Lower decker sets and triple points for surface-knots

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Surface-knots

- 1957 R. H. Fox and J. W. Milnor gave an example of a 2-knot.
- 1965 E. C. Zeeman introduced a construction method of a 2-knot called an m-**twist spinning**.
- 1982 A. Kawauchi, T. Shibuya and S. Suzuki described a surface-knot with a **normal form**.
- 1980's Roseman proposed geometric approach to describe surface in 4-space and introduced elementary deformations called **Roseman moves**.
- 1990's with the development of algebraic structure for this area such as racks and quandles, geometric approaches have been used and developed.
- 1999 J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford and M. Saito applied **quandle co-homology** to knots and surface-knots.

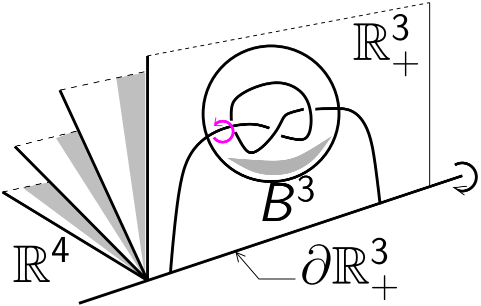
Surface-knots

- 2002 S. Satoh and A. Shima determined triple point numbers for 2-twist and 3-twist spun trefoils.
- 2005 E. Hatakenaka gave a lower bound of triple point number for 2-twist spun $(2,5)$-torus knot.

In this talk we discuss about the number of non-degenerate triple points of a surface diagram.

Zeeman's twist spinning

Let B^3 be a 3–ball in $\mathbb{R}^3_+$ such that $\partial B^3 \cap T(K)$ is the pair of antipodal points of ∂B^3.
An m-**twist-spun knot** obtained from K is defined by rotating the tangle $B^3 \cap T(K)$ about the axis through the antipodal points m times while $\mathbb{R}^3_+$ spins. We denote this 2-knot by $T_m(K)$.



Theorem (Zeeman, 1965)

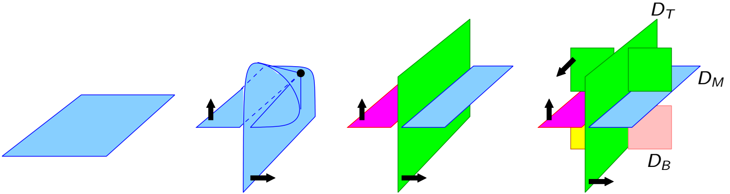
Every m-twist spun knot $T_m(K)$ obtained from K is fibred ($m \geq 1$); the fibre is the one-punctured m-fold branched covering space of S^3 along K.

Corollary (Zeeman, 1965)

For any knot K, 1-twist spun knot obtained from K is trivial.

Surface Diagrams

A **surface-knot** is a connected oriented closed surface embedded in 4-space. Let $F \subset \mathbb{R}^4$ be a surface-knot. Let $\pi : \mathbb{R}^4 \to \mathbb{R}^3$; $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3)$, be the orthogonal projection. A **surface diagram** of F is a union of the following local diagrams.

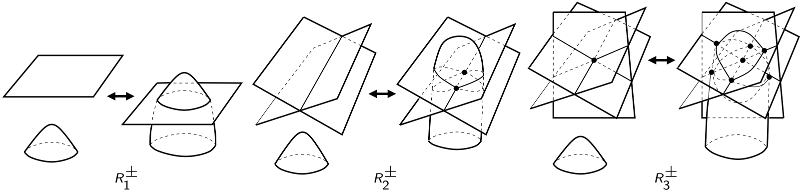


Branch point

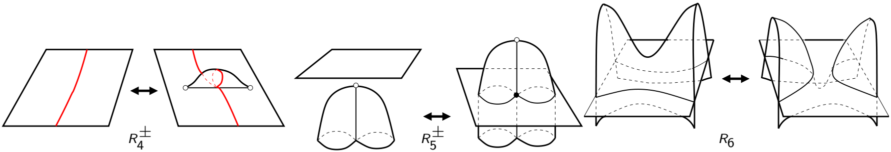
Triple point

Roseman moves

Two surface diagrams are equivalent if they are projected image of the same type of a surface-knot. Two equivalent surface diagrams are modified from one to the other by a **finite sequence of local moves** called Roseman moves.



Roseman moves



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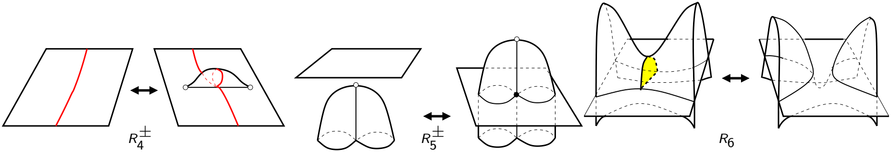

Triple point numbers

For a surface-knot F, the minimal number of triple points for all possible surface diagrams is called the **triple point number** of F denoted by $t(F)$. A surface diagram D_F of F with $t(F)$ triple points is called a **t-minimal surface diagram**.

Theorem (T. Y. 2005)

Let K be the $(2, k)$-torus knot. Then the following holds.

$$t(T_m(K)) \leq m(k-1), (m > 1).$$

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Diagrams of Twist Spun Trefoil

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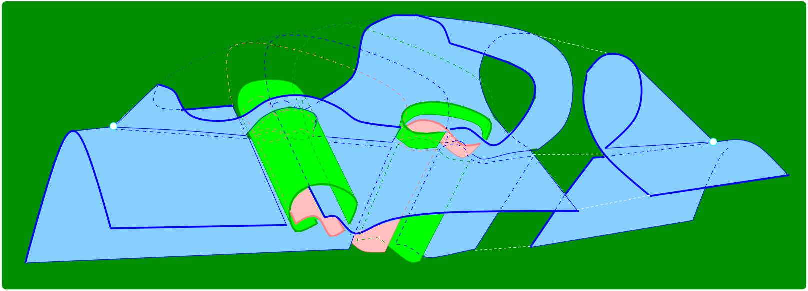

