On a surface-knot invariant obtained from surface-knot diagrams

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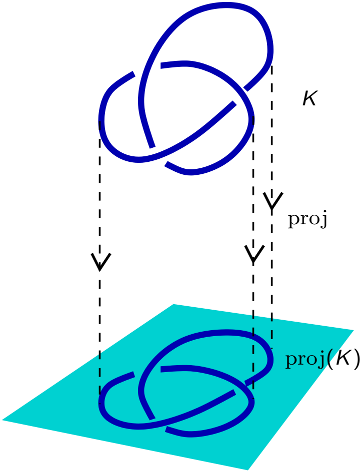
Classical Knots and their diagrams

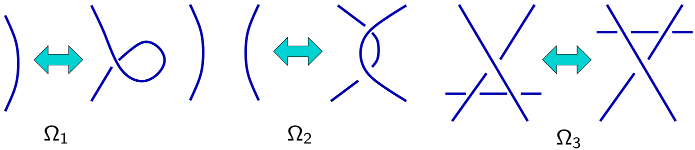
A **knot** is a closed 1-dimensional manifold ($\mathbb{S}^1$) embedded in $\mathbb{R}^3$.

A **knot diagram** D_K is the image of K under the orthogonal projection $\mathrm{proj}(x_1, x_2, x_3) = (x_1, x_2)$ with crossing information.



Reidemeister moves and knot deformation

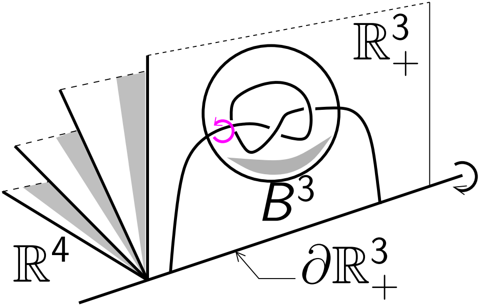
Redemeister Moves



Zeeman's twist spinning

Let B^3 be a 3–ball in $\mathbb{R}^3_+$ such that it contains a tangle $T(K)$ of a knot K, and $\partial B^3 \cap T(K)$ is the pair of antipodal points of ∂B^3.

An m-**twist-spun knot** obtained from K is defined by rotating $B^3 \cap T(K)$ about the axis through the antipodal points m times while $\mathbb{R}^3_+$ spins denoted by $T_m(K)$.



Theorem (Zeeman, 1965)

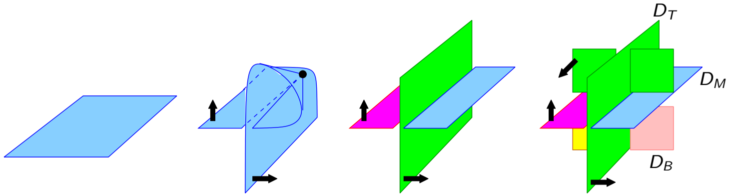
Every m-twist spun knot $T_m(K)$ obtained from K is fibred ($m \geq 1$); the fibre is the one-punctured m-fold branched covering space of S^3 along K.

Corollary (Zeeman, 1965)

For any knot K, 1-twist spun knot obtained from K is trivial.

Surface-knot Diagrams

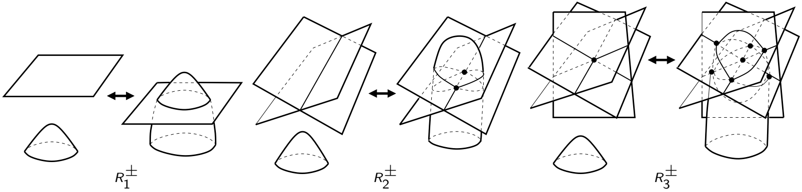
A **surface-knot** is a connected oriented closed surface embedded in 4-space. Let $F \subset \mathbb{R}^4$ be a surface-knot. Let $\mathrm{proj} : \mathbb{R}^4 \to \mathbb{R}^3$; $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3)$, be the orthogonal projection. A **surface-knot diagram** of F is a union of the following local diagrams.



