Stochastic thermodynamics of a non-equilibrium Sherrington-Kirkpatrick model

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Stochastic thermodynamics unveils the exchange of energy and matter in physical, chemical, and biological processes with the environment. Furthermore, it elucidates the information-theoretic bounds of different thermodynamics quantities. However, when system's elements are numerous, it becomes challenging to characterize nonequilibrium states due to the expansion of the state space. Conceived as a model of magnetic materials, the disordered Ising or Sherrington-Kirkpatrick model has been a standard tool to study interacting systems such as neural networks in the thermodynamic limit, revealing rich repertoires of equilibrium dynamics, including the celebrated spin-glass phases and transitions between them. Nevertheless, researchers have not yet theoretically underpinned the nonequilibrium properties of spin kinetics caused by asymmetric, heterogeneous connectivities of the Ising model with large system size.

Here, we study the kinetics of asymmetric Sherrington-Kirkpatrick systems as a prototypical model of nonlinear and nonequilibrium processes. We apply a path integral approach to calculate exactly a generating functional over the system's trajectories capturing the system's sufficient statistics as well as quantities related with its nonequilibrium thermodynamics. As a result, we derive the exact solutions of the order parameters (means and correlations) of the system, the conditional entropy of the dynamics, and the entropy flow of the system (equivalent to the entropy production in steady state conditions) in the limit of infinitely large networks. Unlike the replica method, the path integral approach for fully asymmetric networks gives the exact solutions in thermodynamic limit without additional ansatzes such as the analytic continuation and replica symmetry breaking.

As expected, the order parameters reveal that the kinetic model exhibits order-disordered nonequilibrium phase transitions analogous to the paramagnetic-ferromagnetic phase transitions in the equilibrium systems but no dynamics akin to the spin-glass phases (which vanish due to coupling asymmetry). In addition, we show that the entropy production is maximized near the nonequilibrium phase transition points, being the first derivative discontinuous at phase transitions (Fig. 1C, dashed line). However, the entropy production can be more prominent outside the critical regime, especially in the disordered systems with low entropy rates, i.e., if the connections are heterogeneous and strong enough to make the dynamics disordered but highly deterministic (Fig. 1C, top right).

Our results indicate that a non-smooth change of the entropy flow or entropy production can be a useful indicator for nonequilibrium phase transitions. At the same time, our results suggest that, even if the system shows large entropy flow/production, it does not necessarily indicate the system is near a phase transition. Instead, a combination of the order parameters, entropy rate and entropy production yield a more precise picture of the behaviour of complex, disordered systems and its phase transitions.

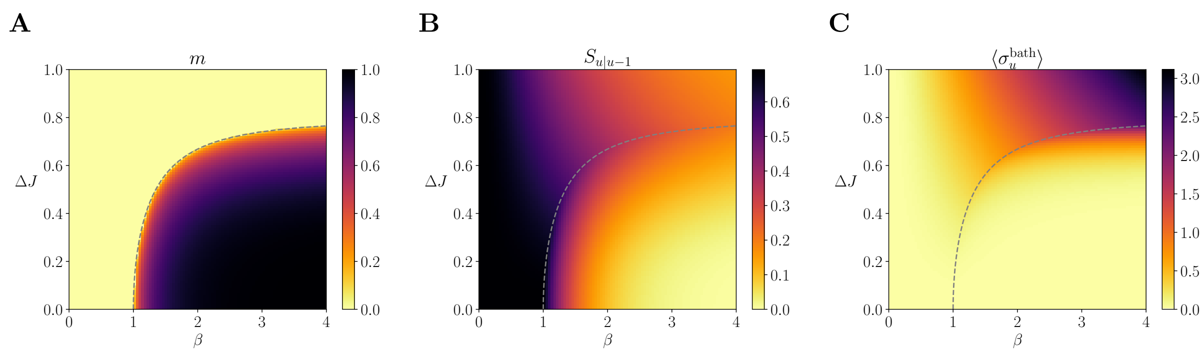

Figure 1: **Analytical results of the behaviour of the asymmetric, kinetic SK model**. Analytical solution of a system with asymmetric couplings following a distribution $J_{ij} \sim \mathcal{N}(\frac{1}{N}, \frac{\Delta J^2}{N})$ and no internal fields, showing the mean magnetization (**A**), conditional entropy (**B**) and entropy production (**C**).

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