Impossible Objects of Your Choice: Designing Any 3D Objects from a 2D Line Drawing

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***Abstract*—There is a class of line drawings, called *impossible objects*, that are perceived as 3D structures but are impossible to completely construct in 3D space. Sugihara [1] proposes a systematic method for creating a type of impossible objects. This method provides a way to judge the existence of possible three-dimensional (3D) coordinate for a line drawing, and a way to compute it, if possible. There are, however, some technical difficulties in using Sugihara's method. Firstly, Sugihara's method requires to introduce a large number of variables for use in a set of equations, which requires some intense labor for a designer of impossible objects when programming. Secondly, in theory, there are an infinite number of possible 3D coordinates for the same line drawing, but Sugihara's method can only determine one of them for a pre-defined set of parameters. In practice, a designer of impossible objects may also wish to arrange the 3D coordinates' undetermined degree of freedom for a given two-dimensional (2D) line drawing. Given these technical issues in Sugihara's method, we propose a new method for explore not just some but *all* 3D coordinates for a 2D line drawing that the designer can use at will. The proposed method requires both a minimal number of variables in its computation, resulting in it being computationally cheap, and less manual programming. Moreover, the proposed method provides a user interface that the designer can use to manually adjust the degree of freedom in the class of constructible impossible objects. This allows the designer to create impossible objects that reflect their tastes.**

***Index Terms*—impossible objects, line drawing, three dimensional (3D) perception, computational geometry**

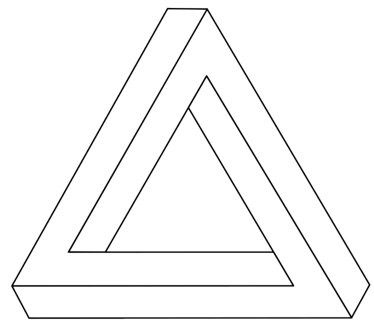

Fig. 1. Penrose triangle

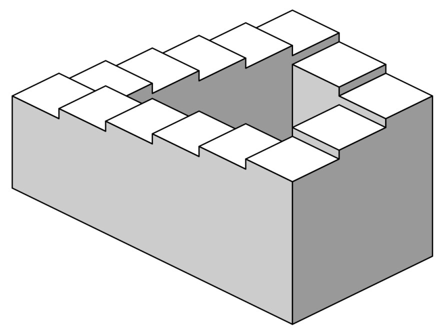

Fig. 2. Penrose staircase

I. Introduction

We can perceive the shape of a three-dimensional (3D) object from a two-dimensional (2D) picture on a flat surface. There is a special class of 2D drawings, called *impossible objects*, that can be perceived as a 3D object locally but cannot be perceived as a 3D object as a whole. Each part of the picture is drawn as a 3D structure that makes sense locally, resulting in it being perceived as 3D, but, as a whole, there are contradictions that cause it to be seen as an impossible 3D structure. Many artists have used impossible objects as motifs in their works, including "Penrose triangle" (Figure 1) and "Penrose stairs" (Figure 2), which were formulated by the mathematician Penrose, who was inspired by the works of Escher.

Although these impossible objects often induce the perception of impossible 3D structures, some of them can be reconstructed as 3D objects. By replacing right angles in a figure with acute or obtuse angles or by changing sections that appear to be continuous so that they appear discontinuous, it is possible to create 3D objects that appear to be impossible objects but only when viewed from a certain perspective. These impossible line drawings, that can be built as 3D objects, are called *impossible objects*.

On the other hand, it is not clearly understood whether these impossible objects people perceive can be created using the coordinates for 3D space. The method for constructing them has been treated as a geometric problem [1] and has also been studied as a model for the 3D perceptual mechanism [2].

For impossible objects, Sugihara devised a method for labeling line drawings that is based on Huffman's vertex dictionary [3][3], in which the feasibility of creating impossible objects is determined by linear programming problems [4]. However,

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this method has some technical and practical issues. Firstly, Sugihara's method requires to introduce an additional set of variables that are more than the minimum requirement to represent a 3D object. These additional variables obscure the essential mathematical structure used for designing a 3D object and results in more manual effort being required to design the impossible object. Secondly, Sugihara's method can be used to obtain one 3D object out of an infinite number of possible 3D objects that can havea 2D line drawing as their projections. Thus, the designer of the 3D object cannot freely adjust and choose a possible configuration of the impossible object.

