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On the partial order competition dimensions of chordal graphs

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ABSTRACT

Choi et al. (2016) introduced the notion of the partial order competition dimension of a graph. It was shown that complete graphs, interval graphs, and trees, which are chordal graphs, have partial order competition dimensions at most three.

In this paper, we study the partial order competition dimensions of chordal graphs. We show that chordal graphs have partial order competition dimensions at most three if the graphs are diamond-free. Moreover, we also show the existence of chordal graphs containing diamonds whose partial order competition dimensions are greater than three.

1. Introduction

The *competition graph* $C(D)$ of a digraph D is an undirected graph which has the same vertex set as D and which has an edge xy between two distinct vertices x and y if and only if for some vertex $z \in V$, the arcs (x, z) and (y, z) are in D.

Let d be a positive integer. For $x = (x_1, x_2, \dots, x_d), y = (y_1, y_2, \dots, y_d) \in \mathbb{R}^d$, we write $x \prec y$ if $x_i < y_i$ for each $i = 1, \dots, d$. For a finite subset S of $\mathbb{R}^d$, let D_S be the digraph defined by $V(D_S) = S$ and $A(D_S) = \{(x, v) \mid v, x \in S, v \prec x\}$. A digraph D is called a *d-partial order* if there exists a finite subset S of $\mathbb{R}^d$ such that D is isomorphic to the digraph D_S. A 2-partial order is also called a *doubly partial order*. Cho and Kim [2] studied the competition graphs of doubly partial orders and showed that interval graphs are exactly the graphs having partial order competition dimensions at most two. Especially, Wu and Lu [10] answered an open problem posed by Cho and Kim [2] of characterizing competition graphs of d-partial orders for $d \leq 2$. Several variants of competition graphs of doubly partial orders also have been studied (see [4–9]).

Choi et al. [3] introduced the notion of the partial order competition dimension of a graph, which had been also implicitly introduced by Wu and Lu [10] (refer to Remark 3.4 in [3] for further details).

Definition. For a graph G, the *partial order competition dimension* of G, denoted by $\dim_{\mathrm{poc}}(G)$, is the smallest nonnegative integer d such that G together with k isolated vertices is the competition graph of a d-partial order D for some nonnegative integer k, i.e.,

$$\dim_{\mathrm{poc}}(G) := \min\{d \in \mathbb{Z}_{\geq 0} \mid \exists k \in \mathbb{Z}_{\geq 0}, \exists S \subseteq \mathbb{R}^d \text{ s.t. } G \cup I_k = C(D_S)\},$$

where $\mathbb{Z}_{\geq 0}$ is the set of nonnegative integers and I_k is a set of k isolated vertices.

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Choi et al. [3] studied graphs having small partial order competition dimensions, and gave characterizations of graphs with partial order competition dimension 0, 1, or 2 as follows.

Proposition 1.1. *Let G be a graph. Then,* $\dim_{\text{poc}}(G) = 0$ *if and only if* $G = K_1$.

Proposition 1.2. *Let G be a graph. Then,* $\dim_{\text{poc}}(G) = 1$ *if and only if* $G = K_{t+1}$ *or* $G = K_t \cup K_1$ *for some positive integer* t.

Proposition 1.3. *Let G be a graph. Then,* $\dim_{\text{poc}}(G) = 2$ *if and only if G is an interval graph which is neither* K_s *nor* $K_t \cup K_1$ *for any positive integers s and* t.

Choi et al. [3] also gave some families of graphs with partial order competition dimension three.

Proposition 1.4. *If G is a cycle of length at least four, then* $\dim_{\text{poc}}(G) = 3$.

A *caterpillar* is a tree the removal of whose pendant vertices results in a path.

Theorem 1.5. *Let T be a tree. Then* $\dim_{\text{poc}}(T) \leq 3$, *and the equality holds if and only if T is not a caterpillar.*

In this paper, we study the partial order competition dimensions of chordal graphs. We thought that most likely candidates for the family of graphs having partial order competition dimension at most three are chordal graphs since both trees and interval graphs, which are chordal graphs, have partial order competition dimensions at most three. In fact, we show that chordal graphs have partial order competition dimensions at most three if the graphs are diamond-free. However, contrary to our presumption, we could show the existence of chordal graphs with partial order competition dimensions greater than three.

2. Preliminaries

We say that two sets in $\mathbb{R}^d$ are *homothetic* if they are related by a geometric contraction or expansion. Choi et al. [3] gave a characterization of the competition graphs of d-partial orders. We state it in the case where $d = 3$.

Theorem 2.1 ([3]). *A graph G is the competition graph of a* 3*-partial order if and only if there exists a family* $\mathcal{F}$ *of homothetic open equilateral triangles contained in the plane* $\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$ *and there exists a one-to-one correspondence* $A : V(G) \to \mathcal{F}$ *such that*

($\star$) *two vertices* v *and* w *are adjacent in G if and only if two elements* $A(v)$ *and* $A(w)$ *have the intersection containing the closure* $\triangle(x)$ *of an element* $A(x)$ *in* $\mathcal{F}$.

Choi et al. [3] also gave a sufficient condition for a graph being the competition graph of a d-partial order. We state their result in the case where $d = 3$.

