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The double competition hypergraph of a digraph

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ABSTRACT

In this paper, we introduce the notion of the double competition hypergraph of a digraph. We give characterizations of the double competition hypergraphs of arbitrary digraphs, loopless digraphs, reflexive digraphs, and acyclic digraphs in terms of hyperedge labelings of the hypergraphs.

1. Introduction

A *digraph* D is a pair $(V(D), A(D))$ of a set $V(D)$ of *vertices* and a set $A(D)$ of ordered pairs of vertices, called *arcs*. An arc of the form (v, v) is called a *loop*. For a vertex x in a digraph D, we denote the *out-neighborhood* of x in D by $N_D^+(x)$ and the *in-neighborhood* of x in D by $N_D^-(x)$, i.e., $N_D^+(x) := \{v \in V(D) \mid (x, v) \in A(D)\}$ and $N_D^-(x) := \{v \in V(D) \mid (v, x) \in A(D)\}$. A *graph* G is a pair $(V(G), E(G))$ of a set $V(G)$ of *vertices* and a set $E(G)$ of unordered pairs of vertices, called *edges*. The *competition graph* of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if $N_D^+(x) \cap N_D^+(y) \neq \emptyset$. In other words, the competition graph of a digraph is the intersection graph of the family of the out-neighborhoods of the vertices of the digraph (see [4] for intersection graphs). This notion was introduced by J.E. Cohen [1] in 1968 in connection with a problem in ecology, and several variants and generalizations of competition graphs have been studied.

In 1987, D.D. Scott [8] introduced the notion of double competition graphs as a variant of the notion of competition graphs. The *double competition graph* (or the *competition-common enemy graph* or the *CCE graph*) of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if both $N_D^+(x) \cap N_D^+(y) \neq \emptyset$ and $N_D^-(x) \cap N_D^-(y) \neq \emptyset$ hold. See [2,3,7,13] for recent results on double competition graphs.

A *hypergraph* $\mathcal{H}$ is a pair $(V(\mathcal{H}), E(\mathcal{H}))$ of a set $V(\mathcal{H})$ of *vertices* and a set $E(\mathcal{H})$ of nonempty subsets of $V(\mathcal{H})$, called *hyperedges*. A hyperedge of the form $\{v\}$ is called a *loop*. We assume that all hypergraphs in this paper have no loops. So all the hyperedges of a hypergraph contain at least two vertices. The notion of competition hypergraphs was introduced by M. Sonntag and H.-M. Teichert [9] in 2004 as another variant of the notion of competition graphs. The *competition hypergraph* of a digraph D is the hypergraph which has the same vertex set as D and in which $e \subseteq V(D)$ is a hyperedge if and only if $|e| \geq 2$ and there exists a vertex v of D such that $e = N_D^-(v)$. See [5,6,10–12] for recent results on competition hypergraphs.

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In this paper, we introduce the notion of the double competition hypergraph of a digraph, and we give characterizations of the double competition hypergraphs of arbitrary digraphs, loopless digraphs, reflexive digraphs, and acyclic digraphs in terms of hyperedge labelings of the hypergraphs.

2. Main results

We define the double competition hypergraph of a digraph as follows.

Definition 1. Let D be a digraph. The *double competition hypergraph* of D is the hypergraph which has the same vertex set as D and in which $e \subseteq V(D)$ is a hyperedge if and only if $|e| \geq 2$ and there exist vertices u and v of D such that $e = N_D^+(u) \cap N_D^-(v)$. □

For a positive integer n, let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

Theorem 1. *Let $\mathcal{H}$ be a hypergraph with n vertices. Then, $\mathcal{H}$ is the double competition hypergraph of an arbitrary digraph if and only if there exist an ordering $(v_1, \ldots, v_n)$ of the vertices of $\mathcal{H}$ and an injective labeling $L : E(\mathcal{H}) \to [n] \times [n]$ of the hyperedge set of $\mathcal{H}$ such that the following condition holds:*

($\star$) *for any $i, j \in [n]$, if $|X_i \cap Y_j| \geq 2$, then $X_i \cap Y_j = e_{ij}$,*

where e_{ij} denotes the hyperedge e such that $L(e) = (i, j)$ if such e exists, and $e_{ij} = \emptyset$ otherwise, and X_i and Y_j are the sets defined by

$$X_i := \left(\bigcup_{p \in [n]} e_{ip} \right) \cup \{v_b \mid v_i \in e_{ab}\ (a, b \in [n])\}, \tag{1}$$

$$Y_j := \left(\bigcup_{q \in [n]} e_{qj} \right) \cup \{v_a \mid v_j \in e_{ab}\ (a, b \in [n])\}. \tag{2}$$