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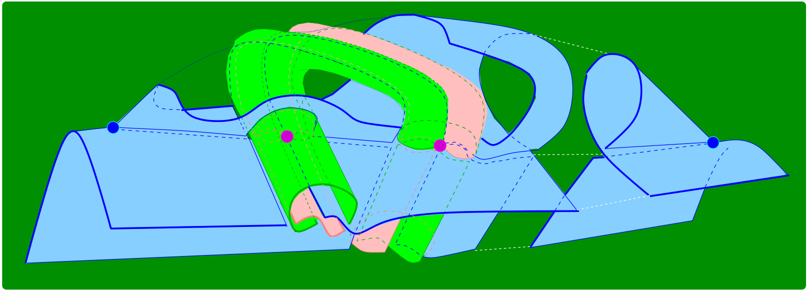

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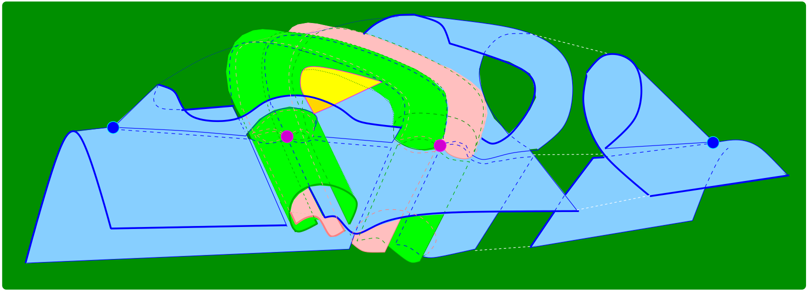

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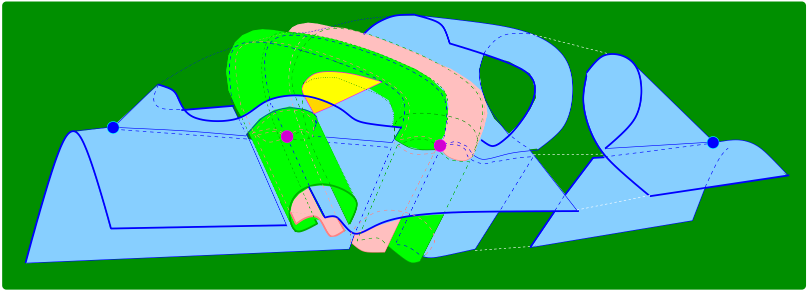

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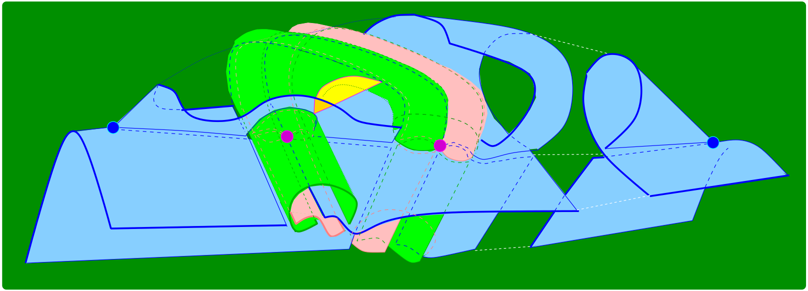

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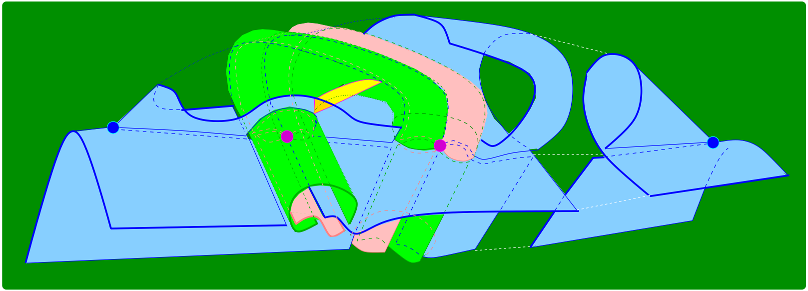

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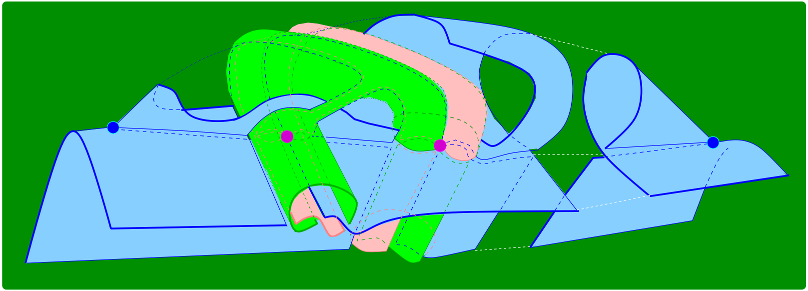

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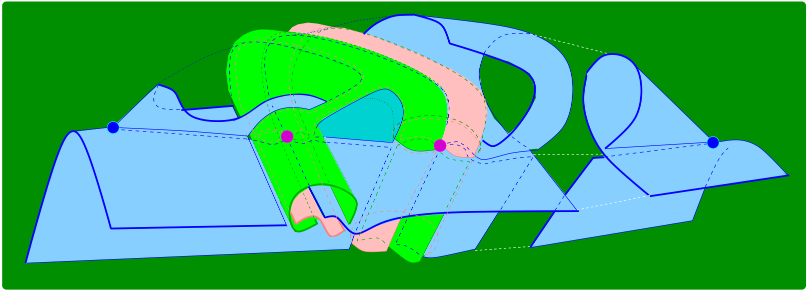

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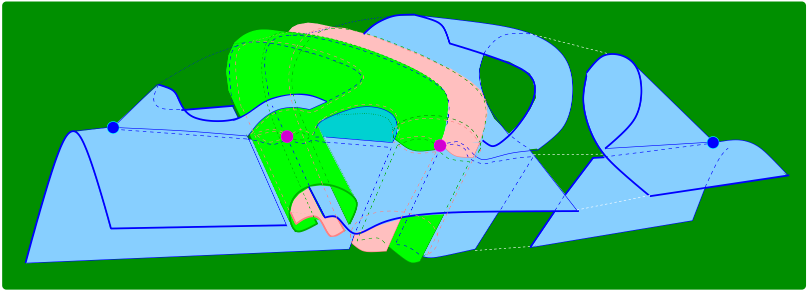

