On the phylogeny graphs of degree-bounded digraphs

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ABSTRACT

Hefner et al. (1991) characterized acyclic digraphs each vertex of which has indegree and outdegree at most two and whose competition graphs are interval. They called acyclic digraphs each vertex of which has indegree and outdegree at most two $(2, 2)$ digraphs. In this paper, we study the phylogeny graphs of $(2, 2)$ digraphs. Especially, we give a sufficient condition and necessary conditions for $(2, 2)$ digraphs having chordal phylogeny graphs. Phylogeny graphs are also called moral graphs in Bayesian network theory. Our work is motivated by problems related to evidence propagation in a Bayesian network for which it is useful to know which acyclic digraphs have their moral graphs being chordal.

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1. Introduction

Throughout this paper, we deal with simple graphs and simple digraphs. In a digraph, we sometimes represent an arc (u, v) by $u \to v$.

Given an acyclic digraph D, the *competition graph* of D, denoted by $C(D)$, is the graph having vertex set $V(D)$ and edge set $\{uv \mid (u, w), (v, w) \in A(D)$ for some $w \in V(D)\}$. A graph G is called an *interval graph* if we can assign to each vertex x of G a real interval $J(x)$ so that, whenever $x \neq y$, $xy \in E(G)$ if and only if $J(x) \cap J(y) \neq \emptyset$. Cohen [1] introduced the notion of competition graphs in the study on predator–prey concepts in ecological food webs. Cohen's empirical observation that real-world competition graphs are usually interval graphs had led to a great deal of research on the structure of competition graphs and on the relationship between the structure of digraphs and their corresponding competition graphs. In the same vein, various variants of competition graphs have been introduced and studied. For recent work related to competition graphs, see [4,7,8,10,19].

Steif [18] showed that it might be difficult to find the structural properties of acyclic digraphs whose competition graphs are interval. In that respect, Hefner et al. [6] placed restrictions on the indegree and the outdegree of vertices of acyclic digraphs to obtain the list of forbidden subdigraphs for acyclic digraphs whose competition graphs are interval.

The notion of phylogeny graphs was introduced by Roberts and Sheng [13] as a variant of competition graphs. (See also [5,11,14–16,20] for study on phylogeny graphs.) Given an acyclic digraph D, the *underlying graph* of D, denoted by $U(D)$, is the graph with vertex set $V(D)$ and edge set $\{xy \mid (x, y) \in A(D)$ or $(y, x) \in A(D)\}$. The *phylogeny graph* of an acyclic digraph D, denoted by $P(D)$, is the graph with vertex set $V(D)$ and edge set $E(U(D)) \cup E(C(D))$. For example, given an acyclic digraph D in Fig. 1(a), the competition graph of D is the graph $C(D)$ in Fig. 1(b), and the phylogeny graph of D is the graph $P(D)$ in Fig. 1(c).

"Moral graphs", having arisen from studying Bayesian networks, are the same as phylogeny graphs. One of the best-known problems, in the context of Bayesian networks, is related to the propagation of evidence. It consists of the assignment

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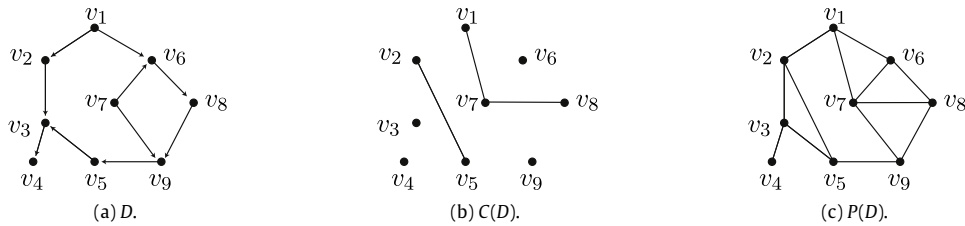
Fig. 1. (a) An acyclic digraph D, (b) The competition graph $C(D)$ of D, (c) The phylogeny graph $P(D)$ of D.

of probabilities to the values of the rest of the variables, once the values of some variables are known. Cooper [2] showed that this problem is NP-hard. Most noteworthy algorithms for this problem are given by Pearl [12], Shachter [17] and by Lauritzen and Spiegelhalter [9]. Those algorithms include a step of triangulating a moral graph, that is, adding edges to a moral graph to form a chordal graph.

A graph G is said to be *chordal* if every cycle in G of length greater than 3 has a chord, namely, an edge joining two nonconsecutive vertices on the cycle, that is, G does not contain a cycle of length at least 4 as an induced subgraph. Throughout this paper, we call a cycle C of length at least 4 as an induced subgraph a *hole*. A necessary and sufficient condition for a graph being interval is that the graph does not contain a cycle of length at least 4 as an induced subgraph and the complement of the graph is transitively orientable. This implies that an interval graph is chordal.

As triangulations of moral graphs play an important role in algorithms for propagation of evidence in a Bayesian network, studying chordality of the phylogeny graphs of acyclic digraphs is meaningful. Yet, characterizing the acyclic digraphs whose phylogeny graphs are chordal seems to be more difficult than characterizing the acyclic digraphs whose competition graphs are interval. In this respect, hoping to provide insights for the further research, we begin with "(2, 2) digraphs" to attack the problem. A (2, 2) *digraph* is an acyclic digraph such that each vertex has both outdegree and indegree at most two. Hefner et al. [6] characterized (2, 2) digraphs whose competition graphs are interval.

In this paper, we study the phylogeny graphs of (2, 2) digraphs. Especially, we give a sufficient condition and necessary conditions for (2, 2) digraphs having chordal phylogeny graphs.

2. Preliminaries

2.1. Properties of chordal graphs

We first see some properties of chordal graphs which will be used in characterizing the (2, 2) digraphs whose phylogeny graphs are chordal. The following propositions are easy to check.

Proposition 2.1. *Any induced subgraph of a chordal graph is also a chordal graph.*

Proposition 2.2. *A chordal graph containing a cycle of length n has at least $2n - 3$ edges.*

For a graph G and a vertex v of G, a *neighbor* of v in G is a vertex adjacent to v in G, and the set of neighbors of v in G is denoted by $N_G(v)$. Then the following holds.

