Pseudo-cycles of surface-knots and their applications

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Motivation

H. Poincaré stated the following conjecture.

Conjecture (Poincaré (1904))

Every simply connected orientable closed 3-manifold is homeomorphic to $\mathbb{S}^3$.

It is no longer a conjecture. G. Perelman (2002, 2003) proved the Geometric Conjecture and thus the Poincaré conjecture is true.

Motivation

In 1980s many approaches were tried to tackle the conjecture. One of them is the following: Let D^2 be a disc and let Δ^3 be a homotopy 3-ball. Consider the map

$$f : \partial(D^2 \times I) \to \Delta^3$$

in which $\partial(D^2 \times I)$ is mapped onto the parallel to the boundary of Δ^3. Extending f into $D^2 \times I$, we obtain a homotopy $F : D^2 \times I \to \Delta^3$. Then modify the homotopy F into an isotopy.

Motivation

We can see this idea in Homma-Nagase (1985, 1987) and others. The singularities of $F(D^2 \times \{t\})$ consist of double curves, triple points and sometimes branch points. If all singularities of $F(D^2 \times \{t\})$ are elminated, it is done.

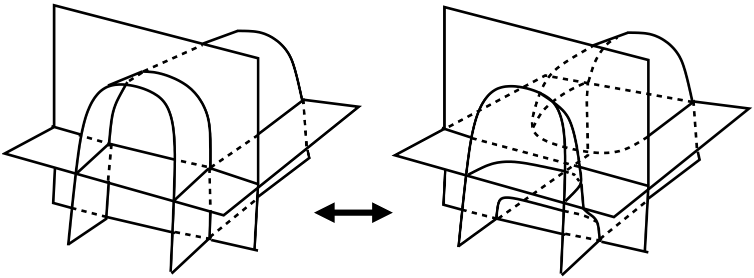
It is known that if the sigularities are only double curves and branch points, then it is done. Therefore, if triple points are eliminated, then it will be done.

Motivation

However, this approach faced the following problem:

Problem

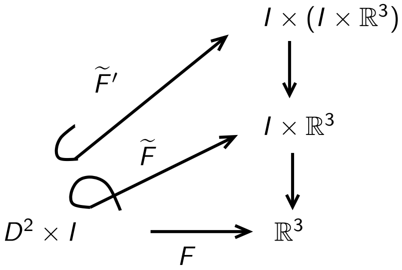
How can we characterize the cancelling pair of triple points?



To characterize the pre-image of triple point is not easy.

Motivation

One of approaches to this problem was:



The obstruction of regular homotopy of $\widetilde{F}$ relates to the existence of triple points.
We can view the image $\widetilde{F}'(D^2 \times \{t\})$ is an embedded disc in $\{s\} \times I \times \mathbb{R}^3$.

Motivation

Reagarding to deal with triple points in generic surfaces, surface-knot diagrams have some advantages:

1. The singular set consists of two families: Upper and Lower decker sets (Carter-Saito (1998) characterized the singular set.)
2. Algebraic structures are associated with the diagram (the fundamental group, quandles etc.)

Our approach to deal with triple points is:

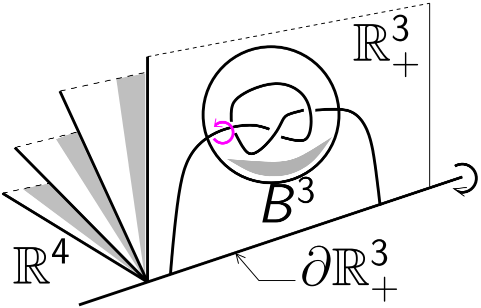
Our Approach

To look at the pre-images (singular sets).

Zeeman's twist spinning

Let B^3 be a 3–ball in $\mathbb{R}^3_+$ such that it contains a tangle $T(K)$ of a knot K, and $\partial B^3 \cap T(K)$ is the pair of antipodal points of ∂B^3.