In this study, we propose a new method of creating a 3D structure that addresses these technical problems. In this proposed method, the only unknown set of variables is the set of 3D coordinates of the vertices, which are the minimal set of variables, so there is no need to introduce unnecessary variables. Specifically, the equations representing the vertices on the same plane are expressed as determinants, and, by using singular value decomposition, the number of vertices N of a 3D object is divided into the $(N-K)$-dimensional constrained subspace and the K-dimension that can be freely designed. Here, the $(N-K)$-dimension is constrained by the given planer projection of the 3D object, and the other K–dimension is the degree of freedom in a 3D figure that keeps the specific 2D line drawing invariant. Therefore, by adjusting these degrees of freedom manually, the designer can draw an object that reflects their tastes, independent of the constraints imposed.

II. Related Works

A. Modeling of Impossible Objects

In their work on impossible Objects, Wu et al. [5] propose an optimization method for modeling and rendering impossible objects. The method is inspired by the ways in which users build physical 3D models to generate impossible objects. When provided with a 3D section of a locally possible object, this method automatically optimizes a 3D model according to the 3D constraints needed to render the impossible object in the desired viewpoint.

Although this method allows for the construction of a 3D object from a 2D image of an impossible object, it can only create a discontinuous object by cutting some part that are perceptually connected on the 2D drawing. Our method, in contrast, has no such limitations and can design a 3D impossible object without any additional cutting of perceptually connected components.

B. Converting a 2D Image to 3D

There is research on creating 3D images. Hu et al. [6] propose a method for generating a 3D scene from a single image that can reproduce depth and move the viewpoint back and forth. Synthesizing a 3D structure from a single 2D-RGB image is difficult because it requires an understanding of the 3D structure of the 2D image and texture mapping to generate both visible and invisible regions in the relevant viewpoint. In this study, a convolutional neural network was used to understand the 3D structure by attaching planar mesh in a way that matched the estimated depth. The texture was mapped to the shape of the mesh, allowing the texture to be placed naturally, even when the viewpoint changed.

This technique is important in its own right, but it requires shading or textural structure of a surface in order to reconstruct a 3D structure. Thus, without these visual cues, this type of method would not work optimally for the abstract line drawings.

C. Gestalt Psychology

Gestalt psychology is a field of psychology that focuses on the wholeness and structure of human psychology rather than on a collection of parts or elements [7]. *Gestalt* refers to a coherent structure that has a sense of wholeness. According to Gestalt psychology, when people look at something, they tend to grasp the whole from the beginning; they understand the figure as simply and coherently as possible rather than recognizing the whole by accumulating an understanding of the parts. An impossible object is a figure that looks 3D because each part of the picture is drawn as a 3D structure makes sense locally but contains contradictions as a whole. As object perception, including of impossible figures, is one of central subjects of Gestault psychology, our proposed method may serve as a tool to create and explore such local and global nature of the human perception.

III. Theoretical Formulation

In this chapter, we briefly describe Sugihara's method [4], which can be used to create impossible objects from a 2D line drawing. Next, the proposed method for solving its technical problems is described. In this paper, we focus on line drawings, which are defined as figures that are drawn only with line segments. We do not deal with color shades or tones; only with the presence or absence of points (black or white). The same method is employed in [4].

The line drawing used in this chapter is in the xy-plane, and the line drawing is an orthogonal projection of an object placed in 3D space that is coordinated by $(x, y, z) \in \mathbb{R}^3$. The set of nodes in the line drawing is denoted by U, and the nodes are numbered serially from 1 to $|\mathrm{U}|$; for example, $\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_{|\mathrm{U}|}$. $\boldsymbol{u}_i = (x_i, y_i) \in \mathbb{R}^2$ are the coordinates of the ith node $\boldsymbol{u_i}$ on the line drawing. As the line drawing is provided, x_i andy_i are known real values. $\boldsymbol{v}_i = (x_i, y_i, z_i)$ denotes the 3D coordinate of the i^{th} vertex of an object that may have this node in its projection. Since the values of x_i and y_i are fixed, only z_i is inderterminate.

A. Sugihara's Method

This subsection introduces Sugihara's method [4].