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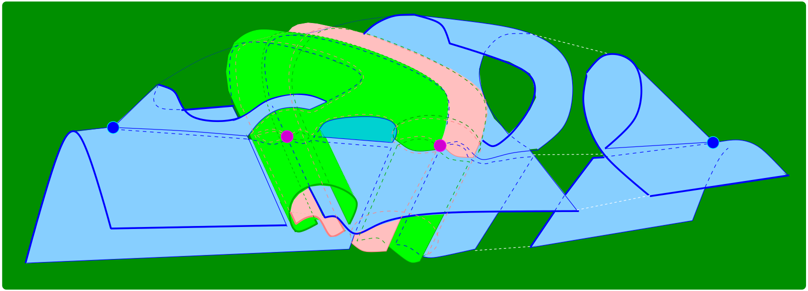

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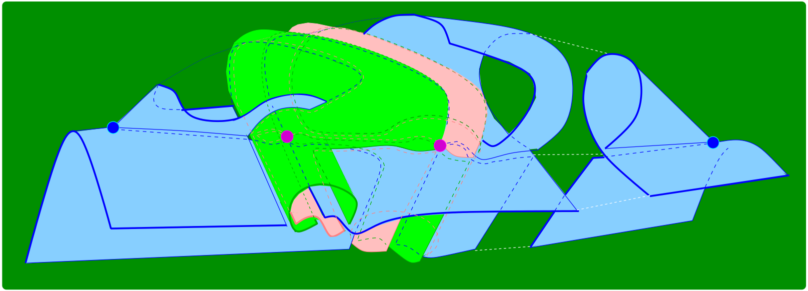

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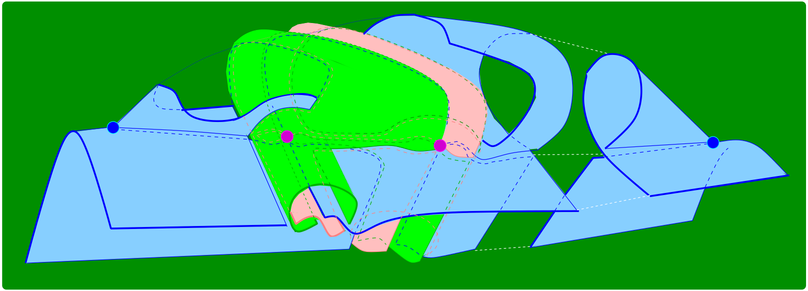

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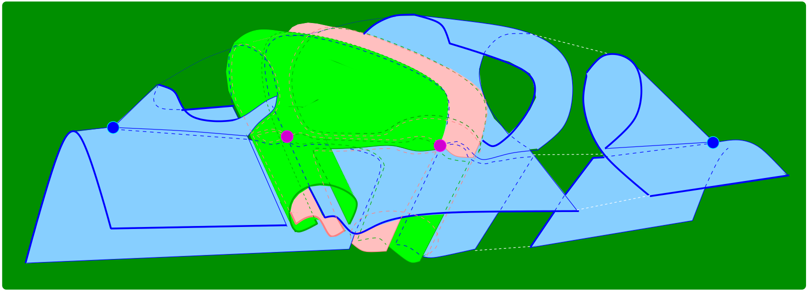

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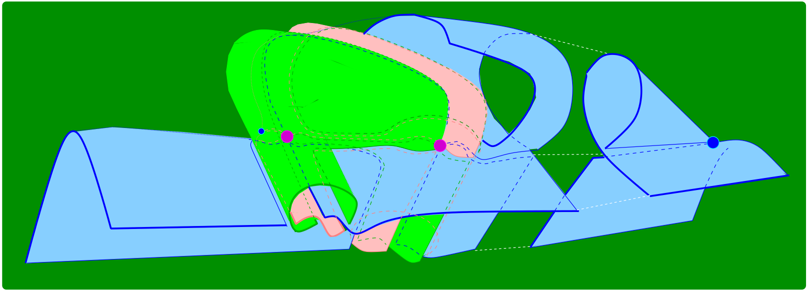

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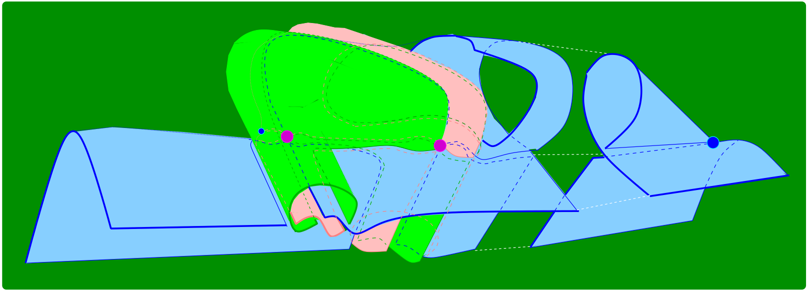

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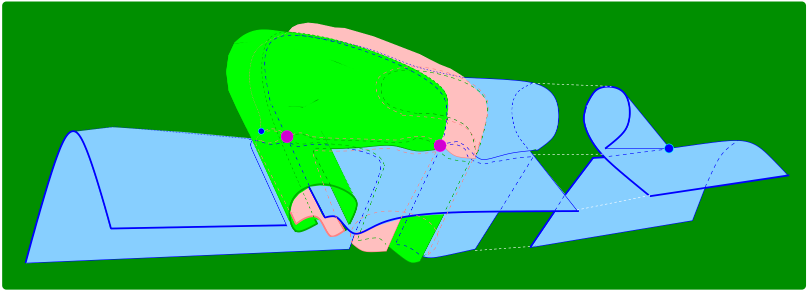

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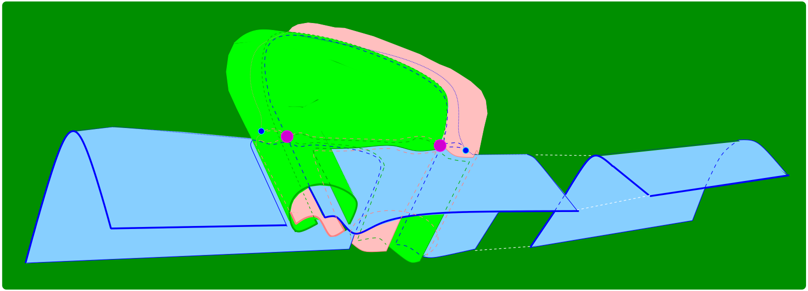

Lower Bounds of Triple Point Numbers

Theorem (Satoh-Shima 2002, 2004)

Let K be a trefoil knot. Let $T_m(K)$ be m-twist-spinning of K. Then $t(T_2(K)) = 4, \ t(T_3(K)) = 6$.