An **m-twist-spun knot** obtained from K is defined by rotating $B^3 \cap T(K)$ about the axis through the antipodal points m times while $\mathbb{R}^3_+$ spins denoted by $T_m(K)$.



Theorem (Zeeman, 1965)

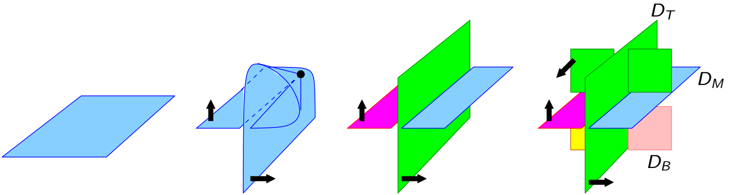
Every m-twist spun knot $T_m(K)$ obtained from K is fibred ($m \geq 1$); the fibre is the one-punctured m-fold branched covering space of S^3 along K.

Corollary (Zeeman, 1965)

For any knot K, 1-twist spun knot obtained from K is trivial.

Surface-knot Diagrams

A **surface-knot** is a connected oriented closed surface embedded in 4-space. Let $F \subset \mathbb{R}^4$ be a surface-knot. Let $\mathrm{proj} : \mathbb{R}^4 \to \mathbb{R}^3$; $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3)$, be the orthogonal projection. A **surface-knot diagram** of F is a union of the following local diagrams.

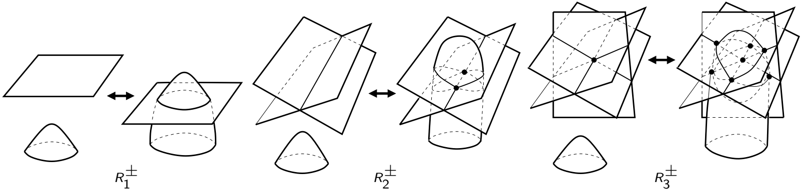


Branch point

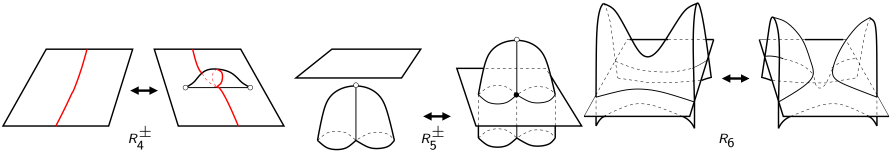
Triple point

Roseman moves

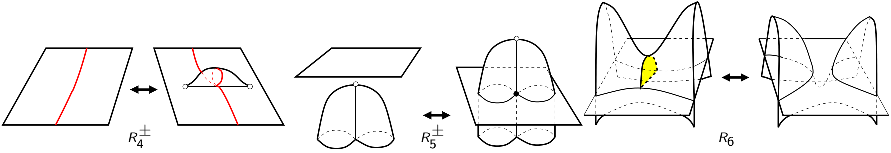
Two surface-knot diagrams are equivalent if they are projected image of the same type of a surface-knot. Two equivalent surface-knot diagrams are modified from one to the other by a **finite sequence of local moves** called Roseman moves.



Roseman moves



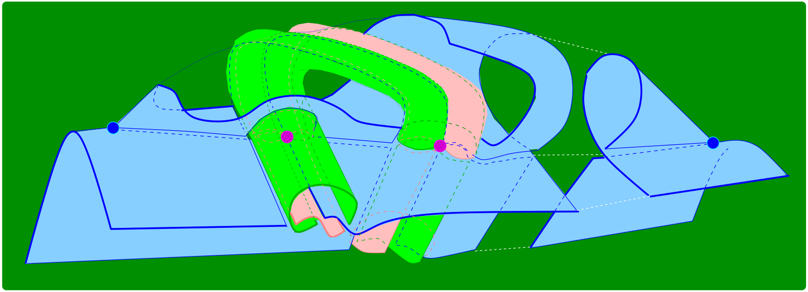
Roseman moves



These six moves have some height variations (T.Y. 2005).
Roseman's seven moves can be expressed by other six moves. (K. Kawamura 2015).

