Coverings of surface-knots and their pseudo-cycles

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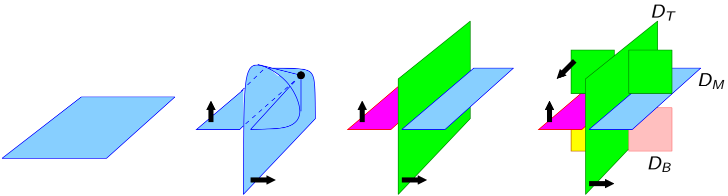
Contents

Motivation

A **surface-knot** is a connected oriented closed surface smoothly embedded in 4-space. If the surface has genus zero, then it is called a **2-knot**. A **surface-knot diagram** is a projected image under the orthogonal projection $\mathrm{proj} : \mathbb{R}^4 \to \mathbb{R}^3$, $(x_1, x_2, x_3, c_4) \mapsto (x_1, x_2, x_3)$. with the crossing information. A **surface-knot diagram** of F is a finite union of copies of the following local diagrams.



Branch point Triple point

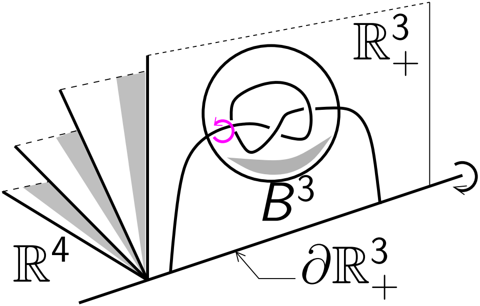
Motivation

The “knottedness of a surface” appears in the singularity set.
We are interested in what is the “knottedness”.

Motivation

Let B^3 be a 3–ball in $\mathbb{R}^3_+$ such that it contains a tangle $T(K)$ of a knot K, and $\partial B^3 \cap T(K)$ is the pair of antipodal points of ∂B^3.

An ***m*-twist-spun knot** obtained from K is defined by rotating $B^3 \cap T(K)$ about the axis through the antipodal points m times while $\mathbb{R}^3_+$ spins denoted by $T_m(K)$.



Motivation

Zeeman (1965) proved that the m-twist spun knot obtained from a knot K is fibred, and the fibre is the one-punctured m-fold branched covering space of S^3 along K.
As a consequence, a 1-twist spun knot obtained from K is trivial.
A **triple point number** of a surface-knot is the minimal number of triple points for all possible surface-knot diagrams. This is a geometric surface-knot invariant. Only few triple point numbers have been determined (Satoh, Shima et.al.)

Motivation

Our approach to deal with knottedness is:

Our Approach

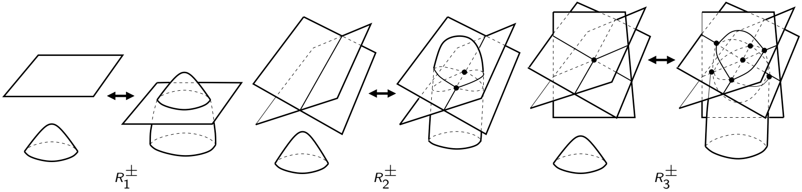
To look at the pre-images (singular sets).

Viewing a surface-knot diagram as a generic surface, the singular set (pre-image of the multiple point set) characterizes the surface-knot:

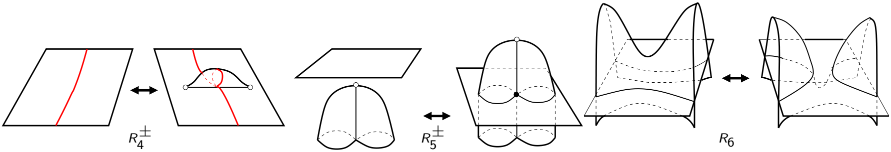
1. The singular set consists of two families: Upper and Lower decker sets (Carter-Saito (1998) characterized the singular set.)
2. Algebraic structures are associated with the diagram (the fundamental group, quandles etc.)

