# On connected components of the lower decker sets of surface diagrams

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# Background and motivation

- 1957 R. H. Fox and J. W. Milnor gave an example of a 2-knot.
- 1965 E. C. Zeeman introduced a construction method of a 2-knot called an *m*-**twist spinning**.
- 1982 A. Kawauchi, T. Shibuya and S. Suzuki described a surface-knot with a **normal form**.
- 1980s Roseman proposed diagrammatic approach to describe surface in 4-space and introduced elementary deformations called **Roseman moves** (1998).
- 1980s-1990s with the development of algebraic structure for this area such as racks and quandles, geometric approaches have been used and developed.
- 1992 S. Kamada introduced braid surfaces and charts.
- 1998 J. S. Carter and M. Saito introduced the double decker set.

# Surface-knots

- 1999 J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford and M. Saito applied **quandle co-homology** to knots and surface-knots.
- 2002 S. Satoh and A. Shima determined triple point numbers for 2-twist and 3-twist spun trefoils.
- 2005 E. Hatakenaka gave a lower bound of triple point number for 2-twist spun $(2,5)$-torus knot.

## Motivation

- Can we symbolize geometric objects? (example: tangles, surface braid charts, etc.).

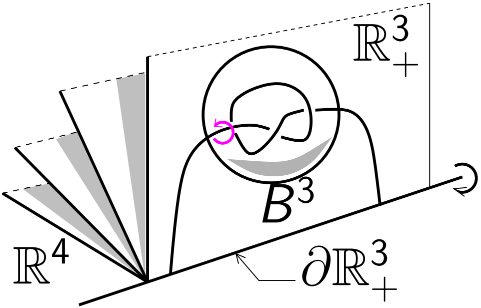
## In this talk

Discuss about the number of essential connected components of the lower decker set of a surface diagram.

# Zeeman's twist spinning

Let $B^3$ be a 3–ball in $\mathbb{R}^3_+$ such that it contains a tangle $T(K)$ of a knot $K$, and $\partial B^3 \cap T(K)$ is the pair of antipodal points of $\partial B^3$.

An $m$-**twist-spun knot** obtained from $K$ is defined by rotating $B^3 \cap T(K)$ about the axis through the antipodal points $m$ times while $\mathbb{R}^3_+$ spins denoted by $T_m(K)$.



**Theorem (Zeeman, 1965)**

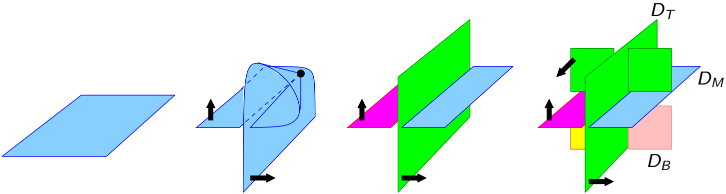
*Every m-twist spun knot $T_m(K)$ obtained from $K$ is fibred ($m \geq 1$); the fibre is the one-punctured m-fold branched covering space of $S^3$ along $K$.*

**Corollary (Zeeman, 1965)**

*For any knot $K$, 1-twist spun knot obtained from $K$ is trivial.*

# Surface Diagrams

A **surface-knot** is a connected oriented closed surface embedded in 4-space. Let $F \subset \mathbb{R}^4$ be a surface-knot. Let $\mathrm{proj} : \mathbb{R}^4 \to \mathbb{R}^3$; $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3)$, be the orthogonal projection. A **surface diagram** of $F$ is a union of the following local diagrams.

