On a surface-knot invariant obtained from the lower decker sets

Tsukasa Yashiro

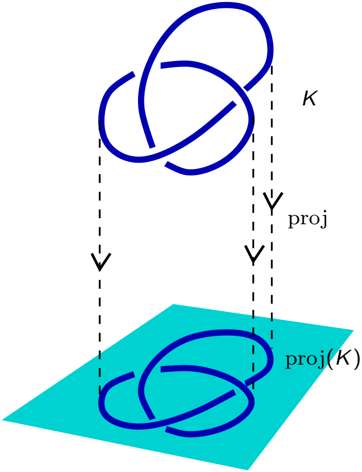
Department of Mathematics and Statistics
Sultan Qaboos University

DOMAS Seminar,
Sultan Qaboos University, Muscat, Oman
5 November 2015

1. Introduction (Background and Motivation)
2. Surfaces in 4-space
3. Pre-image of Multiple Points
4. Roseman moves
5. Pseudo cycles
6. Coloring homomorphisms
7. Matrices of boundary mappings

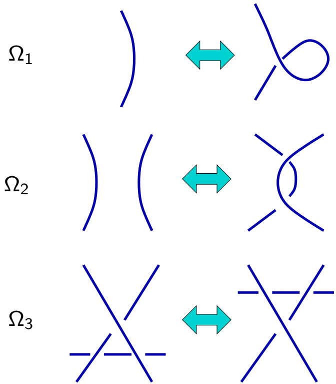
Classical Knots and their diagrams

A **knot** is a closed 1-dimensional manifold ($\mathbb{S}^1$) embedded in $\mathbb{R}^3$.
A **knot diagram** D_K is the image of K under the orthogonal projection $\mathrm{proj}(x_1, x_2, x_3) = (x_1, x_2)$ with crossing information.

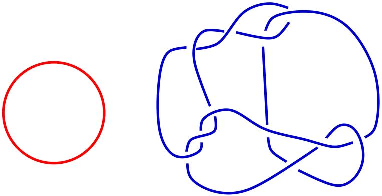


Reidemeister moves and knot deformation

Redemeister Moves

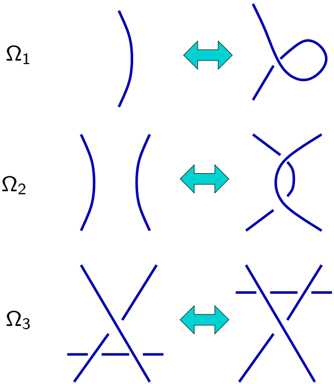


Equivalent knot diagrams are deformed by Reidemeister moves.

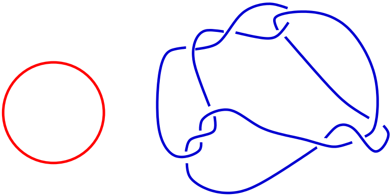


Reidemeister moves and knot deformation

Redemeister Moves

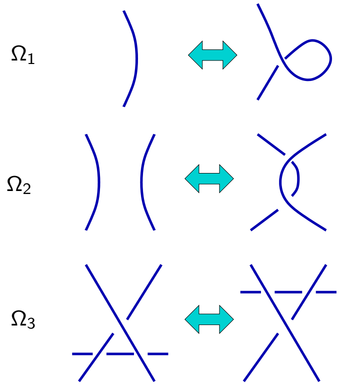


Equivalent knot diagrams are deformed by Reidemeister moves.

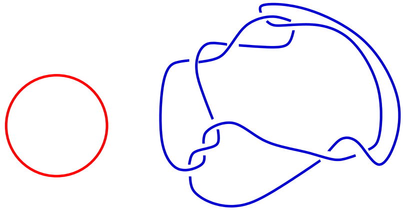


Reidemeister moves and knot deformation

Redemeister Moves

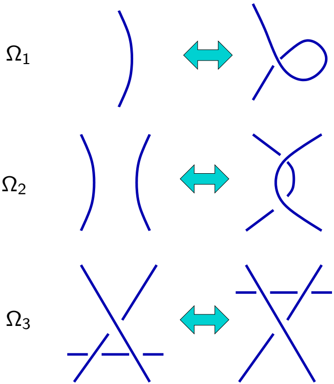


Equivalent knot diagrams are deformed by Reidemeister moves.

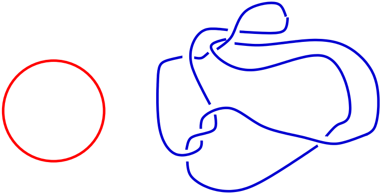


Reidemeister moves and knot deformation

Redemeister Moves

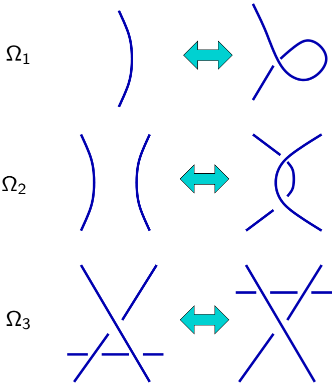


Equivalent knot diagrams are deformed by Reidemeister moves.

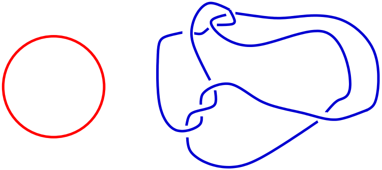


Reidemeister moves and knot deformation

Redemeister Moves

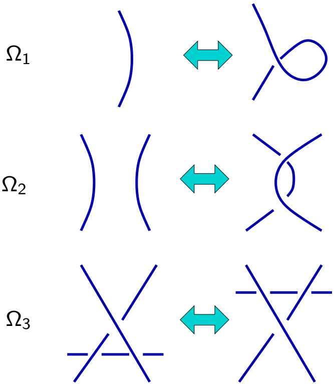


Equivalent knot diagrams are deformed by Reidemeister moves.