1) Interpretation as a Planar-Panel Scene: The set of faces in the line drawing is denoted by F; the faces are numbered serially from 1 to $|\mathrm{F}|$, and they are denoted by $f_1, f_2, \cdots, f_{|\mathrm{F}|}$. The j^{th} plane f_j is a planar polygon. If there is an assumption that the viewpoint is generally positioned, in that the lines and faces in a 3D space are the lines and faces on a 2D view, respectively, a point (x, y, z) on a plane f_j satisfies the following equation:

$$a_j x + b_j y + z + c_j = 0, \tag{1}$$

where $a_j, b_j, c_j \in \mathbb{R}$ $(1 \le j \le |\mathrm{F}|)$ are unknown constants specific to f_j. In this equation, we assume that there is no plane parallel to the z-axis, so the coefficient of z can be set to 1. When the i^{th} vertex V_i is on the j-face F_j, the equation is:

$$a_j x_i + b_j y_i + z_i + c_j = 0. \tag{2}$$

The same linear equation can be obtained for all pairs of vertices and the faces on which they are located. Putting them all together, we have

$$\mathbf{A}\boldsymbol{w} = \mathbf{0}, \tag{3}$$

where $\mathbf{A}$ is a constant matrix and $\boldsymbol{w}$ is a vector of the unknowns:

$$\boldsymbol{w} = (z_1, \cdots, z_{|\mathrm{U}|}, a_1, b_1, c_1, \cdots, a_{|\mathrm{F}|}, b_{|\mathrm{F}|}, c_{|\mathrm{F}|})^{\mathrm{T}}. \tag{4}$$

Next, the constraints imposed by the occlusion between faces of the object are considered. Using the vector of unknowns $\boldsymbol{w}$ outlined in Equation (4), the linear inequality without equality

$$\mathbf{B}\boldsymbol{w} > \mathbf{0} \tag{5}$$

or linear inequality with an equal sign

$$\mathbf{C}\boldsymbol{w} \ge \mathbf{0} \tag{6}$$

are used in the form employed by Sugihara, which he derived from the labels of line drawings that were based on Huffman's vertex dictionary [3].

2) Computation as a Linear Programming Problem: Whether a line drawing represents an object can be determined by the presence or absence of solutions to equations (3), (5), and (6). In order to adapt this problem to the standard form of a linear programming problem, the inequality (5) can be written without the equality sign for a positive number e, as in:

$$\mathbf{B}\boldsymbol{w} \ge (e \cdots e)^{\mathrm{T}}. \tag{7}$$

In this case, the existence of a solution that satisfies equations (3), (5), and (6) is equivalent to the existence of a solution that satisfies equations (3), (6), and (7). Equations (3), (6), and (7) are constraints on the linear programming problem, and the unknowns a_i, b_i, c_i, z_j can be used to make it a standard problem in linear programming, where the slack variables are:

$$a_j^+, a_j^-, b_j^+, b_j^-, c_j^+, c_j^-, z_i^+, z_i^- \ge 0 \tag{8}$$

and the set is:

$$\begin{aligned} a_j = \quad & a_j^+ - a_j^-, \quad b_j = \quad b_j^+ - b_j^-, \\ c_j = \quad & c_j^+ - c_j^-, \quad z_i = \quad z_i^+ - z_i^-. \end{aligned} \tag{9}$$

The constraint obtained by such a transformation is denoted as:

$$\sum_{j=1}^{n} p_{ij} s_j = q_i \ge 0, \quad i = 1, 2, \cdots, m \tag{10}$$

$$s_j \ge 0, j = 1, 2, \cdots, n \tag{11}$$

In this case, it will be rewritten in standard form, wherein the number of equality constraints and the number of variables are represented by m and n, respectively. The existence of a solution that satisfies this constraint is equivalent to the original system of restoring equations and perspective inequalities having a solution.

Lastly, we introduce the new artificial variables:

$$s_{n+1}, s_{n+2}, \cdots, s_{n+m}$$

to this constraint equation to solve the linear programming problem.

$$\text{Minimize:} \quad w \equiv \sum_{j=1}^{m} s_{n+j} \tag{12}$$

$$\text{Subject to:} \quad \sum_{j=1}^{n} p_{ij} s_j + s_{n+i} = q_i, \quad i = 1, 2, \cdots, n \tag{13}$$

$$s_j \ge 0, \quad j = 1, 2, \cdots, n + m \tag{14}$$

If we solve this linear programming problem and obtain the optimal solution value for w_* minimizing (12), if $w_* > 0$, then the original linear programming problem has no feasible solution. In other words, there is no object with a line drawing as its basis. On the other hand, if $w_* = 0$, there are several solutions, with all of them constituting a feasible solution to the original linear programming problem. Thus, the object can be recovered. In this way, we can create an artificial linear programming problem and determine the possibility of recovering the object from the value of the optimal solution.