Theorem (S. Satoh 2005)

For every 2-knot F with $t(F) \neq 0$, $4 \leq t(F)$.

Theorem (E. Hatakenaka (2004))

For a 2-twist spun $(2, 5)$-torus knot F, $6 \leq t(F)$.

$$6 \leq t(F) \leq 8.$$

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Quandles

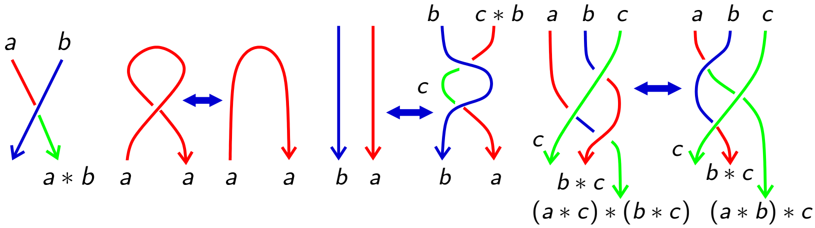
In 1980s to 1990s concepts of **racks** and **quandles** were introduced by many people, Joyce (1982), Matveev (1988), Brieskorn (1988), Fenn and Rourke (1992).

- (Co)homology theory for racks and quandles were introduced by Fenn, Rouuke and Sanderson (1995) and
- the state-sum invariants for knots and surface-knots were defined by Carter, Jelsovsky, Kamada, Langford, Saito (1999).

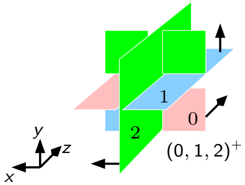
Quandles

A **quandle** X is a non-empty set with a binary operation $(a, b) \mapsto a * b$ such that

1. For any $a \in X$, $a * a = a$,
2. For any $a, b \in X$, there is a unique $c \in X$ such that $c * b = a$.
3. For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.



Example



The **dihedral quandle** $(X, *)$ of order $n > 0$ denoted by R_n is a quandle $X = \{0, \ldots, n-1\}$ with the binary operation
$(i, j) \mapsto 2j - i \ (\mathrm{mod} n)$.
The triple point in the left diagram is coloured by R_3; $(0, 1, 2)$ and the orientation is determined by orientation normals to D_T, D_M, D_B respectively.

Quandle Chain Complex

Let $C_n(X)$ $(n \geq 1)$ be a free abelian group generated by n-tuples $(x_1, \dots, x_n) \in X^n$. Let $C_n^D(X)$ be a sub group of $C_n(X)$ generated by $(x_1, \dots, x_n)$ such that $x_i = x_j$ for some $1 \leq i, j, \leq n$ and $(|i - j| = 1)$. We denote the quotient group $C_n(X)/C_n^D(X)$ by $C_n^Q(X)$.

Quandle Cocylce

We fix the colouring $\mathcal{C}$ on the surface diagram. Let A be an Abelian group. A mapping $\theta : C_3(X) \to A$ is a quandle 3 cocycle if for any $p, q, r, s \in X$, the following holds.

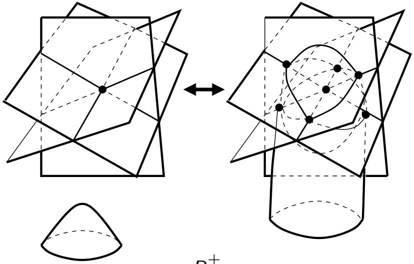
1. $\theta(p, p, r) = \theta(p, q, q) = 0$
2. $\theta(q, r, s) \cdot \theta(p * q, r, s)^{-1} \cdot \theta(p, q, s)^{-1} \cdot$
   $\theta(p * r, q * r, s) \cdot \theta(p, q, r) \cdot \theta(p * s, q * s, r * s)^{-1} = 1$

Let τ be a triple point in D_F coloured by the quandle X. Let $A = \langle t \rangle$. Define $B(\tau, \mathcal{C}) = \theta(\tau)^{\epsilon(\tau)}$. Then we define the following

$$\Phi_\theta(F) = \sum_{\mathcal{C}} \prod_{\tau} B(\tau, \mathcal{C}) \in \mathbb{Z}\langle t \rangle$$

This is a surface-knot invariant called a **quandle cocycle invariant**.

Roseman move R_3^+ create six triple point around a triple point (x, y, z).

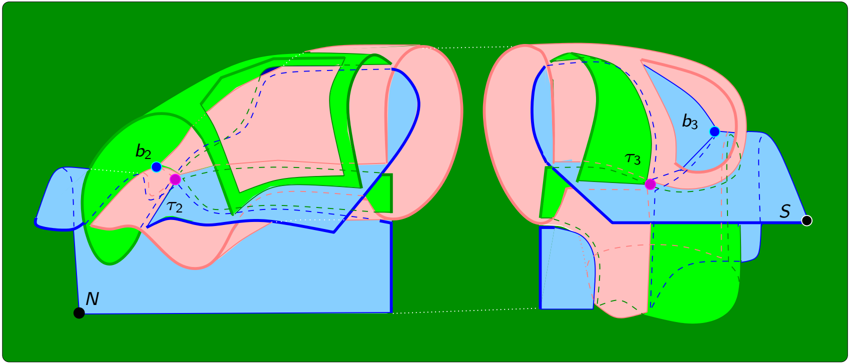


$R_3^{\pm}$

Suppose the colour of the moving disc is d. Then the six triple points are given by either $\partial(d, x, y, z)$ or $\partial(x, d, y, z)$ or $\partial(x, y, d, z)$ or $\partial(x, y, z, d)$.