Theorem 2.2 ([3]). *If G is the intersection graph of a finite family of homothetic closed equilateral triangles, then G together with sufficiently many new isolated vertices is the competition graph of a* 3*-partial order.*

By the definition of the partial order competition dimension of a graph, we have the following:

Corollary 2.3. *If G is the intersection graph of a finite family of homothetic closed equilateral triangles, then* $\dim_{\text{poc}}(G) \leq 3$.

Note that the converse of Corollary 2.3 is not true by an example given by Choi et al. [3] (see Fig. 1). In this context, one can guess that it is not so easy to show that a graph has partial order competition dimension greater than three.

The correspondence A in Theorem 2.1 can be precisely described as follows: Let $\mathcal{H} := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1+x_2+x_3 = 0\}$ and $\mathcal{H}_+ := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1+x_2+x_3 > 0\}$. For a point $v = (v_1, v_2, v_3) \in \mathcal{H}_+$, let $p_1^{(v)}$, $p_2^{(v)}$, and $p_3^{(v)}$ be points in $\mathbb{R}^3$ defined by $p_1^{(v)} := (-v_2 - v_3, v_2, v_3)$, $p_2^{(v)} := (v_1, -v_1 - v_3, v_3)$, and $p_3^{(v)} := (v_1, v_2, -v_1 - v_2)$, and let $\triangle(v)$ be the convex hull of the points $p_1^{(v)}$, $p_2^{(v)}$, and $p_3^{(v)}$, i.e., $\triangle(v) := \text{Conv}(p_1^{(v)}, p_2^{(v)}, p_3^{(v)}) = \left\{\sum_{i=1}^3 \lambda_i p_i^{(v)} \mid \sum_{i=1}^3 \lambda_i = 1, \lambda_i \geq 0\, (i = 1, 2, 3)\right\}$. Then it is easy to check that $\triangle(v)$ is an closed equilateral triangle which is contained in the plane $\mathcal{H}$. Let $A(v)$ be the relative interior of the closed triangle $\triangle(v)$, i.e., $A(v) := \text{rel.int}(\triangle(v)) = \left\{\sum_{i=1}^3 \lambda_i p_i^{(v)} \mid \sum_{i=1}^3 \lambda_i = 1, \lambda_i > 0\, (i = 1, 2, 3)\right\}$. Then $A(v)$ and $A(w)$ are homothetic for any $v, w \in \mathcal{H}_+$.

For $v \in \mathcal{H}_+$ and $(i, j) \in \{(1, 2), (2, 3), (1, 3)\}$, let $l_{ij}^{(v)}$ denote the line through the two points $p_i^{(v)}$ and $p_j^{(v)}$, i.e., $l_{ij}^{(v)} := \{x \in \mathbb{R}^3 \mid x = \alpha p_i^{(v)} + (1 - \alpha) p_j^{(v)}, \alpha \in \mathbb{R}\}$, and let $R_{ij}(v)$ denote the following region:

$$R_{ij}(v) := \{x \in \mathbb{R}^3 \mid x = (1 - \alpha - \beta) p_k^{(v)} + \alpha p_i^{(v)} + \beta p_j^{(v)}, 0 \leq \alpha \in \mathbb{R}, 0 \leq \beta \in \mathbb{R}, \alpha + \beta \geq 1\},$$

where k is the element in $\{1, 2, 3\} \setminus \{i, j\}$; for $k \in \{1, 2, 3\}$, let $R_k(v)$ denote the following region:

$$R_k(v) := \{x \in \mathbb{R}^3 \mid x = (1 + \alpha + \beta) p_k^{(v)} - \alpha p_i^{(v)} - \beta p_j^{(v)}, 0 \leq \alpha \in \mathbb{R}, 0 \leq \beta \in \mathbb{R}\},$$

where i and j are elements such that $\{i, j, k\} = \{1, 2, 3\}$. (See Fig. 2 for an illustration.)

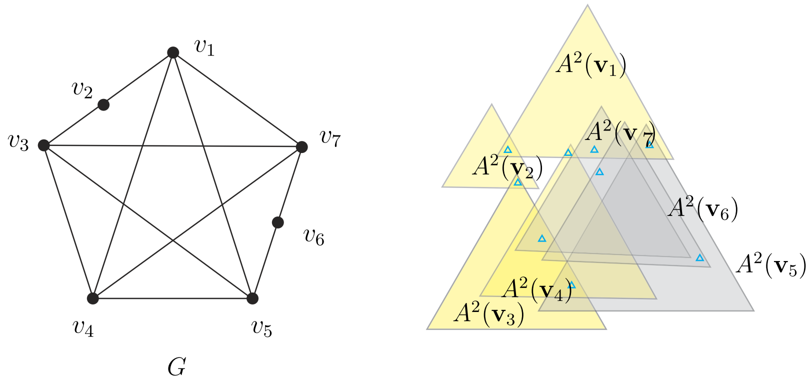

Fig. 1. A subdivision G of K_5 and a family of homothetic equilateral triangles making G together with 9 isolated vertices into the competition graph of a 3-partial order, which is given in [3].