Proof. First, we show the only-if part. Let $\mathcal{H}$ be the double competition hypergraph of an arbitrary digraph D. Let $(v_1, \ldots, v_n)$ be an ordering of the vertices of D. For $i, j \in [n]$, we define

$$e_{ij} := N_D^+(v_i) \cap N_D^-(v_j). \tag{3}$$

Then e_{ij} is a hyperedge of $\mathcal{H}$ if $|e_{ij}| \geq 2$. Let E^* be the family of e_{ij}'s whose sizes are at least two, i.e.,

$$E^* := \{e_{ij} \mid i, j \in [n], |e_{ij}| \geq 2\}. \tag{4}$$

By the definition of a double competition hypergraph, E^* is the hyperedge set of $\mathcal{H}$. Let $L : E(\mathcal{H}) \to [n] \times [n]$ be the map defined by $L(e_{ij}) = (i, j)$. Then L is injective.

We show that condition ($\star$) holds. Fix i and j in $[n]$ and let X_i and Y_j be sets as defined in (1) and (2). Let

$$V_{i*} := \bigcup_{p \in [n]} e_{ip}, \qquad W_i^+ := \{v_b \mid v_i \in e_{ab}\ (a, b \in [n])\},$$
$$V_{*j} := \bigcup_{q \in [n]} e_{qj}, \qquad W_j^- := \{v_a \mid v_j \in e_{ab}\ (a, b \in [n])\},$$

for convenience. Then $X_i = V_{i*} \cup W_i^+$ and $Y_j = V_{*j} \cup W_j^-$. Since $e_{ij} \subseteq X_i$ and $e_{ij} \subseteq Y_j$, it holds that $e_{ij} \subseteq X_i \cap Y_j$. Now we assume that $|X_i \cap Y_j| \geq 2$ and take any vertex $v_k \in X_i \cap Y_j$. There are four cases for v_k arising from the definitions of X_i and Y_j as follows: (i) $v_k \in V_{i*} \cap V_{*j}$; (ii) $v_k \in V_{i*} \cap W_j^-$; (iii) $v_k \in W_i^+ \cap V_{*j}$; (iv) $v_k \in W_i^+ \cap W_j^-$. To show $X_i \cap Y_j \subseteq e_{ij}$, we will check that $v_k \in e_{ij}$ for each case. Consider the case (i). Since $v_k \in V_{i*}$, there exists $p \in [n]$ such that $v_k \in e_{ip}$. Since $v_k \in V_{*j}$, there exists $q \in [n]$ such that $v_k \in e_{qj}$. By (3), we have $v_k \in e_{ip} \cap e_{qj} = N_D^+(v_i) \cap N_D^-(v_p) \cap N_D^+(v_q) \cap N_D^-(v_j) \subseteq N_D^+(v_i) \cap N_D^-(v_j) = e_{ij}$. Consider the case (ii). Since $v_k \in V_{i*}$, there exists $p \in [n]$ such that $v_k \in e_{ip}$. Since $v_k \in W_j^-$, there exists $b \in [n]$ such that $v_j \in e_{kb}$. By (3), we have $v_k \in e_{ip} = N_D^+(v_i) \cap N_D^-(v_p) \subseteq N_D^+(v_i)$ and $v_j \in e_{kb} = N_D^+(v_k) \cap N_D^-(v_b) \subseteq N_D^+(v_k)$, i.e., $v_k \in N_D^-(v_j)$. Therefore $v_k \in N_D^+(v_i) \cap N_D^-(v_j) = e_{ij}$. Consider the case (iii). Since $v_k \in W_i^+$, there exists $a \in [n]$ such that $v_i \in e_{ak}$. Since $v_k \in V_{*j}$, there exists $q \in [n]$ such that $v_k \in e_{qj}$. By (3), we have $v_i \in e_{ak} = N_D^+(v_a) \cap N_D^-(v_k) \subseteq N_D^-(v_k)$, i.e., $v_k \in N_D^+(v_i)$, and $v_k \in e_{qj} = N_D^+(v_q) \cap N_D^-(v_j) \subseteq N_D^-(v_j)$. Therefore $v_k \in N_D^+(v_i) \cap N_D^-(v_j) = e_{ij}$. Consider the case (iv). Since $v_k \in W_i^+$, there exists $a \in [n]$ such that $v_i \in e_{ak}$. Since $v_k \in W_j^-$, there exists $b \in [n]$ such that $v_j \in e_{kb}$. By (3), we have $v_i \in e_{ak} = N_D^+(v_a) \cap N_D^-(v_k) \subseteq N_D^-(v_k)$, i.e., $v_k \in N_D^+(v_i)$, and $v_j \in e_{kb} = N_D^+(v_k) \cap N_D^-(v_b) \subseteq N_D^+(v_k)$, i.e., $v_k \in N_D^-(v_j)$. Therefore $v_k \in N_D^+(v_i) \cap N_D^-(v_j) = e_{ij}$. Thus we obtain $X_i \cap Y_j \subseteq e_{ij}$, and so $X_i \cap Y_j = e_{ij}$. Hence condition ($\star$) holds.