Proposition 2.3. *Let G be a chordal graph. For any cycle C in G and any edge xy on C, there exists a vertex on C that is a common neighbor of x and y in G.*

Proof. We show this by contradiction. Suppose that there exists a cycle C in G which has an edge $xy \in E(C)$ satisfying $N_G(x) \cap N_G(y) \cap V(C) = \emptyset$. We take a cycle of the shortest length among such cycles. Let $C := v_0 v_1 \ldots v_{k-1} v_0$ be such a cycle. If $k = 3$, then $N_G(x) \cap N_G(y) \cap V(C) \neq \emptyset$ for any $xy \in E(C)$. Thus $k \geq 4$. Since G is chordal, there exists a chord $v_i v_j$ of C, where $i < j$. Let P_1 and P_2 be the two (v_i, v_j)-sections of C and consider the cycles $C_1 := P_1 + v_i v_j$ and $C_2 := P_2 + v_i v_j$. Since both C_1 and C_2 have lengths shorter than k, $N_G(x) \cap N_G(y) \cap V(C_t) \neq \emptyset$ for any $xy \in E(C_t)$ and $t = 1, 2$. This implies that $N_G(x) \cap N_G(y) \cap V(C) \neq \emptyset$ for any $xy \in E(C)$, which is a contradiction. $\square$

We say that a vertex v on a cycle C of length at least 4 in a chordal graph G is a *vertex opposite to a chord of C* if the two vertices immediately following and immediately preceding it, respectively, in the sequence of C are adjacent. (For example, each of the vertices v_2 and v_5 in Fig. 2 is a vertex opposite to a chord.)

Proposition 2.4. *Each cycle of length at least 4 in a chordal graph has at least two nonconsecutive vertices each of which is opposite to a chord of the cycle.*
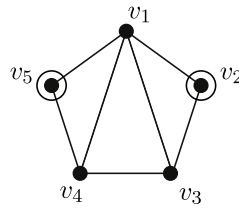
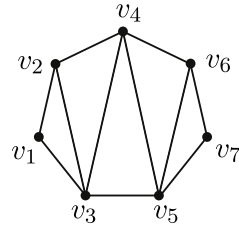**Fig. 2.** Each of v_2 and v_5 is a vertex opposite to a chord of the cycle $v_1 v_2 v_3 v_4 v_5 v_1$.
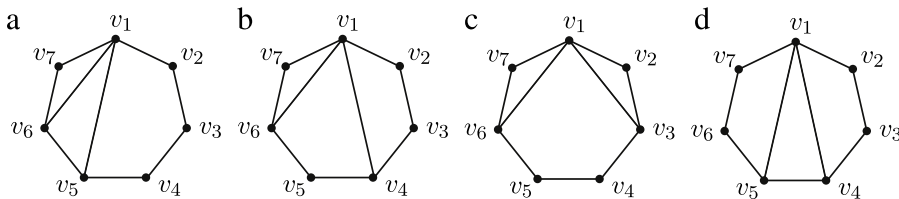

Fig. 3. A W-configuration.

Fig. 4. The possible neighbors of v_1.

Proof. Let G be a chordal graph, C be a cycle of length at least 4 in G, and G' be the subgraph of G induced by the set of vertices of C, that is, $G' = G[V(C)]$. Then G' is chordal by Proposition 2.1. If G' is complete, then the statement is trivially true. Suppose that G' is not complete. As Dirac [3] showed that every non-complete chordal graph has at least two nonadjacent simplicial vertices (a simplicial vertex means a vertex whose neighbors form a clique), there exist two nonadjacent simplicial vertices u and v in G'. Then each of u and v is obviously opposite to a chord of C. Since u and v are not adjacent in the induced subgraph G' of G, u and v are not consecutive on C. □

We call a graph isomorphic to the graph G defined by $V(G) = \{v_1, \ldots, v_7\}$ and $E(G) = \{v_i v_j \mid 1 \leq i < j \leq 7, j - i \leq 2\}$ a W-configuration (see Fig. 3).

Proposition 2.5. *If a chordal graph with the degree of each vertex at most four contains a cycle of length 7, then it contains a W-configuration as a subgraph.*

Proof. Let G be a chordal graph with the degree of each vertex at most four and let $C = v_1 \cdots v_7 v_1$ be a cycle of length 7 of G, and let H be the subgraph of G induced by the vertex set of C. Since G is chordal, so is H by Proposition 2.1. Since H is a chordal graph containing a cycle of length 7, $|E(H)| \geq 11$ by Proposition 2.2. Then

$$\sum_{v \in V(H)} \deg_H(v) = 2|E(H)| \geq 2 \times 11 = 22.$$

Therefore there exists a vertex on C of degree 4 in H by the pigeonhole principle and the hypothesis. Without loss of generality, we may assume that v_1 is a vertex of degree 4. By symmetry, it is sufficient to consider the following cases for the possible pairs of neighbors of v_1 other than v_2 and v_7: (a) $\{v_5, v_6\}$; (b) $\{v_4, v_6\}$; (c) $\{v_3, v_6\}$; (d) $\{v_4, v_5\}$ (see Fig. 4).

As a matter of fact, the cases (b) and (c) cannot happen. Suppose to the contrary that (b) happened. Then each of the 4-cycles $v_1 v_4 v_5 v_6 v_1$ and $v_1 v_2 v_3 v_4 v_1$ must contain a chord. By the way, v_1 already has degree 4, so $v_4 v_6$ and $v_4 v_2$ must be the chords of $v_1 v_4 v_5 v_6 v_1$ and $v_1 v_2 v_3 v_4 v_1$, respectively. Then $\deg_G(v_4) \geq 5$, which is a contradiction. Thus the case (b) cannot happen. Suppose that (c) happened. Then the 5-cycle $v_1 v_3 v_4 v_5 v_6 v_1$ must contain a chord since G is chordal. By the hypothesis that each vertex of G has degree at most 4, $v_3 v_5$, $v_4 v_6$, $v_3 v_6$ are the only possible chords. If $v_3 v_6$ is a chord, then $v_3 v_4 v_5 v_6 v_3$ is a

hole, which is impossible. Thus v_3 and v_6 are not joined. Now if v_3v_5 is a chord, then $v_1v_3v_5v_6v_1$ is a hole, which is impossible again. Thus v_4v_6 is a chord. Then $v_1v_3v_4v_6v_1$ is a hole and we reach a contradiction. Hence the case (c) cannot happen.