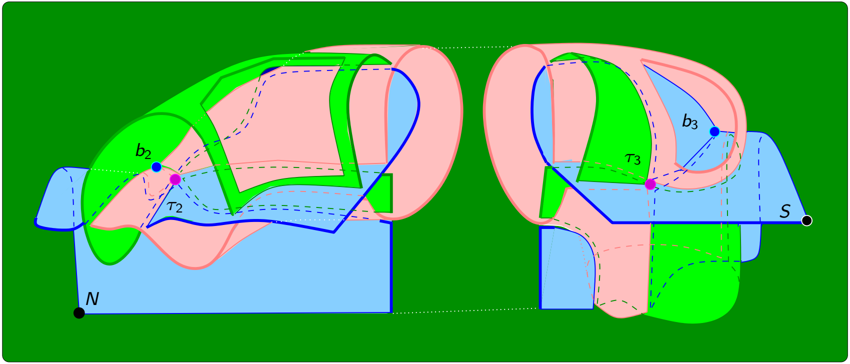



Branch point

Triple point

# Example(Double twist spun trefoil)

S. Satoh (2002) constructed a diagram of twist span knots. The following is a reduced diagram obtained by a sequence of Roseman moves from Satoh's construction.



The preimage of singularities of the projection $\mathrm{proj}$ is:

$$S = \{x \in F \mid \#((\mathrm{proj}|_F)^{-1}(\mathrm{proj}(x)) > 1\}$$

The set $S$ is the union of two families of immersed circles and immersed open intervals:

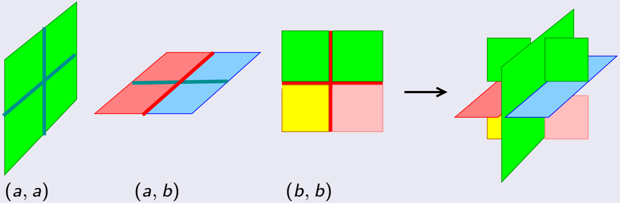
$$\begin{aligned} \mathcal{S}_a &= \{s_{a1}, s_{a2}, \dots, s_{al}\} \\ \mathcal{S}_b &= \{s_{b1}, s_{b2}, \dots, s_{bl}\} \end{aligned}$$

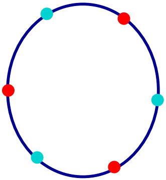
where for $x \in s_{ai}$, $y \in s_{bi}$ $(i = 1, 2, \dots, l)$, if $\mathrm{proj}(x) = \mathrm{proj}(y)$, then $h(x) > h(y)$.

## Lemma (Carter-Saito (1998))

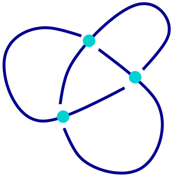
*Let $F$ be a closed orientable surface. Let $f : F \to \mathbb{R}^3$ be a generic map. Then there is an embedding $g : F \to \mathbb{R}^4$ such that* $\mathrm{proj} \circ g = f$ *if and only if*

1. $S(f) = \bigcup \mathcal{S}_a \cup \bigcup \mathcal{S}_b$.
2. *For each triple point, the pre-images are crossings of types* $(a, a)$, $(a, b)$ *and* $(b, b)$.

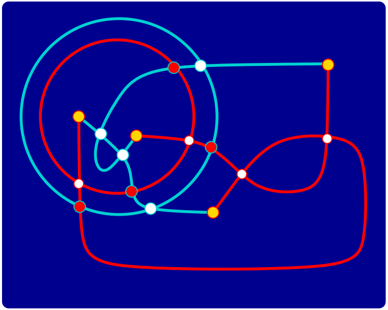




Pre-image of $D_K$

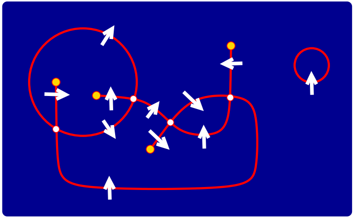


$D_K$



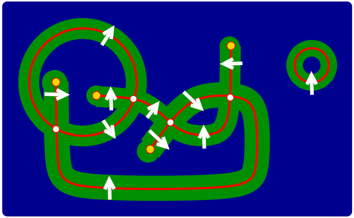
The closure of the pre-image of double curves in $D_F$ is a union of two families of arcs called the **double decker set** (Carter-Saito). The blue arcs represent the **upper decker set** and the red arcs represent the **lower decker set**.