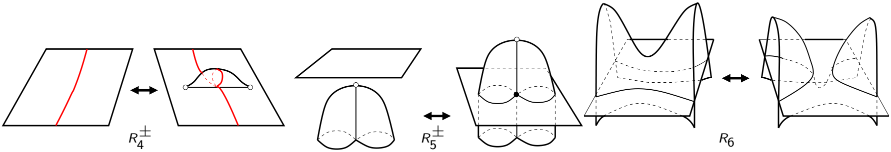
Branch point   Triple point

Roseman moves

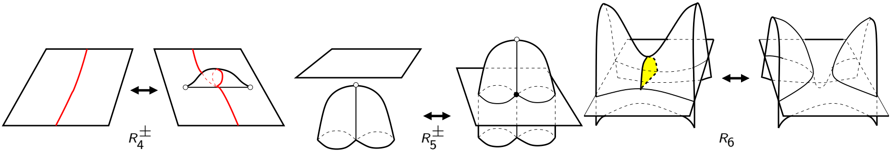
Two surface-knot diagrams are equivalent if they are projected image of the same type of a surface-knot. Two equivalent surface-knot diagrams are modified from one to the other by a **finite sequence of local moves** called Roseman moves.



Roseman moves

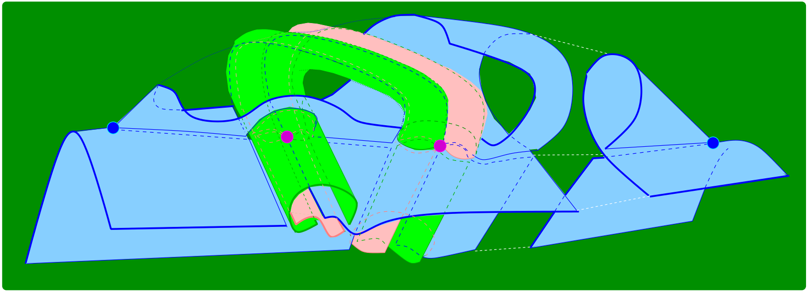


Roseman moves



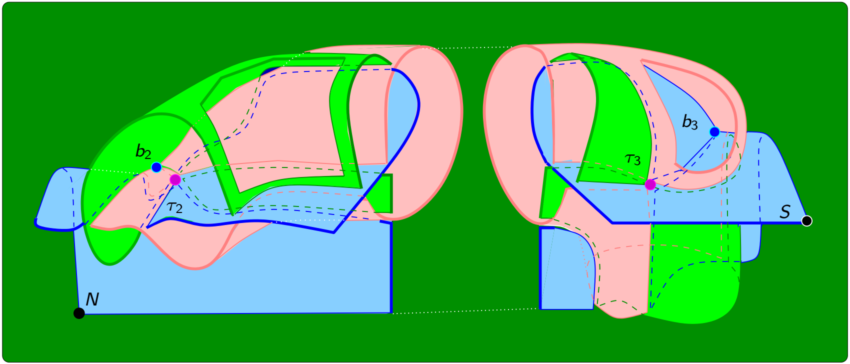
Example(Double twist spun trefoil)

S. Satoh (2002) constructed a diagram of twist span knots.



Example(Double twist spun trefoil)

The following is a reduced diagram obtained by a sequence of Roseman moves from Satoh's construction.



The preimage of singularities of the projection proj is:

$$S = \{x \in F \mid \#((\mathrm{proj}|_F)^{-1}(\mathrm{proj}(x)) > 1\}$$

The set S is the union of two families of immersed circles and immersed open intervals:

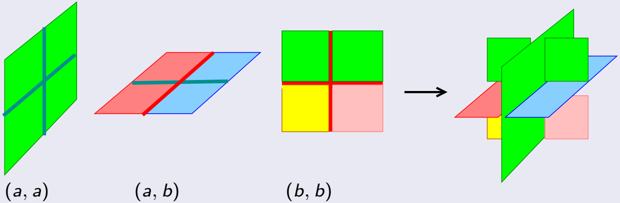
$$\begin{aligned} \mathcal{S}_a &= \{s_{a1}, s_{a2}, \ldots, s_{al}\} \\ \mathcal{S}_b &= \{s_{b1}, s_{b2}, \ldots, s_{bl}\} \end{aligned}$$

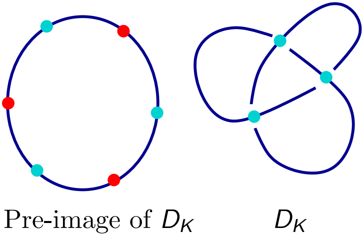
where for $x \in s_{ai}$, $y \in s_{bi}$ $(i = 1, 2, \ldots, l)$, if $\mathrm{proj}(x) = \mathrm{proj}(y)$, then $h(x) > h(y)$.