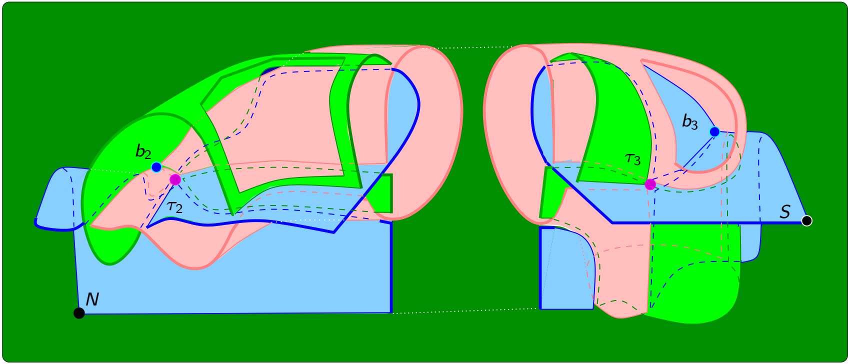
Double Decker Sets

S. Satoh (2002) constructed a diagram of twist span knots.



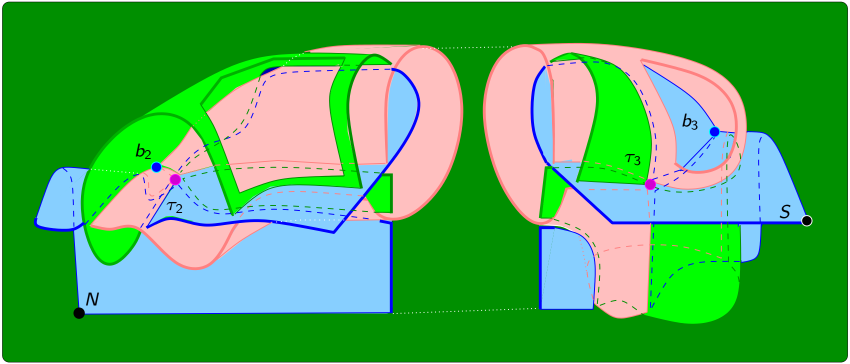
Double Decker Sets

The following is a reduced diagram obtained by a sequence of Roseman moves from Satoh's construction (t-minimal diagram).



Double Decker Sets

This diagram does not have non-trivial discendent discs.
(d-minimal diagram (A. Al Kharusi and TY)).



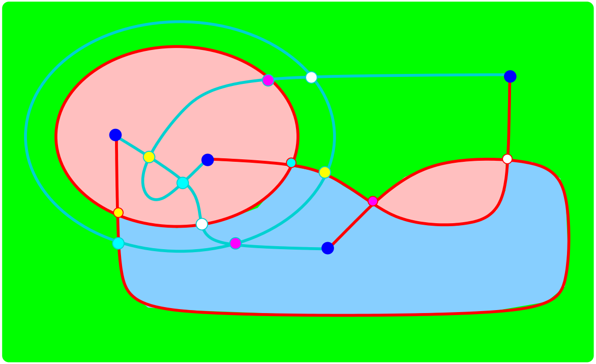
The preimage of singularities of the projection proj is:

$$S = \{x \in F \mid \#((\mathrm{proj}|_F)^{-1}(\mathrm{proj}(x)) > 1\}$$

The set S is the union of two families of immersed circles and immersed open intervals:

$$\begin{aligned} \mathcal{S}_a &= \{s_{a1}, s_{a2}, \ldots, s_{al}\} \\ \mathcal{S}_b &= \{s_{b1}, s_{b2}, \ldots, s_{bl}\} \end{aligned}$$

where for $x \in s_{ai}$, $y \in s_{bi}$ $(i = 1, 2, \ldots, l)$, if $\mathrm{proj}(x) = \mathrm{proj}(y)$, then $h(x) > h(y)$.