# Rectangular-cell complexes



We denote the lower decker set by $S_b$.
$F \setminus S_b = \{R_0, \ldots, R_n\}$. Let $N(S_b)$ be a small neighbourhood of $S_b$ in $F$.
$F \setminus N(S_b) = \{V_0, \ldots, V_n\}$;
$V_i \subset R_i$ $(i = 0, \ldots, n)$.

# Rectangular-cell complexes



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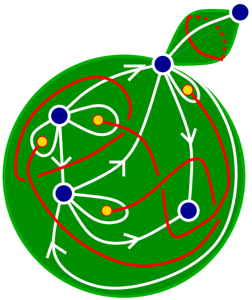
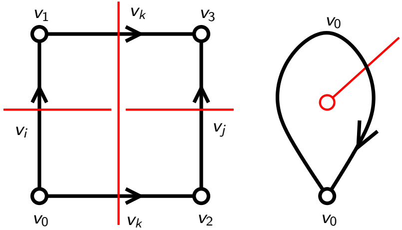


The quotient map $q : F \to F/_\sim$ is defined by $q(V_i) = v_i$, $(i = 0, \ldots, n)$. The quotient space is a 2-dimensional complex. We will denote the complex by $K_{D_F}$. A subcomplex of $K_{D_F}$ induced from a simple closed curve in $S_b$ is called a **bubble**.

# Rectangular-cell complexes

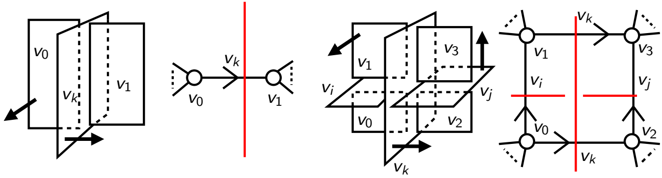
A subcomplex of $K_{D_F}$ corresponding to a connected component of the lower decker set $S_b$ is called a **parcel**. Each parcel is a bubble or a subcomplex consisting of some rectangles and loop discs:




We denote the rectangle by $(v_0; v_0v_1, v_0v_2; v_3)$ and the loop by $\widehat{v_0v_0}$.

# Rectangular-cell complexes

Each double segment corresponds to an edge of the complex $K_{D_F}$.
Each edge has a **weight**, which is a vertex of the complex.



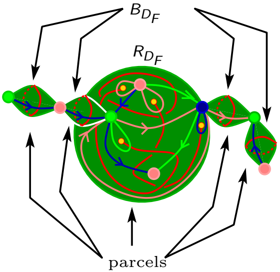

The lower decker set $S_b \subset |K_{D_F}|$ is a union of edges of $K_{D_F}$.

# Rectangular-cell complexes

$K_{D_F}$ can be decomposed into parcels $K_1, \ldots, K_n$ such that

$$\begin{aligned} K_{D_F} &= K_1 + \cdots + K_n, \\ &= Rec_{D_F} + Bub_{D_F}. \end{aligned}$$

where $Rec_{D_F}$ is the union of rectangles and loop discs, and $Bub_{D_F}$ be the union of bubbles.

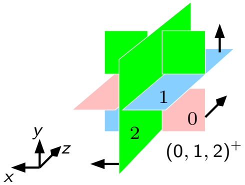


# Quandle colorings

A **quandle** $X$ is a non-empty set with a binary operation $(a, b) \mapsto a * b$ such that

1. For any $a \in X$, $a * a = a$,
2. For any $a, b \in X$, there is a unique $c \in X$ such that $c * b = a$.
3. For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

# Quandle colorings




The **dihedral quandle** $(X, *)$ of order $n > 0$ denoted by $R_n$ is a quandle $X = \{0, \ldots, n-1\}$ with the binary operation $i * j = 2j - i \pmod{n}$.

The triple point in the left diagram is coloured by $R_3$; $(0, 1, 2)$ and the orientation is determined by orientation normals to $D_T, D_M, D_B$ respectively.