B. Proposed Method

Our method reduces the amount of computational procedures in Sugihara's method by focusing only on the vertex-face relationship; some of the vertices labelled as the set U, V are on a plane in F. However, we do not introduce unknown coefficients a_j, b_j, c_j for each face f_j as in (2) but represent it in an equivalent form without these coefficients.

If and only if there are four points

$$v_1 = (x_1, y_1, z_1), \ldots, v_4 = (x_4, y_4, z_4) \in \mathbb{R}^3$$

in three dimensions that are on the same plane, we have the following identity:

$$\Delta_{1,2,3,4} := \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0. \tag{15}$$

Expanding the above determinant by a cofactor series results in:

$$\Delta_{1,2,3,4} = \quad z_1\Delta_{2,3,4} - z_2\Delta_{1,3,4} + z_3\Delta_{1,2,4} - z_4\Delta_{1,2,3}, \tag{16}$$

where

$$\Delta_{i,j,k} := \begin{vmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ 1 & 1 & 1 \end{vmatrix}.$$

For each quadraple (i, j, k, l) sharing the same plane, Equation (16) becomes:

$$(\Delta_{j,k,l}, -\Delta_{i,k,l}, \Delta_{i,j,l}, -\Delta_{i,j,k})(z_i, z_j, z_k, z_l)^{\mathrm{T}} = 0. \tag{17}$$

Therefore, for each plane i $(i = 1, \cdots, m)$ with no three colinear points, we provide the $(|f_i| - 3)$ equations of (17). The number of equations denote the following:

$$\hat{m} := \sum_{i=1}^{m}(|f_i| - 3). \tag{18}$$

Including all the $\hat{m}$ equations, the plane equations as a whole can be written as:

$$\mathbf{Q}\boldsymbol{z} = 0, \tag{19}$$

where $\mathbf{Q} \in \mathbb{R}^{\hat{m}\times n}$ is a constant matrix and $\boldsymbol{z} \in \mathbb{R}^n$ is $\boldsymbol{z} = (z_1, z_2, \ldots, z_n)$. For this $\mathbf{Q}$, there exists an orthogonal matrix $\mathbf{G} \in \mathbb{R}^{\hat{m}\times\hat{m}}, \mathbf{H} \in \mathbb{R}^{n\times n}$ that satisfies the following singular value decomposition:

$$\mathbf{Q} = \mathbf{G}\mathbf{S}\mathbf{H}^{\mathrm{T}}. \tag{20}$$

Here, $\mathbf{S} \in \mathbb{R}^{\hat{m}\times n}$ is denoted as:

$$\mathbf{S} = \begin{pmatrix} \sum & \mathbf{0} \end{pmatrix}, \quad \sum = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_r), \tag{21}$$

where $\sigma_k (k = 1, \cdots, r)$ is the singular value of $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0, \quad r \leq \mathrm{rank}\mathbf{Q}$. This results in the following:

$$\mathbf{G}\mathbf{S}\mathbf{H}^{\mathrm{T}}\boldsymbol{z} = 0, \tag{22}$$

where each of $\mathbf{G}, \mathbf{H}$ is a unitary matrix. Let $\hat{\mathbf{H}} \in \mathbb{R}^{n\times(n-\hat{m})}$ be the submatrix from $\hat{m} + 1$ to n columns of $\mathbf{H}$, that spans the kernel of $\mathbf{Q}$. Using the vector $\boldsymbol{r} \in \boldsymbol{R}^{n-\hat{m}}$ with arbitrary values, the solution $\boldsymbol{z} = (z_1, z_2, \ldots, z_n) \in \boldsymbol{R}^n$ of (19) can be expressed as follows:

$$\boldsymbol{z} = \hat{\mathbf{H}}\boldsymbol{r}. \tag{23}$$

By changing the value of $\boldsymbol{r}$, we can change the 3D structure of the object while maintaining the projection on the constrained plane.

The algorithm of the proposed method is summarized in TABLE I.