Twist Spun Trefoil (Reduced diagram)

The following diagram is coloured by R_3.



The total diagram of the double twist-spun trefoil has triple points $(1,2,0)^-$, $(0,2,1)^+$, $(1,0,2)^-$ and $(2,0,1)^+$.
Define $\theta \in Z^3(X;\mathbb{Z}_3)$ by

$$\theta = t^{-\chi_{(0,1,0)}-\chi_{(0,2,1)}+\chi_{(0,2,0)}+\chi_{(1,0,1)}+\chi_{(1,0,2)}+\chi_{(2,0,2)}+\chi_{(2,1,2)}},$$

where $\chi_\alpha(\beta) = 1$ if $\alpha = \beta$ otherwise 0. Then

$$\prod B(\tau, \mathcal{C}) = \prod_\tau \theta(\tau)^{\epsilon(\tau)} = t.$$

The numbers of non-trivial and trivial colourings are 6 and 3 respectively. Thus

$$\Phi_\theta(F) = 3 + 6t$$

Pre-images of Multiple Points

Let F be a closed orientable surface and let $f : F \to \mathbb{R}^3$ be a generic map. The pre-image of the singular set of f is:

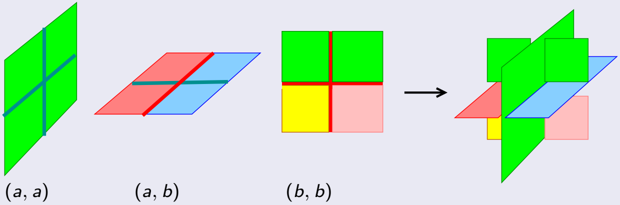
$$S(f) = \{x \in F \mid \#f^{-1}(f(x)) > 1 \}$$

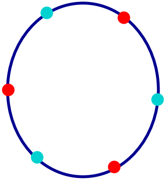
$S(f)$ the union of two families of immersed circles or immersed intervals: $\mathcal{S}_a = \{s_a^1, s_a^2, \ldots, s_a^k\}$ and $\mathcal{S}_b\{s_b^1, s_b^2, \ldots, s_b^k\}$ with $f(s_a^i) = f(s_b^i)$, $(i = 1, 2, \ldots, k)$.
The pre-image of a triple point consists of three cossings. If the crossing is formed by two curves $s_x \in \mathcal{S}_x$ and $s_y \in \mathcal{S}_y$ $x, y \in \{a, b\}$, then the crossing is of type (x, y).

Lemma (Carter-Saito (1998))

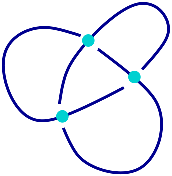
Let F be a closed orientable surface. Let $f : F \to \mathbb{R}^3$ *be a generic map. Then f has an embedding* $g : F \to \mathbb{R}^4$ *such that* $\mathrm{proj} \circ g = f$ *if and only if*

1. $S(f) = \bigcup \mathcal{S}_a \cup \bigcup \mathcal{S}_b$.
2. *For each triple point, the pre-images are crossings of types* (a, a), (a, b) *and* (b, b).

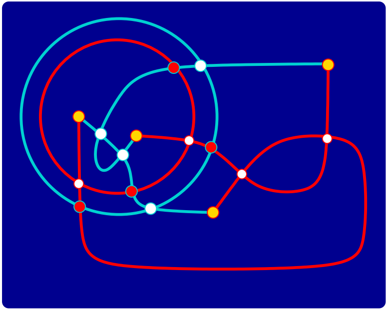




Pre-image of D_K

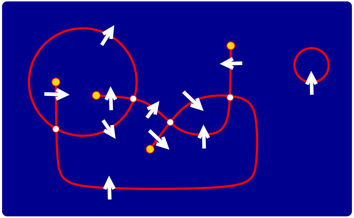


D_K



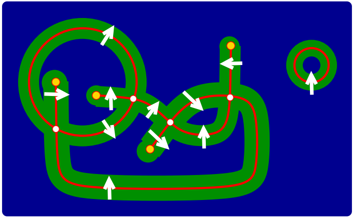
The closure of the pre-image of double curves in D_F is a union of two families of arcs called the **double decker set** (Carter-Saito). The blue arcs represent the **upper decker set** and the red arcs represent the **lower decker set**.

Pre-images of Multiple Points



We denote the lower decker set by S_b. $F \setminus S_b = \{R_0, \ldots, R_n\}$. Let $N(S_b)$ be a small neighbourhood of S_b in F.
$F \setminus N(S_b) = \{V_0, \ldots, V_n\}$; $V_i \subset R_i$ $(i = 0, \ldots, n)$.

Pre-images of Multiple Points



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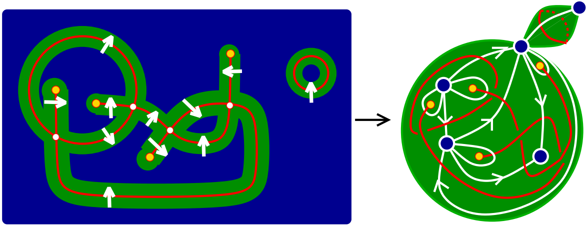
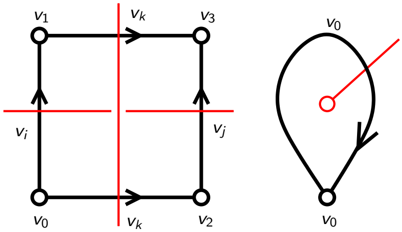


The quotient map $q : F \to F/_\sim$ is defined by $q(V_i) = v_i$, $(i = 0, \ldots, n)$.
The quotient space is a 2-dimensional complex. We will denote the complex by K_{D_F}. A subcomplex of K_{D_F} induced from a simple closed curve in S_b is called a **bubble**.