Next, we show the if part. Let $\mathcal{H}$ be a hypergraph with n vertices, and suppose that there exist an ordering $(v_1, \dots, v_n)$ of the vertices of $\mathcal{H}$ and an injective labeling $L : E(\mathcal{H}) \to [n] \times [n]$ such that condition $(\star)$ holds. We define a digraph D by

$$V(D) := V(\mathcal{H}) \quad \text{and} \quad A(D) := \bigcup_{i,j\in[n]} \left(\bigcup_{v_k\in e_{ij}} \{(v_i, v_k), (v_k, v_j)\} \right). \tag{5}$$

Then, it holds that $e_{ij} = N_D^+(v_i) \cap N_D^-(v_j)$ for any $i, j \in [n]$. Thus it follows from the definition of a double competition hypergraph and the assumption that hypergraphs have no loops that the double competition hypergraph of D is the hypergraph $\mathcal{H}$. □

A digraph D is said to be *loopless* if D has no loops, i.e., $(v, v) \notin A(D)$ holds for any $v \in V(D)$.

Theorem 2. *Let $\mathcal{H}$ be a hypergraph with n vertices. Then, $\mathcal{H}$ is the double competition hypergraph of a loopless digraph if and only if there exist an ordering $(v_1, \dots, v_n)$ of the vertices of $\mathcal{H}$ and an injective labeling $L : E(\mathcal{H}) \to [n] \times [n]$ of the hyperedge set of $\mathcal{H}$ such that the following conditions hold:*

$(\star)$ *for any $i, j \in [n]$, if $|X_i \cap Y_j| \geq 2$, then $X_i \cap Y_j = e_{ij}$,*
(L) *for any $i, j \in [n]$, $v_i \notin e_{ij}$ and $v_j \notin e_{ij}$,*

where e_{ij} denotes the hyperedge e such that $L(e) = (i, j)$ if such e exists, and $e_{ij} = \emptyset$ otherwise, and X_i and Y_j are the sets defined by (1) and (2).

Proof. First, we show the only-if part. Let $\mathcal{H}$ be the double competition hypergraph of a loopless digraph D. Let $(v_1, \dots, v_n)$ be an ordering of the vertices of D. Let e_{ij} $(i, j \in [n])$ be the sets defined by (3), and let E^* be the set defined by (4). Then e_{ij} is a hyperedge of $\mathcal{H}$ if $|e_{ij}| \geq 2$, and E^* is the hyperedge set of $\mathcal{H}$. Let $L : E(\mathcal{H}) \to [n] \times [n]$ be the map defined by $L(e_{ij}) = (i, j)$. Then L is injective. Moreover, we can show, as in the proof of Theorem 1, that condition $(\star)$ holds. Now we show that condition (L) holds. Take any vertex $v_k \in e_{ij}$. Then $v_k \in N_D^+(v_i) \cap N_D^-(v_j)$, i.e., $(v_i, v_k), (v_k, v_j) \in A(D)$. Since D is loopless, we have $v_i \neq v_k$ and $v_j \neq v_k$. Therefore it follows that $v_i \notin e_{ij}$ and $v_j \notin e_{ij}$. Thus condition (L) holds.