Now we consider the case (a). By applying Proposition 2.3 to the edge v_1v_2 on the cycle $v_1v_2v_3v_4v_5v_1$, we conclude that one of v_3, v_4, v_5 is a vertex that is adjacent to both v_1 and v_2. Since each vertex has degree at most 4 by the hypothesis, it must be v_5. By the same proposition applied to the edge v_2v_3 on the cycle $v_2v_3v_4v_5v_2$, both of v_2 and v_3 are adjacent to v_4 or v_5. Since v_5 is adjacent to four vertices, the possibility of v_5 is eliminated and so v_4 is adjacent to v_2 and v_3. Thus G contains a W-configuration.

We consider the case (d). By applying Proposition 2.3 to the edge v_1v_5 on the cycle $v_1v_5v_6v_7v_1$, we conclude that v_6 or v_7 is a vertex that is adjacent to both v_1 and v_5. Since v_1 is already adjacent to four vertices other than v_6, v_6 is excluded and so v_7 is adjacent to both v_1 and v_5. By applying Proposition 2.3 to the cycle $v_1v_2v_3v_4v_1$ and the edge v_1v_4, we conclude that v_2 is adjacent to both v_1 and v_4. Consequently we obtain a W-configuration. □

2.2. Induced edges of the phylogeny graphs of (i, j) digraphs

We call an edge in the phylogeny graph $P(D)$ a *cared edge* in $P(D)$ if the edge belongs to the competition graph $C(D)$ but not to the underlying graph $U(D)$. For a cared edge xy in $P(D)$, there is a common out-neighbor v of x and y in D by definition. The vertex v is called a *vertex taking care of the edge xy* and it is said that *xy is taken care of by v* or that *v takes care of xy*. A vertex in D is called an *caring vertex* if an edge of $P(D)$ is taken care of by the vertex. For example, the edges v_1v_7, v_7v_8, and v_2v_5 of $P(D)$ in Fig. 1(c) are cared edges and the vertices v_6, v_9, and v_3 are vertices taking care of v_1v_7, v_7v_8, and v_2v_5, respectively.

We call an acyclic digraph D an (i, j) *digraph* if each vertex of D has indegree at most i and outdegree at most j. In the following, we study the structure of the phylogeny graph of a (i, j) digraph.

Lemma 2.6. *Given an (i, j) digraph D, there is no vertex that takes care of more than $\frac{1}{2}i(i-1)$ edges in the phylogeny graph of D.*

Proof. Suppose to the contrary that there exists a vertex x taking care of t different edges for $t \geq \frac{1}{2}i(i-1)+1$. Then these edges belong to the clique K in the phylogeny graph of D formed by the in-neighbors of x in D. Thus K contains at least $i+1$ vertices. Hence the indegree of x is greater than i, which contradicts the hypothesis that D is an (i, j) digraph. □

The following is a consequence of Lemma 2.6 when $(i, j) = (2, 2)$.

Corollary 2.7. *Given a $(2, 2)$ digraph D, there is no vertex that takes care of more than one cared edge in the phylogeny graph of D.*

Lemma 2.8. *Given an (i, j) digraph D, there is no vertex that is incident to more than $\frac{1}{2}i(i-1)j$ distinct cared edges in the phylogeny graph of D.*

Proof. Take a vertex x incident to at least one cared edge and let $e_1, \ldots, e_t$ be the cared edges in the phylogeny graph $P(D)$ of D incident to x, where t is a positive integer. Let $w_1, \ldots, w_s$ be distinct vertices in D taking care of $e_1, \ldots, e_t$, where s is a positive integer. Then $w_1, \ldots, w_s$ are out-neighbors of x, so $s \leq j$. By Lemma 2.6, each of the vertices $w_1, \ldots, w_s$ can take care of at most $\frac{1}{2}i(i-1)$ edges in $P(D)$. Therefore $t \leq \frac{1}{2}i(i-1)s \leq \frac{1}{2}i(i-1)j$ and thus the lemma holds. □

The following is a consequence of Lemma 2.8 when $(i, j) = (2, 2)$.

Corollary 2.9. *Given a $(2, 2)$ digraph D, there is no vertex that is incident to three cared edges in the phylogeny graph of D.*

3. Main results

3.1. The phylogeny graphs of $(2, 2)$ digraphs are K_5-free

In this subsection, we show that there is no $(2, 2)$ digraph whose phylogeny graph contains a complete graph K_n for $n \geq 5$.

Note that we can construct a $(2, 2)$ digraph whose phylogeny graph contains a complete graph K_4 as a subgraph as shown in Example 3.1.

Example 3.1. Let D be a digraph defined by $V(D) = \{v_1, v_2, v_3, v_4, v_5\}$ and $A(D) = \{(v_1, v_2), (v_1, v_4), (v_2, v_3), (v_2, v_5), (v_3, v_4), (v_4, v_5)\}$ (see Fig. 5(a)). Then D is a $(2, 2)$ digraph, and the subgraph of $P(D)$ induced by $\{v_1, v_2, v_3, v_4\}$ is isomorphic to K_4 (see Fig. 5(b)).

Theorem 3.2. *For any $(2, 2)$ digraph D, the phylogeny graph of D is K_5-free.*
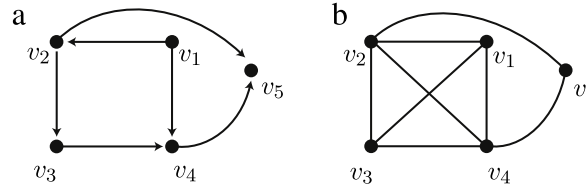
Fig. 5. (a) A $(2, 2)$ digraph D, (b) The phylogeny graph of D contains K_4.
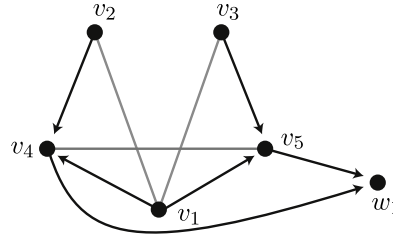

Fig. 6. The vertex w_1 taking care of v_4v_5. The gray edges are cared edges.

