Cross-exchangeable curves and d-minimal surface diagrams

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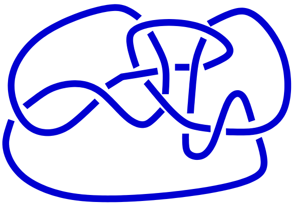
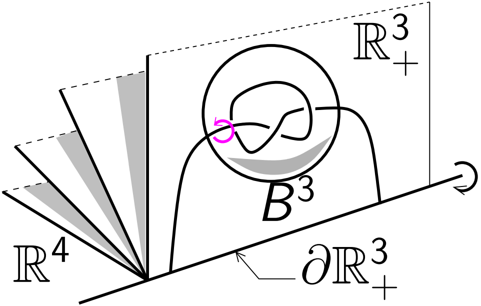
Knots

A **classical knot** is a circle embedded in $\mathbb{R}^3$ (or $\mathbb{S}^3$). The projected image of a knot with crossing information is called a **knot diagram**.



High-dimensional Knots

Let B^3 be a 3–ball in $\mathbb{R}^3_+$ such that $\partial B^3 \cap T(K)$ is the pair of antipodal points of ∂B^3.
An *m*-**twist-spun knot** obtained from K is defined by rotating the tangle $B^3 \cap T(K)$ about the axis through the antipodal points m times while $\mathbb{R}^3_+$ spins. We denote this 2-knot by $T_m(K)$.



High-dimensional Knots

Theorem (Zeeman, 1965)

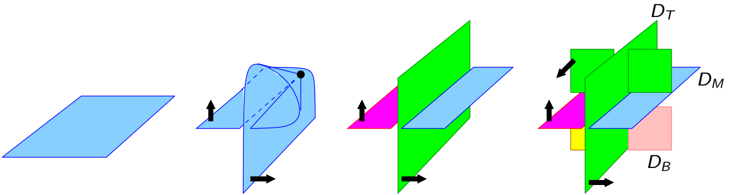
Every m-twist spun knot $T_m(K)$ obtained from K is fibred $(m \geq 1)$; the fibre is the one-punctured m-fold branched covering space of S^3 along K.

Corollary (Zeeman, 1965)

For any knot K, 1-twist spun knot obtained from K is trivial.

Surface Diagrams

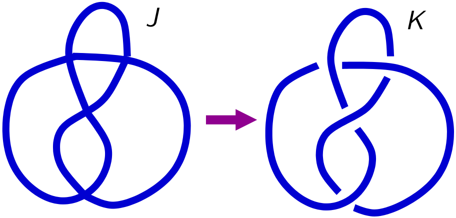
A **surface-knot** is a connected oriented closed surface embedded in 4-space. Let $F \subset \mathbb{R}^4$ be a surface-knot. Let $\pi : \mathbb{R}^4 \to \mathbb{R}^3$; $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3)$, be the orthogonal projection. A **surface diagram** of F is a union of the following local diagrams.



Branch point Triple point

Liftability

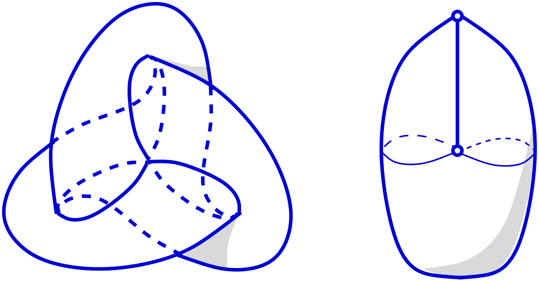
It is known that for every immersed circle J in the plane, there exists a knot K (trivial or non-trivial) such that the projected image of K is J.



An immersed circle is said to be **liftable** into $\mathbb{R}^3$. There is a natural question:
Can we generalise the statement for high-dimensional cases?

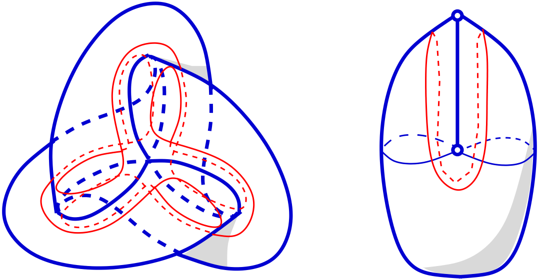
Liftability

For a surface immersed in 3-space, a lift into 4-space does not always exist. One of examples is a **Boy surface** [Boy, Math. Ann. **57** (1903)].



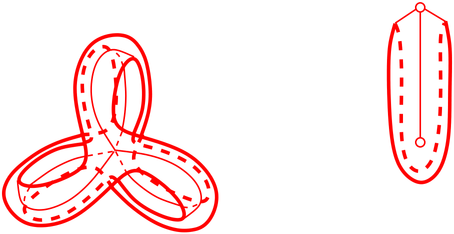
Boy surface and Cross Cap.

Liftability



Möbius Bands

Liftability



Möbius Bands

Liftability

This is an immersed projected plane which is non-orientable closed surface and is **not** liftable. An oriented version can be constructed from the Boy surface as its double cover. This is an immersed sphere in 3-space [Giller, Illinois J. Math. **67** (1982)].
On the other hand, the cross-cap, which represents the projective plane with two branch points, is liftable.