Roseman moves

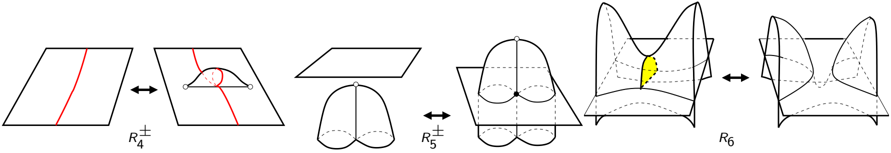
Two surface-knot diagrams are **equivalent** if they are projected image of the same type of a surface-knot. Two equivalent surface-knot diagrams are modified from one to the other by a finite sequence of local moves called **Roseman moves**.



Roseman moves



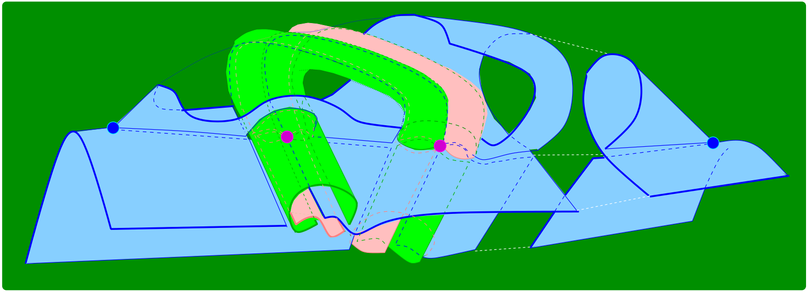
Roseman moves



These six moves have some height variations (T.Y. 2005).
Roseman's seven moves can be expressed by other six moves. (K. Kawamura 2015).

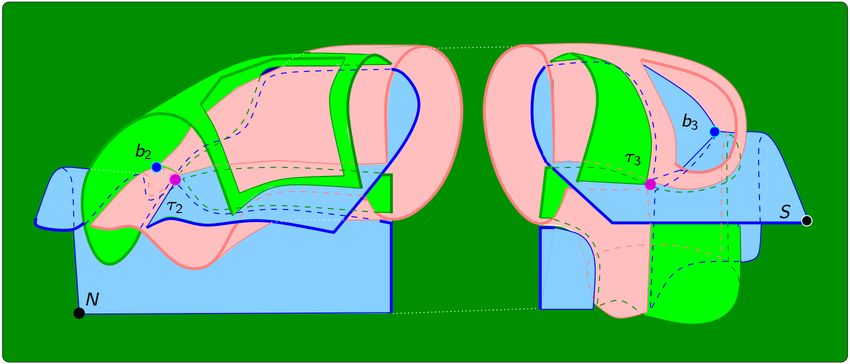
Double Decker Sets

S. Satoh (2002) constructed a diagram of twist span knots.



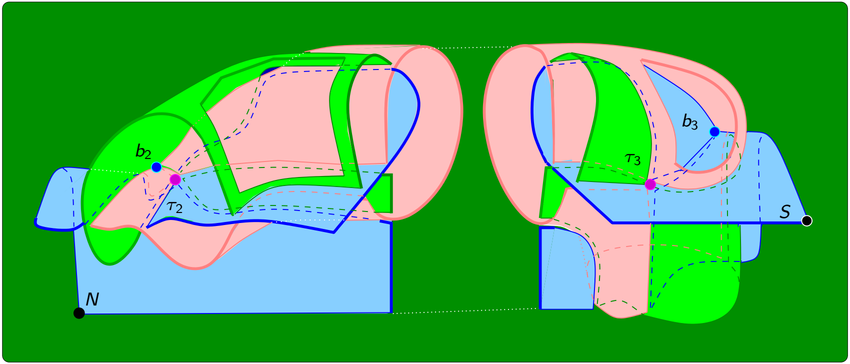
Double Decker Sets

The following is a reduced diagram obtained by a sequence of Roseman moves from Satoh's construction (t-minimal diagram).



Double Decker Sets

This diagram does not have non-trivial descendent discs.
(d-minimal diagram (A. Al Kharusi and TY)).