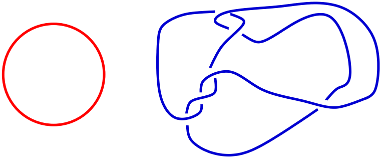


Reidemeister moves and knot deformation

Redemeister Moves

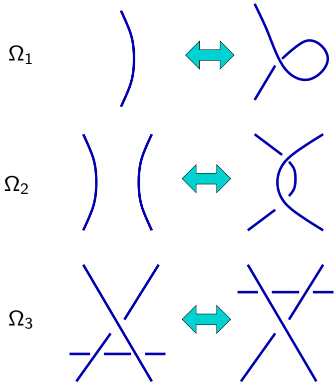


Equivalent knot diagrams are deformed by Reidemeister moves.

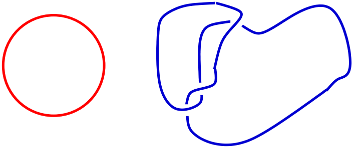


Reidemeister moves and knot deformation

Redemeister Moves

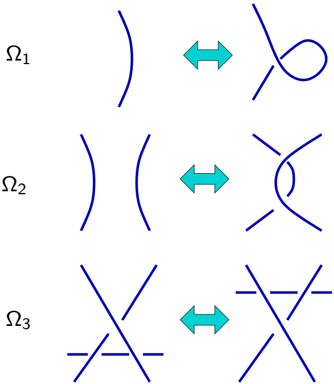


Equivalent knot diagrams are deformed by Reidemeister moves.

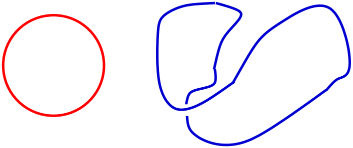


Reidemeister moves and knot deformation

Redemeister Moves

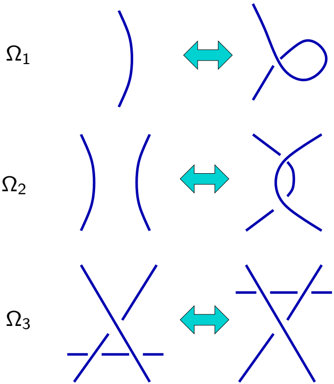


Equivalent knot diagrams are deformed by Reidemeister moves.

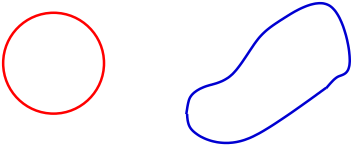


Reidemeister moves and knot deformation

Redemeister Moves



Equivalent knot diagrams are deformed by Reidemeister moves.

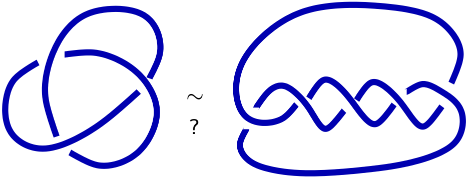


Reidemeister moves and knot deformation

To distinguish two knots, we need an albebraic invariant.

- Knot groups,
- Alexander, Jones polynomials,
- Quandles

Are they different?

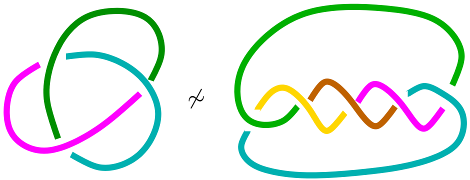


Reidemeister moves and knot deformation

To distinguish two knots, we need an albebraic invariant.

- Knot groups,
- Alexander, Jones polynomials,
- Quandles

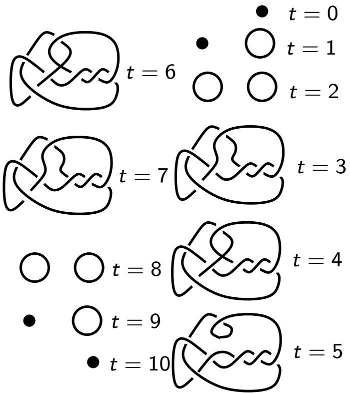
Yes, they are. The Alexander polynomials are different.



$t^{-1} - 1 + t^2$ $\qquad$ $t^{-2} - t^{-1} - t + t^2$

Surfaces in 4-space

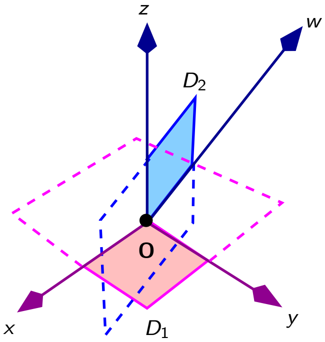
Knots; closed 1-manifolds embedded in $\mathbb{R}^3$ can be generalized as closed surfaces embedded in $\mathbb{R}^4$. One way to describe the surface F in $\mathbb{R}^4$ is to take intersections with the hyperplanes: $(\mathbb{R}^3 \times [t]) \cap F$.



Surfaces in 4-space

$$D_1 = \{(x, y, 0, 0) \mid |x| \le 1, |y| \le 1\},$$
$$D_2 = \{(0, 0, z, w) \mid |x| \le 1, |y| \le 1\}.$$

Then $D_1 \cap D_2 = \mathbf{O}$.
This intersection cannot be removed by a small isotopy move.



Background

- 1957 R. H. Fox and J. W. Milnor gave an example of a 2-knot.
- 1965 E. C. Zeeman introduced a construction method of a 2-knot called an ***m*-twist spinning**.
- 1982 A. Kawauchi, T. Shibuya and S. Suzuki described a surface-knot with a **normal form**.
- 1980s Roseman proposed diagrammatic approach to describe surface in 4-space and introduced elementary deformations called **Roseman moves** (1998).
- 1980s-1990s with the development of algebraic structure for this area such as racks and quandles, geometric approaches have been used and developed.