Proof. Suppose to the contrary that there is a $(2, 2)$ digraph D whose phylogeny graph $P(D)$ contains K_5 as a subgraph. Let v_1, v_2, v_3, v_4, v_5 be the vertices of K_5. Let D_1 be the subdigraph of D induced by $\{v_1, \ldots, v_5\}$. Since D_1 is acyclic, there is a vertex of indegree 0 in D_1. Without loss of generality, we may assume that v_1 has indegree 0. Now consider the edges $v_1v_2, v_1v_3, v_1v_4, v_1v_5$ in $P(D)$. At most two of them are cared edges by Corollary 2.9. Therefore, at least two of $v_1v_2, v_1v_3, v_1v_4, v_1v_5$ belong to $U(D)$. By the way, they must be arcs outgoing from v_1 since v_1 has indegree 0 in D_1. Since v_1 has outdegree at most two in D, v_1 has outdegree exactly two in D. Therefore exactly two of $v_1v_2, v_1v_3, v_1v_4, v_1v_5$ belong to $U(D)$ and, consequently, the remaining two edges are cared edges in $P(D)$.

Without loss of generality, we may assume that v_1v_2 and v_1v_3 are cared edges. Then v_4 and v_5 are the out-neighbors of v_1 in D. Since a vertex taking care of v_1v_2 (resp. v_1v_3) is an out-neighbor of v_1, v_1v_2 (resp. v_1v_3) is taken care of by v_4 or v_5. Without loss of generality, we may assume that v_4 is a common out-neighbor of v_1 and v_2 in D. Then, by Corollary 2.7, v_5 is a common out-neighbor of v_1 and v_3. However, since both v_4 and v_5 have indegree two, the edge v_4v_5 cannot belong to $U(D)$ and so is a cared edge in $P(D)$. Since $(v_1, v_4), (v_2, v_4), (v_1, v_5)$, and (v_3, v_5) are arcs of D, none of v_1, v_2, v_3 is a common out-neighbor of v_4 and v_5 by the acyclicity of D. Therefore, the edge v_4v_5 is taken care of by a vertex w_1 distinct from v_1, v_2, v_3 (see Fig. 6).

We show that at least one of v_3v_4, v_2v_5 is a cared edge. Suppose to the contrary that both v_3v_4 and v_2v_5 are not cared edges. Then v_3v_4 and v_2v_5 are inherited from arcs (v_4, v_3) and (v_5, v_2), respectively, since the indegrees of v_4 and v_5 are at most two. Therefore $v_2 \to v_4 \to v_3 \to v_5 \to v_2$ is a directed cycle, which contradicts the hypothesis that D is acyclic. Thus at least one of v_3v_4, v_2v_5 is a cared edge. Without loss of generality, we may assume that v_3v_4 is a cared edge. Since none of v_2 and v_5 can be an out-neighbor of v_4, neither v_2 nor v_5 takes care of v_3v_4. Since the indegree of v_1 is 0, v_1 does not take care of v_3v_4 either. Since w_1 has in-neighbors v_4 and v_5, w_1 cannot take care of the edge v_3v_4 due to the degree condition imposed on D. Therefore the edge v_3v_4 is taken care of by a vertex w_2 distinct from v_1, v_2, v_5, w_1. Recall that v_3v_4 and v_3v_1 are cared edges. Since v_3 cannot be incident to three cared edges by Corollary 2.9, the edge v_2v_3 belongs to $U(D)$. Then v_2 is an in-neighbor of v_3 since the outdegree of v_3 is at most two.

Finally we take a look at the edge v_2v_5. Since the indegree of v_5 is two that is achieved by v_1 and v_3, v_2 cannot be an in-neighbor of v_5. If v_2 is an out-neighbor of v_5, then it results in the directed cycle $v_2 \to v_3 \to v_5 \to v_2$, which is a contradiction. Therefore v_2v_5 is a cared edge. Since v_3 and v_4 are the only out-neighbors of v_2, v_3 or v_4 takes care of v_2v_5. Since v_5 is an out-neighbor of v_3, v_3 cannot take care of the edge v_2v_5. Since the edge v_4v_5 is a cared edge, v_4 cannot take care of the edge v_2v_5 either. Hence we have reached a contradiction. □

3.2. A necessary condition for the phylogeny graph of a $(2, 2)$ digraph being chordal

Properties of $(2, 2)$ digraphs make us speculate that a sufficiently long hole in the underlying graph of a $(2, 2)$ digraph D might give rise to a hole in the phylogeny graph of D as chords cannot be produced enough to fill in it. This motivates us to find the length of a shortest hole among holes in the underlying graph of a $(2, 2)$ digraph whose phylogeny graph is chordal.

Lemma 3.3. *Let D be a $(2, 2)$ digraph. If the underlying graph of D contains a hole H of length at least 7, then the subgraph of the phylogeny graph of D induced by $V(H)$ is not chordal.*
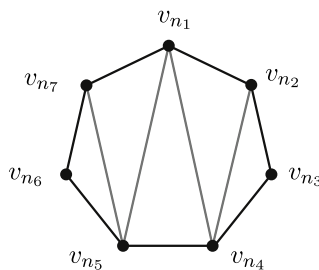
Fig. 7. A W-configuration of G^*. The gray edges are cared edges.

Proof. Let $H = v_1 v_2 \cdots v_n v_1$ ($n \geq 7$) be a hole of length at least 7 in $U(D)$ and let G_1 be the subgraph of $P(D)$ induced by $V(H) = \{v_1, \ldots, v_n\}$. Note that no edges on H are cared edges while the edges of G_1 not on H are cared edges in $P(D)$. Suppose to the contrary that G_1 is chordal. If there exists a vertex v with $\deg_{G_1}(v) \geq 5$, then v is incident to at least three cared edges in G_1 and so in $P(D)$, which contradicts Corollary 2.9. Therefore $d_{G_1}(v) \leq 4$ for every vertex v in G_1.