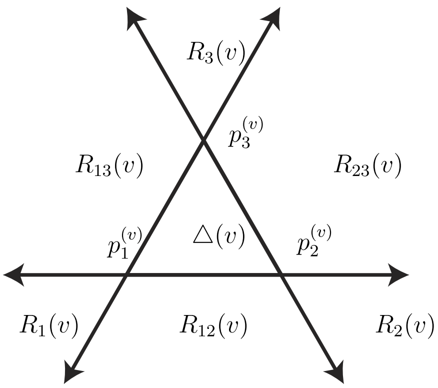

Fig. 2. The regions determined by v. By our assumption, for any vertex u of a graph considered in this paper, $p_1^{(u)}, p_2^{(u)}, p_3^{(u)}$ correspond to $p_1^{(v)}, p_2^{(v)}, p_3^{(v)}$ respectively.

If a graph G satisfies $\dim_{\mathrm{poc}}(G) \le 3$, then, by Theorem 2.1, we may assume that $V(G) \subseteq \mathcal{H}_+$ by translating each of the vertices of G in the same direction and by the same amount.

Lemma 2.4. *Let D be a 3-partial order and let G be the competition graph of D. Suppose that G contains an induced path uvw of length two. Then neither $A(u) \cap A(v) \subseteq A(w)$ nor $A(v) \cap A(w) \subseteq A(u)$.*

Proof. We show by contradiction. Suppose that $A(u) \cap A(v) \subseteq A(w)$ or $A(v) \cap A(w) \subseteq A(u)$. By symmetry, we may assume without loss of generality that $A(u) \cap A(v) \subseteq A(w)$. Since u and v are adjacent in G, there exists a vertex $a \in V(G)$ such that $\triangle(a) \subseteq A(u) \cap A(v)$ by Theorem 2.1. Therefore $\triangle(a) \subseteq A(w)$. Since $\triangle(a) \subseteq A(u)$, u and w are adjacent in G by Theorem 2.1, which is a contradiction to the assumption that u and w are not adjacent in G. Hence the lemma holds. □

Definition. For $v, w \in \mathcal{H}_+$, we say that v and w are *crossing* if $A(v) \cap A(w) \neq \emptyset$, $A(v) \setminus A(w) \neq \emptyset$, and $A(w) \setminus A(v) \neq \emptyset$.

Lemma 2.5. *Let D be a 3-partial order and let G be the competition graph of D. Suppose that G contains an induced path $xuvw$ of length three. Then u and v are crossing.*

Proof. Since u and v are adjacent in G, there exists a vertex $a \in V(G)$ such that $\triangle(a) \subseteq A(u) \cap A(v)$ by Theorem 2.1. Therefore $A(u) \cap A(v) \neq \emptyset$. If $A(v) \subseteq A(u)$, then $A(v) \cap A(w) \subseteq A(u)$, which contradicts Lemma 2.4. Thus $A(v) \setminus A(u) \neq \emptyset$. If $A(u) \subseteq A(v)$, then $A(x) \cap A(u) \subseteq A(v)$, which contradicts Lemma 2.4. Thus $A(u) \setminus A(v) \neq \emptyset$. Hence u and v are crossing. □

Lemma 2.6. *If v and w in $\mathcal{H}_+$ are crossing, then $p_k^{(x)} \in \triangle(y)$ for some $k \in \{1, 2, 3\}$ where $\{x, y\} = \{v, w\}$.*

Proof. Since v and w are crossing, we have $A(v) \cap A(w) \neq \emptyset$, $A(v) \setminus A(w) \neq \emptyset$, and $A(w) \setminus A(v) \neq \emptyset$. Then one of the vertices of the triangles $\triangle(v)$ and $\triangle(w)$ is contained in the other triangle, thus the lemma holds. □

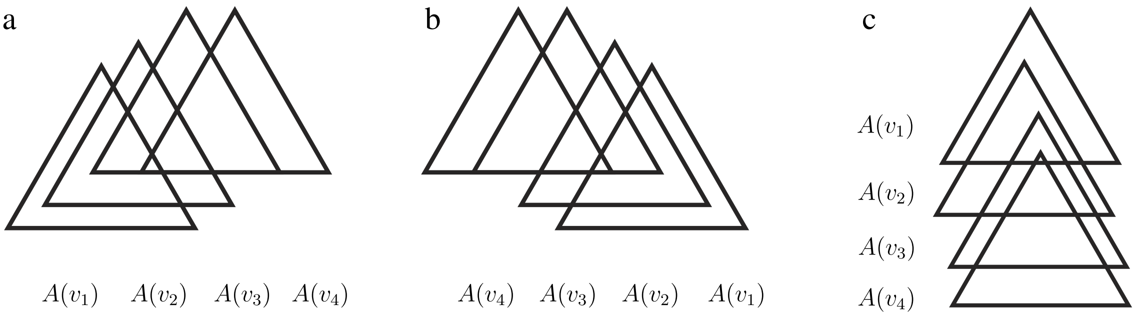

Fig. 3. The sequences (v_1, v_2, v_3, v_4) in (a), (b), (c) are consecutively tail-biting of Type 1, 2, 3, respectively.

Definition. For $k \in \{1, 2, 3\}$, we define a binary relation $\overset{k}{\to}$ on $\mathcal{H}_+$ by

$$x \overset{k}{\to} y \quad \Leftrightarrow \quad x \text{ and } y \text{ are crossing, and } p_k^{(y)} \in \triangle(x)$$

for any $x, y \in \mathcal{H}_+$.