TABLE I
THE ALGORITHM FOR DESIGNING THE IMPOSSIBLE OBJECT

Algorithm 1 Designing the impossible object

Input: $\mathbf{U} = (x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$
Output: $\boldsymbol{z} = (z_1, z_2, \cdots, z_n)$
1: Compute the matrix $\mathbf{Q}$ in (19) for $\mathbf{U}$.
2: Compute the matrix $\hat{\mathbf{H}}$ in (23) by singular decomposition of $\mathbf{Q}$.
3: Create a graphical user interface to manipulate the parameter $\mathbf{r}$ and visualize the 3D structure of V, with $\mathbf{z} = \hat{\mathbf{H}}\mathbf{r}$ for the parameter $\mathbf{r}$, as specified by the user's choice.
4: Output the $\mathbf{z}$ of the user's choice.

IV. DEMONSTRATION

Here, we discuss two cases in which the proposed method was used. Firstly, we consider an "impossible triangle," the line drawing of which is shown in Figure 3. Figure 4 shows an impossible triangle, the faces of which are colored to specify which face is occluded. This figure consists of 15 vertices, 19 line segments, and five faces. For this input, the matrix $\mathbf{Q} \in \mathbb{R}^{33\times 15}$ has rank$Q = 9$ and $\hat{\mathbf{H}} \in \mathbb{R}^{6\times 15}$. Thus, there are six degrees of freedom for the designer to manually adjust $\mathbf{r} \in \mathbb{R}^6$. To demonstrate the impact of a designer's choice of parameter, a 3D instantiation of the impossible triangle with a different parameter is provided, as shown in Figure 5 and Figure 6. In terms of the parameter controlling the depth of each vertex, the brown face is over the pink face, resulting in a "possible" triangle.

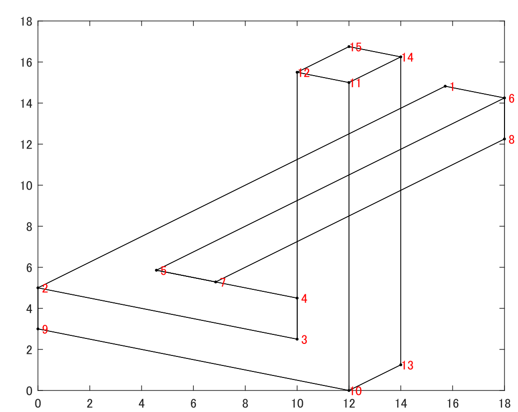

Fig. 3. Input line drawing of the impossible triangle.

Next, we demonstrate the application of the proposed method in another example: the "impossible cube," the line drawing of which is shown in Figure 7 and the colored version of which is shown in Figure 8. The impossible cube consists of 38 vertices, 54 line segments, and 15 faces. For this input, the matrix $\mathbf{Q} \in \mathbb{R}^{222\times 38}$ has rank$Q = 27$ and $\hat{\mathbf{H}} \in \mathbb{R}^{11\times 38}$. Thus, there are 11 degrees of freedom for the designer to manually adjust; $\mathbf{r} \in \mathbb{R}^{11}$. To demonstrate the impact of a designer's choice of parameter, a 3D instantiation of the

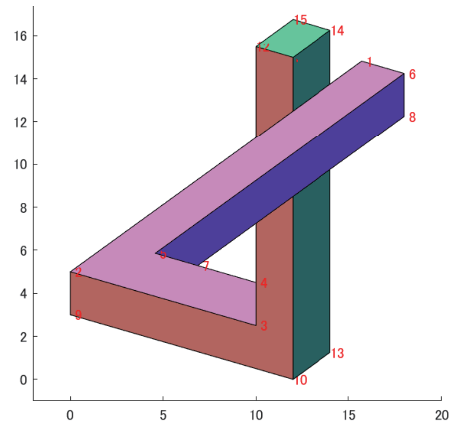

Fig. 4. Line drawing of the impossible triangle with its faces in color.