Liftability

Although we relax the condition from immersed surfaces to generic surfaces (allowing existence of branch points), a lift does not always exist.

Let D be a generic surface in $\mathbb{R}^3$; the image of a closed surface under a generic map $f : F \to \mathbb{R}^3$. Let $U(x)$ denote a small neighbourhood a point $x \in D$ in $\mathbb{R}^3$. For a triple point t, the pre-image of $U(t) \cap D$ is a union of three discs D_B, D_M and D_T. $D_X \cap S$ $(X = B, M, T)$ is a cross shape. If it consists of subarcs of S_x and S_y, $x, y \in \{a, b\}$ then we call the disc type (xy).

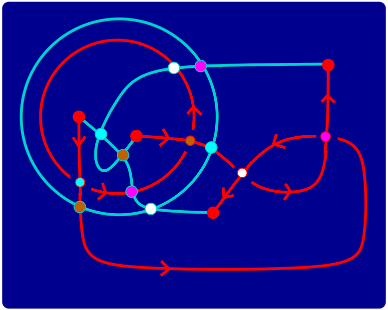
Liftability

Lemma (S. J. Carter and M. Saito (1998))

A generic surface D is liftable if and only if

- *The singular set S of f is a union of the union of two families $S_a = \{s_a^1, s_a^2, \ldots, s_a^n\}$ and such that $f(s_a^i) = f(s_b^i)$ $(i = 1, 2, \ldots, n)$. $S_b = \{s_b^1, s_b^2, \ldots, s_b^n\}$ of immersed arcs and circles in F*
- *for every triple point, types of D_B, D_M and D_T are (bb), (ab) and (aa) respectively.*
- *at a branch point b, pre-image of $U(b) \cap D$ is a disc with subarcs of S_a and S_b joined at b.*

Pre-images of Multiple Points



Carter-Saito's Lemma implies that if a generic surface does not have any triple points, then it is liftable into 4-space. The closure of S_a is called an **upper decker set** and the closure of S_b is called a **lower decker set**.

In the left diagram, the blue arcs represent the upper decker set and the red arcs represent the lower decker set.

Unknotting Number

- It is known that if we change some crossings of a knot diagram, then we obtain a trivial knot.

The least number of crossing changes to obtain a trivial knot for all diagrams is called an **unknotting number**.

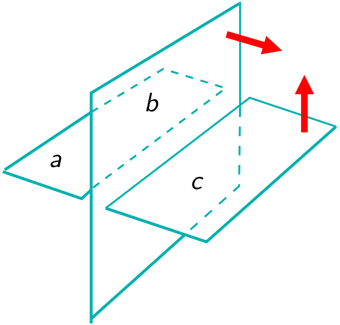
- The unknotting number is a knot invariant.
- It is not known whether or not we can generalise this concept for high-dimensional knots.
- There is an "unknotting number" for a surface-knot which is the least number of handles attached to obtain a trivial surface-knot.

Quandles

A **quandle** X is a non-empty set with a binary operation $(a, b) \mapsto a * b$ such that

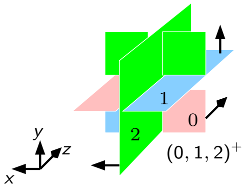
1. For any $a \in X$, $a * a = a$,
2. For any $a, b \in X$, there is a unique $c \in X$ such that $c * b = a$.
3. For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

Quandle colouring



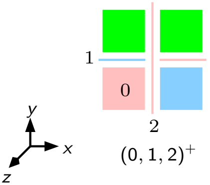
Set $\mathcal{R} = F \setminus \bigcup \mathcal{S}_b$. A quandle colouring is a map $\mathcal{C} : \mathcal{R} \to X$ satisfying $\mathcal{C}(a) * \mathcal{C}(b) = \mathcal{C}(c)$ with respect to the local diagram:

Dihedral Quandle



The **dihedral quandle** $(X, *)$ of order $n > 0$ denoted by R_n is a quandle $X = \{0, \ldots, n-1\}$ with the binary operation $(i, j) \mapsto 2j - i \pmod{n}$.

Dihedral Quandle



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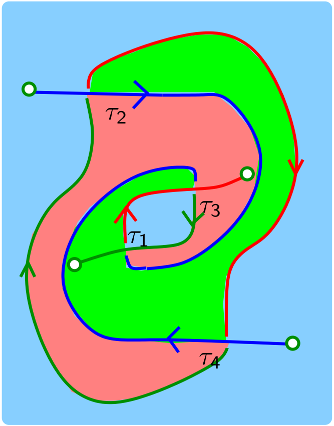
$(i, j) \mapsto 2j - i \pmod{n}$.