Lemma 2.7. *Let $x, y, z \in \mathcal{H}_+$. Suppose that $x \overset{k}{\to} y$ and $y \overset{k}{\to} z$ for some $k \in \{1, 2, 3\}$ and that x and z are crossing. Then $x \overset{k}{\to} z$.*

Proof. Since $x \overset{k}{\to} y$, $p_l^{(x)} \notin R_i(y) \cup R_{ij}(y) \cup R_j(y)$ for each $l \in \{1, 2, 3\}$, where $\{i, j, k\} = \{1, 2, 3\}$ Since $y \overset{k}{\to} z$, $p_l^{(z)} \in R_i(y) \cup R_{ij}(y) \cup R_j(y)$ for each $l \in \{i, j\}$. Since x and z are crossing, $p_k^{(z)} \in \triangle(x)$. □

Definition. For $k \in \{1, 2, 3\}$, a sequence $(v_1, \ldots, v_m)$ of m points in $\mathcal{H}_+$, where $m \geq 2$, is said to be *consecutively tail-biting in Type k* if $v_i \overset{k}{\to} v_j$ for any $i < j$ (see Fig. 3). A finite set V of points in $\mathcal{H}_+$ is said to be *consecutively tail-biting* if there is an ordering $(v_1, \ldots, v_m)$ of V such that $(v_1, \ldots, v_m)$ is consecutively tail-biting.

3. The partial order competition dimensions of diamond-free chordal graphs

A *diamond* of a graph G is an induced subgraph G isomorphic to the complete tripartite graph $K_{1,1,2}$. We call the edge connecting the partite sets of size 1 the *diagonal* of the diamond. A graph G is said to be *diamond-free* if G does not contain a diamond.

In this section, we show that a chordal graph has partial order competition dimension at most three if it is diamond-free.

A *block graph* is a graph such that each of its maximal 2-connected subgraphs is a complete graph. The following is well-known.

Lemma 3.1 (*[1, Proposition 1]*). *A graph is a block graph if and only if the graph is a diamond-free chordal graph.*

Note that a block graph having no cut vertex is a disjoint union of complete graphs. For block graphs having cut vertices, the following lemma holds.

Lemma 3.2. *Let G be a block graph having at least one cut vertex. Then G has a maximal clique that contains exactly one cut vertex.*

Proof. Let H be the subgraph induced by the cut vertices of G. By definition, H is obviously a block graph, so H is chordal and there is a simplicial vertex v in H. Since v is a cut vertex of G, v belongs to at least two maximal cliques of G. Suppose that each maximal clique containing v contains another cut vertex of G. Take two maximal cliques X_1 and X_2 of G containing v and let x and y be cut vertices of G belonging to X_1 and X_2, respectively. Then both x and y are adjacent to v in H. Since G is a block graph, $X_1 \setminus \{v\}$ and $X_2 \setminus \{v\}$ are contained in distinct connected components of $G - v$. This implies that x and y are not adjacent in H, which contradicts the choice of v. Therefore there is a maximal clique X containing v without any other cut vertex of G. □

Lemma 3.3. *Every block graph G is the intersection graph of a family $\mathcal{F}$ of homothetic closed equilateral triangles in which every clique of G is consecutively tail-biting.*

Proof. We show by induction on the number of cut vertices of G. If a block graph has no cut vertex, then it is a disjoint union of complete graphs and the statement is trivially true as the vertices of each complete subgraph can be formed as a sequence which is consecutively tail-biting (refer to Fig. 3).

Assume that the statement is true for any block graph G with m cut vertices where $m \geq 0$. Now we take a block graph G with $m+1$ cut vertices. By Lemma 3.2, there is a maximal clique X that contains exactly one cut vertex, say w. By definition, the vertices of X other than w are simplicial vertices.

Deleting the vertices of X other than w and the edges adjacent to them, we obtain a block graph G^* with m cut vertices. Then, by the induction hypothesis, G^* is the intersection graph of a family $\mathcal{F}^*$ of homothetic closed equilateral triangles satisfying the statement. We consider the triangles corresponding to w. Let C and C' be two maximal cliques of G^* containing w. By the induction hypothesis, the vertices of C and C' can be ordered as $v_1, v_2, \ldots, v_l$ and $v'_1, v'_2, \ldots, v'_{l'}$, respectively, so that $v_i \xrightarrow{k} v_j$ if $i < j$, for some $k \in \{1, 2, 3\}$ and that $v'_{i'} \xrightarrow{k'} v'_{j'}$ if $i' < j'$, for some $k' \in \{1, 2, 3\}$.

Suppose that $\triangle(v_i) \cap \triangle(v'_j) \neq \emptyset$ for v_i and v'_j which are distinct from w. Then v_i and v'_j are adjacent in G^*, which implies the existence of a diamond in G since maximal cliques have size at least two. We have reached a contradiction to Lemma 3.1 and so $\triangle(v_i) \cap \triangle(v'_j) = \emptyset$ for any i, j. Therefore there is a segment of a side on $\triangle(w)$ (with a positive length) that does not intersect with the triangle assigned to any vertex in G^* other than w since there are finitely many maximal cliques in G^* that contain w. If the side belongs to $l^{(w)}_{ij}$ for $i, j \in \{1, 2, 3\}$, then we may order the deleted vertices and assign the homothetic closed equilateral triangles with sufficiently small sizes to them so that the closed neighborhood of v is consecutively tail-biting in Type k for $k \in \{1, 2, 3\} \setminus \{i, j\}$ and none of the triangles intersects with the triangle corresponding to any vertex other than w in G^*. It is not difficult to see that the set of the triangles in $\mathcal{F}^*$ together with the triangles just obtained is the one desired for $\mathcal{F}$. □

As block graphs are not necessarily interval graphs, the following result extends a known family of graphs with partial order competition dimension three.