The preimage of singularities of the projection proj is:

$$S = \{x \in F \mid \#((\mathrm{proj}|_F)^{-1}(\mathrm{proj}(x)) > 1\}$$

The set S is the union of two families of immersed circles and immersed open intervals:

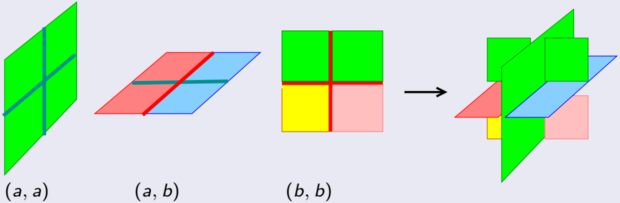
$$\begin{aligned} \mathcal{S}_a &= \{s_{a1}, s_{a2}, \ldots, s_{al}\} \\ \mathcal{S}_b &= \{s_{b1}, s_{b2}, \ldots, s_{bl}\} \end{aligned}$$

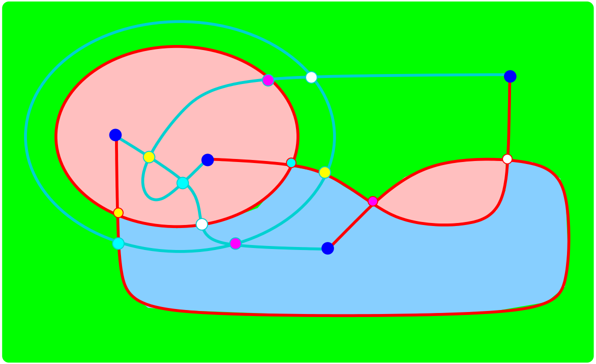
where for $x \in s_{ai}$, $y \in s_{bi}$ $(i = 1, 2, \ldots, l)$, if $\mathrm{proj}(x) = \mathrm{proj}(y)$, then $h(x) > h(y)$.

Lemma (Carter-Saito (1998))

Let F be a closed orientable surface. Let $f : F \to \mathbb{R}^3$ be a generic map. Then there is an embedding $g : F \to \mathbb{R}^4$ such that $\mathrm{proj} \circ g = f$ *if and only if*

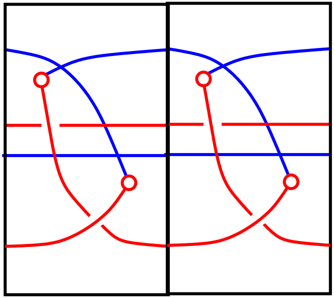
1. $S(f) = \bigcup \mathcal{S}_a \cup \bigcup \mathcal{S}_b$.
2. *For each triple point, the pre-images are crossings of types* (a, a), (a, b) *and* (b, b).





The closure of $S(\mathrm{proj}_F)$ is called the **double decker set** (Carter-Saito). The closure of $\bigcup \mathcal{S}_a$ (blue arcs denoted by S_a) is called an **upper decker set** and the closure of $\bigcup calS_b$ (red arcs denoted by S_b) is called a **lower decker set**.

It is known that (TY, 2016) some tri-colourable and d-minimal surface-knot diagram can be obtained by pasting copies of the following primitive diagram

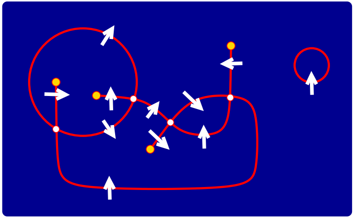


Triple point numbers of twist-spun knots

The minimal number of triple points for all possible surface-knot diagrams of a surface-knot F is called a **triple point number** denoted by $t(F)$, which is a surface-knot invariant.

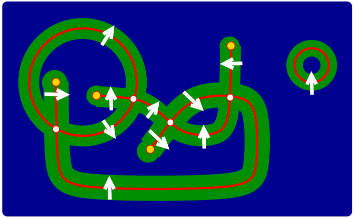
- 1999 J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford and M. Saito applied **quandle co-homology** to knots and surface-knots.
- 2002 S. Satoh and A. Shima determined triple point numbers for 2-twist and 3-twist spun trefoils as **4** and **6** respectively.
- 2005 E. Hatakenaka gave a lower bound **6** of the triple point number for 2-twist spun $(2, 5)$-torus knot.