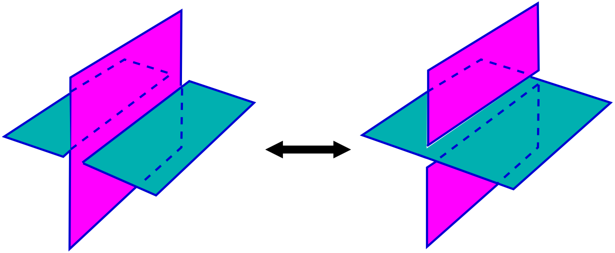
Quandle Colouring



The left diagram is coloured by the dihedral quandle of order 3: $\{0, 1, 2\}$. In the diagram, the complement of the lower decker set is coloured by $0 =$pink, $1 =$blue and $2 =$ green. Also the lower decker set itself is coloured.

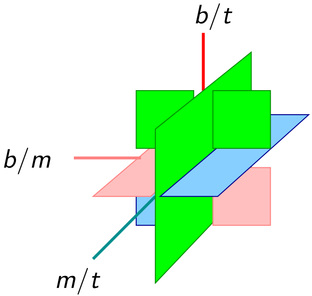
Crossing change operations

We also need to generalise the concept "Crossing Change" for surface-knots. For a classical knot, in its diagram, it is always possible to change the crossing information at a crossing point of the diagram. This is not true for surface-knots.



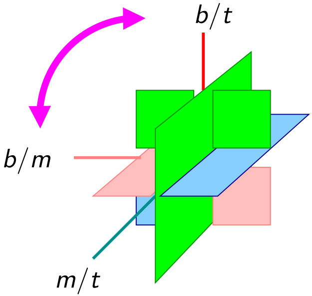
Crossing change operations

Around a triple point there are three types of double arcs. The double arc formed by the top sheet and the middle sheet called a m/t-curve, by the middle sheet and the bottom sheet called an b/m-curve and by the top sheet and the bottom sheet called a b/t-curve.



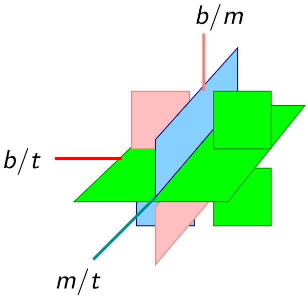
Crossing change operations

One can easily change the crossing information along m/t and b/m-curves.



Crossing change operations

One can change crossing information along the b/t curve if crossing information of other two curves is changed.



Cross-Exchangeable curves

We define a crossing change operation along a double curve. Let $\gamma \subset D_F$ be a closed double curve without triple points. Take a point p on γ. There is a small ball neighbourhood $B(p)$ of p in $\mathbb{R}^3$ such that there exist two discs D_U and D_L in $F \subset \mathbb{R}^4$ such that

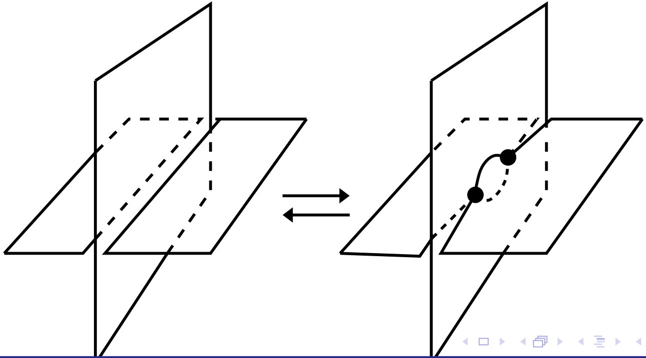
$$\mathrm{proj}^{-1}(B(p)) \cap F = D_U \cup D_L \quad ,$$

$$h(D_L) < h(D_U)$$

$$\mathrm{proj}(D_U) \cap \mathrm{proj}(D_L) \subset \gamma \qquad \text{(connected segment.)}$$

Cross-Exchangeable curves

By a homotopy move, push D_U into D_L so that the deformed disc and D_L have a pair of singular points above γ. We call this a *push down operation*. This local move can be applied at any double point.



Suppose that one traverses along γ it meets a triple point and it is on the (b/t)-edge. First we apply push down operation along the (b/m)-edge and then apply push down operation along the main line. We call the resulting triple point a **semi-saturated triple point** (The 3rd diagram below). If we apply one more push down operation on the last edge, then we call the resulting triple point **saturated triple point** (the 4th diagram below).

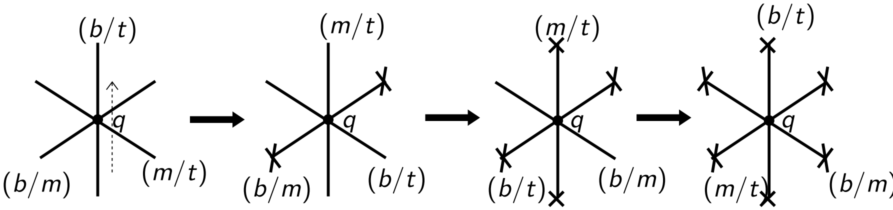


Figure : Saturation of a triple point.

Let γ be a double curve in D_F. A **singular operation** is defined for each of the following case.

Case 1 There is no triple point on γ. Apply the push down operation at any point in γ.

Case 2 There is some triple points along γ. Traverse along γ and

(i) if one meets the edge with singular points, then skip the triple point and continue,

(ii) if one meets (m/t)- or (b/m)-edge, then apply the push down operation along the edge and continue.