Theorem 3.4. *For any diamond-free chordal graph G, $\dim_{\text{poc}}(G) \leq 3$.*

Proof. The theorem follows from Corollary 2.3 and Lemma 3.3. □

4. Chordal graphs having partial order competition dimension greater than three

In this section, we present infinitely many chordal graphs G with $\dim_{\text{poc}}(G) > 3$.

We first show two lemmas which will be repeatedly used in the proof of the theorem in this section.

Lemma 4.1. *Let D be a 3-partial order and let G be the competition graph of D. Suppose that G contains a diamond $K_4 - e$ as an induced subgraph, where u, v, w, x are the vertices of the diamond and $e = vx$. If the sequence (u, v, w) is consecutively tail-biting in Type k for some $k \in \{1, 2, 3\}$, then $p^{(x)}_i \in R_i(v)$ and $p^{(x)}_j \notin R_j(v)$ hold or $p^{(x)}_i \notin R_i(v)$ and $p^{(x)}_j \in R_j(v)$ hold where $\{i, j, k\} = \{1, 2, 3\}$.*

Proof. Without loss of generality, we may assume that $k = 3$. We first claim that $p^{(x)}_1 \in R_1(v) \cup R_2(v) \cup R_{12}(v)$. Suppose not. Then $p^{(x)}_1 \in R := \mathcal{H} \setminus (R_1(v) \cup R_2(v) \cup R_{12}(v))$. Since $A(x)$ and $A(v)$ are homothetic, $A(x) \subseteq R$. Thus $A(w) \cap A(x) \subseteq A(w) \cap R$. Since (u, v, w) is consecutively tail-biting in Type 3, $A(w) \cap R \subseteq A(v)$. Therefore $A(w) \cap A(x) \subseteq A(v)$, which contradicts Lemma 2.4. Thus $p^{(x)}_1 \in R_1(v) \cup R_2(v) \cup R_{12}(v)$. By symmetry, $p^{(x)}_2 \in R_1(v) \cup R_2(v) \cup R_{12}(v)$.

Suppose that both $p^{(x)}_1$ and $p^{(x)}_2$ are in $R_{12}(v)$. Since $A(x)$ and $A(v)$ are homothetic, $A(x) \cap R \subseteq A(v)$. By the hypothesis that (u, v, w) is consecutively tail-biting in Type 3, we have $A(u) \subseteq R$. Therefore $A(x) \cap A(u) \subseteq A(x) \cap R$. Thus $A(x) \cap A(u) \subseteq A(v)$, which contradicts Lemma 2.4. Therefore $p^{(x)}_1 \in R_1(v) \cup R_2(v)$ or $p^{(x)}_2 \in R_1(v) \cup R_2(v)$. Since $p^{(x)}_1 \in R_2(v)$ (resp. $p^{(x)}_2 \in R_1(v)$) implies $p^{(x)}_2 \in R_2(v)$ (resp. $p^{(x)}_1 \in R_1(v)$), which is impossible, we have $p^{(x)}_1 \in R_1(v)$ or $p^{(x)}_2 \in R_2(v)$.

Suppose that both $p^{(x)}_1 \in R_1(v)$ and $p^{(x)}_2 \in R_2(v)$ hold. Then $A(v) \subseteq A(x)$ since $A(v)$ and $A(x)$ are homothetic. Then $A(u) \cap A(v) \subseteq A(x)$, which contradicts Lemma 2.4. Hence $p^{(x)}_1 \in R_1(v)$ and $p^{(x)}_2 \notin R_2(v)$ hold or $p^{(x)}_1 \notin R_1(v)$ and $p^{(x)}_2 \in R_2(v)$ hold. □

Let $\overline{\mathrm{H}}$ be the graph on vertex set $\{t, u, v, w, x, y\}$ such that $\{t, u, v, w\}$ forms a complete graph K_4, x is adjacent to only t and v, and y is adjacent to only u and w in $\overline{\mathrm{H}}$ (see Fig. 4 for an illustration).