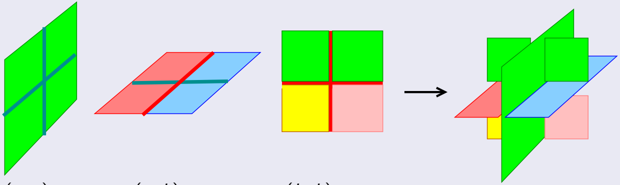


The closure of the pre-image of double curves in D_F is a union of two families of arcs called the **double decker set** (Carter-Saito). The blue arcs represent the **upper decker set** and the red arcs represent the **lower decker set**.

Lemma (Carter-Saito (1998))

Let F be a closed orientable surface. Let $f : F \to \mathbb{R}^3$ be a generic map. Then there is an embedding $g : F \to \mathbb{R}^4$ such that $\mathrm{proj} \circ g = f$ *if and only if*

1. $S(f) = \bigcup \mathcal{S}_a \cup \bigcup \mathcal{S}_b$.
2. *For each triple point, the pre-images are crossings of types* (a, a), (a, b) *and* (b, b).



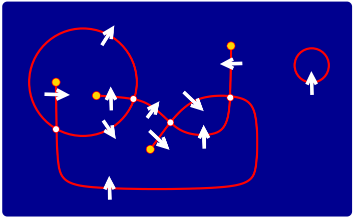
(a, a) (a, b) (b, b)

Triple point numbers of twist-spun knots

The minimal number of triple points for all possible surface-knot diagrams of a surface-knot F is called a **triple point number** denoted by $t(F)$, which is a surface-knot invariant.

- 1999 J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford and M. Saito applied **quandle co-homology** to knots and surface-knots.
- 2002 S. Satoh and A. Shima determined triple point numbers for 2-twist and 3-twist spun trefoils as **4** and **6** respectively.
- 2005 E. Hatakenaka gave a lower bound **6** of the triple point number for 2-twist spun $(2, 5)$-torus knot.

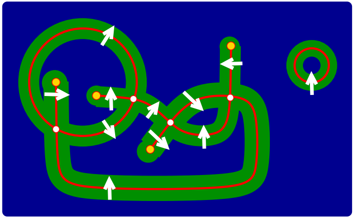
Rectangular-cell complexes



We denote the lower decker set by S_b.
$F \setminus S_b = \{R_0, \ldots, R_n\}$. Let $N(S_b)$ be a small neighbourhood of S_b in F.
$F \setminus N(S_b) = \{V_0, \ldots, V_n\}$; $V_i \subset R_i$ $(i = 0, \ldots, n)$.

Proof is done by analysing the double decker sets.

Rectangular-cell complexes



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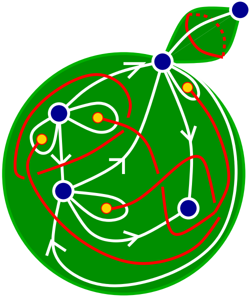
Rectangular-cell complexes

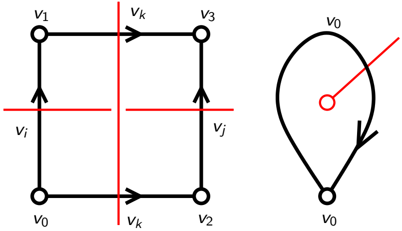


The quotient map $q : F \to F/_\sim$ is defined by $q(V_i) = v_i$, $(i = 0, \ldots, n)$. The quotient space is a 2-dimensional complex. We will denote the complex by K_{D_F}. A subcomplex of K_{D_F} induced from a simple closed curve in S_b is called a **bubble**.

Proof is done by analysing the double decker sets.