(iii) if one meets the (b/t)-edge at a triple point, then semi-saturate the triple point and continue.

We continue these operations on all double curves which have singular points.

We define the crossing change operation along a set of double curves as follows: Choose a pair (γ_0, p_0) of a double curve γ_0 and an initial point p_0 on γ_0 in D_F. Apply the singular operations along (γ_0, p_0). There are double curves $\gamma_1, \gamma_2, \ldots, \gamma_k$ such that each γ_i intersects γ_0 and it has singular points. Choose initial point p_i for γ_i $i = 1, 2, \ldots, k$. Apply the singular operation on $(\gamma_1, p_1), (\gamma_2, p_2), \ldots, (\gamma_k, p_k)$. Then for each (γ_i, p_i), we repeat the procedure above.

Double curves which receive singular operations are called **related double curves**. When all related double curves are done, we have a set of double curves with singular points. They can be eliminated by push down operation or by pairing with branch points. Crossing information on the related double curves are changed.

From the observation above, we obtain the following lemma.

Lemma

For every surface diagram D_F of a surface-knot F, there is a union of double curves such that the crossing information along them can be exchanged.

We have the following question.

Question:

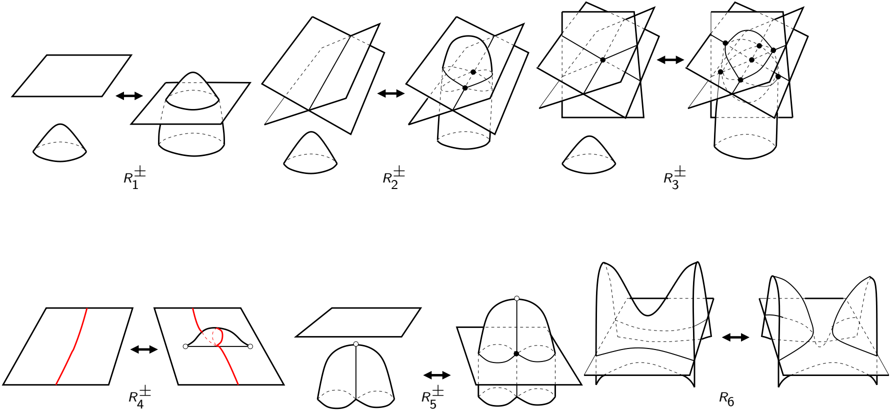
For every surface diagram of a surface-knot, can we obtain a trivial surface diagram by crossing change along cross-exchangeable curves?

In this talk we introduces a ***d*-minimal** surface diagram. We will construct a family of non-trivial d-minimal surface diagrams which have triple points and some cross-exchangeable curves.

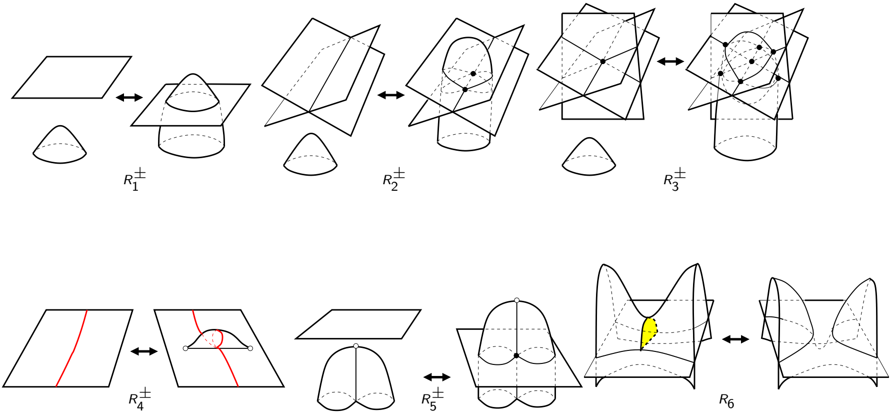
Definition

Two surface diagrams are **equivalent** if one is modified to the other by a **finite sequence of local moves** called Roseman moves.

Roseman moves



Roseman moves



Roseman moves

Roseman moves are introduced by D. Roseman in 1980s. The number of original moves are seven. It is known that these six moves above can describe seven moves.

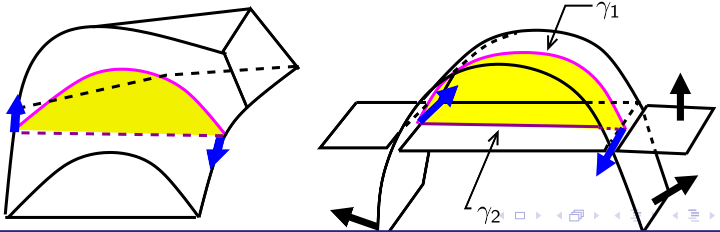
Simple diagrams

Let F be a surface-knot and let D_F and D'_F be its surface diagrams. D_F and D'_F are in a **simple class** if D_F is deformed into D'_F so that Roseman moves $R_1^{\pm}$, $R_4^{\pm}$.
The complement $\mathbb{R}^3 \setminus |D_F|$ is a union of 3-dimensional connected components. The closure of each component is bounded by a union of closures of components of $F \setminus (S_a \cup S_b)$.