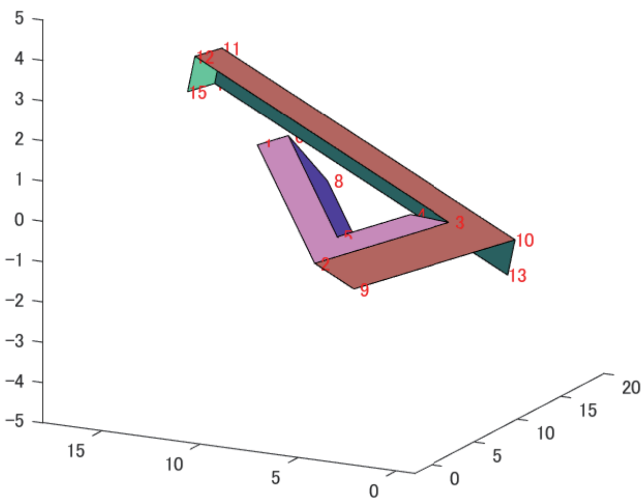

Fig. 5. A 3D impossible triangle.

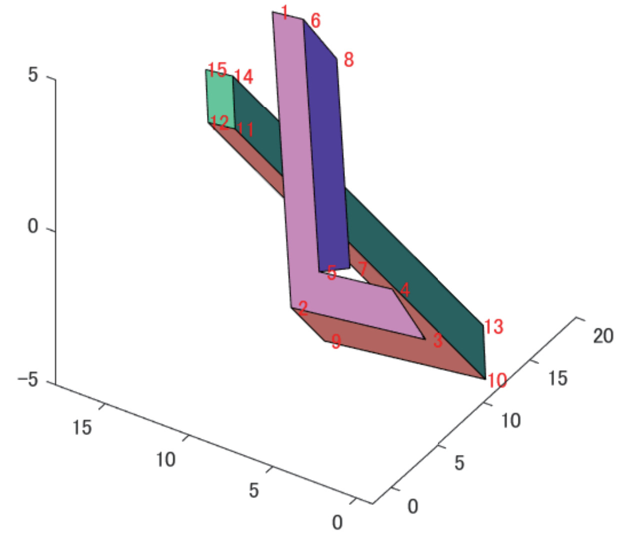

Fig. 6. The 3D impossible triangle with a different parameter to that shown in Figure 5.

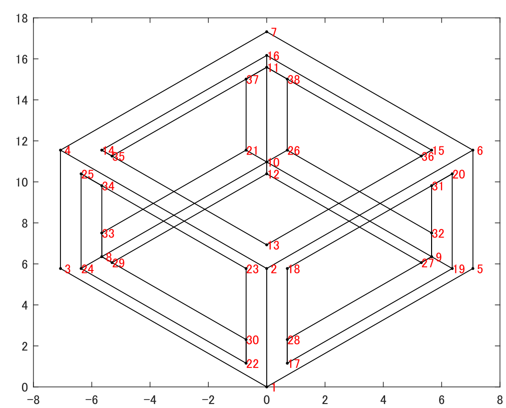

Fig. 7. Input line drawing of the impossible cube.

impossible triangle is shown in Figure 9, and an example of 3D instantiation of the impossible cube with a different parameter is shown in Figure 10.

Note that our method does not require unnecessary cutting (discontinuation) of the 3D object, unlike the study previously discussed [5]. The connected faces in the line drawing (Figure 7) are also connected in 3D example (Figure 5 and Figure 6).

V. Discussion

In this study,motivated by Sugihara's work, we propose a minimally reduced method or designing impossible objects [4]. This method introduces a minimal number of variables for the depth coordinate of each vertex and provides a manually controllable set of parameters of impossible objects from which the designer can choose. This amount of control offers the possibility of designing some figural pattern of the same 3D object from another view, and it also makes it easier to control a single 3D object that is in a certain context *of the designer's choice*. Thus, the proposed method is not just another method for building an impossible object, but it is also a new computational tool that allows the further exploration of 3D planer-faced objects.

With an unnecessary cut on the 3D structure, such as the previous method [5], introduces multiple points in 3D, which are projected onto the same point on the constrained plane. Our method holds both 3D object and its 2D projection have the same number of points, line segments, and planes. Due to this difference, our method would give a more robust planer view against the change of view point than the previous one.