Background

- 1992 S. Kamada introduced braid surfaces and charts.
- 1998 J. S. Carter and M. Saito introduced the double decker set.
- 1999 J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford and M. Saito applied **quandle co-homology** to knots and surface-knots.
- 2002 S. Satoh and A. Shima determined triple point numbers for 2-twist and 3-twist spun trefoils.
- 2005 E. Hatakenaka gave a lower bound of triple point number for 2-twist spun $(2, 5)$-torus knot.

Background

- 2005 T. Y. showed Roseman moves can be described by six types of local moves.
- 2012 Jabonowski proved that there is a finite sequecne of Roseman moves between pseudo-ribbon surface-knot diagrams which must have some triple points on the way.
- 2012 A. Mohamad and T. Y. proved if lower decker set is connected and the number of triple points is at most two, then the knot group is isomorphic to $\mathbb{Z}$.
- 2015 K. Kawamura proved that one seven types of Roseman moves can be induced from other six.

Motivation

- Can we symbolize geometric objects? (example: tangles, surface braid charts, etc.) so that we can describe mathematics by these symbols.

Our research work (2015):

- **Surface-knots and their diagrams**
  (with Abdul Mohamad (Nizwa), Amal Al-Kharusi (SQU))
- **Surface-links** (with Zainab Al-Maamari (SQU)).
- **Topological Model for DNA Replications**
  (with A. Mohamad (Nizwa)).

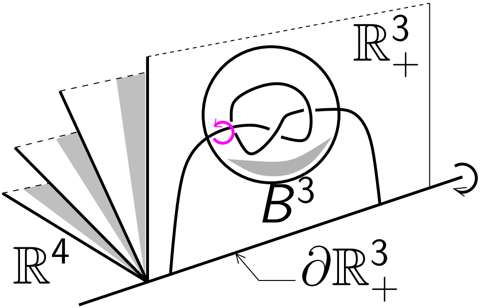
In this talk

We discuss about the number of essential connected components of the lower decker set of a surface diagram and invariants induced from them.

Zeeman's twist spinning

Let B^3 be a 3–ball in $\mathbb{R}^3_+$ such that it contains a tangle $T(K)$ of a knot K, and $\partial B^3 \cap T(K)$ is the pair of antipodal points of ∂B^3.

An m-**twist-spun knot** obtained from K is defined by rotating $B^3 \cap T(K)$ about the axis through the antipodal points m times while $\mathbb{R}^3_+$ spins denoted by $T_m(K)$.



Theorem (Zeeman, 1965)

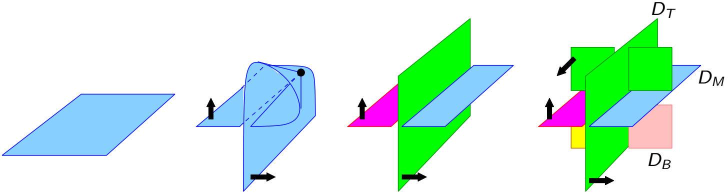
Every m-twist spun knot $T_m(K)$ obtained from K is fibred $(m \geq 1)$; the fibre is the one-punctured m-fold branched covering space of S^3 along K.

Corollary (Zeeman, 1965)

For any knot K, 1-twist spun knot obtained from K is trivial.

Surface Diagrams

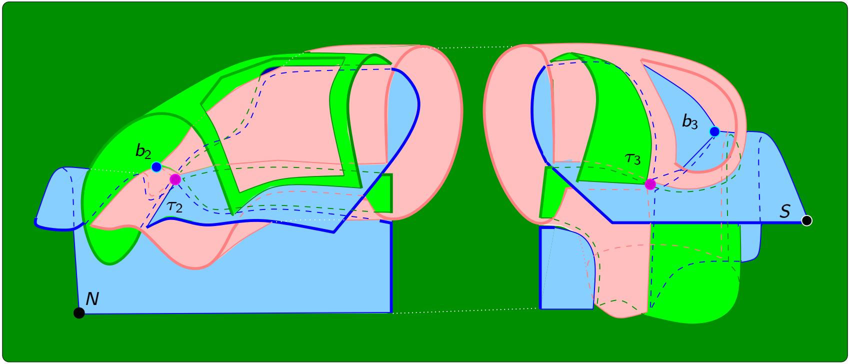
A **surface-knot** is a connected oriented closed surface embedded in 4-space. Let $F \subset \mathbb{R}^4$ be a surface-knot. Let $\mathrm{proj} : \mathbb{R}^4 \to \mathbb{R}^3$; $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3)$, be the orthogonal projection. A **surface diagram** of F is a union of the following local diagrams.



Branch point
Triple point

Example(Double twist spun trefoil)

S. Satoh (2002) constructed a diagram of twist span knots. The following is a reduced diagram obtained by a sequence of Roseman moves from Satoh's construction.



The preimage of singularities of the projection proj is:

$$S = \{x \in F \mid \#((\mathrm{proj}|_F)^{-1}(\mathrm{proj}(x)) > 1\}$$

The set S is the union of two families of immersed circles and immersed open intervals:

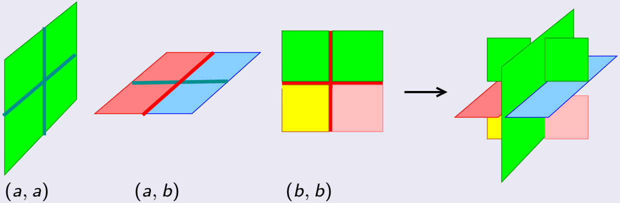
$$\begin{aligned} \mathcal{S}_a &= \{s_{a1}, s_{a2}, \ldots, s_{al}\} \\ \mathcal{S}_b &= \{s_{b1}, s_{b2}, \ldots, s_{bl}\} \end{aligned}$$

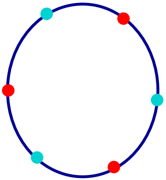
where for $x \in s_{ai}$, $y \in s_{bi}$ $(i = 1, 2, \ldots, l)$, if $\mathrm{proj}(x) = \mathrm{proj}(y)$, then $h(x) > h(y)$.