Simple diagrams

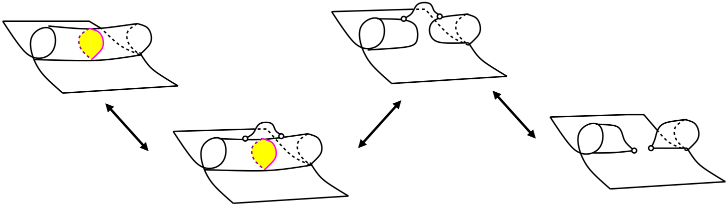
A properly embedded disc in the connected 3-d component is called a **descendent disc** if it is bounded by two proper arcs γ_1 and γ_2 such that

1. γ_1 is a properly embedded arc in the closure of a connected component of $F \setminus (S_a \cup S_b)$.
2. γ_1 is bounded by a pair of points on distinct components of S_a and γ_2 is bounded by a pair of points on distinct components of S_b.



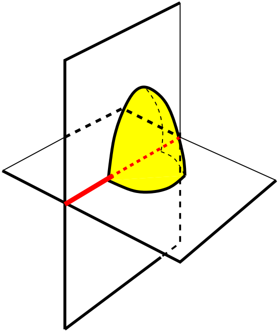
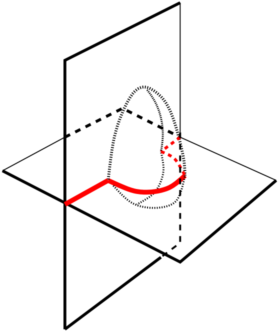
Simple diagrams

A properly embedded disc in the connected 3-d component is called a **pinch disc** if it is bounded by a simple arc on the closure of a connected component of $F \setminus (S_a \cup S_b)$. The move below is called **pinch move**.



Simple diagrams

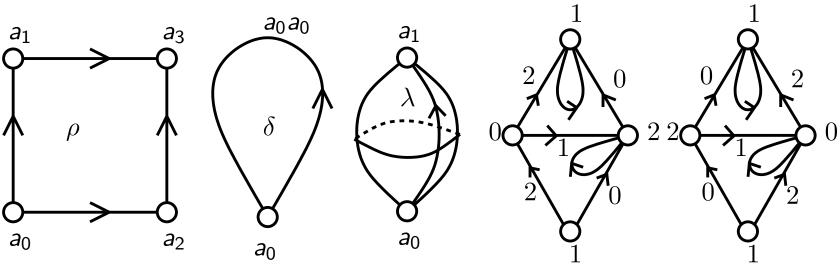
Note that at every double segment, locally we can find a descendent disc. But the modified diagram is in the simple class.

Simple diagrams

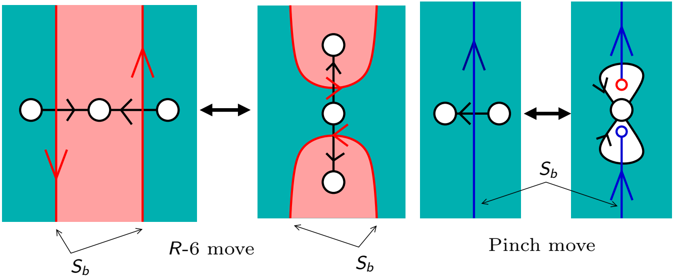
For a surface diagram, applying R_6 move to any descendent disc, or pinch move to any pinch disc, we obtain a simple diagram, then the diagram is called a ***d*-minimal surface diagram**.

Complexes for surface diagrams



For each surface diagram, we obtain a 2-dimensional complex which may consist of rectangles, loop discs and bubbles.

R_6 move and Pinch move

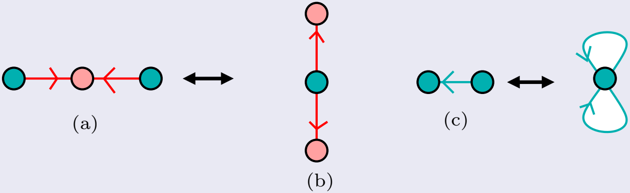


Detection of Descendent and Pinch Discs

Lemma

Let D_F be a surface diagram of a surface-knot F. Let K_{D_F} be the induced complex from D_F.

1. *If there is a discendent disc for D_F, then K_{D_F} contains a subcomplex of type (a) or (b).*
2. *If there is a pinch disc for D_F, then K_{D_F} contains a subcomplex of type (c).*



Main Theorems

Theorem (1)

Let F be a closed oriented surface embedded in $\mathbb{R}^3$ and let α be a simple closed curve on F. Then F can be embedded in $\mathbb{R}^4$ as a tri-colourable surface-knot F satisfying the followings:

(1) *it has a d-minimal surface diagram D_F,*

(2) *there exists a simple closed curve δ such that $\gamma = \mathrm{proj}(\delta)$ is an c-e curve and δ is isotopic to α in F, and*

(3) *$D_F(\gamma)$ is trivial.*

Main Theorems

Theorem (2)

For any integer $n > 0$, there exist a tri-colourable surface-knot F_n and its surface diagram D_n such that

(1) $F_m \neq F_n$ *if* $m \neq n$.