Rectangular-cell complexes

A subcomplex of K_{D_F} corresponding to a connected component of the lower decker set S_b is called a **parcel**. Each parcel is a bubble or a subcomplex consisting of some rectangles and loop discs:

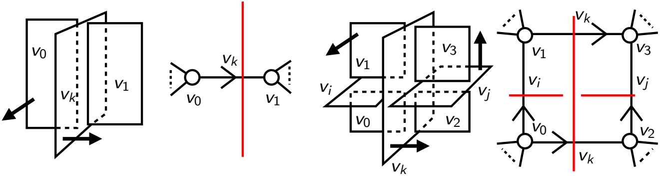


We denote the rectangle by $(v_0; v_0v_1, v_0v_2; v_3)$ and the loop by $\widehat{v_0v_0}$.

Proof is done by analysing the double decker sets.

Rectangular-cell complexes

Each double segment corresponds to an edge of the complex K_{D_F}.
Each edge has a **weight**, which is a vertex of the complex.

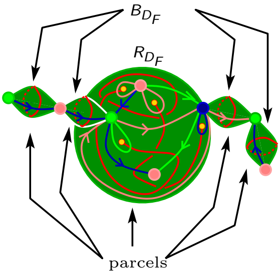


The lower decker set $S_b \subset |K_{D_F}|$ is a union of edges of K_{D_F}.
Proof is done by analysing the double decker sets.

Rectangular-cell complexes

The Numer of Parcels

For every surface-knot, we can make the number of parcels one.



Proof is done by analysing the double decker sets.

Rectangular-cell complexes

S. Satoh (2005) proved that for every 2-knot S,

$$4 \leq t(S)$$

Theorem (A. Al Kharusi-TY 2016 preprint)

Let F be a surface-knot with genus 1 and not pseudo-ribbon. Then

$$3 \leq t(F)$$

Proof is done by analysing the double decker sets.

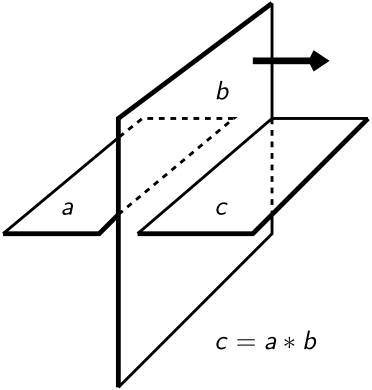
Quandle colorings

A **quandle** X is a non-empty set with a binary operation $(a, b) \mapsto a * b$ such that

1. For any $a \in X$, $a * a = a$,
2. For any $a, b \in X$, there is a unique $c \in X$ such that $c * b = a$.
3. For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

Quandle colorings

Let $\mathcal{R} = \{R_1, R_2, \ldots, R_n\}$ be the set of closures of connected components of $F - S_b$. For a quandle X, a **quandle coloring** of a diagram is a mapping $\mathrm{Col} : \mathcal{R} \to X$ such that



Quandle colorings

The coloring Col can be interpreted in terms of the rectangular-cell complex K_{D_F}:

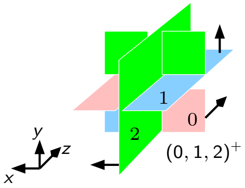
$$\mathrm{Col} : K_{D_F}^{(0)} \to X$$

If an edge e is incident with vertices v_1 and v_2, oriented from v_1 to v_2 and with weight v_3, then the mapping from the 1-skeleton to X

$$\mathrm{Col} : K_{D_F}^{(1)} \to X$$

is defined satisfying $\mathrm{Col}(v_1) * \mathrm{Col}(v_3) = \mathrm{Col}(v_2)$. We call this mapping also a **quandle coloring**.

Quandle colorings



The **dihedral quandle** $(X, *)$ of order $n > 0$ denoted by R_n is a quandle $X = \{0, \dots, n-1\}$ with the binary operation $i * j = 2j - i \pmod{n}$.