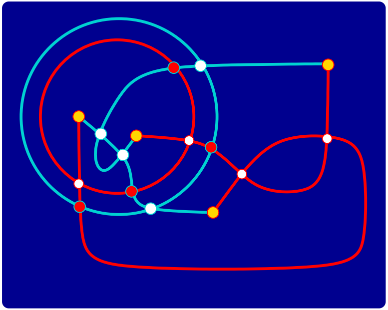
Lemma (Carter-Saito (1998))

Let F be a closed orientable surface. Let $f : F \to \mathbb{R}^3$ be a generic map. Then there is an embedding $g : F \to \mathbb{R}^4$ such that $\mathrm{proj} \circ g = f$ if and only if

1. $S(f) = \bigcup \mathcal{S}_a \cup \bigcup \mathcal{S}_b$.
2. *For each triple point, the pre-images are crossings of types (a, a), (a, b) and (b, b).*

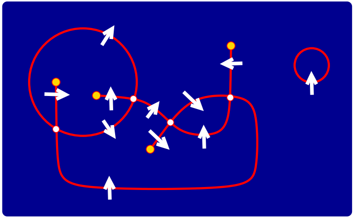






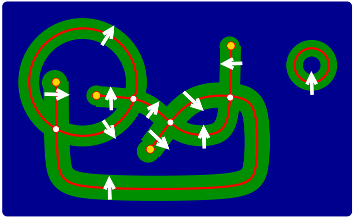
The closure of the pre-image of double curves in D_F is a union of two families of arcs called the **double decker set** (Carter-Saito). The blue arcs represent the **upper decker set** and the red arcs represent the **lower decker set**.

Rectangular-cell complexes



We denote the lower decker set by S_b.
$F \setminus S_b = \{R_0, \ldots, R_n\}$. Let $N(S_b)$ be a small neighbourhood of S_b in F.
$F \setminus N(S_b) = \{V_0, \ldots, V_n\}$;
$V_i \subset R_i$ $(i = 0, \ldots, n)$.

Rectangular-cell complexes



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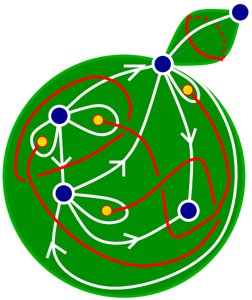
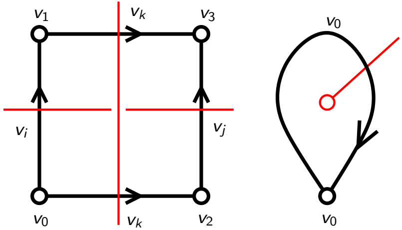
Rectangular-cell complexes



The quotient map $q : F \to F/_{\sim}$ is defined by $q(V_i) = v_i$, $(i = 0, \ldots, n)$. The quotient space is a 2-dimensional complex. We will denote the complex by K_{D_F}. A subcomplex of K_{D_F} induced from a simple closed curve in S_b is called a **bubble**.

Rectangular-cell complexes

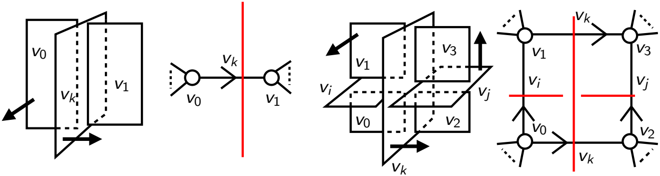
A subcomplex of K_{D_F} corresponding to a connected component of the lower decker set S_b is called a **parcel**. Each parcel is a bubble or a subcomplex consisting of some rectangles and loop discs:



We denote the rectangle by $(v_0; v_0v_1, v_0v_2; v_3)$ and the loop by $\widehat{v_0v_0}$.