# Quandle colorings

Let $\mathcal{R}$ be the set of connected components of $F - S_b$. For a quandle $X$, a **quandle coloring** of a diagram is a mapping $\mathrm{Col} : \mathcal{R} \to X$ such that if

1. $V_1$ and $V_2$ in $\mathcal{R}$ have a common boundary arc $c$ corresponding to an upper sheet $V_3 \in \mathcal{R}$ and
2. the orientation normal to $\mathrm{proj}(V_3)$ directs from $\mathrm{proj}(V_1)$ to $\mathrm{proj}(V_2)$,

then $\mathrm{Col}(V_1) * \mathrm{Col}(V_3) = \mathrm{Col}(V_2)$.

# Quandle colorings

The coloring $\mathrm{Col}$ can be interpreted in terms of the rectangular-cell complex $K_{D_F}$. If an edge $e$ is incident with vertices $v_1$ and $v_2$, oriented from $v_1$ to $v_2$ and with weight $v_3$, then the mapping from the 1-skeleton to $X$

$$\mathrm{Col} : K_{D_F}^{(1)} \to X$$

is defined satisfying $\mathrm{Col}(v_1) * \mathrm{Col}(v_3) = \mathrm{Col}(v_2)$. We call this mapping also a **quandle coloring**.

# Chain groups

## Quandle chain groups

Let $C_n(X)$ $(n \geq 1)$ be a free abelian group generated by $n$-tuples $(x_1, \ldots, x_n) \in X^n$. Let $C_n^D(X)$ be a sub group of $C_n(X)$ generated by $(x_1, \ldots, x_n)$ such that $x_i = x_j$ for some $1 \leq i, j, \leq n$ and $(|i - j| = 1)$. We denote the quotient group $C_n(X)/C_n^D(X)$ by $C_n^Q(X)$.

## Chain groups of $K_{D_F}$

The chain group $C_k(K_{D_F})$ is defined as a free abelian group generated by $k$-dimensional elements of $K_{D_F}$. For $k = 2$, it is generated by the rectangular cells, loop discs and bubbles in $K_{D_F}$. For $k = 1$, it is generated by edges in $K_{D_F}$. For $k = 0$, it is generated by vertices of $K_{D_F}$.

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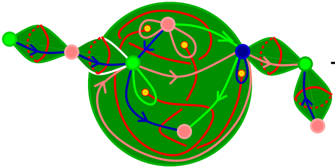
# Coloring homomorphisms

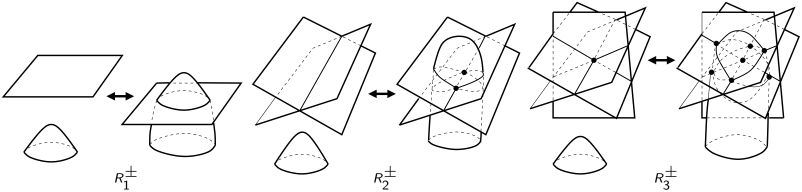
The quandle coloring $\mathrm{Col}$ can be extended to a homomorphism $\mathrm{Col}_\sharp : C_2(K_{D_F}) \to C_3^Q(X)$ defined as follows. For
$\sigma = (v_0; v_0v_1, v_0v_2; v_3)$,
$\mathrm{Col}_\sharp(\sigma) = (\mathrm{Col}(v_0), \mathrm{Col}(v_0v_1), \mathrm{Col}(v_0v_2) \in C_3^Q(X)$.



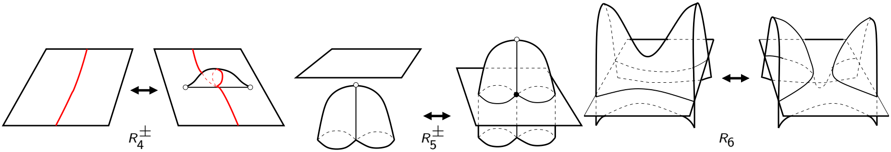
$$\xrightarrow{\mathrm{Col}_\sharp} C_3^Q(X)$$

# Roseman moves

Two surface diagrams are equivalent if they are projected image of the same type of a surface-knot. Two equivalent surface diagrams are modified from one to the other by a **finite sequence of local moves** called Roseman moves.