Chain groups

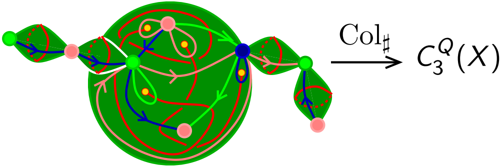
Quandle chain groups

Let $C_n(X)$ $(n \geq 1)$ be a free abelian group generated by n-tuples $(x_1, \ldots, x_n) \in X^n$. Let $C_n^D(X)$ be a sub group of $C_n(X)$ generated by $(x_1, \ldots, x_n)$ such that $x_i = x_j$ for some $1 \leq i, j, \leq n$ and $(|i-j| = 1)$. We denote the quotient group $C_n(X)/C_n^D(X)$ by $C_n^Q(X)$.

Chain groups of K_{D_F}

The chain group $C_k(K_{D_F})$ is defined as a free abelian group generated by k-dimensional elements of K_{D_F}. For $k = 2$, it is generated by the rectangular cells, loop discs and bubbles in K_{D_F}. For $k = 1$, it is generated by edges in K_{D_F}. For $k = 0$, it is generated by vertices of K_{D_F}.

Chain groups



The quandle coloring Col can be extended to a homomorphism $\mathrm{Col}_\sharp : C_2(K_{D_F}) \to C_3^Q(X)$ defined as follows.
For $\sigma = (v_0; v_0 v_1, v_0 v_2; v_3)$,

$$\mathrm{Col}_\sharp(\sigma) = (\mathrm{Col}(v_0), \mathrm{Col}(v_0 v_1), \mathrm{Col}(v_0 v_2)) \in C_3^Q(X).$$

Pseudo-cycles

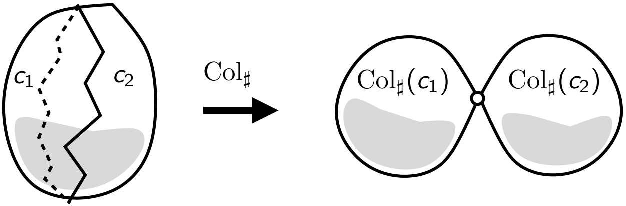
Definition

Let c be a chain of $C_2(K_{D_F})$. If c satisfies the following conditions,

(i) $\mathrm{Col}_\sharp\partial(c) = 0$ and

(ii) $[\mathrm{Col}_\sharp(c)] \neq 0 \in H_3^Q(X)$,

then c is called a **pseudo-cycle**.



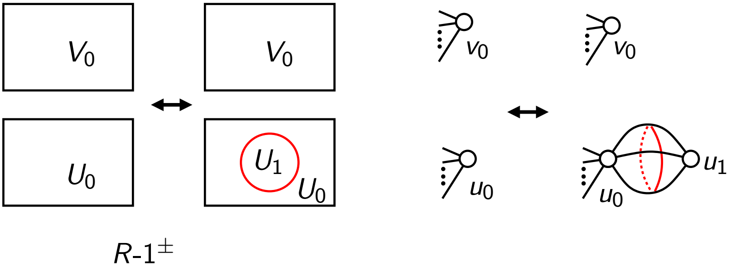
Pseudo-cycles

Theorem

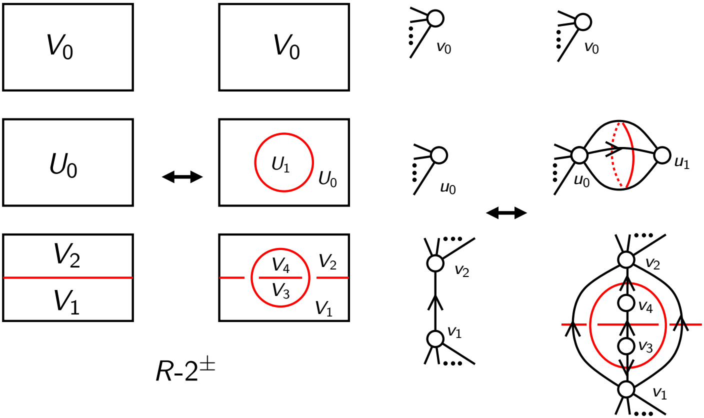
For a surface-knot diagram D_F, the maximal number of pseudo-cycles in K_{D_F} is an invariant under Roseman moves up to quandle homology.