Rectangular-cell complexes

Each double segment corresponds to an edge of the complex K_{D_F}.
Each edge has a **weight**, which is a vertex of the complex.



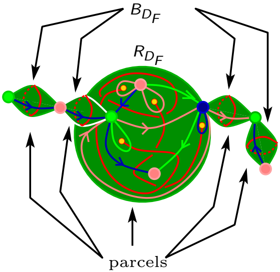
The lower decker set $S_b \subset |K_{D_F}|$ is a union of edges of K_{D_F}.

Rectangular-cell complexes

K_{D_F} can be decomposed into parcels $K_1, \ldots, K_n$ such that

$$\begin{aligned} K_{D_F} &= K_1 + \cdots + K_n, \\ &= Rec_{D_F} + Bub_{D_F}. \end{aligned}$$

where Rec_{D_F} is the union of rectangles and loop discs, and Bub_{D_F} be the union of bubbles.

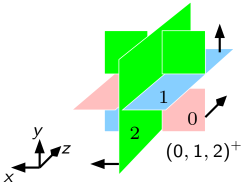


Quandle colorings

A **quandle** X is a non-empty set with a binary operation $(a, b) \mapsto a * b$ such that

1. For any $a \in X$, $a * a = a$,
2. For any $a, b \in X$, there is a unique $c \in X$ such that $c * b = a$.
3. For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

Quandle colorings



The **dihedral quandle** $(X, *)$ of order $n > 0$ denoted by R_n is a quandle $X = \{0, \ldots, n-1\}$ with the binary operation $i * j = 2j - i \pmod{n}$.

The triple point in the left diagram is coloured by R_3; $(0, 1, 2)$ and the orientation is determined by orientation normals to D_T, D_M, D_B respectively.

Quandle colorings

Let $\mathcal{R} = \{R_1, R_2, \ldots, R_n\}$ be the set of closures of connected components of $F - S_b$. For a quandle X, a **quandle coloring** of a diagram is a mapping $\mathrm{Col} : \mathcal{R} \to X$ such that if

1. R_1 and R_2 in $\mathcal{R}$ have a common boundary arc c corresponding to an upper sheet $R_3 \in \mathcal{R}$ and
2. the orientation normal to $\mathrm{proj}(R_3)$ directs from $\mathrm{proj}(R_1)$ to $\mathrm{proj}(R_2)$,

then $\mathrm{Col}(R_1) * \mathrm{Col}(R_3) = \mathrm{Col}(R_2)$.

Quandle colorings

The coloring Col can be interpreted in terms of the rectangular-cell complex K_{D_F}. If an edge e is incident with vertices v_1 and v_2, oriented from v_1 to v_2 and with weight v_3, then the mapping from the 1-skeleton to X

$$\mathrm{Col} : K_{D_F}^{(1)} \to X$$

is defined satisfying $\mathrm{Col}(v_1) * \mathrm{Col}(v_3) = \mathrm{Col}(v_2)$. We call this mapping also a **quandle coloring**.

Chain groups

Quandle chain groups

Let $C_n(X)$ $(n \geq 1)$ be a free abelian group generated by n-tuples $(x_1, \ldots, x_n) \in X^n$. Let $C_n^D(X)$ be a sub group of $C_n(X)$ generated by $(x_1, \ldots, x_n)$ such that $x_i = x_j$ for some $1 \leq i, j, \leq n$ and $(|i - j| = 1)$. We denote the quotient group $C_n(X)/C_n^D(X)$ by $C_n^Q(X)$.

Chain groups of K_{D_F}

The chain group $C_k(K_{D_F})$ is defined as a free abelian group generated by k-dimensional elements of K_{D_F}. For $k = 2$, it is generated by the rectangular cells, loop discs and bubbles in K_{D_F}. For $k = 1$, it is generated by edges in K_{D_F}. For $k = 0$, it is generated by vertices of K_{D_F}.