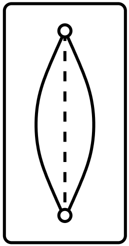
(2) D_n *is d-minimal.*

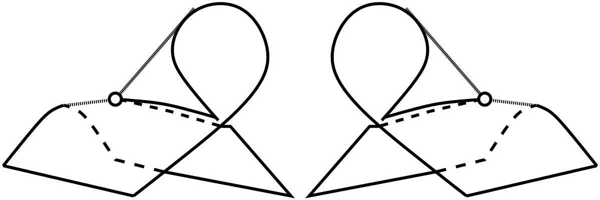
(3) D_n *has disjoint n cross-exchangeable curves* $\gamma_1, \gamma_2, \ldots, \gamma_n$.

(4) $D_n(\gamma_1, \gamma_2, \ldots, \gamma_n)$ *is trivial.*

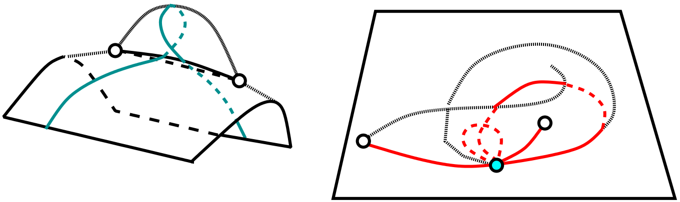
Bug construction

A bug is an immersed disc with one simple double arc bounded by branch points with crossing information. One can construct a trivial surface diagram by adding some bugs on a closed oriented surface.



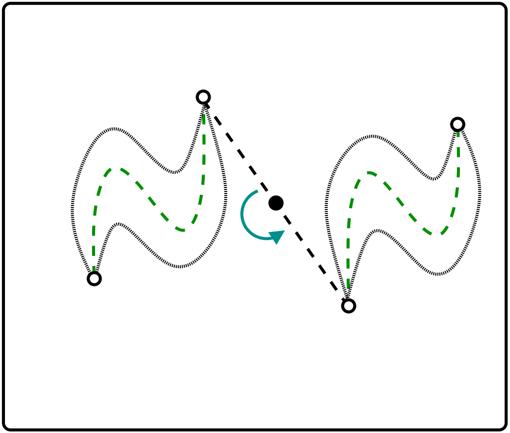


Bug construction



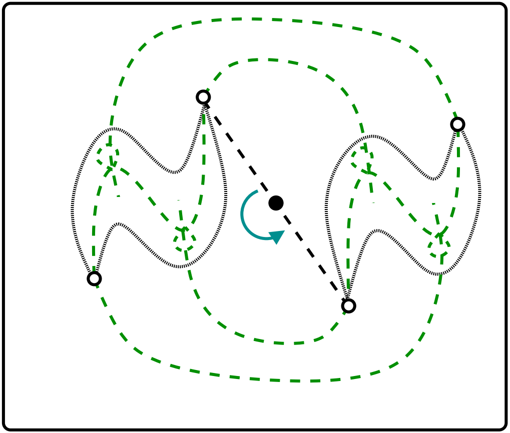
Note that a branch point can move through a double curve to create a triple point.

Bug construction



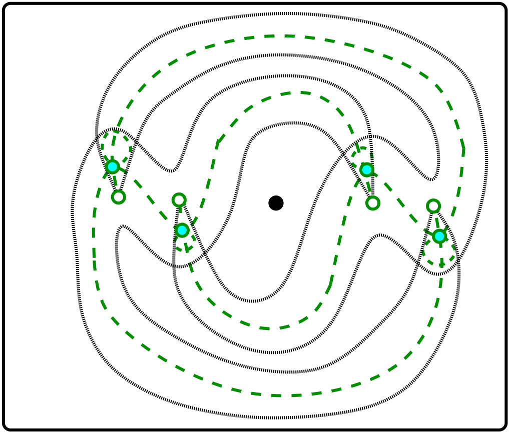
Put a bug on a surface. Take a copy and rotate it for 180 degrees.

Bug construction



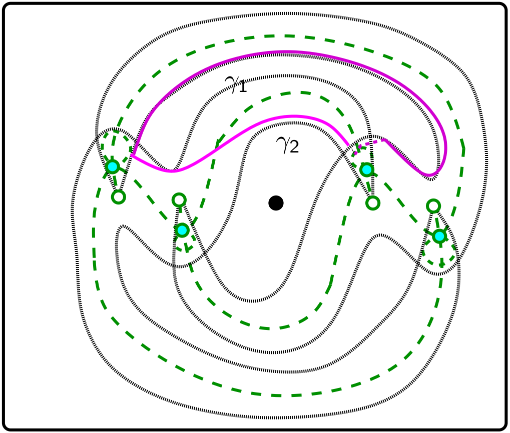
Move each branch point to the other bug so that a triple point is created.