It is proved by checking that each Roseman move does not change the number of pseudo-cycles:

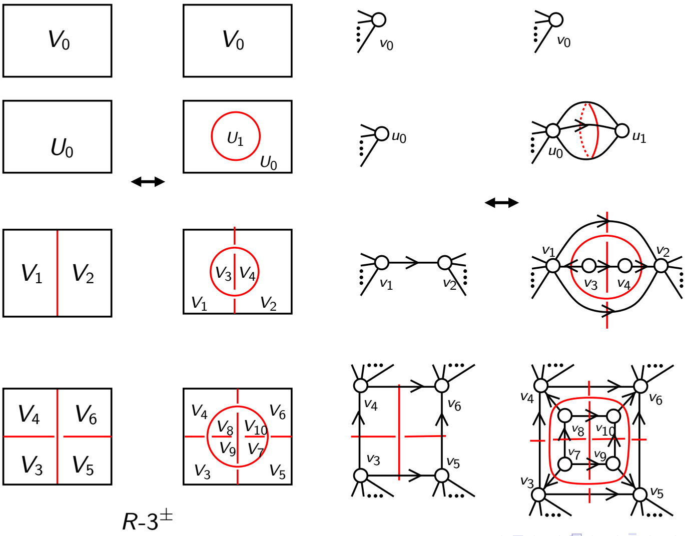
Pseudo-cycles



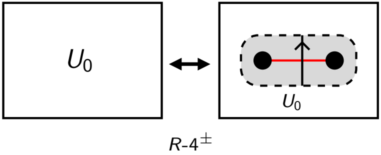
Pseudo-cycles

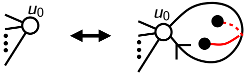


Pseudo-cycles



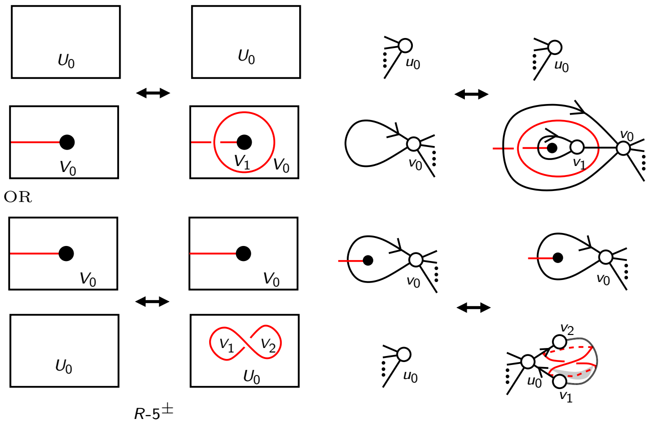
Pseudo-cycles





The Roseman move R-4^{+} creates two branch points corresponding to two loop discs. These loop discs are homologically zero.