Rectangular-cell complexes



We denote the lower decker set by S_b.
$F \setminus S_b = \{R_0, \ldots, R_n\}$. Let $N(S_b)$ be a small neighbourhood of S_b in F.
$F \setminus N(S_b) = \{V_0, \ldots, V_n\}$;
$V_i \subset R_i$ $(i = 0, \ldots, n)$.

Rectangular-cell complexes



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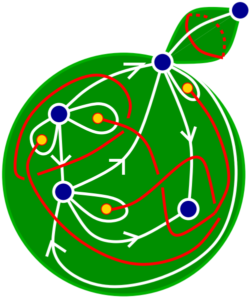
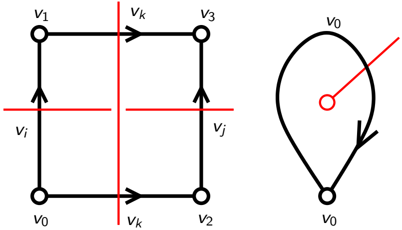
Rectangular-cell complexes



The quotient map $q : F \to F/_{\sim}$ is defined by $q(V_i) = v_i$, $(i = 0, \ldots, n)$. The quotient space is a 2-dimensional complex. We will denote the complex by K_{D_F}. A subcomplex of K_{D_F} induced from a simple closed curve in S_b is called a **bubble**.

Rectangular-cell complexes

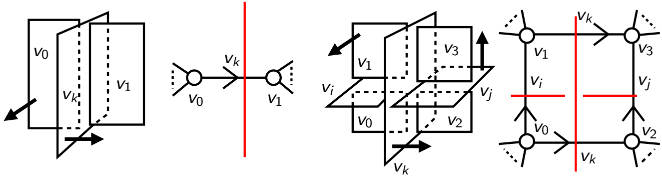
A subcomplex of K_{D_F} corresponding to a connected component of the lower decker set S_b is called a **parcel**. Each parcel is a bubble or a subcomplex consisting of some rectangles and loop discs:



We denote the rectangle by $(v_0; v_0v_1, v_0v_2; v_3)$ and the loop by $\widehat{v_0v_0}$.

Rectangular-cell complexes

Each double segment corresponds to an edge of the complex K_{D_F}.
Each edge has a **weight**, which is a vertex of the complex.

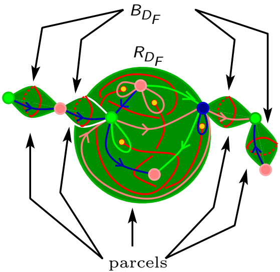


The lower decker set $S_b \subset |K_{D_F}|$ is a union of edges of K_{D_F}.

Rectangular-cell complexes

The Number of Parcels

For every surface-knot, we can make the number of parcels one.



Rectangular-cell complexes

Theorem (S. Satoh (2005))

For every 2-knot S,

$$4 \leq t(S)$$

Theorem (A. Al Kharusi-TY preprint)

Let F be a surface-knot with genus 1 and not pseudo-ribbon. Then

$$3 \leq t(F)$$

Proof is done by analysing the double decker sets.

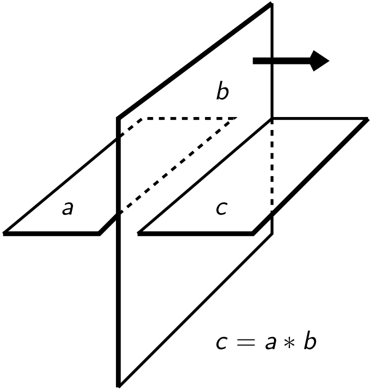
Quandle colorings

A **quandle** X is a non-empty set with a binary operation $(a, b) \mapsto a * b$ such that

1. For any $a \in X$, $a * a = a$,
2. For any $a, b \in X$, there is a unique $c \in X$ such that $c * b = a$.
3. For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

Quandle colorings

Let $\mathcal{R} = \{R_1, R_2, \ldots, R_n\}$ be the set of closures of connected components of $F - S_b$. For a quandle X, a **quandle coloring** of a diagram is a mapping $\mathrm{Col} : \mathcal{R} \to X$ such that



Quandle colorings

The coloring Col can be interpreted in terms of the rectangular-cell complex K_{D_F}:

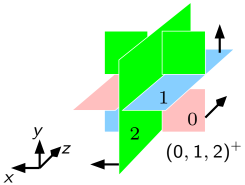
$$\mathrm{Col} : K_{D_F}^{(0)} \to X$$

If an edge e is incident with vertices v_1 and v_2, oriented from v_1 to v_2 and with weight v_3, then the mapping from the 1-skeleton to X

$$\mathrm{Col} : K_{D_F}^{(1)} \to X$$

is defined satisfying $\mathrm{Col}(v_1) * \mathrm{Col}(v_3) = \mathrm{Col}(v_2)$. We call this mapping also a **quandle coloring**.