One issue that needs to be addressed in future studies is the choice of coordinate systems for the kernel space of $\hat{\mathbf{Q}}$, spanned by the bases $\hat{\mathbf{H}}$. One variable in the parameter $\hat{\mathbf{r}}$ that

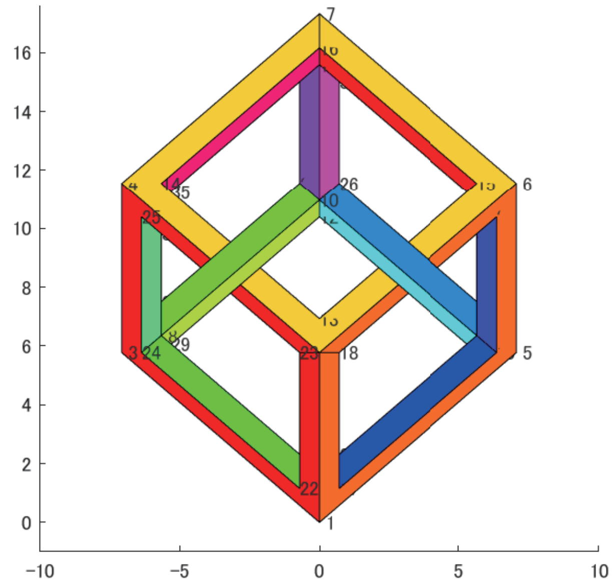

Fig. 8. Line drawing of the impossible cube with its faces colored.

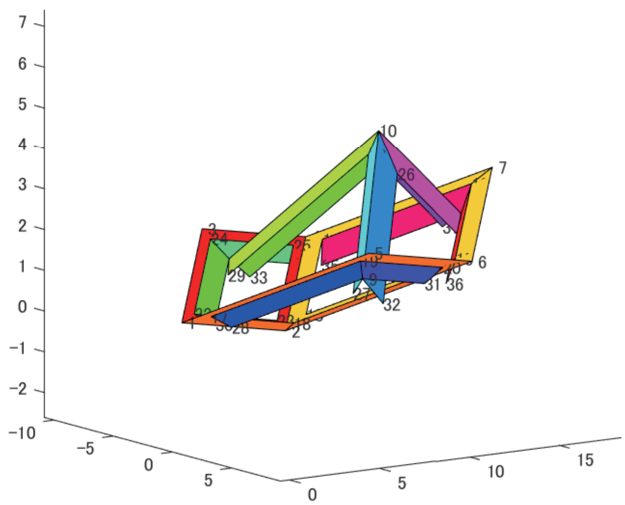

Fig. 9. A 3D impossible cube.

moves all the planes of a 3D object while designing is a "bad" choice in this coordinate system. In contrast, a "good" choice in this coordinate system results in simpler manual adjustment of the parameter, in that one variable in the parameter $\hat{\mathbf{r}}$ moves only one or a few of the impossible object's planes. It is crucial to select a good coordinate system for a complex object, such as an impossible cube with a large number of degrees of freedom, as shown in Figure 8. If the coordinate system is not natural, the manual designing efforts increase exponentially according to the degrees of freedom.

One of the challenges when choosing the coordinate system results from each line drawing (as the input) possibly having different degrees of freedom and a different structure that is dependent on the variables. A good user interface for

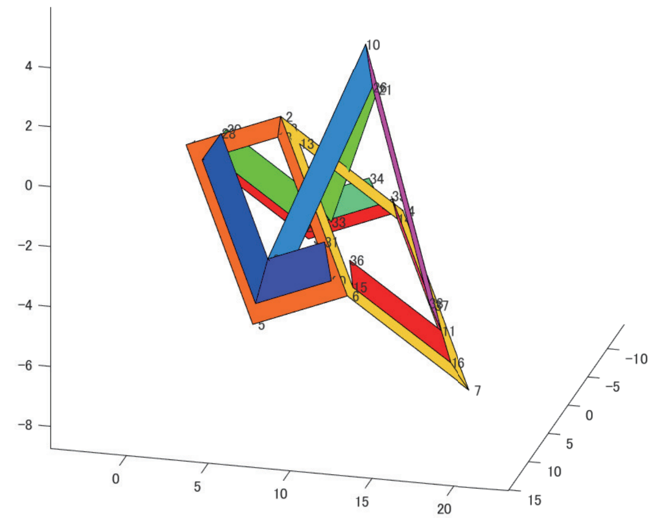

Fig. 10. The 3D impossible cube with a different parameter to that shown in Figure 9.

manipulating 3D objects would provide a good understanding of its 3D structure. Thus, it is necessary to understand humans perceive and understand 3D objects in order to allow for the automatic construction of a custom-made natural interface for each 3D object.

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