# Roseman moves



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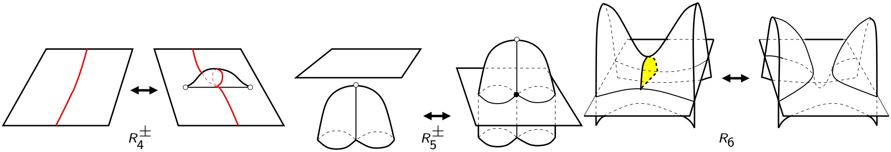
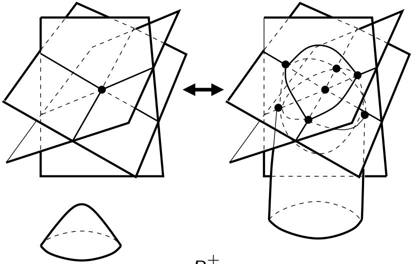


Roseman move $R_3^{+}$ create six triple point around a triple point $(x, y, z)$.



$R_3^{\pm}$

Suppose the colour of the moving disc is $d$. Then the six triple points are given by either $\partial(d, x, y, z)$ or $\partial(x, d, y, z)$ or $\partial(x, y, d, z)$ or $\partial(x, y, z, d)$.

# Pseudo cycles

## Definition

Let $c$ be a chain of $C_2(K_{D_F})$. If $c$ satisfies the following conditions,

(i) $\partial \mathrm{Col}_\sharp(c) = 0$ and

(ii) $[\mathrm{Col}_\sharp(c)] \neq 0 \in H_3^Q(X)$,

then $c$ is called a **pseudo cycle**.

## Theorem

*The number of pseudo-cycles in $K_{D_F}$ is an invariant under Roseman moves up to homology.*

Proof. A proof can by done by checking that each Roseman move does not change the number of pseudo cycles.

# Pseudo cycles

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# Coloring homomorphisms

Let $D_F$ be a surface diagram of a surface-knot $F$ and let $K_{D_F}$ be the rectangular complex induced from $D_F$. For a coloring homomorphism $\mathrm{Col}_* : H_2(K_{D_F}) \to H_3^Q(X)$, determined by the number of non-degenerate pseudo cycles. Thus the following holds.

**Theorem**

*Let $D_F$ be a surface diagram of a surface-knot $F$ and let $K_{D_F}$ be a rectangular-cell complex induced from $D_F$ colored by a quandle $X$. The number of coloring homomorphisms*

$$\mathrm{Col}_* : H_2(K_{D_F}) \to H_3^Q(X)$$

*is a surface-knot invariant.*

The number of pseudo cycles in $D_F$ will be denoted by $\nu(F)$.

### Theorem

*Let $F$ be a double twist spun of $(2, k)$-torus knot for odd $k > 1$. Then*

$$\nu(F) = 1$$

# The matrix of the boundary mapping

Suppose the complex $K_{D_F}$ contains a pseudo 2-cycle $\sum_{i=1}^{m} \tau_i$, where $\tau_i$ is a rectangle of $K_{D_F}$ and also edges $\zeta_1, \zeta_2, \ldots, \zeta_n$. For the homomorphism

$$\mathrm{Col} \circ \partial_2 : C_2(K_{D_F}) \to C_1(K_{D_F}) \to C_2^Q(X),$$

we can define an $(n \times m)$-matrix for the ordered non-degenerate generators of $C_2(K_{D_F})$ and $C_2^Q(X)$ denoted by $M(D_F)$.