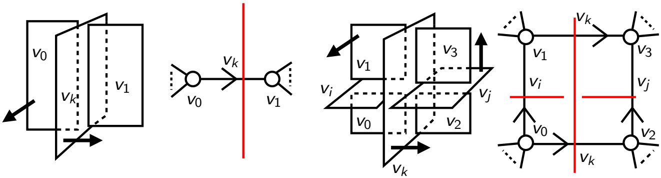
Pre-images of Multiple Points

A subcomplex of K_{D_F} corresponding to a connected component of the lower decker set S_b is called a **parcel**.
Each parcel is a bubble or a subcomplex consisting of some rectangles and loop discs.



Pre-images of Multiple Points

Each double segment corresponds to an edge of the complex K_{D_F}. Each edge has a **weight**, which is a vertex of the complex.



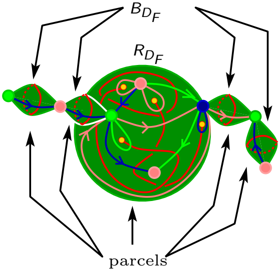
The lower decker set $S_b \subset |K_{D_F}|$ is a union of edges of K_{D_F}.

2-dimensional complexes for surface diagrams

K_{D_F} can be decomposed into parcels $K_1, \ldots, K_n$ such that

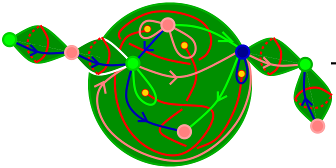
$$\begin{aligned} K_{D_F} &= K_1 + \cdots + K_n, \\ &= R_{D_F} + B_{D_F}. \end{aligned}$$

where R_{D_F} is the union of rectangles and B_{D_F} be the union of bubbles.



2-dimensional complexes for surface diagrams

We define a chain group $C_2(K_{D_F})$ of K_{D_F}. A homomorphism $\mathrm{Col}_\sharp : C_2(K_{D_F}) \to C_3^Q(X)$ is induced from the colouring of D_F.



$$\xrightarrow{\mathrm{Col}_\sharp} C_3^Q(X)$$

Connectedness of the Lower Decker Set

Theorem (A.M.-T. Y. (2011))

Let F be a surface-knot. Let D_F be a surface diagram of F. If the lower decker set S_b is connected and the number of triple points of D_F is at most two, then $\pi F \cong \mathbb{Z}$.

Self-contained parcels

Let D_F be a surface diagram of a surface-knot F. Let K_{D_F} be a cell-complex induced from D_F:

$$K_{D_F} = R_1 + R_2 + \cdots + R_r + B_1 + \cdots + B_s,$$

where R_i consists of rectangles and B_j is a bubble.
Each of $|R_i|$ and $|B_j|$ $(i = 1, \ldots, r, j = 1, \ldots, s)$ contains a connected component of the lower decker set S_b.
The connected component $s_i \subset S_b$ induces a 1-dimensional subcomplex $L(s_i)$ of K_{D_F}. A parcel R_i is **self-contained** if

$$e \in L(s_i) \Longrightarrow e \in R_i.$$

For a parcel K as a chain in $C_2(K_{D_F})$, if $\mathrm{Col}_\sharp(K) = 0$, then K is said to be degenerate, otherwise non-degenerate.
The number of non-degenerate parcels of K_{D_F} will be denoted by $\nu(K_{D_F})$.

Theorem

Let F be a surface-knot and let D_F be a surface diagram of F. Let $K_{D_F} = R_1 + R_2 + \cdots + R_r + B_{D_F}$ be a cell-complex induced from D_F. If each of R_i $i = 1, \ldots, r$ is self contained, then the following holds:

$$4\nu(K_{D_F}) \leq t(D_F)$$

For a parcel K of K_{D_F}, if $[K] \in H_2(K_{D_F})$, $\mathrm{Col}_*[K] \in H_3^Q(X)$ vanishes, then $[K]$ is homologically degenerate otherwise homologically non-degenerate. Let $\nu(F)$ denote the number of homologically non-degenerate parcels of K_{D_F}.

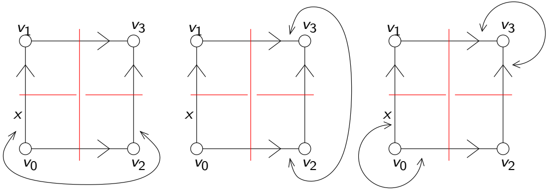
Theorem

Let F be a surface-knot and let D_F be a surface diagram of F with coloured by some quandle X. Then

$$4\nu(F, X) \leq t(F)$$

One rectangle

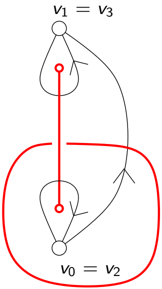
Let R be a parcel consisting of rectangles in K_{D_F}. If there is only one triangle (rectangle + loop disc), then it is not closed. So, it must be a rectangle. There are 3 cases:



For the left case, $v_0 = v_2$, $v_1 = v_3$.

One rectangle

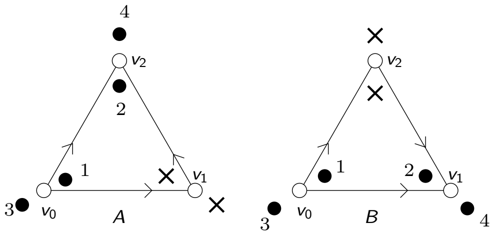
R contains two loop discs; two branch ponits are joined by simple arc in s_b. This shows $v_0 = v_1$. Thus all vertices are the same, so $\mathrm{Col}_\sharp(K_i) = 0$.



Therefore, there is no self-contained parcel with exactly one rectangle.

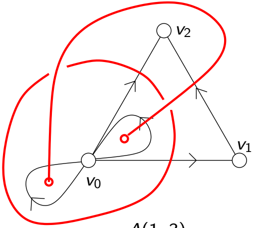
Two rectangles

If there are two rectangles, then there is no possibility to have one rectangle and one triangle as the number of all edges of the parts is odd $(3+4=7)$.
There are two cases:

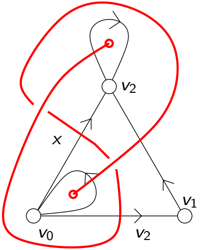


Black dots are places where a loop disc can be placed otherwise it has the cross.