Since G_1 is a hamiltonian chordal graph with at least seven vertices, there exists a vertex on G_1 opposite to a chord by Proposition 2.4. Without loss of generality, we may assume that v_2 is such a vertex. Let G_2 be the graph obtained by deleting v_2 from G_1. Since v_2 is a vertex opposite to a chord, G_2 is a hamiltonian chordal graph with $n-1$ vertices with a hamiltonian cycle $v_1 v_3 v_4 \cdots v_n v_1$. Since $n - 1 \geq 6$, we may apply Proposition 2.4 again to have a vertex on G_2 which is opposite to a chord. We delete one of such vertices from G_2 to obtain a hamiltonian chordal graph with $n-2$ vertices. We continue this process until we obtain a hamiltonian chordal graph G^* with 7 vertices. Let $v_{n_1} v_{n_2} \cdots v_{n_7} v_{n_1}$ be a hamiltonian cycle of G^* with $n_1 < n_2 < \cdots < n_7$, which exists by the definition of G^*. By Proposition 2.5, G^* is a W-configuration. Without loss of generality, we may assume that it is labeled as in Fig. 7.

If the end vertices of an edge in G^* are not on the hamiltonian cycle $v_{n_1} v_{n_2} \cdots v_{n_7} v_{n_1}$, then the index difference of them is neither 1 nor $n-1$ since $n_1 < n_2 < \cdots < n_7$. Noting that an edge $v_m v_k$ ($1 \leq m, k \leq n$) is on H if and only if $|m - k| = 1$ or $|m - k| = n - 1$, we may conclude that $v_{n_4} v_{n_1}$, $v_{n_4} v_{n_2}$, $v_{n_5} v_{n_1}$, and $v_{n_5} v_{n_7}$ are cared edges. Let w_1, w_2, w_3, w_4 be caring vertices of the edges $v_{n_4} v_{n_1}$, $v_{n_4} v_{n_2}$, $v_{n_5} v_{n_1}$, $v_{n_5} v_{n_7}$, respectively. Then w_1, w_2, w_3, w_4 are all distinct by Corollary 2.7, and w_1, w_2 (resp. w_3, w_4) are out-neighbors of v_{n_4} (resp. v_{n_5}). Since D is a $(2, 2)$ digraph, there cannot exist an arc between v_{n_4} and v_{n_5}. Therefore $v_{n_4} v_{n_5}$ should be a cared edge. Then v_{n_4} is incident to three cared edges, which contradicts Corollary 2.9. Hence G_1 is not chordal. □

By Lemma 3.3 and Proposition 2.1, the following theorem holds.

Theorem 3.4. *Let D be a $(2, 2)$ digraph. If the underlying graph of D contains a hole of length at least 7, then the phylogeny graph of D is not a chordal graph.*

3.3. Forbidden subdigraphs for the class of $(2, 2)$ digraphs whose phylogeny graphs are chordal

Let $\mathcal{D}$ be a class of digraphs. A digraph D_0 is called a *forbidden subdigraph* for $\mathcal{D}$ if $D \notin \mathcal{D}$ holds for any digraph D containing D_0 as an induced subdigraph. Note that, by Lemma 3.3, the orientations of cycles of length at least 7 are forbidden subdigraphs for the class $\mathcal{D}^*$ of $(2, 2)$ digraphs whose phylogeny graphs are chordal. In this subsection, we determine the non-isomorphic orientations of cycles of length 4 or 5 or 6 that are forbidden subdigraphs for $\mathcal{D}^*$.

Theorem 3.5. *Let $\mathcal{D}^*$ be the class of $(2, 2)$ digraphs whose phylogeny graphs are chordal. Then the digraphs given in Fig. 8 are the forbidden subdigraphs among orientations of cycles of length at most six for $\mathcal{D}^*$.*

Proof. Suppose that a $(2, 2)$ digraph D contains an orientation C of a cycle with length six given in Fig. 8 as an induced subdigraph. (We provided the chordal phylogeny graph of a $(2, 2)$ digraph containing each of orientations of cycles of length 4 or 5 or 6 in Figs. 9–11 other than the ones given in Fig. 8.) Let S be the subgraph of the phylogeny graph $P(D)$ of D induced by $V(C)$. To reach a contradiction, suppose that $P(D)$ is chordal. Then, in case of (a)–(c), the subgraph of $P(D)$ induced by the vertex set of a cycle H_1 of length five is contained in S, so it has at least two adjacent chords which are taken care of by two caring vertices. Since each vertex in D has indegree at most two, the two caring vertices must be distinct. In case of (d), (e), S contains the subgraph of $P(D)$ induced by the vertex set of a cycle H_2 of length four, so it has a chord which is taken care of by a caring vertex. Since C is an induced subdigraph of D, neither the vertices taking care of chords of H_1 nor the vertices taking care of chords of H_2 can be on C. Therefore, in case of (a)–(c), the vertex common to the two adjacent chords of H_1 must have two out-neighbors not on C and in case of (d), (e), there are two nonadjacent vertices on H_2 each of which has an out-neighbor not on C. However, by the structure of C, each vertex on H_1 has an out-neighbor on C and especially in case of (d), (e), there are two adjacent vertices on H_2 each of which has two out-neighbors on C in D. Hence, in either case, we obtain a vertex of outdegree at least three, which is a contradiction. □
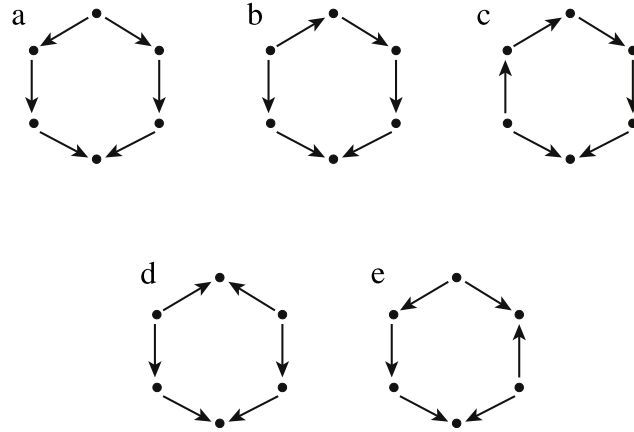
Fig. 8. The forbidden subdigraphs among orientations of cycles of length at most six for the class of $(2, 2)$ digraphs whose phylogeny graphs are chordal.
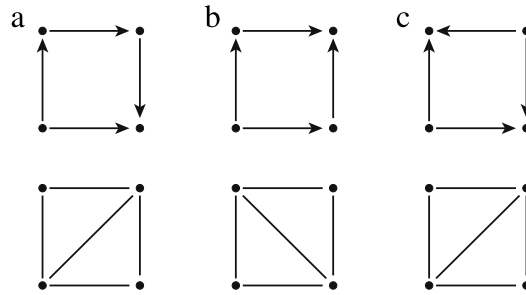