Lemma (Carter-Saito (1998))

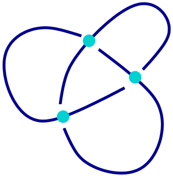
Let F be a closed orientable surface. Let $f : F \to \mathbb{R}^3$ be a generic map. Then there is an embedding $g : F \to \mathbb{R}^4$ such that $\mathrm{proj} \circ g = f$ *if and only if*

1. $S(f) = \bigcup \mathcal{S}_a \cup \bigcup \mathcal{S}_b$.
2. *For each triple point, the pre-images are crossings of types* (a, a), (a, b) *and* (b, b).

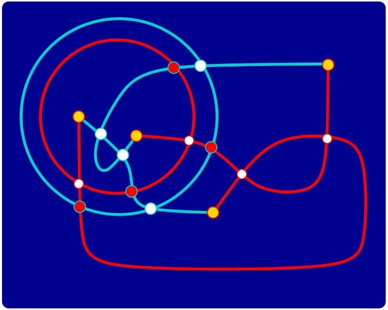




Pre-image of D_K

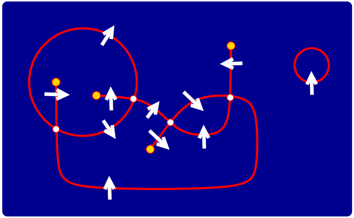


D_K



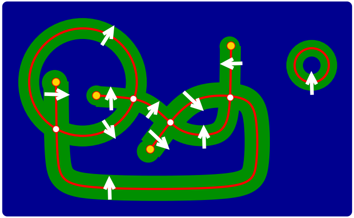
The closure of the pre-image of double curves in D_F is a union of two families of arcs called the **double decker set** (Carter-Saito). The blue arcs represent the **upper decker set** and the red arcs represent the **lower decker set**.

Rectangular-cell complexes



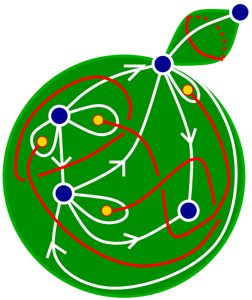
We denote the lower decker set by S_b.
$F \setminus S_b = \{R_0, \ldots, R_n\}$. Let $N(S_b)$ be a small neighbourhood of S_b in F.
$F \setminus N(S_b) = \{V_0, \ldots, V_n\}$;
$V_i \subset R_i$ $(i = 0, \ldots, n)$.

Rectangular-cell complexes



We denote the lower decker set by S_b.
$F \setminus S_b = \{R_0, \ldots, R_n\}$. Let $N(S_b)$ be a small neighbourhood of S_b in F.
$F \setminus N(S_b) = \{V_0, \ldots, V_n\}$;
$V_i \subset R_i$ $(i = 0, \ldots, n)$.

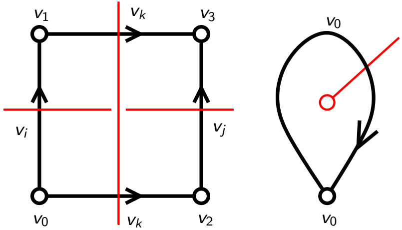
Rectangular-cell complexes



The quotient map $q : F \to F/_{\sim}$ is defined by $q(V_i) = v_i$, $(i = 0, \dots, n)$. The quotient space is a 2-dimensional complex. We will denote the complex by K_{D_F}. A subcomplex of K_{D_F} induced from a simple closed curve in S_b is called a **bubble**.

Rectangular-cell complexes

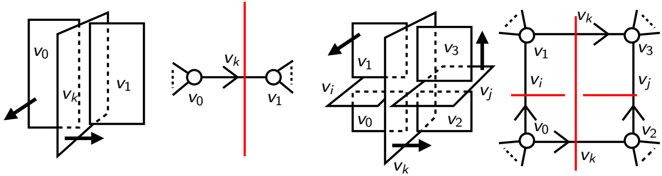
A subcomplex of K_{D_F} corresponding to a connected component of the lower decker set S_b is called a **parcel**. Each parcel is a bubble or a subcomplex consisting of some rectangles and loop discs:



We denote the rectangle by $(v_0; v_0v_1, v_0v_2; v_3)$ and the loop by $\widehat{v_0v_0}$.

Rectangular-cell complexes

Each double segment corresponds to an edge of the complex K_{D_F}.
Each edge has a **weight**, which is a vertex of the complex.



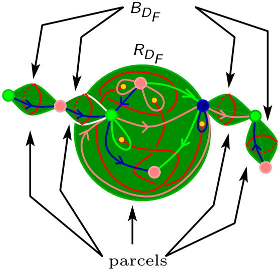
The lower decker set $S_b \subset |K_{D_F}|$ is a union of edges of K_{D_F}.

Rectangular-cell complexes

K_{D_F} can be decomposed into parcels $K_1, \ldots, K_n$ such that

$$\begin{aligned} K_{D_F} &= K_1 + \cdots + K_n, \\ &= Rec_{D_F} + Bub_{D_F}. \end{aligned}$$

where Rec_{D_F} is the union of rectangles and loop discs, and Bub_{D_F} be the union of bubbles.

