is a d -partial order. By Lemma 3.6, the competition graph of D_S is a subgraph of $H \cup (I \setminus V(H)) \cup J$.

Now take two adjacent vertices \mathbf{x} and \mathbf{y} in H . Then, since they are adjacent in G , there exists a vertex $\mathbf{v} \in V(D)$ such that $\mathbf{v} < \mathbf{x}$ and $\mathbf{v} < \mathbf{y}$. If \mathbf{v} is isolated in G or $\mathbf{v} \in J$, then (\mathbf{x}, \mathbf{v}) and (\mathbf{y}, \mathbf{v}) belong to $A(D_S)$ by definition. Suppose that $\mathbf{v} \notin I \cup J$. Then, by Lemma 3.7, there exists a vertex \mathbf{w} in $I \cup J$ such that $\mathbf{w} < \mathbf{v}$. Then $\mathbf{w} < \mathbf{x}$ and $\mathbf{w} < \mathbf{y}$ and so (\mathbf{x}, \mathbf{w}) and (\mathbf{y}, \mathbf{w}) belong to $A(D_S)$. Thus $H \cup (I \setminus V(H)) \cup J$ is a subgraph of the competition graph of D_S and we have shown that it is the competition graph of D_S . Hence $\dim_{\text{poc}}(H) \leq d$ and the proposition holds. \square

It does not seem to be easy to compute the partial order competition dimension of a graph in general. In this context, we first characterize graphs having small partial order competition dimensions. In such a way, we wish to have a better idea to settle the problem.

Let K_n denote the complete graph with n vertices.

Proposition 3.9. *Let G be a graph. Then, $\dim_{\text{poc}}(G) = 0$ if and only if $G = K_1$.*

Proof. The proposition immediately follows from the definition of 0-partial order. \square

Proposition 3.10. *Let G be a graph. Then, $\dim_{\text{poc}}(G) = 1$ if and only if $G = K_{t+1}$ or $G = K_t \cup K_1$ for some positive integer t .*

Proof. First we remark that if D is a 1-partial order with $V(D) \subseteq \mathbb{R}^1$ and $v^* \in V(D)$ is the minimum among $V(D)$, then the competition graph of D is the disjoint union of a clique $V(D) \setminus \{v^*\}$ and an isolated vertex v^* . Therefore, if $\dim_{\text{poc}}(G) = 1$, then we obtain $G = K_{t+1}$ or $G = K_t \cup K_1$ for some nonnegative integer t . By Proposition 3.9, $G \neq K_1$ and thus t is a positive integer.

If $G = K_{t+1}$ or $G = K_t \cup K_1$ for some positive integer t , then we obtain $\dim_{\text{poc}}(G) \leq 1$. By Proposition 3.9, since $G \neq K_1$, we have $\dim_{\text{poc}}(G) = 1$. \square

Lemma 3.11. *Let G be a graph such that $\dim_{\text{poc}}(G) \geq 2$ and let G' be a graph obtained from G by adding isolated vertices. Then $\dim_{\text{poc}}(G) = \dim_{\text{poc}}(G')$.*

Proof. Let a_1, \dots, a_k be the isolated vertices added to G to obtain G' . Let $d = \dim_{\text{poc}}(G)$. Then G can be made into the competition graph a d -partial order D by adding sufficiently many isolated vertices. Since $d \geq 2$, we can locate k points $\mathbf{a}_1, \dots, \mathbf{a}_k$ in \mathbb{R}^d corresponding to a_1, \dots, a_k so that no two points in $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ are related by $<$ and that no point in $V(D)$ and no point in $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ are related by $<$. Indeed, we can do this in the following way: for $i = 1, \dots, k$, let \mathbf{a}_i be a point in \mathbb{R}^d defined by

$$(\mathbf{a}_i)_1 = r_1 + i; \quad (\mathbf{a}_i)_2 = r_2 - i; \quad (\mathbf{a}_i)_j = 0 \quad (j = 3, \dots, d),$$

where $r_1 := \max\{(\mathbf{v})_1 \mid \mathbf{v} \in V(D)\}$ and $r_2 := \min\{(\mathbf{v})_2 \mid \mathbf{v} \in V(D)\}$. Then G' is the competition graph of D together with $\mathbf{a}_1, \dots, \mathbf{a}_k$ and thus $\dim_{\text{poc}}(G') \leq \dim_{\text{poc}}(G)$.

Since G is an induced subgraph of G' , by Proposition 3.8, $\dim_{\text{poc}}(G) \leq \dim_{\text{poc}}(G')$. Hence $\dim_{\text{poc}}(G) = \dim_{\text{poc}}(G')$. \square

Proposition 3.12. *Let G be a graph. Then, $\dim_{\text{poc}}(G) = 2$ if and only if G is an interval graph which is neither K_s nor $K_t \cup K_1$ for any positive integers s and t .*

Proof. Suppose that $\dim_{\text{poc}}(G) = 2$. By Theorem 1.1, G is an interval graph. By Propositions 3.9 and 3.10, G is neither K_s nor $K_t \cup K_1$ for any positive integers s and t .