Fig. 9. The non-isomorphic orientations of cycles of length 4 and their corresponding phylogeny graphs which are chordal.
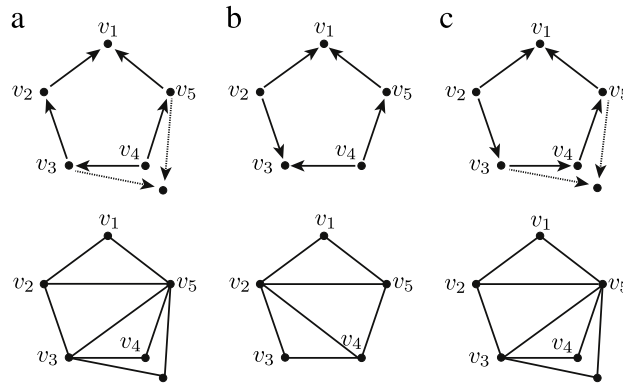

Fig. 10. Acyclic digraphs including the non-isomorphic (acyclic) orientations of the 5-cycle $v_1v_2v_3v_4v_5v_1$ as induced subdigraphs and their chordal phylogeny graphs.

3.4. Holes in the underlying graph and the phylogeny graph of a $(2, 2)$ digraph

As the edges on holes in the underlying graph of a digraph are inherited to its phylogeny graph, one may expect that the phylogeny graph cannot have a hole longer than the ones in the underlying graph. Contrary to this expectation, each hole in the underlying graph of D in Fig. 12(a) has length 4 while the hole $H = v_1v_2v_3v_6v_7v_4v_1$ in its phylogeny graph $P(D)$ in Fig. 12(b) has length 6. However, the phylogeny graph of a $(2, 2)$ digraph lives up to the expectation as long as its underlying graph is chordal. Before we prove it, we derive the following statements to be used in the proof.

Proposition 3.6. *Suppose that the phylogeny graph of a $(2, 2)$ digraph D contains a hole H. If v is a vertex taking care of an edge on H, then v is not on H.*
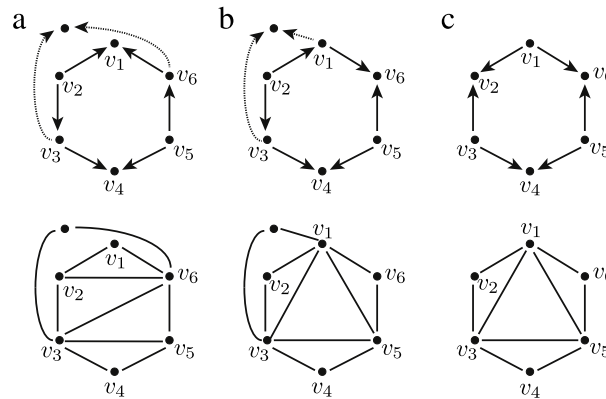
Fig. 11. The non-isomorphic acyclic orientations of the 6-cycle $v_1 v_2 v_3 v_4 v_5 v_6 v_1$ satisfying the property that there exists an acyclic digraph including one of them as an induced subdigraph such that its phylogeny graph is chordal. Each of the bottom graphs is the phylogeny graph of the digraph above it.
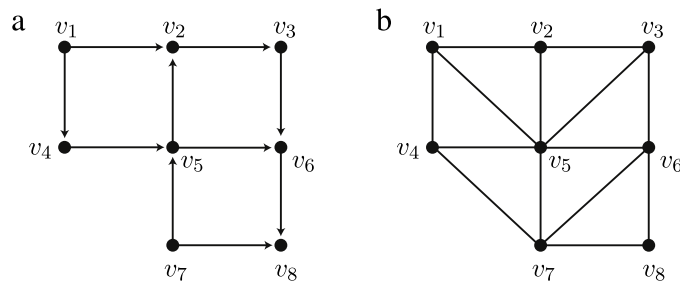

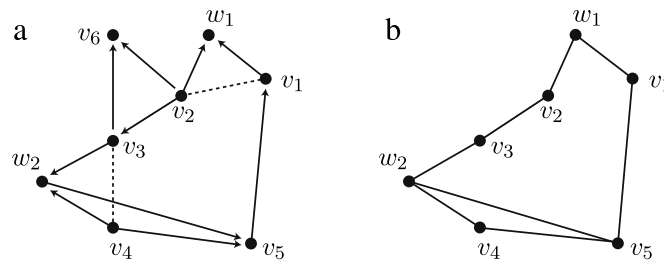**Fig. 12.** (a) A (2, 2) digraph D, (b) The phylogeny graph of D.

Fig. 13. (a) A (2, 2) digraph D having a hole $H = v_1 v_2 v_3 v_4 v_5 v_1$ in $P(D)$. (b) The subgraph obtained from H by $\{w_1, w_2\}$.

Proof. Suppose to the contrary that there exists a vertex v on H taking care of an edge xy on H. Then the edge xv or the edge yv is a chord of H, which is a contradiction. □

Given a (2, 2) digraph D, suppose that the phylogeny graph $P(D)$ has a hole H of length n for $n \geq 4$ and $e_1, e_2, \ldots, e_m$ are the cared edges of H. Let $w_1, w_2, \ldots, w_m$ be vertices taking care of $e_1, e_2, \ldots, e_m$, respectively, and $W = \{w_1, w_2, \ldots, w_m\}$. We call W a *set extending H*. Then $W \subseteq V(D) - V(H)$ by Proposition 3.6. We may obtain a cycle in $U(D)$ from H by replacing each edge e_i with a path of length two from one end of e_i to the other end of e_i with the interior vertex w_i. We call such a cycle the *cycle obtained from H by W*. Let L be the subgraph of $U(D)$ induced by $V(H) \cup W$. We call L the *subgraph of $U(D)$ obtained from H by W*. By definition, the cycle obtained from H by W is a hamiltonian cycle of the subgraph obtained from H by W. For example the graph in Fig. 13(b) is the subgraph obtained from H by $\{w_1, w_2\}$ (see Fig. 13(a) for the corresponding digraph).