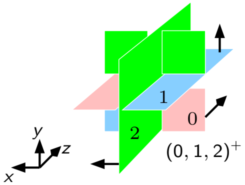


Quandle colorings

A **quandle** X is a non-empty set with a binary operation $(a, b) \mapsto a * b$ such that

1. For any $a \in X$, $a * a = a$,
2. For any $a, b \in X$, there is a unique $c \in X$ such that $c * b = a$.
3. For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

Quandle colorings



The **dihedral quandle** $(X, *)$ of order $n > 0$ denoted by R_n is a quandle $X = \{0, \ldots, n-1\}$ with the binary operation $i * j = 2j - i \pmod{n}$.

The triple point in the left diagram is coloured by R_3; $(0, 1, 2)$ and the orientation is determined by orientation normals to D_T, D_M, D_B respectively.

Quandle colorings

Let $\mathcal{R}$ be the set of connected components of $F - S_b$. For a quandle X, a **quandle coloring** of a diagram is a mapping $\mathrm{Col} : \mathcal{R} \to X$ such that if

1. V_1 and V_2 in $\mathcal{R}$ have a common boundary arc c corresponding to an upper sheet $V_3 \in \mathcal{R}$ and
2. the orientation normal to $\mathrm{proj}(V_3)$ directs from $\mathrm{proj}(V_1)$ to $\mathrm{proj}(V_2)$,

then $\mathrm{Col}(V_1) * \mathrm{Col}(V_3) = \mathrm{Col}(V_2)$.

Quandle colorings

The coloring Col can be interpreted in terms of the rectangular-cell complex K_{D_F}. If an edge e is incident with vertices v_1 and v_2, oriented from v_1 to v_2 and with weight v_3, then the mapping from the 1-skeleton to X

$$\mathrm{Col} : K_{D_F}^{(1)} \to X$$

is defined satisfying $\mathrm{Col}(v_1) * \mathrm{Col}(v_3) = \mathrm{Col}(v_2)$. We call this mapping also a **quandle coloring**.

Chain groups

Quandle chain groups

Let $C_n(X)$ $(n \geq 1)$ be a free abelian group generated by n-tuples $(x_1, \ldots, x_n) \in X^n$. Let $C_n^D(X)$ be a sub group of $C_n(X)$ generated by $(x_1, \ldots, x_n)$ such that $x_i = x_j$ for some $1 \leq i, j, \leq n$ and $(|i - j| = 1)$. We denote the quotient group $C_n(X)/C_n^D(X)$ by $C_n^Q(X)$.

Chain groups of K_{D_F}

The chain group $C_k(K_{D_F})$ is defined as a free abelian group generated by k-dimensional elements of K_{D_F}. For $k = 2$, it is generated by the rectangular cells, loop discs and bubbles in K_{D_F}. For $k = 1$, it is generated by edges in K_{D_F}. For $k = 0$, it is generated by vertices of K_{D_F}.

Chain groups

Quandle chain groups

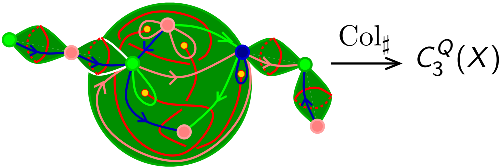
Let $C_n(X)$ $(n \geq 1)$ be a free abelian group generated by n-tuples $(x_1, \ldots, x_n) \in X^n$. Let $C_n^D(X)$ be a sub group of $C_n(X)$ generated by $(x_1, \ldots, x_n)$ such that $x_i = x_j$ for some $1 \leq i, j, \leq n$ and $(|i-j| = 1)$. We denote the quotient group $C_n(X)/C_n^D(X)$ by $C_n^Q(X)$.

Chain groups of K_{D_F}

The chain group $C_k(K_{D_F})$ is defined as a free abelian group generated by k-dimensional elements of K_{D_F}. For $k = 2$, it is generated by the rectangular cells, loop discs and bubbles in K_{D_F}. For $k = 1$, it is generated by edges in K_{D_F}. For $k = 0$, it is generated by vertices of K_{D_F}.

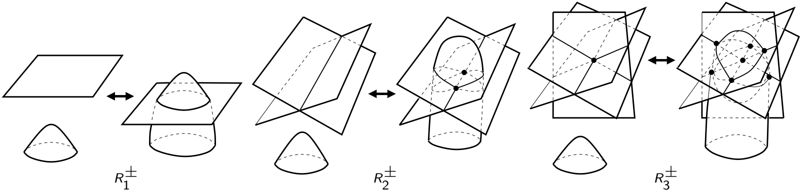
Coloring homomorphisms

The quandle coloring Col can be extended to a homomorphism $\mathrm{Col}_\sharp : C_2(K_{D_F}) \to C_3^Q(X)$ defined as follows. For
$\sigma = (v_0; v_0 v_1, v_0 v_2; v_3)$,
$\mathrm{Col}_\sharp(\sigma) = (\mathrm{Col}(v_0), \mathrm{Col}(v_0 v_1), \mathrm{Col}(v_0 v_2) \in C_3^Q(X)$.

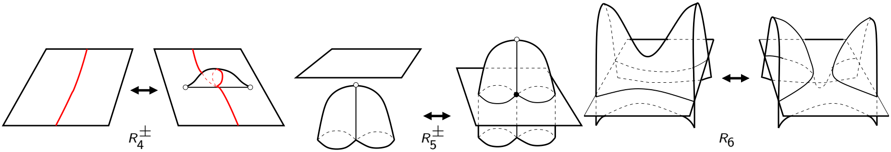


Roseman moves

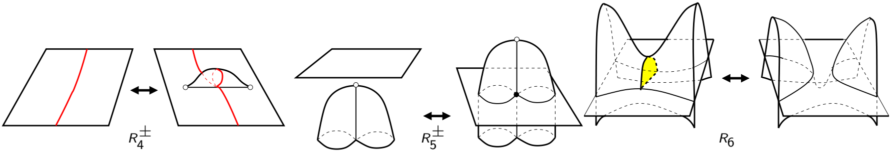
Two surface diagrams are equivalent if they are projected image of the same type of a surface-knot. Two equivalent surface diagrams are modified from one to the other by a **finite sequence of local moves** called Roseman moves.