Next, we show the if part. Let $\mathcal{H}$ be a hypergraph with n vertices, and suppose that there exists an ordering $(v_1, \dots, v_n)$ of the vertices of $\mathcal{H}$ and an injective labeling $L : E(\mathcal{H}) \to [n] \times [n]$ such that conditions $(\star)$ and (L) hold. We define a digraph D by (5). By condition (L), it follows from the definition of D that $(v_i, v_i) \notin A(D)$ for any $i \in [n]$. Therefore D is a loopless digraph. Moreover we can show, as in the proof of Theorem 1, that $\mathcal{H}$ is the double competition hypergraph of D. □

A digraph D is said to be *reflexive* if all the vertices of D have loops, i.e., $(v, v) \in A(D)$ holds for any $v \in V(D)$.

Theorem 3. *Let $\mathcal{H}$ be a hypergraph with n vertices. Then, $\mathcal{H}$ is the double competition hypergraph of a reflexive digraph if and only if there exist an ordering $(v_1, \dots, v_n)$ of the vertices of $\mathcal{H}$ and an injective labeling $L : E(\mathcal{H}) \to [n] \times [n]$ of the hyperedge set of $\mathcal{H}$ such that the following conditions hold:*

$(\star)$ *for any $i, j \in [n]$, if $|X_i \cap Y_j| \geq 2$, then $X_i \cap Y_j = e_{ij}$,*
(R) *for any $i \in [n]$, $v_i \in \left(\bigcup_{p\in[n]} e_{ip}\right) \cup \left(\bigcup_{p\in[n]} e_{pi}\right)$,*

where e_{ij} denotes the hyperedge e such that $L(e) = (i, j)$ if such e exists, and $e_{ij} = \emptyset$ otherwise, and X_i and Y_j are the sets defined by (1) and (2).

Proof. First, we show the only-if part. Let $\mathcal{H}$ be the double competition hypergraph of a reflexive digraph D. Let $(v_1, \dots, v_n)$ be an ordering of the vertices of D. Let e_{ij} $(i, j \in [n])$ be the sets defined by (3), and let E^* be the family defined by (4). Then e_{ij} is a hyperedge of $\mathcal{H}$ if $|e_{ij}| \geq 2$, and E^* is the hyperedge set of $\mathcal{H}$. Let $L : E(\mathcal{H}) \to [n] \times [n]$ be the map defined by $L(e_{ij}) = (i, j)$. Then L is injective. Moreover, we can show, as in the proof of Theorem 1, that condition $(\star)$ holds. Now we show that condition (R) holds. Since D is reflexive, we have $(v_i, v_i) \in A(D)$ for any $i \in [n]$. Then it follows that $v_i \in N_D^+(v_i) \cap N_D^-(v_i) = e_{ii}$. Therefore $v_i \in \left(\bigcup_{p\in[n]} e_{ip}\right) \cup \left(\bigcup_{p\in[n]} e_{pi}\right)$. Thus condition (R) holds.

Next, we show the if part. Let $\mathcal{H}$ be a hypergraph with n vertices, and suppose that there exist an ordering $(v_1, \dots, v_n)$ of the vertices of $\mathcal{H}$ and an injective labeling $L : E(\mathcal{H}) \to [n] \times [n]$ such that conditions $(\star)$ and (R) hold. We define a digraph D by (5). Fix any $i \in [n]$. By condition (R), there exists $p \in [n]$ such that $v_i \in e_{ip}$ or $v_i \in e_{pi}$. Then it follows from the definition of D that $(v_i, v_i) \in A(D)$. Therefore D is a reflexive digraph. Moreover we can show, as in the proof of Theorem 1, that $\mathcal{H}$ is the double competition hypergraph of D. □

A digraph D is said to be *acyclic* if D has no directed cycles. An ordering $(v_1, \dots, v_n)$ of the vertices of a digraph D, where n is the number of vertices of D, is called an *acyclic ordering* of D if $(v_i, v_j) \in A(D)$ implies $i < j$. It is well known that a digraph D is acyclic if and only if D has an acyclic ordering.