Lemma 3.7. *Suppose that the phylogeny graph of a (2, 2) digraph D contains a hole H. If L is the subgraph of the underlying graph $U(D)$ of D obtained from H by a set W extending H, then there is no edge joining two vertices belonging to W in $U(D)$.*
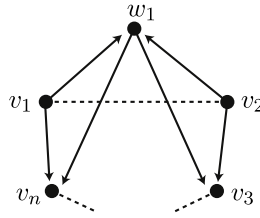
Fig. 14. The arcs of D corresponding to v_1v_n, v_1w_1, v_2v_3, v_2w_1, w_1v_n, w_1v_3 in $U(D)$.

Proof. Suppose to the contrary that there exist vertices w_1, $w_2 \in W$ that are adjacent in $U(D)$. Without loss of generality, we may assume that $(w_1, w_2) \in A(D)$. Since L is obtained from H by W, w_2 is a vertex taking care of an edge on H. Thus w_2 has two in-neighbors in D which belong to $V(H)$. Since $(w_1, w_2) \in A(D)$, w_2 has indegree at least three, which is a contradiction. $\square$

Lemma 3.8. *Let H be a hole in the phylogeny graph $P(D)$ of a $(2, 2)$ digraph D, and L be the subgraph of the underlying graph $U(D)$ of D obtained from H by a set W extending H. If L is chordal and $xy \in E(H)$ is an edge in $P(D)$ taken care of by $w \in W$, then there exists a vertex z on H such that z is adjacent to both x and w in L.*

Proof. Let C be the cycle obtained from H by W. Then C is a hamiltonian cycle of L. Since L is obtained from H by W containing w, the edge xw is on C. Since L is chordal, there exists a vertex $z \in V(C) - \{x, w\}$ that is adjacent to both x and w in L by Proposition 2.3. Since L is a subgraph of $U(D)$, w and z are adjacent in $U(D)$. By Lemma 3.7, $z \notin W$ and so z belongs to H, which completes the proof. $\square$

Theorem 3.9. *Let H be a hole of the phylogeny graph $P(D)$ of a $(2, 2)$ digraph D. Then there is a hole $\phi(H)$ in the underlying graph $U(D)$ of D such that*

- *$\phi(H)$ equals H if H is a hole in $U(D)$;*
- *$\phi(H)$ is a hole in $U(D)$ only containing vertices in the subgraph obtained from H by a set extending H otherwise.*

Moreover, if the holes of $P(D)$ are mutually vertex-disjoint and no hole in $U(D)$ has length 4 or 6, then there exists an injective map from the set of holes in $P(D)$ to the set of holes in $U(D)$.

Proof. Let $H = v_1v_2 \cdots v_nv_1$ be a hole in $P(D)$. If no edge of H is taken care of, then H is a hole in $U(D)$ and we let $\phi(H) = H$.

Suppose that at least one edge of H is taken care of. Let $e_1, \ldots, e_m$ be the cared edges of H and let w_i be a vertex taking care of e_i for each $i = 1, \ldots, m$. Let L be the subgraph of $U(D)$ obtained from H by $\{w_1, \ldots, w_m\}$.

To reach a contradiction, suppose that L is chordal. Without loss of generality, we may assume that v_1 and v_2 are the end vertices of e_1. By Lemma 3.8, v_1 and w_1 have a common neighbor in L, say z, on H. Since v_1v_2 is a cared edge, $z \neq v_2$. Since H is a hole in $P(D)$, the edge v_1z cannot be a chord of H and so $z = v_n$. Therefore v_nw_1 and v_nv_1 are edges in L. Since L is a subgraph of $U(D)$, v_nw_1 and v_nv_1 are edges in $U(D)$. Since w_1 has v_1 and v_2 as in-neighbors and D is a $(2, 2)$ digraph, v_n must be an out-neighbor of w_1 in D. Since D is acyclic, v_n is an out-neighbor of v_1. Similarly, v_3 is a common out-neighbor of v_2, w_1 in D (see Fig. 14).

Now we consider the graph L^* obtained by deleting v_1 and v_2 from L. We note that $H^* := w_1v_3v_4 \cdots v_nw_1$ is a hole in $P(D)$ and that L^* is the subgraph of $U(D)$ obtained from H^* by $\{w_2, w_3, \ldots, w_n\}$. By applying Proposition 2.1 to L, we can conclude that the subgraph L^* is chordal. For an edge w_1v_3 on L^*, there exists a vertex $z^* \in V(H^*)$ that is adjacent to both w_1 and v_3 in $U(D)$ by Lemma 3.8. Since z^*w_1 and z^*v_3 are edges of L^* and L^* is a subgraph of $U(D)$, they are edges in $U(D)$. Then $z^* = v_n$ since $z^* \in N_{U(D)}(w_1) = \{v_1, v_2, v_3, v_n\}$ and $z^* \notin \{v_1, v_2, v_3\}$. Therefore the edge z^*v_3 in $U(D)$ is now v_nv_3. Thus either (v_3, v_n) or (v_n, v_3) is an arc in D. However $v_n \notin \{v_2, w_1\} = N_D^-(v_3)$ and $v_3 \notin \{v_1, w_1\} = N_D^-(v_n)$, which is a contradiction. Thus L contains a hole, that is, there exists a hole all of whose vertices are in L. We take one of such holes as $\phi(H)$. Then ϕ defines a map from the set of holes in $P(D)$ to the set of holes in $U(D)$.