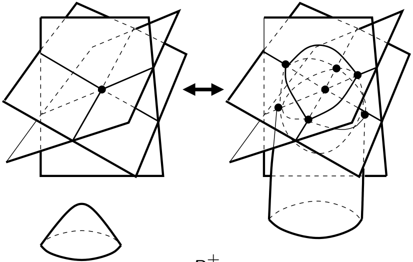
Roseman moves



Roseman moves



Roseman move R_3^+ create six triple point around a triple point (x, y, z).



Suppose the colour of the moving disc is d. Then the six triple points are given by either $\partial(d, x, y, z)$ or $\partial(x, d, y, z)$ or $\partial(x, y, d, z)$ or $\partial(x, y, z, d)$.

Pseudo cycles

Definition

Let c be a chain of $C_2(K_{D_F})$. If c satisfies the following conditions,

(i) $\partial \mathrm{Col}_\sharp(c) = 0$ and

(ii) $[\mathrm{Col}_\sharp(c)] \neq 0 \in H_3^Q(X)$,

then c is called a **pseudo cycle**.

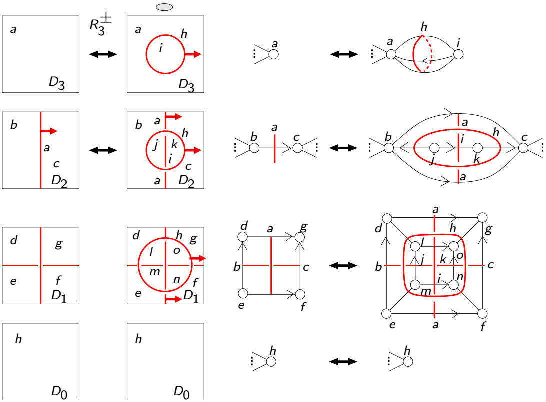
Pseudo cycles

Theorem

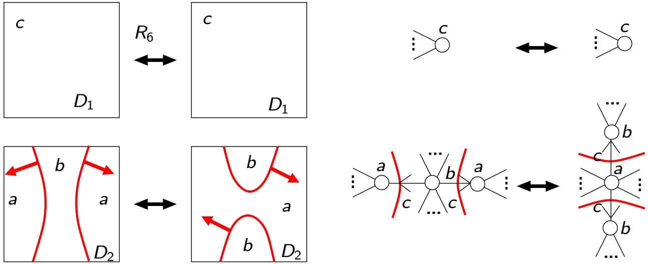
For a surface-knot diagram D_F, the maximal number of pseudo-cycles in K_{D_F} is an invariant under Roseman moves up to quandle homology.

Proof. It is suffised to check each Roseman move does not change the number of pseudo cycles.

Pseudo cycles



Pseudo cycles



Coloring homomorphisms

Let D_F be a surface diagram of a surface-knot F and let K_{D_F} be the rectangular complex induced from D_F. For a coloring homomorphism $\mathrm{Col}_* : H_2(K_{D_F}) \to H_3^Q(X)$, determined by the number of non-degenerate pseudo cycles. Thus the following holds.

Theorem

Let D_F be a surface diagram of a surface-knot F and let K_{D_F} be a rectangular-cell complex induced from D_F colored by a quandle X. The number of coloring homomorphisms

$$\mathrm{Col}_* : H_2(K_{D_F}) \to H_3^Q(X)$$

is a surface-knot invariant.

The number of pseudo cycles in D_F will be denoted by $\nu(F)$.

Theorem

Let F be a double twist spun of $(2, k)$-torus knot for odd prime $k > 1$. Then

$$\nu(F) = 1$$

The matrix of the boundary mapping

Suppose the complex K_{D_F} contains a pseudo 2-cycle $\sum_{i=1}^{m} \tau_i$, where τ_i is a rectangle of K_{D_F} and also edges $\zeta_1, \zeta_2, \ldots, \zeta_n$. For the homomorphism

$$\mathrm{Col} \circ \partial_2 : C_2(K_{D_F}) \to C_1(K_{D_F}) \to C_2^Q(X),$$

we can define an $(n \times m)$-matrix for the ordered non-degenerate generators of $C_2(K_{D_F})$ and $C_2^Q(X)$ denoted by $M(D_F)$.

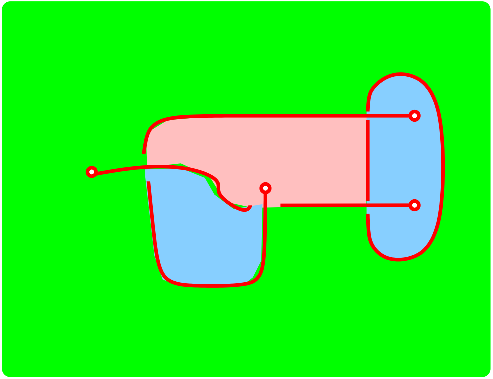
Lemma

Let F be a surface diagram coloured by a quandle X. Then

$$\mathrm{rank}(M_{D_F}) \leq m - 1.$$

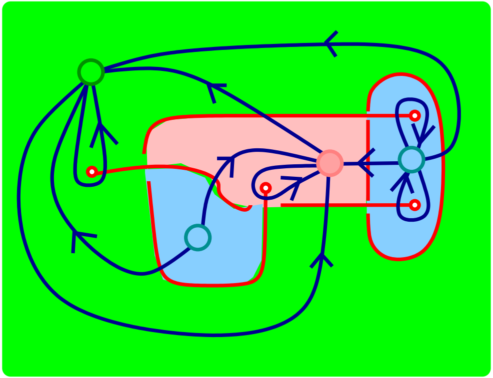
M_{D_F} of double twist spun $(2, p)$-torus knots