Quandle colorings



The **dihedral quandle** $(X, *)$ of order $n > 0$ denoted by R_n is a quandle $X = \{0, \ldots, n-1\}$ with the binary operation $i * j = 2j - i \pmod{n}$.

Chain groups

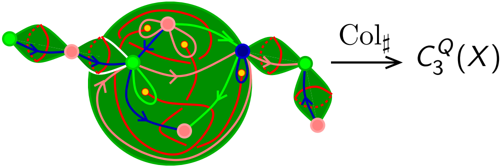
Quandle chain groups

Let $C_n(X)$ $(n \geq 1)$ be a free abelian group generated by n-tuples $(x_1, \ldots, x_n) \in X^n$. Let $C_n^D(X)$ be a sub group of $C_n(X)$ generated by $(x_1, \ldots, x_n)$ such that $x_i = x_j$ for some $1 \leq i, j, \leq n$ and $(|i-j| = 1)$. We denote the quotient group $C_n(X)/C_n^D(X)$ by $C_n^Q(X)$.

Chain groups of K_{D_F}

The chain group $C_k(K_{D_F})$ is defined as a free abelian group generated by k-dimensional elements of K_{D_F}. For $k = 2$, it is generated by the rectangular cells, loop discs and bubbles in K_{D_F}. For $k = 1$, it is generated by edges in K_{D_F}. For $k = 0$, it is generated by vertices of K_{D_F}.

Chain groups



The quandle coloring Col can be extended to a homomorphism

$$\mathrm{Col}_\sharp : C_2(K_{D_F}) \to C_3^Q(X)$$

$$(v_0; v_0v_1, v_0v_2; v_3) \mapsto (\mathrm{Col}(v_0), \mathrm{Col}(v_0v_1), \mathrm{Col}(v_0v_2))$$

Pseudo-cycles

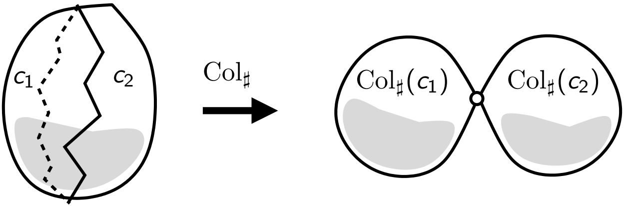
Definition

Let c be a chain of $C_2(K_{D_F})$. If c satisfies the following conditions,

(i) $\mathrm{Col}_\sharp \partial(c) = 0$ and

(ii) $[\mathrm{Col}_\sharp(c)] \neq 0 \in H_3^Q(X)$,

then c is called a **pseudo-cycle**.



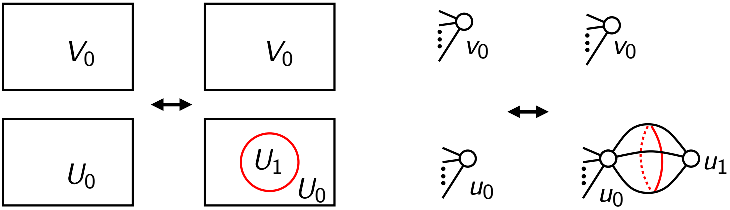
Pseudo-cycles

Theorem (TY preprint)

For a surface-knot diagram D_F coloured by a quandle X, the maximal number of pseudo-cycles in $C_2(K_{D_F})$ for all colourings, is an invariant under Roseman moves up to quandle homology.