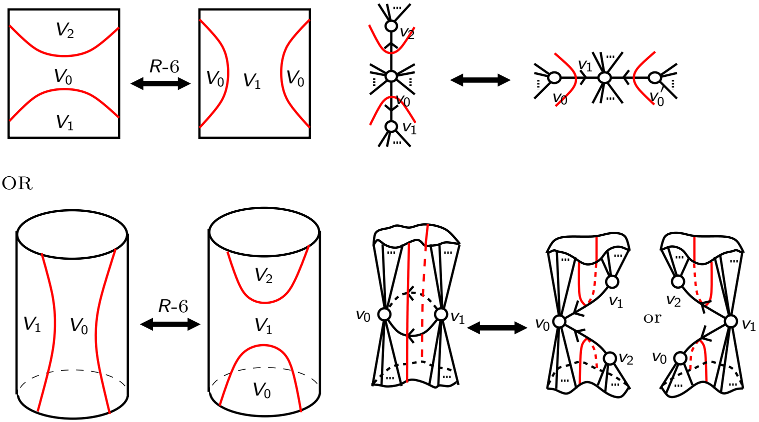
Pseudo-cycles



Pseudo-cycles

The Roseman move R-6 gives two possible cases: one does not change the number of parcels, and the other may change the number of parcels:

Pseudo-cycles



The maximal number of pseudo-cycles in D_F will be denoted by $\nu(F, Q)$.

Theorem

Let F be a double twist spun of $(2, k)$-torus knot for odd prime $k > 1$. Then F is colourable with the dihedral quandle Q of order k and for each Q,

$$\nu(F, Q) = 1$$

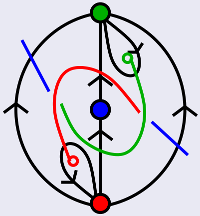
Applications

Theorem

Let F be a tri-colourable surafce-knot. Then

$$2\nu(F, R_3) \leq t(F)$$

Proof.



The pseudo-cycle which has the minimal number of crossings is shown in the left: Therefore, $t(F)$ is bounded below by $2\nu(F, R_3)$.

□

Applications

If a tri-colourable surface-knot F has one pseudo-cycle with two crossings, then it must have more pseudo cycles with at least 2 crossings to construct a diagram. Therefore, we have:

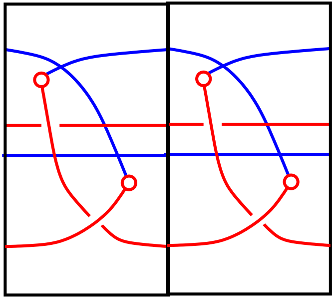
Theorem

Let F be a non-pseudo-ribbon, tri-colourable surface-knot with $\pi F \ncong \mathbb{Z}$. Then

$$4 \leq t(F).$$

Applications

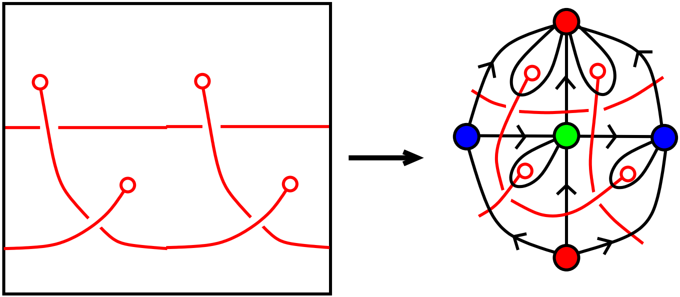
It is known that (T. Yashiro, 2016) some double decker sets of some tri-colorable surface-knot diagram can be obtained by pasting copies of the following primitive diagram and the resulting double decker set is constructible.



It is known that the primitive diagram induces the double twist spun trefoil.

Applications

A surface-knot diagram of the $2k$-twist spun trefoil can be constructed by pasting copies of the primitive diagram. The lower decker set of the primitive diagram induces a pseudo cycle:



Applications

We have the following:

Theorem

For the 2k-twist spun trefoil F,

$$t(F) = 4k \quad (k = 1, 2, \ldots n)$$

Proof. The diagram induces a sum of pseudo cycles:

$$c_1 + c_2 + \cdots c_k.$$

Each c_i is from the primitive double decker set.

- Roseman moves do not change a pseudo-cycle.
- there is no pseudo cycles containing less than four crossings.

Therefore, $4k \leq t(F)$.

Thank You