Let $T_{(2,p)}$ be the double twist spun $(2, p)$-torus knot.
$p = 3$. The number of triple points is 4. The pre-image of the reduced diagram is:

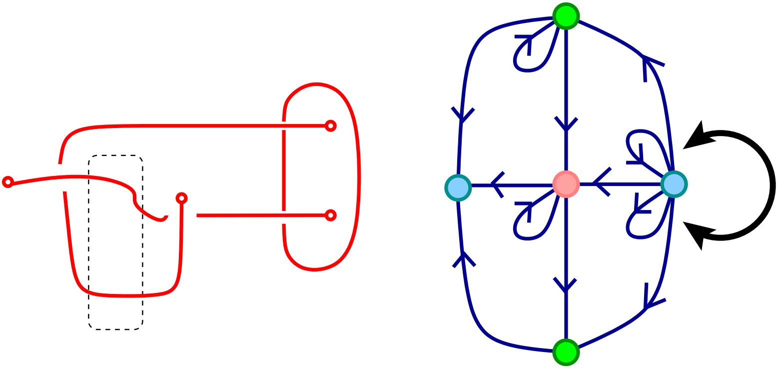


M_{D_F} of double twist spun $(2, p)$-torus knots

The rectangular-cell complex is constructed.



M_{D_F} of double twist spun $(2, p)$-torus knots



M_{D_F} of double twist spun $(2,p)$-torus knots

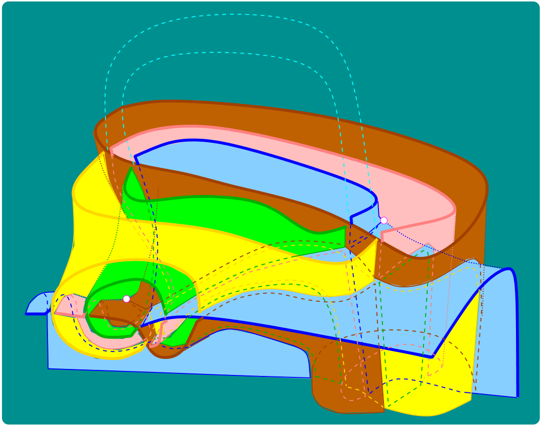
$C_2(KD_{T_{(2,3)}})$ is generated by four rectangles τ_1, τ_2, τ_3 and τ_4 colored as $\{(0,1,0)^+,(1,2,1)^-,(1,0,1)^-,(2,1,2)^+\}$ and the image of the chain is presented by six non-degenerate edges ζ_1, $\zeta_2,\dots,\zeta_6$ colored as $\{(0,1),(0,2),(1,0),(1,2),(2,0),(2,1)\}$.

$$M_{K_{D_{T_{(2,3)}}}} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

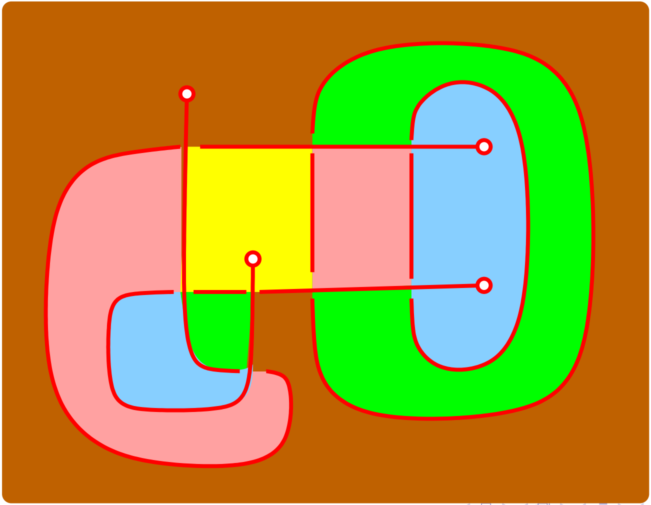
The rank is 3. This implies that there is no proper pseudo 2-cycles in $K_{D_{T_{(2,3)}}}$. Thus $\nu(T_{(2,p)}) = 1$.

M_{D_F} of double twist spun $(2, p)$-torus knots

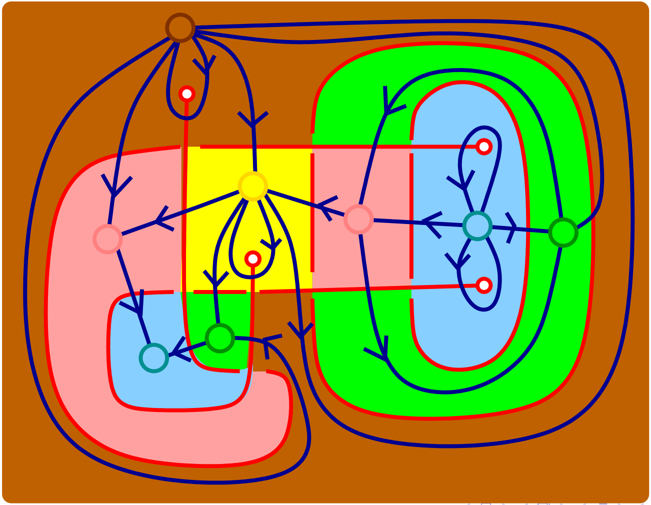
$p = 5$. The number of triple points is 8.