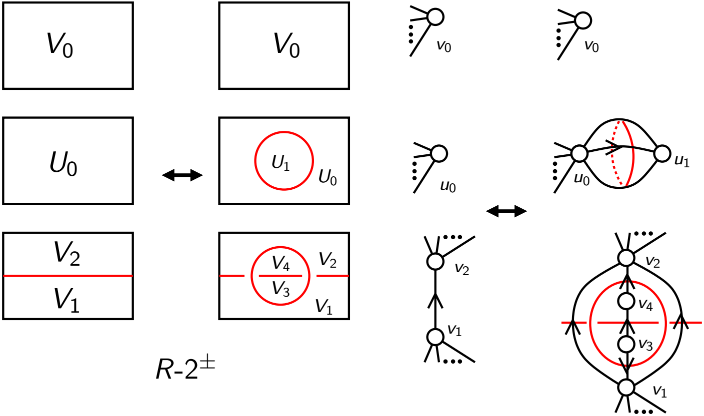
It is proved by checking that each Roseman move does not change the number of pseudo-cycles:

Pseudo-cycles

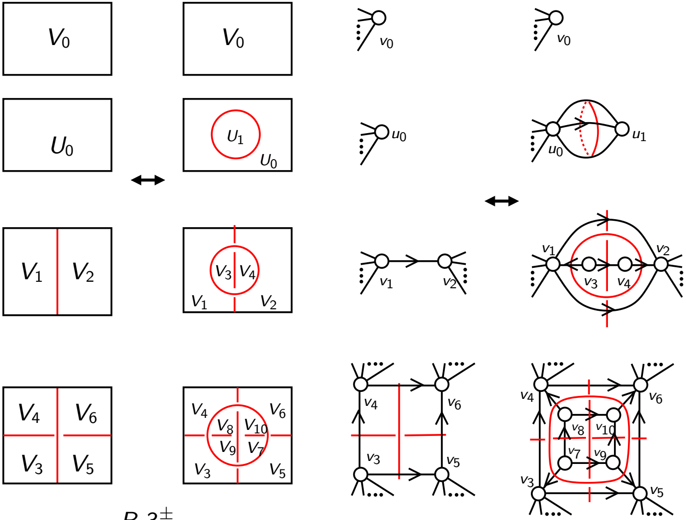


R-$1^{\pm}$

Pseudo-cycles

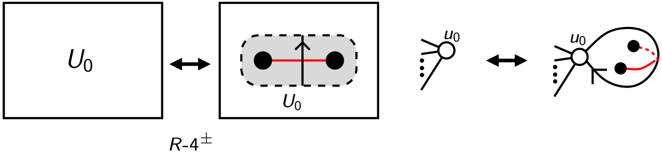


Pseudo-cycles



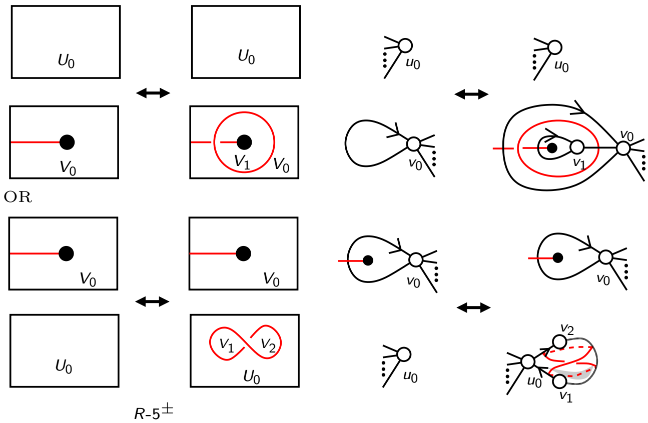
$R\text{-}3^{\pm}$

Pseudo-cycles



The Roseman move R-4^{+} creates two branch points corresponding to two loop discs. These loop discs are homologically zero.