Lemma 4.2. *Let D be a 3-partial order and let G be the competition graph of D. Suppose that G contains the graph $\overline{\mathrm{H}}$ as an induced subgraph and (t, u, v, w) is consecutively tail-biting in Type k for some $k \in \{1, 2, 3\}$. Then, for i, j with $\{i, j, k\} = \{1, 2, 3\}$, $p^{(x)}_i \in R_i(u)$ implies $p^{(y)}_j \in R_j(v)$.*

Proof. Without loss of generality, we may assume that $k = 3$. It is sufficient to show that $p^{(x)}_1 \in R_1(u)$ implies $p^{(y)}_2 \in R_2(v)$. Now suppose that $p^{(x)}_1 \in R_1(u)$. Since (t, u, v, w) is a tail-biting sequence of Type 3, (t, u, v) and (u, v, w) are tail-biting sequences of Type 3. Since $\{t, u, v, x\}$ induces a diamond and (t, u, v) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 4.1 that $p^{(x)}_1 \in R_1(u)$ and $p^{(x)}_2 \notin R_2(u)$ hold or $p^{(x)}_1 \notin R_1(u)$ and $p^{(x)}_2 \in R_2(u)$ hold. Since $p^{(x)}_1 \in R_1(u)$, it must hold that $p^{(x)}_1 \in R_1(u)$ and $p^{(x)}_2 \notin R_2(u)$. Since $A(u)$ and $A(x)$ are homothetic and $p^{(x)}_1 \in R_1(u)$, we have $A(u) \subseteq A(x) \cup R_{23}(x)$.

Since $\{u, v, w, y\}$ induces a diamond and (u, v, w) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 4.1 that $p_1^{(y)} \in R_1(v)$ and $p_2^{(y)} \notin R_2(v)$ hold or $p_1^{(y)} \notin R_1(v)$ and $p_2^{(y)} \in R_2(v)$ hold. We will claim that the latter is true as it implies $p_2^{(y)} \in R_2(v)$. To reach a contradiction, suppose the former, that is, $p_1^{(y)} \in R_1(v)$ and $p_2^{(y)} \notin R_2(v)$. Since $A(v)$ and $A(y)$ are homothetic and $p_1^{(y)} \in R_1(v)$, we have $A(v) \subseteq A(y) \cup R_{23}(y)$. We now show that $A(x) \cap A(v) \subseteq A(y)$. Take any $a \in A(x) \cap A(v)$. Since $A(v) \subseteq A(y) \cup R_{23}(y)$, we have $a \in A(y) \cup R_{23}(y)$. Suppose that $a \notin A(y)$. Then $a \in R_{23}(y)$. This together with the fact that $a \in A(x)$ implies $A(y) \cap R_{23}(x) = \emptyset$. Since $A(u) \subseteq A(x) \cup R_{23}(x)$, we have

$$\begin{aligned} A(u) \cap A(y) &\subseteq (A(x) \cup R_{23}(x)) \cap A(y) \\ &= (A(x) \cap A(y)) \cup (R_{23}(x) \cap A(y)) \\ &= (A(x) \cap A(y)) \cup \emptyset \\ &= A(x) \cap A(y) \subseteq A(x). \end{aligned}$$

Therefore $A(u) \cap A(y) \subseteq A(u) \cap A(x)$. Since u and y are adjacent in G, there exists $b \in V(G)$ such that $\triangle(b) \subseteq A(u) \cap A(y)$. Then $\triangle(b) \subseteq A(u) \cap A(x)$, which is a contradiction to the fact that u and x are not adjacent in G. Thus $a \notin R_{23}(y)$ and so $a \in A(y)$. Hence we have shown that $A(x) \cap A(v) \subseteq A(y)$. Since x and v are adjacent in G, there exists $c \in V(G)$ such that $\triangle(c) \subseteq A(x) \cap A(v)$. Then $\triangle(c) \subseteq A(v) \cap A(y)$, which is a contradiction to the fact that v and y are not adjacent in G. Thus we have $p_1^{(y)} \notin R_1(v)$ and $p_2^{(y)} \in R_2(v)$. Hence the lemma holds. □

Definition. For a positive integer n, let G_n be the graph obtained from the complete graph K_n by adding a path of length 2 for each pair of vertices of K_n, i.e., $V(G_n) = \{v_i \mid 1 \le i \le n\} \cup \{v_{ij} \mid 1 \le i < j \le n\}$ and $E(G_n) = \{v_iv_j \mid 1 \le i < j \le n\} \cup \{v_iv_{ij} \mid 1 \le i < j \le n\} \cup \{v_jv_{ij} \mid 1 \le i < j \le n\}$.

Definition. For a positive integer m, the *Ramsey number* $r(m, m, m)$ is the smallest positive integer r such that any 3-edge-colored complete graph K_r of order r contains a monochromatic complete graph K_m of order m.

Lemma 4.3. *Let m be a positive integer at least 3 and let n be an integer greater than or equal to the Ramsey number $r(m, m, m)$. If $\dim_{\mathrm{poc}}(G_n) \le 3$, then there exists a sequence $(x_1, \ldots, x_m)$ of vertices of G_n such that $\{x_1, \ldots, x_m\}$ is a clique of G_n and that any subsequence $(x_{i_1}, \ldots, x_{i_l})$ of $(x_1, \ldots, x_m)$ is consecutively tail-biting, where $2 \le l \le m$ and $1 \le i_1 < \cdots < i_l \le m$.*

Proof. Since the vertices v_i and v_j of G_n are internal vertices of an induced path of length three by the definition of G_n, it follows from Lemma 2.5 that the vertices v_i and v_j of G_n are crossing. By Lemma 2.6, for any $1 \le i < j \le n$, there exists $k \in \{1, 2, 3\}$ such that $v_i \xrightarrow{k} v_j$ or $v_j \xrightarrow{k} v_i$. Now we define an edge-coloring $c : \{v_iv_j \mid 1 \le i < j \le n\} \to \{1, 2, 3\}$ as follows: For $1 \le i < j \le n$, we let $c(v_iv_j) = k$ so that $v_i \xrightarrow{k} v_j$ or $v_j \xrightarrow{k} v_i$. Then, by the definition of $r(m, m, m)$, K_n contains a monochromatic complete subgraph K with m vertices.