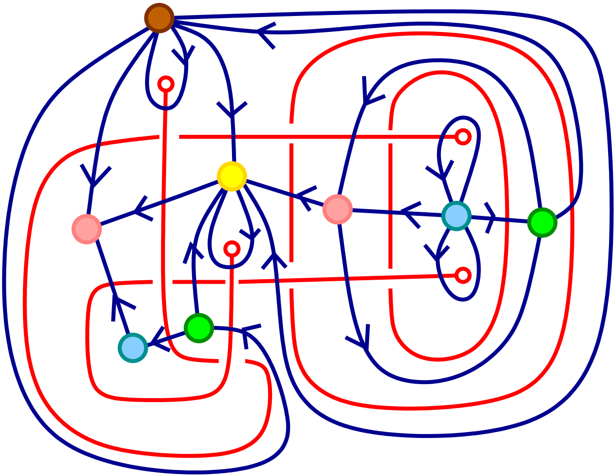
M_{D_F} of double twist spun $(2, p)$-torus knots



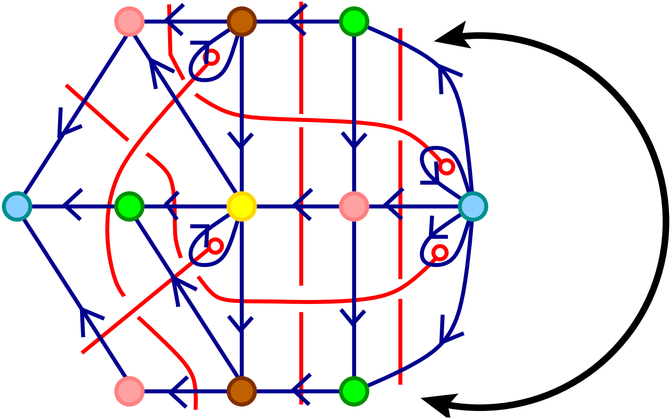
M_{D_F} of double twist spun $(2, p)$-torus knots



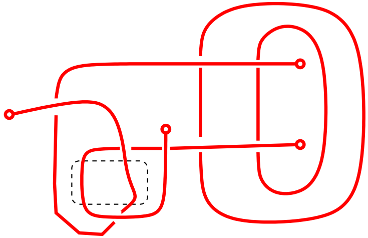
M_{D_F} of double twist spun $(2, p)$-torus knots

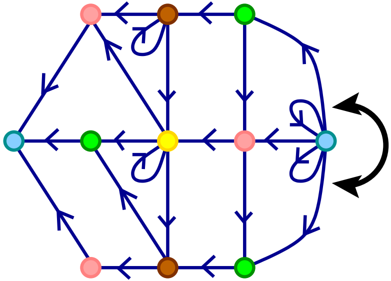


M_{D_F} of double twist spun $(2, p)$-torus knots



M_{D_F} of double twist spun $(2, p)$-torus knots

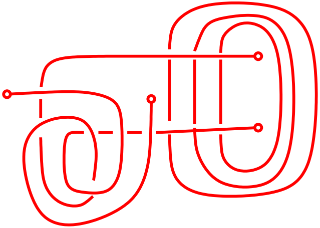


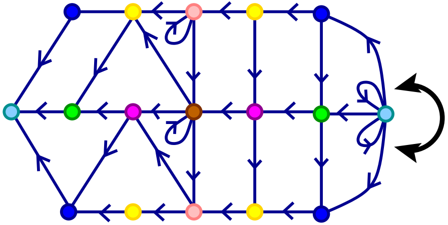


$\mathrm{rank}(M_{D_{T_{(2,5)}}}) = 7$, so $\nu(T_{(2,p)}) = 1$.

M_{D_F} of double twist spun $(2,p)$-torus knots

$p = 7$. The number of triple points is 12.

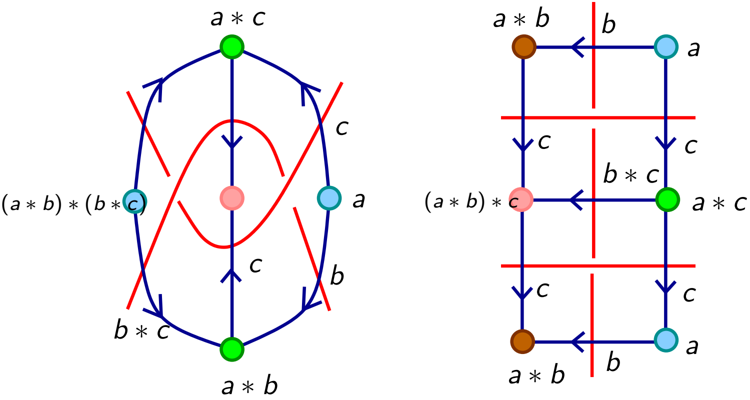




$\mathrm{rank}(M_{D_{T_{(2,p)}}}) = 11$, so $\nu(T_{(2,p)}) = 1$.

M_{D_F} of double twist spun $(2, p)$-torus knots

Four triple points are added: two pairs of rectangles are added.



Let $M = M_{K_{D_{T_{(2,p)}}}}$ and let $N = M_{K_{D_{T_{(2,p+2)}}}}$.

$$M \to N = \left[\begin{array}{ccc|ccccc} & & & * & * & * & * & * \\ & M' & & * & * & * & * & * \\ & & & * & * & * & * & * \\ \hline 0 & \dots & 0 & a & \dots & -a & 0 & \\ 0 & \dots & 0 & 0 & b & -b & \dots & 0 \\ \vdots & \ddots & & & \dots & & & \\ 0 & \dots & 0 & 0 & \dots & 0 & -d & d \end{array}\right].$$

If $\mathrm{rank}(M) = 2p-3$ and $\mathrm{rank}(M') = 2p-2$, then $\mathrm{rank}(N) = 2p+1$.

Thank You