Suppose that G is an interval graph which is neither K_s nor $K_t \cup K_1$ for any positive integers s and t . By Theorem 1.2, $\dim_{\text{poc}}(G) \leq 2$. By Propositions 3.9 and 3.10 $\dim_{\text{poc}}(G) \geq 2$. Thus, $\dim_{\text{poc}}(G) = 2$. \square

Proposition 3.13. *If G is a cycle of length at least four, then $\dim_{\text{poc}}(G) = 3$.*

Proof. Let G be a cycle of length n with $n \geq 4$. Note that G is not an interval graph. By Propositions 3.9, 3.10 and 3.12, we have $\dim_{\text{poc}}(G) \geq 3$. Let \mathcal{F} be the family of n closed triangles given in Fig. 5. Then the intersection graph of \mathcal{F} is the cycle of length n . By Theorem 2.12 with $d = 3$, G together with sufficiently many isolated vertices is the competition graph of a 3-partial order. Thus $\dim_{\text{poc}}(G) \leq 3$. Hence $\dim_{\text{poc}}(G) = 3$. \square

Theorem 3.14. *If a graph G contains an induced cycle of length at least four, then $\dim_{\text{poc}}(G) \geq 3$.*

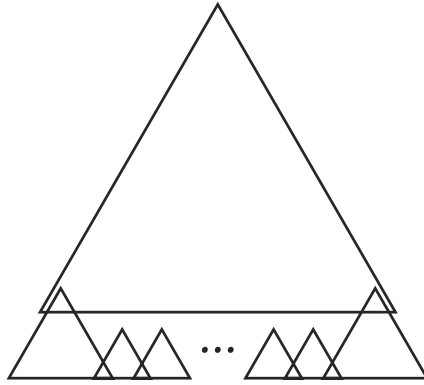


Fig. 5. A family of homothetic equilateral closed triangles.

Proof. The theorem follows from Propositions 3.8 and 3.13. \square

Theorem 3.15. Let T be a tree. Then $\dim_{\text{poc}}(T) \leq 3$, and the equality holds if and only if T is not a caterpillar.

Proof. By Theorem 2.12 with $d = 3$, we need to show that there exists a family of homothetic equilateral closed triangles in \mathbb{R}^2 whose intersection graph is T . As a matter of fact, it is sufficient to find such a family in the xy -plane with the base of each triangle parallel to the x -axis. We call the vertex of a triangle which is opposite to the base the *apex* of the triangle. We show the following stronger statement by induction on the number of vertices:

For a tree T and a vertex v of T , there exists a family $\mathcal{F}_v^T := \{\Delta(x) \mid x \in V(T)\}$ of homothetic equilateral closed triangles whose intersection graph is T such that, for any vertex x distinct from v , the apex and the base of $\Delta(x)$ are below the apex and the base of $\Delta(v)$, respectively.

We call the family \mathcal{F}_v^T in the above statement a *good family* for T and v .

If T is the tree having exactly one vertex, then the statement is vacuously true. Assume that the statement holds for any tree on $n - 1$ vertices, where $n \geq 2$. Let T be a tree with n vertices. We fix a vertex v of T as a root. Let T_1, \dots, T_k ($k \geq 1$) be the connected components of $T - v$. Then T_1, \dots, T_k are trees. For each $i = 1, \dots, k$, T_i has exactly one vertex, say w_i , which is a neighbor of v in T . We take w_i as a root of T_i . By the induction hypothesis, there exists a good family $\mathcal{F}_{w_i}^{T_i}$ for T_i and w_i for each $i = 1, \dots, k$. Preserving the intersection or the non-intersection of two triangles in $\mathcal{F}_{w_i}^{T_i}$ for each $i = 1, \dots, k$, we may translate the triangles in $\mathcal{F}_{w_1}^{T_1} \cup \dots \cup \mathcal{F}_{w_k}^{T_k}$ so that the apexes of $\Delta(w_1), \dots, \Delta(w_k)$ are on the x -axis and any two triangles from distinct families do not intersect. Let δ_i be the distance between the apex of $\Delta(w_i)$ and the apex of a triangle which is the second highest among the apexes of the triangles in $\mathcal{F}_{w_i}^{T_i}$. Now we draw a triangle $\Delta(v)$ in such a way that the base of $\Delta(v)$ is a part of the line $y = -\frac{1}{2} \min\{\delta_1, \dots, \delta_k\}$ and long enough to intersect all of the triangles $\Delta(w_1), \dots, \Delta(w_k)$. Then the family $\mathcal{F}_v^T := \mathcal{F}_{w_1}^{T_1} \cup \dots \cup \mathcal{F}_{w_k}^{T_k} \cup \{\Delta(v)\}$ is a good family for T and v and thus the statement holds. Hence, $\dim_{\text{poc}}(T) \leq 3$ for a tree T .

Since trees which are interval graphs are caterpillars, the latter part of the theorem follows from Propositions 3.9, 3.10 and 3.12. \square

4. Concluding remarks

In this paper, we studied the competition graphs of d -partial orders and gave a characterization by using homothetic open regular simplices. Since any graph can be made into the competition graph of a d -partial order for some positive integer d by adding isolated vertices, we introduced the notion of the partial order competition dimension of a graph. We gave characterizations of graphs having partial order competition dimension 0, 1, and 2. We also showed that cycles and trees have partial order competition dimension at most 3. It would be an interesting research problem to characterize graphs G having partial order competition dimension 3.

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