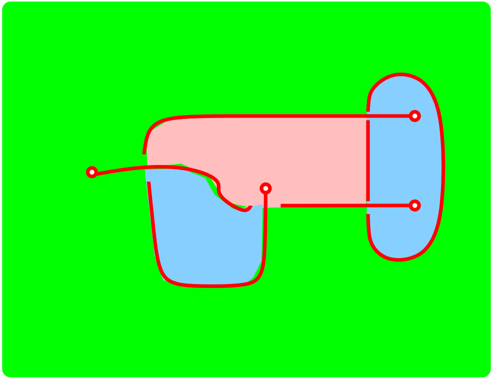
**Lemma**

*Let $F$ be a surface diagram coloured by a quandle $X$. Then*

$$\mathrm{rank}(M_{D_F}) \leq m - 1.$$

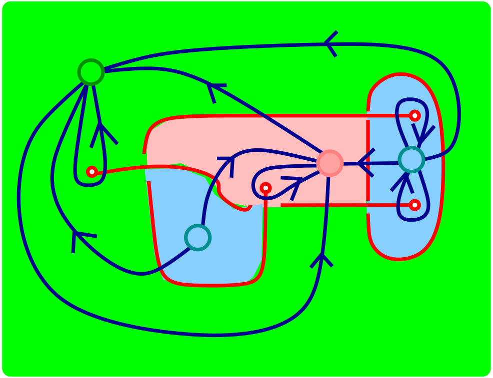
# $M_{D_F}$ of double twist spun $(2, p)$-torus knots

Let $T_{(2,p)}$ be the double twist spun $(2, p)$-torus knot.
$p = 3$. The number of triple points is 4. The pre-image of the reduced diagram is:



# $M_{D_F}$ of double twist spun $(2, p)$-torus knots

The rectangular-cell complex is constructed.



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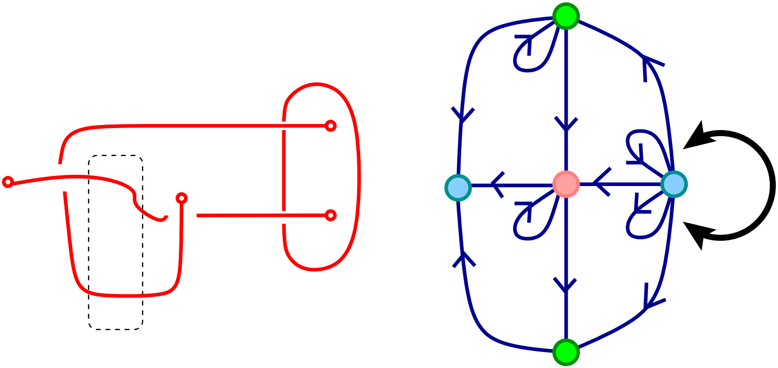


# $M_{D_F}$ of double twist spun $(2,p)$-torus knots

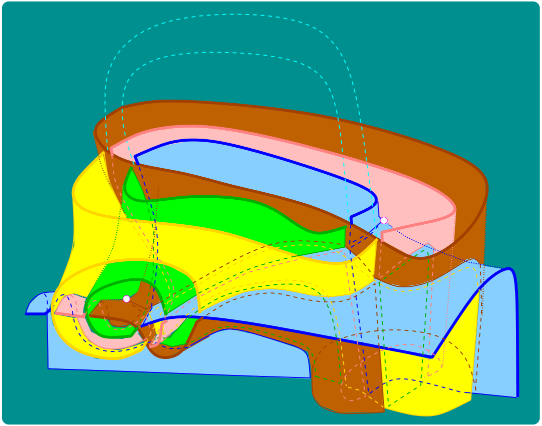
$C_2(KD_{T_{(2,3)}})$ is generated by four rectangles $\tau_1$, $\tau_2$, $\tau_3$ and $\tau_4$ colored as $\{(0,1,0)^+,(1,2,1)^-,(1,0,1)^-,(2,1,2)^+\}$ and the image of the chain is presented by six non-degenerate edges $\zeta_1$, $\zeta_2,\dots,\zeta_6$ colored as $\{(0,1),(0,2),(1,0),(1,2),(2,0),(2,1)\}$.

$$M_{K_{D_{T_{(2,3)}}}} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The rank is 3. This implies that there is no proper pseudo 2-cycles in $K_{D_{T_{(2,3)}}}$. Thus $\nu(T_{(2,p)}) = 1$.