Chain groups

Quandle chain groups

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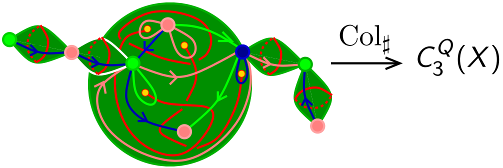
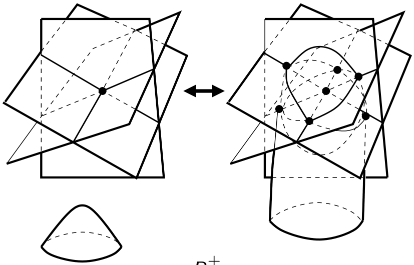
Coloring homomorphisms



The quandle coloring Col can be extended to a homomorphism $\mathrm{Col}_\sharp : C_2(K_{D_F}) \to C_3^Q(X)$ defined as follows.
For $\sigma = (v_0; v_0v_1, v_0v_2; v_3)$,

$$\mathrm{Col}_\sharp(\sigma) = (\mathrm{Col}(v_0), \mathrm{Col}(v_0v_1), \mathrm{Col}(v_0v_2)) \in C_3^Q(X).$$

Roseman move R_3^+ create six triple point around a triple point (x, y, z).



$R_3^{\pm}$

Suppose the colour of the moving disc is d. Then the six triple points are given by either $\partial(d, x, y, z)$ or $\partial(x, d, y, z)$ or $\partial(x, y, d, z)$ or $\partial(x, y, z, d)$.

Pseudo-cycles

Definition

Let c be a chain of $C_2(K_{D_F})$. If c satisfies the following conditions,

(i) $\mathrm{Col}_\sharp\partial(c) = 0$ and

(ii) $[\mathrm{Col}_\sharp(c)] \neq 0 \in H_3^Q(X)$,

then c is called a **pseudo-cycle**.

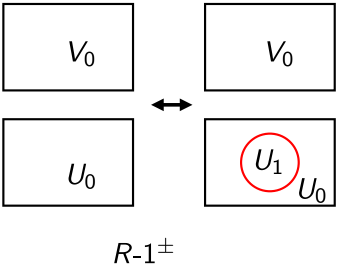
Pseudo-cycles

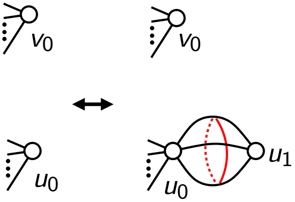
Theorem

For a surface-knot diagram D_F, the maximal number of pseudo-cycles in K_{D_F} is an invariant under Roseman moves up to quandle homology.

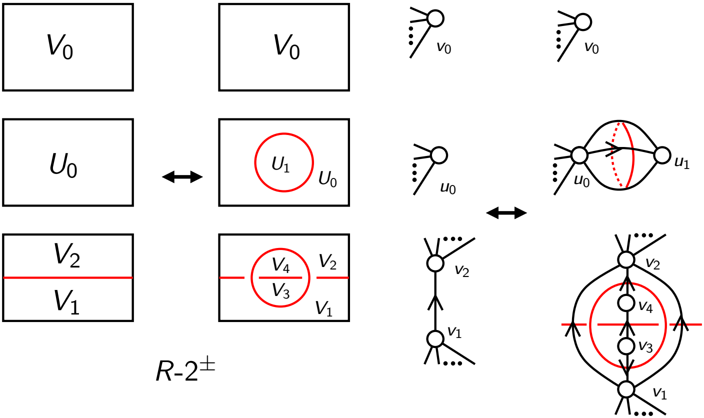
It is suffised to check each Roseman move does not change the number of pseudo-cycles:

Pseudo-cycles





Pseudo-cycles



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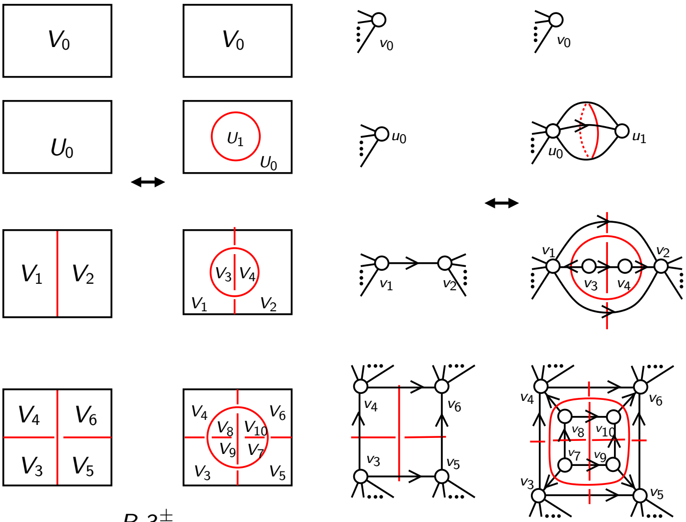
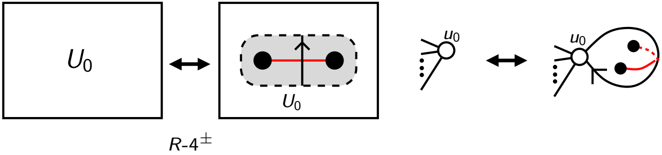


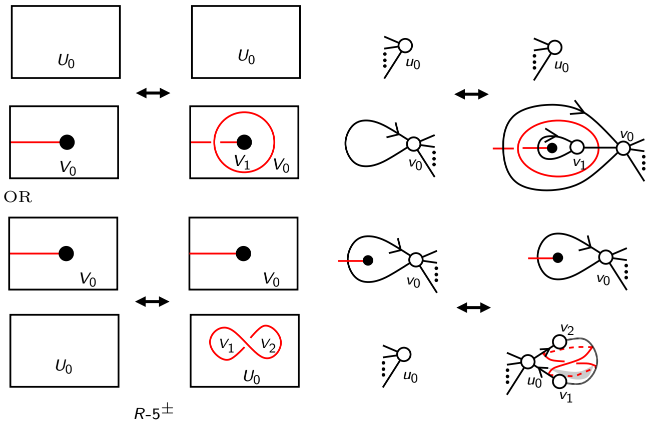
R-$3^{\pm}$

Pseudo-cycles



The Roseman move R-4^{+} creates two branch points corresponding to two loop discs. These loop discs are homologically zero.

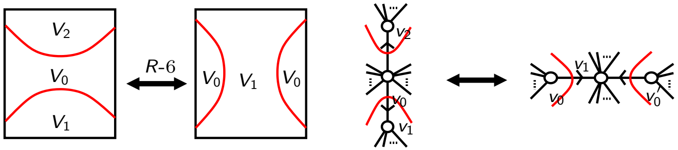
Pseudo-cycles



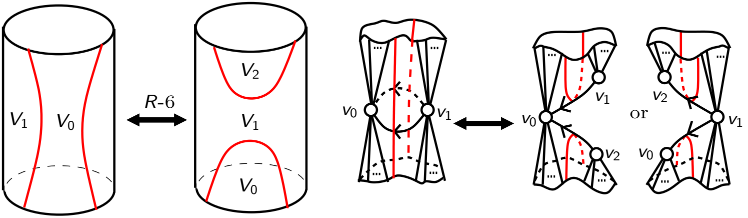
Pseudo-cycles

The Roseman move R-6 gives two possible cases: one does not change the number of parcels, and the other may change the number of parcels:

Pseudo-cycles



OR



The maximal number of pseudo-cycles in D_F will be denoted by $\nu(F)$.

Theorem

Let F be a double twist spun of $(2, k)$-torus knot for odd prime $k > 1$. Then

$$\nu(F) = 1$$

Applications

Let F be a surface-knot. A **triple point number** is the minimal number of triple points for all possible surface-knot diagrams. Let K be a pseudo-cycle and let D be a partial diagram corresponding to K. Then the number of triple points contained in D will be denoted by $t(K)$.

Proposition

Let F be a surface-knot obtained by a connected-sum of some surface-knots $F_1, F_2, \cdots, F_m$ with $\nu(F_i) \neq 0$ for all i. Then

$$0 < t(F_1 \# F_2 \# \cdots \# F_m),$$

where $\#$ means the connected sum.

Applications

Example

Let F be a 2-twit-spun trefoil. Then $F\#F$ is not a ribbon 2-knot.

Thank You