Bug construction



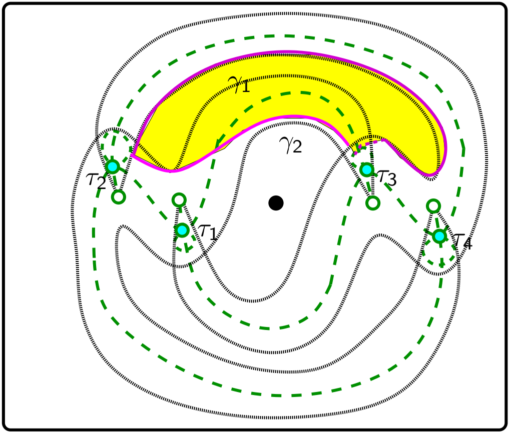
The resulting generic surface is a surface diagram of a trivial surface-knot. This is because that operation consists of finite sequence of Roseman moves.

Applying saddle moves



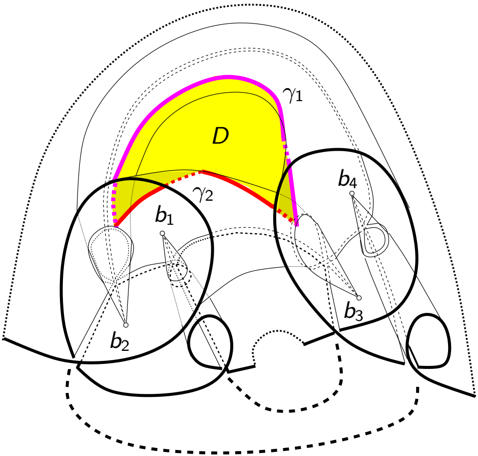
One can find a pair of arcs γ_1, γ_2 on the generic surface such that $\gamma_1 \cup \gamma_2$ is a simple closed curve and it bounds an embedded disc in the complement of the generic surface. This disc is the descendent disc for the Roseman move R-6 (the saddle move).

Applying saddle moves



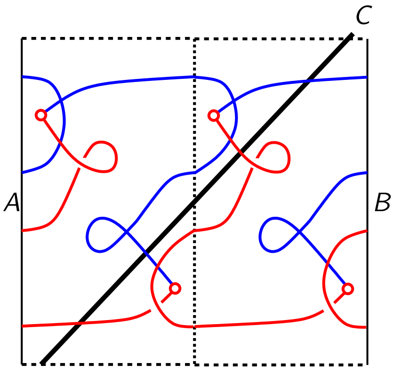
Applying the saddle move to along the descendent disc, a surface diagram with a cross-exchangeable curve is obtained.

Applying saddle moves



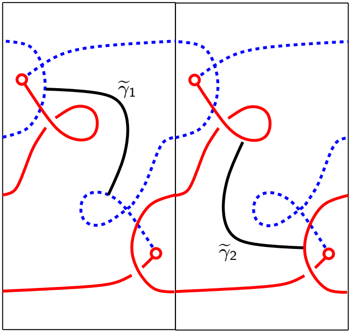
The yellow disc is the descendent disc, which is bounded by two arcs γ_1 and γ_2.

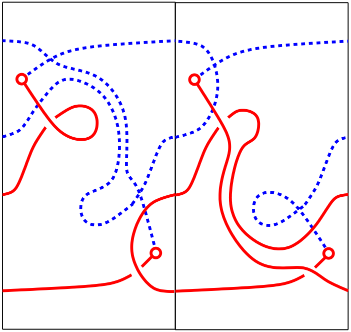
Applying saddle moves



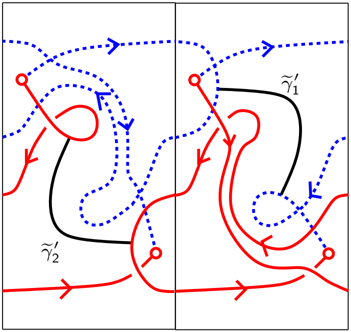
This is the pre-images of the complication with four triple points and four branch points. The thicken line indicate the double looped curve in the previous complicated diagram.

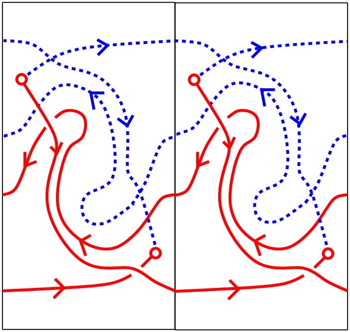
Applying saddle moves



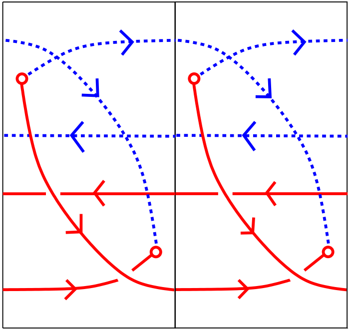


Applying saddle moves



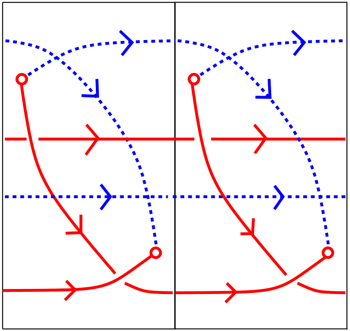


Crossing changes



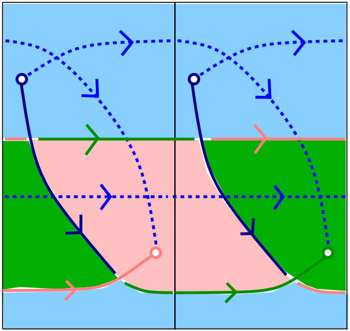
There is the pair of simple upper and lower decker lines. The lower decker line contains crossings corresponding to some triple points. The types of edges at triple points are b/m or m/t. Thus its image is a c-e curve in the diagram.