Suppose that the edges of K have color k, where $k \in \{1, 2, 3\}$. We assign an orientation to each edge xy of K so that x goes toward y if $x \xrightarrow{k} y$. In that way, we obtain a tournament $\overrightarrow{K}$ with m vertices. It is well-known that every tournament has a directed Hamiltonian path. Therefore, $\overrightarrow{K}$ has a directed Hamiltonian path. Let $x_1 \to x_2 \to \cdots \to x_m$ be a directed Hamiltonian path of $\overrightarrow{K}$. Then, by Lemma 2.7, $x_i \xrightarrow{k} x_j$ for any $i < j$. Thus any subsequence $(x_{i_1}, \ldots, x_{i_l})$ of $(x_1, \ldots, x_m)$ is consecutively tail-biting, where $2 \le l \le m$ and $1 \le i_1 < \cdots < i_l \le m$. □

Since the graph G_n is chordal, the following theorem shows the existence of chordal graphs with partial order competition dimensions greater than three. Given a graph G and a set X consisting of six vertices in G, we say that X *induces an* $\overline{\mathrm{H}}$ if it induces a subgraph of G isomorphic to $\overline{\mathrm{H}}$.

Theorem 4.4. *For $n \ge r(5, 5, 5)$, $\dim_{\mathrm{poc}}(G_n) > 3$.*

Proof. We prove by contradiction. Suppose that $\dim_{\mathrm{poc}}(G_n) \le 3$ for some $n \ge r(5, 5, 5)$. By Lemma 4.3, G_n contains a consecutively tail-biting sequence $(v_1, \ldots, v_5)$ of five vertices in Type k such that $\{v_1, \ldots, v_5\}$ is a clique of G_n and that $(v_{i_1}, v_{i_2}, v_{i_3})$ is a consecutively tail-biting sequence for any $1 \le i_1 < i_2 < i_3 \le 5$ and $(v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4})$ is a consecutively tail-biting sequence for any $1 \le i_1 < i_2 < i_3 < i_4 \le 5$. Without loss of generality, we may assume that $k = 3$.

Since $\{v_1, v_2, v_3, v_{13}\}$ induces a diamond and (v_1, v_2, v_3) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 4.1 that $p_1^{(v_{13})} \in R_1(v_2)$ and $p_2^{(v_{13})} \notin R_2(v_2)$ hold or $p_1^{(v_{13})} \notin R_1(v_2)$ and $p_2^{(v_{13})} \in R_2(v_2)$ hold.

We first suppose that $p_1^{(v_{13})} \in R_1(v_2)$ and $p_2^{(v_{13})} \notin R_2(v_2)$. Since $\{v_1, v_2, v_3, v_4, v_{13}, v_{24}\}$ induces an $\overline{\mathrm{H}}$ and (v_1, v_2, v_3, v_4) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 4.2 and $p_1^{(v_{13})} \in R_1(v_2)$ that $p_2^{(v_{24})} \in R_2(v_3)$. Since $\{v_1, v_2, v_3, v_5, v_{13}, v_{25}\}$ induces an $\overline{\mathrm{H}}$ and (v_1, v_2, v_3, v_5) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 4.2 and $p_1^{(v_{13})} \in R_1(v_2)$ that

$$p_2^{(v_{25})} \in R_2(v_3). \tag{1}$$

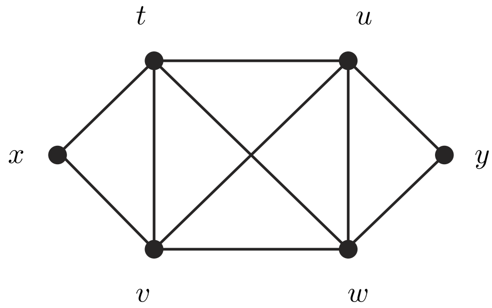

Fig. 4. The graph $\overline{\mathrm{H}}$.

Since $\{v_2, v_3, v_4, v_5, v_{24}, v_{35}\}$ induces an $\overline{\mathrm{H}}$ and (v_2, v_3, v_4, v_5) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 4.2 and $p_2^{(v_{24})} \in R_2(v_3)$ that

$$p_1^{(v_{35})} \in R_1(v_4). \tag{2}$$

Since $\{v_1, v_3, v_4, v_{14}\}$ induces a diamond and (v_1, v_3, v_4) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 4.1 that $p_1^{(v_{14})} \in R_1(v_3)$ and $p_2^{(v_{14})} \notin R_2(v_3)$ hold or $p_1^{(v_{14})} \notin R_1(v_3)$ and $p_2^{(v_{14})} \in R_2(v_3)$ hold. Suppose that $p_1^{(v_{14})} \in R_1(v_3)$ and $p_2^{(v_{14})} \notin R_2(v_3)$. Since $\{v_1, v_3, v_4, v_5, v_{14}, v_{35}\}$ induces an $\overline{\mathrm{H}}$ and (v_1, v_3, v_4, v_5) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 4.2 and $p_1^{(v_{14})} \in R_1(v_3)$ that