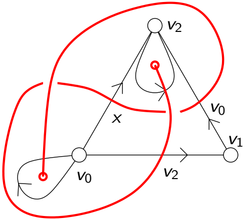
Two rectangles



$A(1,3)$



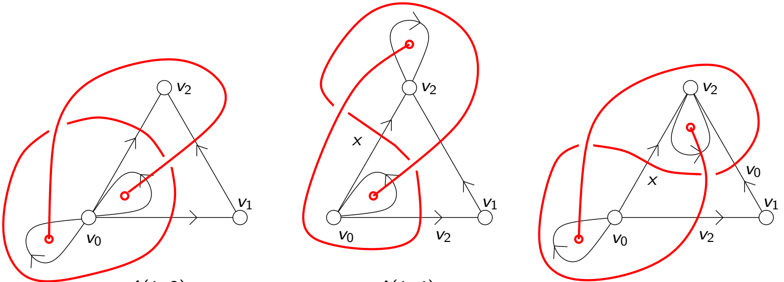
$A(1,4)$



$A(2,3)$

$A(1,3)$ has two triangles sharing the same vertices with opposite orientations; that is the cancelling pair.

Two rectangles

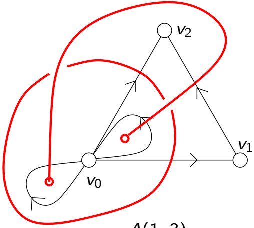


$A(1,4)$ has $c_b = v_2 x + v_0 x$. Then x must be v_2. This implies:

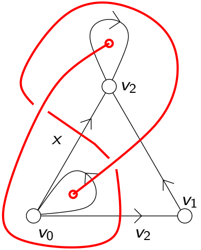
$$\mathrm{Col}(v_0) * \mathrm{Col}(v_2) = \mathrm{Col}(v_2)$$

Thus $\mathrm{Col}(v_0) = \mathrm{Col}(v_2)$. The the parcel is degenerate.

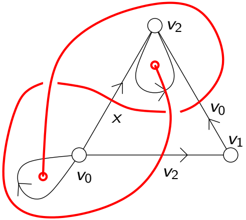
Two rectangles



$A(1,3)$



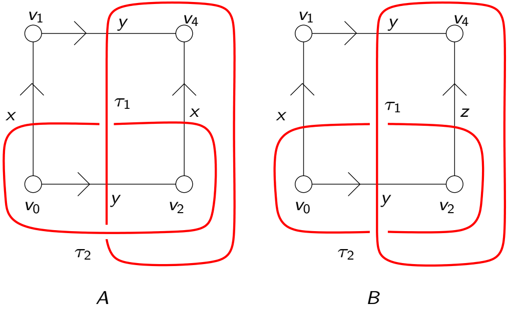
$A(1,4)$



$A(2,3)$

Other cases are similar and there is no self-contained parcel with two rectangles.

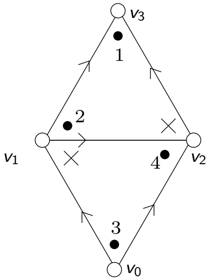
Two rectangles



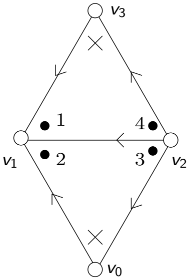
For A, $c_b = xx$ but there is no loop in R. Thus R is not self-contained. For B, $\mathrm{Col}_\sharp(\tau_1 + \tau_2) = 0$. Therefore, there is no such R.

Three rectangles

Two triangles + one rectangle. Conventions:

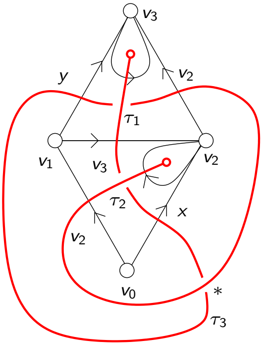


A_1

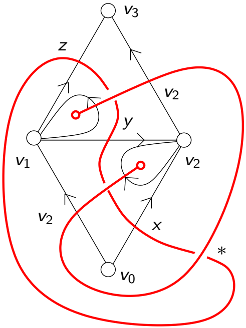


A_2

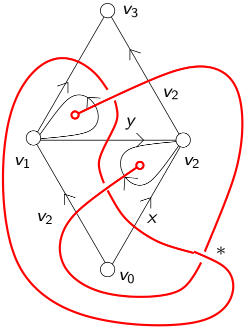
Three rectangles



$A_1(1,4)$



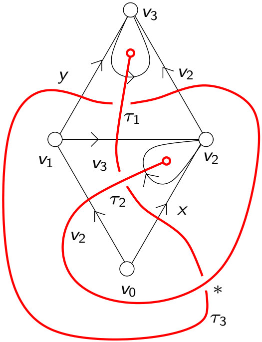
$A_1(2,4)$



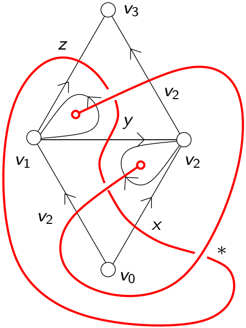
$A_1(2,4)^*$

$A_1(1,4)$. From the diagram,
$\mathrm{Col}(v_3) = \mathrm{Col}(v_2) * \mathrm{Col}(v_2) = \mathrm{Col}(v_2)$.