Theorem 4. *Let $\mathcal{H}$ be a hypergraph with n vertices. Then, $\mathcal{H}$ is the double competition hypergraph of an acyclic digraph if and only if there exist an ordering $(v_1, \dots, v_n)$ of the vertices of $\mathcal{H}$ and an injective labeling $L : E(\mathcal{H}) \to [n] \times [n]$ of the hyperedge set of $\mathcal{H}$ such that the following conditions hold:*

($\star$) *for any* $i, j \in [n]$, *if* $|X_i \cap Y_j| \geq 2$, *then* $X_i \cap Y_j = e_{ij}$,
(A) *for any* $i, j, k \in [n]$, $v_k \in e_{ij}$ *implies* $i < k < j$,

where e_{ij} *denotes the hyperedge* e *such that* $L(e) = (i, j)$ *if such* e *exists, and* $e_{ij} = \emptyset$ *otherwise, and* X_i *and* Y_j *are the sets defined by* (1) *and* (2).

Proof. First, we show the only-if part. Let $\mathcal{H}$ be the double competition hypergraph of an acyclic digraph D. Let $(v_1, \ldots, v_n)$ be an acyclic ordering of the vertices of D. Let e_{ij} $(i, j \in [n])$ be the sets defined by (3), and let E^* be the family defined by (4). Then e_{ij} is a hyperedge of $\mathcal{H}$ if $|e_{ij}| \geq 2$, and E^* is the hyperedge set of $\mathcal{H}$. Let $L : E(\mathcal{H}) \to [n] \times [n]$ be the map defined by $L(e_{ij}) = (i, j)$. Then L is injective. Moreover, we can show, as in the proof of Theorem 1, that condition ($\star$) holds. Now we show that condition (A) holds. Suppose that $v_k \in e_{ij}$. Then $(v_i, v_k), (v_k, v_j) \in A(D)$. Since $(v_1, \ldots, v_n)$ is an acyclic ordering of D, $(v_i, v_k) \in A(D)$ implies $i < k$ and $(v_k, v_j) \in A(D)$ implies $k < j$ Therefore $i < k < j$. Thus condition (A) holds.

Next, we show the if part. Let $\mathcal{H}$ be a hypergraph with n vertices, and suppose that there exist an ordering $(v_1, \ldots, v_n)$ of the vertices of $\mathcal{H}$ and an injective labeling $L : E(\mathcal{H}) \to [n] \times [n]$ such that conditions ($\star$) and (A) hold. We define a digraph D by (5). By condition (A), it follows from the definition of D that $(v_1, \ldots, v_n)$ is an acyclic ordering of D. Therefore D is an acyclic digraph. Moreover we can show, as in the proof of Theorem 1, that $\mathcal{H}$ is the double competition hypergraph of D. □

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References

[1] J.E. Cohen, Interval graphs and food webs. A finding and a problem, RAND Corporation Document 17696-PR, Santa Monica, California, 1968.
[2] S.-J. Kim, S.-R. Kim, Y. Rho, On CCE graphs of doubly partial orders, Discrete Appl. Math. 155 (2007) 971–978.
[3] J. Lu, Y. Wu, Two minimal forbidden subgraphs for double competition graphs of posets of dimension at most two, Appl. Math. Lett. 22 (2009) 841–845.
[4] T.A. McKee, F.R. McMorris, Topics in Intersection Graph Theory, in: SIAM Monographs on Discrete Mathematics and Applications, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.
[5] B. Park, Y. Sano, On the hypercompetition numbers of hypergraphs, Ars Combin. 100 (2011) 151–159.
[6] Y. Sano, Characterizations of competition multigraphs, Discrete Appl. Math. 157 (2009) 2978–2982.
[7] Y. Sano, The competition-common enemy graphs of digraphs satisfying Conditions $C(p)$ and $C'(p)$, Congr. Numer. 202 (2010) 187–194.
[8] D.D. Scott, The competition-common enemy graph of a digraph, Discrete Appl. Math. 17 (1987) 269–280.
[9] M. Sonntag, H.-M. Teichert, Competition hypergraphs, Discrete Appl. Math. 143 (2004) 324–329.
[10] M. Sonntag, H.-M. Teichert, Competition hypergraphs of digraphs with certain properties. I. Strong connectedness, Discuss. Math. Graph Theory 28 (2008) 5–21.
[11] M. Sonntag, H.-M. Teichert, Competition hypergraphs of digraphs with certain properties. II. Hamiltonicity, Discuss. Math. Graph Theory 28 (2008) 23–34.
[12] M. Sonntag, H.-M. Teichert, Competition hypergraphs of products of digraphs, Graphs Combin. 25 (2009) 611–624.
[13] Y. Wu, J. Lu, Dimension-2 poset competition numbers and dimension-2 poset double competition numbers, Discrete Appl. Math. 158 (2010) 706–717.