# $M_{D_F}$ of double twist spun $(2,p)$-torus knots

$p = 5$. The number of triple points is 8.



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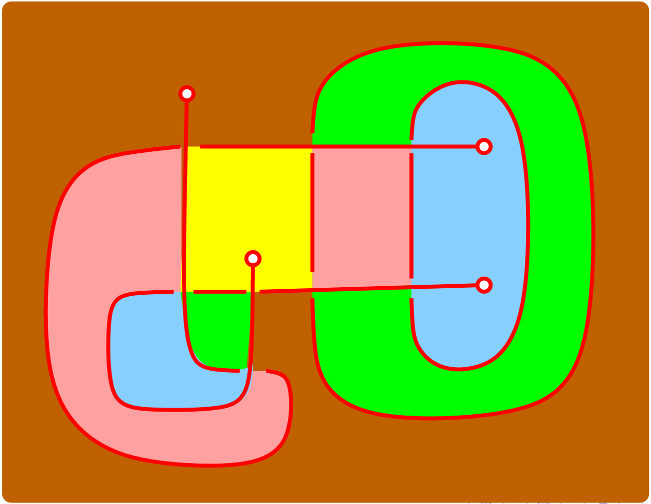

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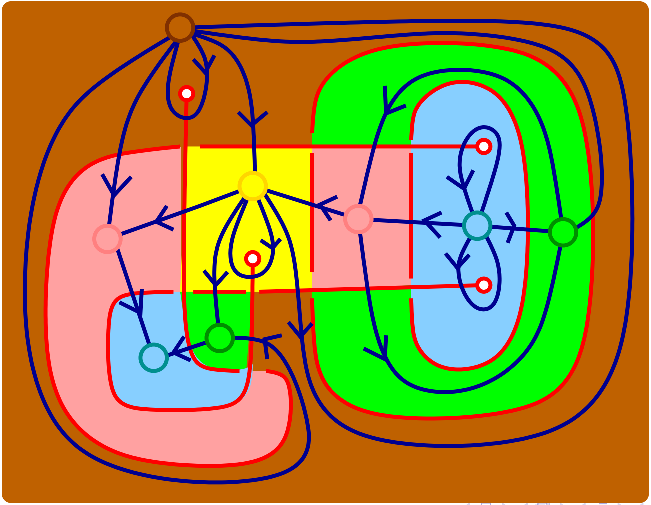

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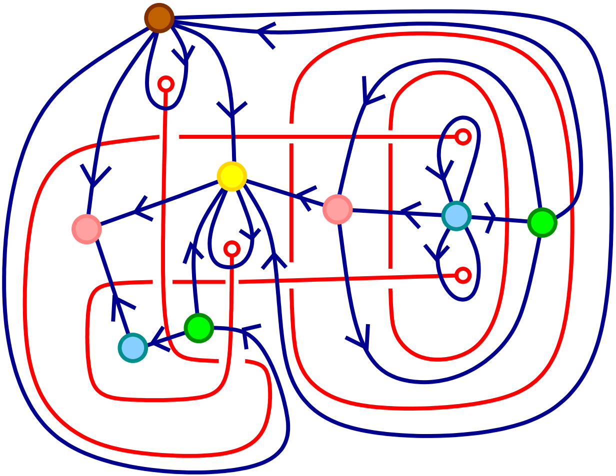

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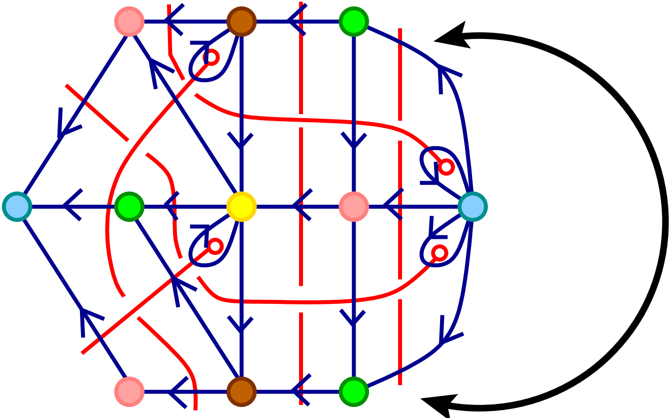