To show the second part of the theorem, we assume that the holes of $P(D)$ are mutually vertex-disjoint and no hole in $U(D)$ has length 4 or 6. We take ϕ^* whose image has the largest size among the maps that can be obtained by the way described in the previous argument. Then we take two distinct holes H_1 and H_2 in $P(D)$. By the hypothesis, H_1 and H_2 are vertex-disjoint. Let L_1 and L_2 be the subgraphs of $U(D)$ obtained from H_1 and H_2 by sets W_1 and W_2 extending H_1 and H_2, respectively. By the above argument, $\phi^*(H_1)$ (resp. $\phi^*(H_2)$) is a hole whose vertices are on L_1 (resp. L_2). Suppose $\phi^*(H_1) = \phi^*(H_2) =: H^*$. If H^* contains a vertex neither on H_1 nor on H_2, then it is a vertex taking care of an edge on H_1 and an edge on H_2 at the same time, which contradicts the hypothesis that D is a $(2, 2)$ digraph. Thus H^* consists of vertices on H_1 or H_2. Suppose that H^* contains two consecutive vertices both of which are on H_1 (resp. H_2). Then they are adjacent by an arc a in D. Since they belong to H_1 (resp. H_2), they are vertices taking care of edges on H_2 (resp. H_1) since H_1 and H_2 are vertex-disjoint. Therefore each of them has two in-neighbors on H_2 (resp. H_1). However, due to a, one of them must have indegree at least three in D and we reach a contradiction. Thus the vertices on H^* belong alternatively to H_1 and H_2 and so H^* is a hole of an even length.

By the hypothesis, H^* has length at least 8. Then, by applying Theorem 3.4 to the subgraph of $P(D)$ induced by $V(H^*)$, there exists a hole in $P(D)$ all of whose vertices are on H^*. Since each vertex on H^* belongs to H_1 or H_2, by the hypothesis that the holes in $P(D)$ are vertex-disjoint, this hole is either H_1 or H_2. Without loss of generality, we may assume that it is H_1. Let C be the cycle obtained from H_2 by W_2. Then

$$V(H_1) \subseteq V(H^*) \subseteq V(C) = V(L_2) = V(H_2) \cup W_2.$$

Now, since H_1 and H_2 are vertex-disjoint, $V(H_1) \subseteq W_2$ and so each vertex on H_1 takes care of an edge on H_2. Moreover, since $|V(H_2) \cap V(H^*)|$ is the same as the number of edges on H_1, each edge on H_1 is taken care of by a vertex in $V(H_2) \cap V(H^*)$.

Take a vertex u on H_1. Then u is adjacent to two vertices, say z_1 and z_2, on H_1. As we claimed that each vertex both on H^* and H_2 takes care of each edge of H_1, there are out-neighbors v and w of u such that v and w are on $V(H^*) \cap V(H_2)$ and (z_1, v), (u, v), (z_2, w), (u, w) are arcs in D. Since v and w are caring vertices, they are not adjacent in $U(D)$ by Lemma 3.7. Since u belongs to H^*, u is a vertex taking care of an edge xy on H_2. Then x and y are in-neighbors of u in D. We take the (v, w)-section P of C that does not contain u. Then x and y, which are consecutive on H_2, do not belong to P since xuy is a section of C by the way in which C is obtained. Thus, by the degree restriction on D, u is not adjacent to any vertex on P other than v and w in $U(D)$. We take a shortest (v, w)-path P^* in the subgraph of $U(D)$ induced by the vertex set of P. Since v and w are not adjacent in $U(D)$, uP^*u is a hole in $U(D)$. Suppose $uP^*u = H^*$. Then the vertex z_1, which is on H^*, is on P^*. If x and v are adjacent in $U(D)$, then, since v has already two in-neighbors z_1 and u, the edge xv in $U(D)$ has orientation (v, x) to form a directed cycle $u \to v \to x \to u$, which is impossible. Thus x and v are not adjacent in $U(D)$. Hence, for the (u, v)-section Q of C containing x, Qu contains a hole H^{**} in $U(D)$ containing u and x since u is not adjacent to any vertex other than x and v on Q. Since x is not on P, it is not on uP^*u. Then, since $uP^*u = H^*$, x does not belong to H^* and therefore H^{**} is distinct from H^*. Therefore we can conclude that uP^*u or H^{**} is a hole different from H^* containing u in $U(D)$. We change $\phi^*(H_2)$ into uP^*u if $H^* \neq uP^*u$ and into H^{**} otherwise. By the degree restriction on D, u belongs to only L_1 and L_2. Thus the new $\phi^*(H_2)$ does not equal any of ϕ^*-values of other holes in $P(D)$ and we have obtained a map from the set of holes in $P(D)$ to a set of holes in $U(D)$ with image larger than ϕ^*, which contradicts the choice of ϕ^*. $\square$

Remark 3.10. The "Moreover" part of Theorem 3.9 does not hold in general. For the digraph D given in Fig. 13, the holes in $P(D)$ are $v_1v_2v_3v_4v_5v_1$ and $v_1v_2v_3w_2v_5v_1$ while the hole in $U(D)$ is $v_1w_1v_2v_3w_2v_5v_1$.

Corollary 3.11. *Let D be a $(2, 2)$ digraph. Suppose that the holes of $P(D)$ are mutually vertex-disjoint and no holes in $U(D)$ has length 4 or 6. Then the number of holes in $U(D)$ is greater than or equal to that of holes in $P(D)$.*

Corollary 3.12. *Let D be a $(2, 2)$ digraph. If $U(D)$ is chordal, then $P(D)$ is also chordal.*

4. Concluding remarks

In this paper, we obtained the complete list of orientations of cycles that are forbidden subdigraphs for the class of $(2, 2)$ digraphs whose phylogeny graphs are chordal. Furthermore, we showed that if the holes of the phylogeny graph $P(D)$ of a $(2, 2)$ digraph D are mutually vertex-disjoint and no holes in the underlying graph $U(D)$ of D has length 4 or 6, then the number of holes in $U(D)$ is greater than or equal to that of holes in $P(D)$, which implies the following: If the underlying graph of a $(2, 2)$ digraph D is chordal, then the phylogeny graph of D is also chordal. It would be interesting to give a good necessary and sufficient condition for $(2, 2)$ digraphs having chordal phylogeny graphs.

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