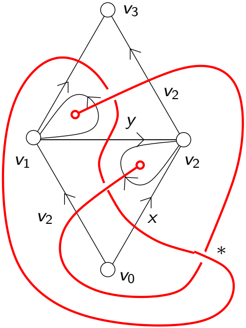
Three rectangles



$A_1(1,4)$



$A_1(2,4)$

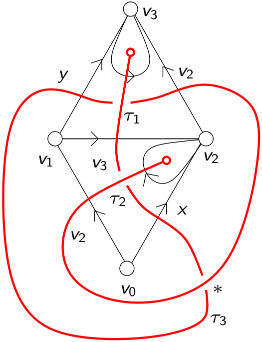


$A_1(2,4)^*$

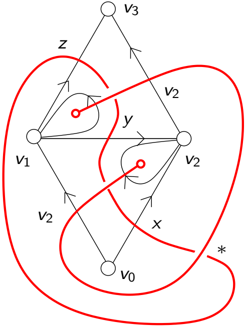
Thus $\mathrm{Col}(v_1) = \mathrm{Col}(v_2)$ also $\mathrm{Col}(v_0) = \mathrm{Col}(v_1)$.

$\therefore\ \mathrm{Col}_\sharp(K) = 0.$

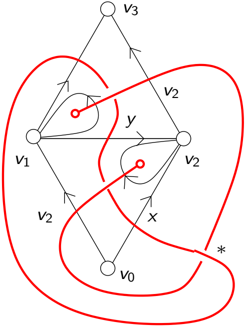
Three rectangles



$A_1(1,4)$



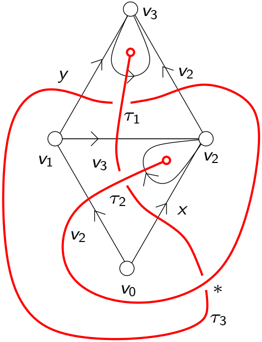
$A_1(2,4)$



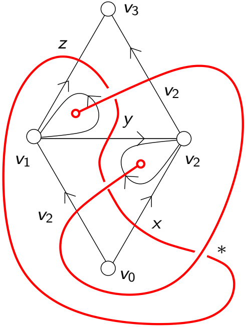
$A_1(2,4)^*$

$$A_1(2,4).\ \mathrm{Col}(v_0) * \mathrm{Col}(v_2) = \mathrm{Col}(v_1) = \mathrm{Col}(v_2).$$

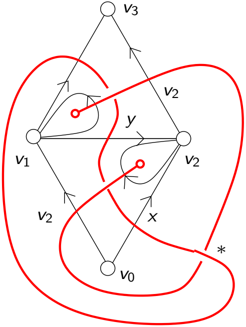
Three rectangles



$A_1(1,4)$



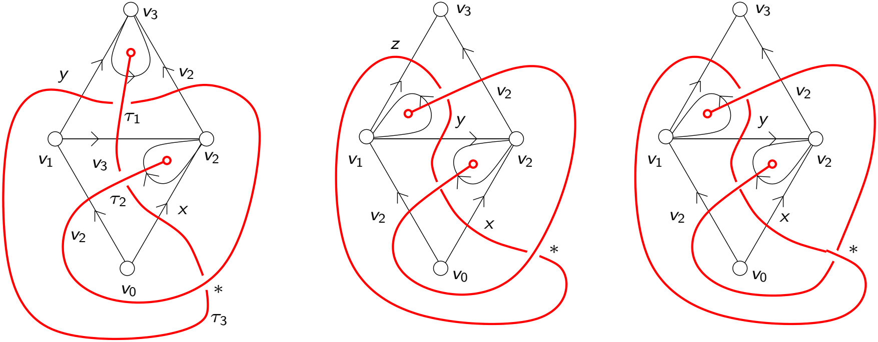
$A_1(2,4)$



$A_1(2,4)^*$

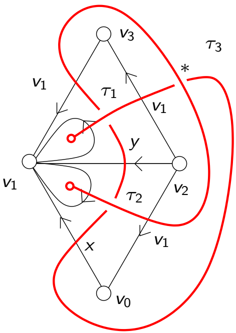
$\mathrm{Col}(v_0) = \mathrm{Col}(v_2)$ and $\mathrm{Col}(v_1) = \mathrm{Col}(v_2) = \mathrm{Col}(v_3)$.

Three rectangles

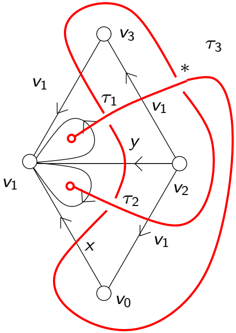


$A_1(1,4)$ $A_1(2,4)$ $A_1(2,4)^*$

$\therefore$ all vertices have the same colour. $\mathrm{Col}_\sharp(R) = 0$.

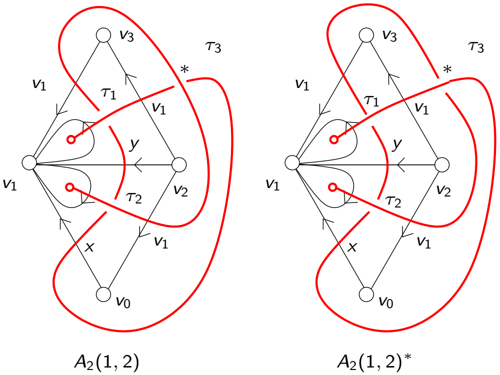


$A_2(1,2)$



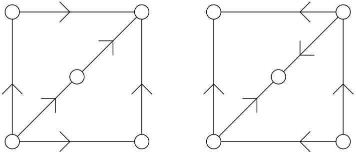
$A_2(1,2)^*$

$A_2(1,2)$. $\mathrm{Col}_\sharp(\tau_1+\tau_2+\tau_3)=0$.

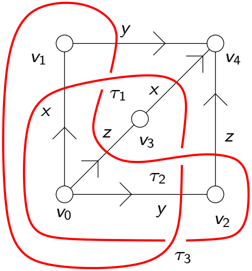


The same argument can be applied to $A_2(1,2)*$, $A_2(3,4)$ and $A_2(3,4)*$. Thus there is no parcel of type A_2.

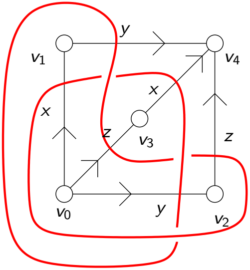
Three rectangles without loop discs. There are two types:



B_1 B_2



B_1



B_2

$c_b = xy + yz + zx$. As R is self-contained, this triangle does not exist in R. Thus there are no parcels of type B_1 and B_2.