Crossing changes



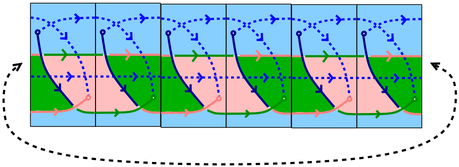
The crossing change operation along the c-e curve exchanges the upper and lower decker curves corresponding to the cross-exchangeable curve and changes crossing information at crossings of lower decker sets.

Crossing changes



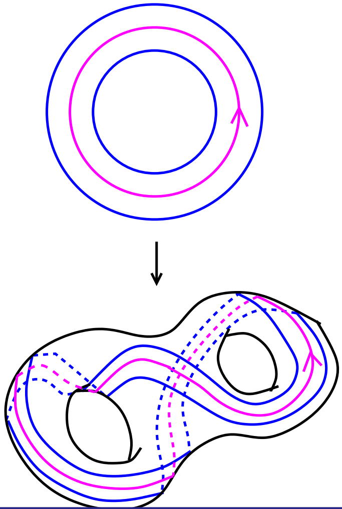
This diagram is non-trivially coloured by the dihedral quandle of order 3.

Crossing changes



Pasting copies of the diagram, we can construct an annulus with the double decker set.

Construction



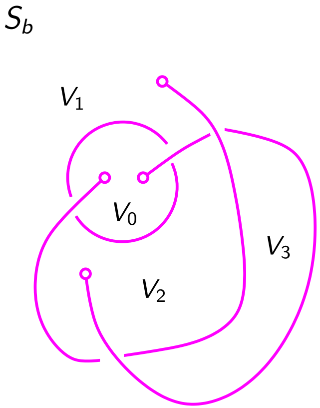
Let F be the boundary of a handle body embedded in $\mathbb{R}^3$.

1. Let α be a simple closed curves on F.
2. Along α, embedded the annulus with the double decker set that we constructed above.

Non-triviality

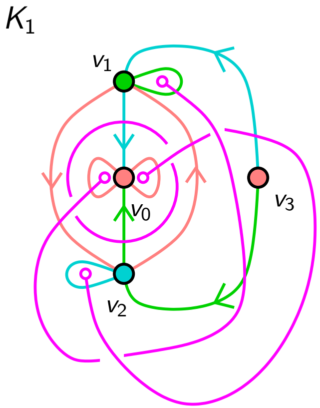
The resulting diagram D is non-trivially coloured by the dihedral quandle of order 3. Thus it is non-trivial.
From the construction, there is a c-e curve γ. Thus $D(\gamma)$ is trivial.

Proof of Theorem (2)



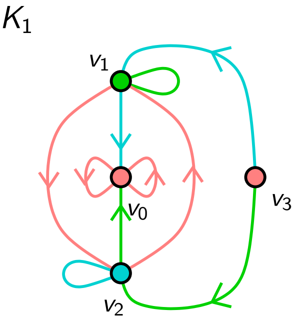
The lower decker set obtained from the constructed diagram with two primitive diagrams. This is also tri-colourable.

Proof of Theorem (2)



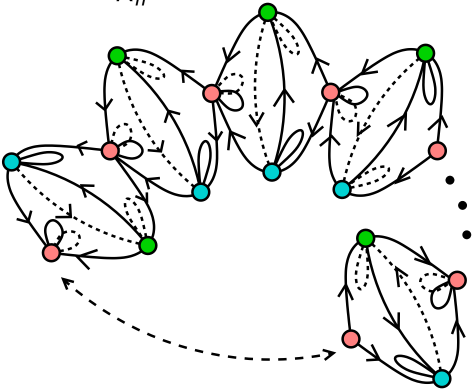
The lower decker set induces a complex, which is spherical.

Proof of Theorem (2)



The colour of v_0 uniquely determines the colour of v_3

Proof of Theorem (2)



Let F be the boundary of a handle body in $\mathbb{R}^3$. Let $L = \{l_1, l_2, \ldots l_n\}$ be a set of parallel n closed curves on F. We denote the constructed surface-knot with L by F_n. Then each l_i $i = 1, 2, \ldots, n$ induces the spherical complex. The number of colourings for K_n is 3^{n+1}. This means $F_i \neq F_j$ if $i \neq j$.

The induced complex shows that there is no sub-complexes of type (a) or (b). Thus the diagram is d-minimal. From the construction $D_n(\gamma_1, \gamma_2, \ldots, \gamma_n)$ is trivial.