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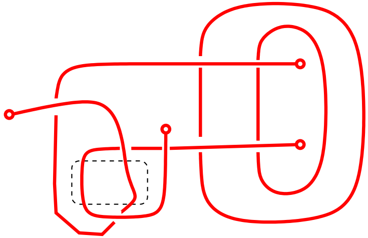
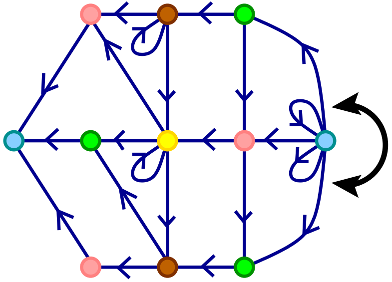
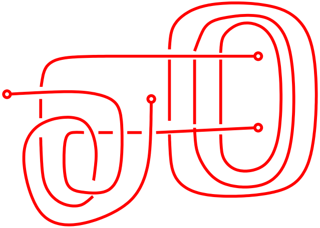


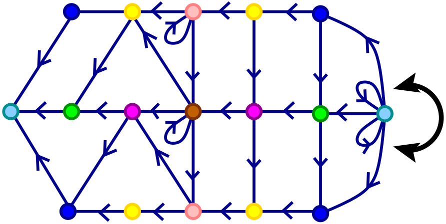


$\mathrm{rank}(M_{D_{T_{(2,5)}}}) = 7$, so $\nu(T_{(2,p)}) = 1$.

# $M_{D_F}$ of double twist spun $(2, p)$-torus knots

$p = 7$. The number of triple points is 12.

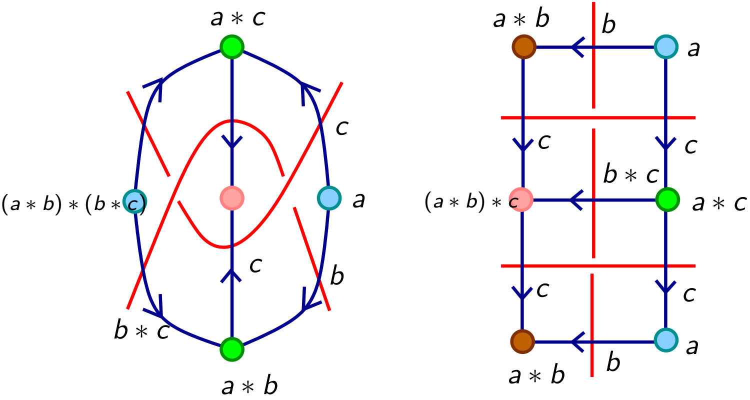




$\mathrm{rank}(M_{D_{T_{(2,p)}}}) = 11$, so $\nu(T_{(2,p)}) = 1$.

# $M_{D_F}$ of double twist spun $(2, p)$-torus knots

Four triple points are added: two pairs of rectangles are added.



Let $M = M_{K_{D_{T_{(2,p)}}}}$ and let $N = M_{K_{D_{T_{(2,p+2)}}}}$.

$$M \to N = \left[\begin{array}{ccc|ccccc} & & & * & * & * & * & * \\ & M' & & * & * & * & * & * \\ & & & * & * & * & * & * \\ \hline 0 & \dots & 0 & a & \dots & -a & 0 & \\ 0 & \dots & 0 & 0 & b & -b & \dots & 0 \\ \vdots & \ddots & & & \dots & & & \\ 0 & \dots & 0 & 0 & \dots & 0 & -d & d \end{array}\right].$$

If $\mathrm{rank}(M) = 2p - 3$ and $\mathrm{rank}(M') = 2p - 2$, then $\mathrm{rank}(N) = 2p + 1$.