$$p_2^{(v_{35})} \in R_2(v_4). \tag{3}$$

Since $\{v_3, v_4, v_5, v_{35}\}$ induces a diamond and (v_3, v_4, v_5) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 4.1 that $p_1^{(v_{35})} \in R_1(v_4)$ and $p_2^{(v_{35})} \notin R_2(v_4)$ hold or $p_1^{(v_{35})} \notin R_1(v_4)$ and $p_2^{(v_{35})} \in R_2(v_4)$ hold, which is a contradiction to the fact that both (2) and (3) hold. Thus

$$p_1^{(v_{14})} \notin R_1(v_3) \text{ and } p_2^{(v_{14})} \in R_2(v_3). \tag{4}$$

Since $\{v_1, v_2, v_4, v_{14}\}$ induces a diamond and (v_1, v_2, v_4) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 4.1 that $p_1^{(v_{14})} \in R_1(v_2)$ and $p_2^{(v_{14})} \notin R_2(v_2)$ hold or $p_1^{(v_{14})} \notin R_1(v_2)$ and $p_2^{(v_{14})} \in R_2(v_2)$ hold. Suppose that $p_1^{(v_{14})} \notin R_1(v_2)$ and $p_2^{(v_{14})} \in R_2(v_2)$. Since $\{v_1, v_2, v_4, v_5, v_{14}, v_{25}\}$ induces an $\overline{\mathrm{H}}$ and (v_1, v_2, v_4, v_5) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 4.2 and $p_2^{(v_{14})} \in R_2(v_2)$ that

$$p_1^{(v_{25})} \in R_1(v_4). \tag{5}$$

By (1) and (5), since $A(v_4)$ and $A(v_{25})$ are homothetic, we have

$$p_2^{(v_{25})} \in R_2(v_4). \tag{6}$$

Since $\{v_2, v_4, v_5, v_{25}\}$ induces a diamond and (v_2, v_4, v_5) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 4.1 that $p_1^{(v_{25})} \in R_1(v_4)$ and $p_2^{(v_{25})} \notin R_2(v_4)$ hold or $p_1^{(v_{25})} \notin R_1(v_4)$ and $p_2^{(v_{25})} \in R_2(v_4)$ hold, which is a contradiction to the fact that both (5) and (6) hold. Thus $p_1^{(v_{14})} \in R_1(v_2)$ and $p_2^{(v_{14})} \notin R_2(v_2)$.

Since $A(v_3)$ and $A(v_{14})$ are homothetic, we have

$$p_1^{(v_{14})} \in R_1(v_3), \tag{7}$$

contradicting (4).

In the case where $p_1^{(v_{13})} \notin R_1(v_2)$ and $p_2^{(v_{13})} \in R_2(v_2)$, we also reach a contradiction by applying a similar argument.

Hence, $\dim_{\mathrm{poc}}(G_n) > 3$ holds for any $n \geq r(5, 5, 5)$. □

Remark 4.5. We have shown that there are chordal graphs with partial order competition dimension greater than three and that there are chordal graphs with partial order competition dimension at most three such as chordal diamond-free graphs. In this vein, it would be interesting to characterize the chordal graphs with partial order competition dimension at most three.

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References

[1] Hans-Jürgen Bandelt, Henry Martyn Mulder, Distance-hereditary graphs, J. Combin. Theory Ser. B 41 (2) (1986) 182–208.
[2] Han Hyuk Cho, Suh-Ryung Kim, A class of acyclic digraphs with interval competition graphs, Discrete Appl. Math. 148 (2) (2005) 171–180.
[3] Jihoon Choi, Kyeong Seok Kim, Suh-Ryung Kim, Jung Yeun Lee, Yoshio Sano, On the competition graphs of d-partial orders, Discrete Appl. Math. 204 (2016) 29–37.
[4] Seog-Jin Kim, Suh-Ryung Kim, Yoomi Rho, On cce graphs of doubly partial orders, Discrete Appl. Math. 155 (8) (2007) 971–978.
[5] Suh-Ryung Kim, Jung Yeun Lee, Boram Park, Won Jin Park, Yoshio Sano, The niche graphs of doubly partial orders, Congr. Numer. 196 (2009) 19–32.
[6] Suh-Ryung Kim, Jung Yeun Lee, Boram Park, Yoshio Sano, The competition hypergraphs of doubly partial orders, Discrete Appl. Math. 165 (2014) 185–191.
[7] Junjie Lu, Yaokun Wu, Two minimal forbidden subgraphs for double competition graphs of posets of dimension at most two, Appl. Math. Lett. 22 (6) (2009) 841–845.
[8] Boram Park, Jung Yeun Lee, Suh-Ryung Kim, The m-step competition graphs of doubly partial orders, Appl. Math. Lett. 24 (6) (2011) 811–816.
[9] Boram Park, Yoshio Sano, The phylogeny graphs of double partial orders, Discuss. Math. Graph Theory 33 (4) (2013) 657–664.
[10] Yaokun Wu, Junjie Lu, Dimension-2 poset competition numbers and dimension-2 poset double competition numbers, Discrete Appl. Math. 158 (6) (2010) 706–717.