Pseudo-cycles



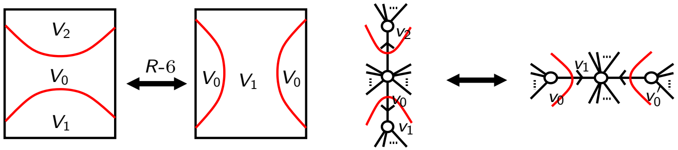
Pseudo-cycles

The Roseman move R-6 gives two possible cases:

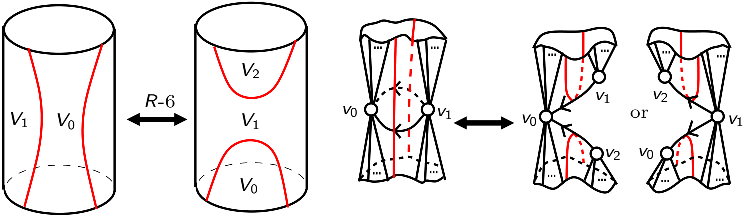
1. it does not change the number of parcels, or
2. it changes the number of parcels.

(1) it is done.
(2) The lower decker set will be split. This means that a parcel will be split; that is, a pseudo cycle will be split into two pseudo-cycles. The maximal number of pseudo-cycles does not change.

Pseudo-cycles



OR



Covering Diagrams

Let $\widetilde{F}$ and F be surface-knots.
Let $\widetilde{\Sigma} \subset \widetilde{F}$ and $\Sigma \subset F$ be the double decker sets of $D_{\widetilde{F}}$ and D_F respectively.
A surface-knot diagram $D_{\widetilde{F}}$ is a **covering diagram** over D_F if

1. $\widetilde{p} : \widetilde{F} \to F$ be a branched covering map.
2. the branch set misses Σ.
3. $\widetilde{p}|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \Sigma$ is a covering map,
4. $\widetilde{p}(\widetilde{S}_x) = S_x$, $x = a, b$, and
5. the following diagram is commutative.

$$\begin{array}{ccc} \widetilde{F} & \xrightarrow{\widetilde{p}} & F \\ \text{proj}\downarrow & & \downarrow\text{proj} \\ D_{\widetilde{F}} & \xrightarrow{p} & D_F \end{array}$$

Covering Diagrams

The degree of the map $\widetilde{p}$ on a regular point is called the degree of $\widetilde{p}$ denoted by $\deg(p)$.
From the definition the map p induces a map $p_* : K_{D_{\widetilde{F}}} \to K_{D_F}$.
The map p_* induces the homomorphism

$$p_\# : C_2(K_{D_{\widetilde{F}}}) \to C_2(K_{D_F})$$

Every pseudo-cycle $\widetilde{c}$ is mapped into a pseudo-cycle $p_\#(\widetilde{c})$.

Covering Diagrams

Lemma

For a covering diagram $D_{\widetilde{F}}$ over D_F with degree n coloured by a quandle X, for every pseudo-cycle c for D_F, there exist pseudo-cycles $c_1, c_2, \ldots, c_n$ such that

$$p_\#(c_i) = c \quad i = 1, 2, \ldots, n.$$

The number of crossing points of S_b contained in a chain c will be denoted by $|c|$

Lemma

A covering diagram $D_{\widetilde{F}}$ over D_F coloured by a quandle X is given. For every pseudo-cycle $\widetilde{c}$ from $K_{D_{\widetilde{F}}}$,

$$|\mathrm{Col}_\#(\widetilde{c})| \leq |\widetilde{c}|, \qquad |p_\#(\widetilde{c})| \leq |\widetilde{c}|.$$

Covering Diagrams

Lemma

For a covering diagram $D_{\widetilde{F}}$ over D_F with degree n coloured by a quandle X,

$$n\mu(F, X) = \mu(\widetilde{F}, X)$$

Proof From the definition of the covering diagram, $\mu(D_{\widetilde{F}}) \leq n\mu(D_F)$. From the previous lemmas, $\mu(D_{\widetilde{F}}) \geq n\mu(D_F)$. For every surface-knot diagram, the maximal number of pseudo-cycles is invariant under Roseman moves up to the quandle homology of X. Thus the result follows.

Applications

Fix a quandle X. For a rectangular cell τ in K_{D_F}, τ is **non-degenerate** if $\mathrm{Col}_{\#}(\tau) \neq 0$
Let c be a pseudo-cycle of K_{D_F} with respect to a dihedral quandle X of order n. If $c = \tau$, then c must be degenerate. Thus the number of non-degenerate rectangular-cells in c is ≥ 2.

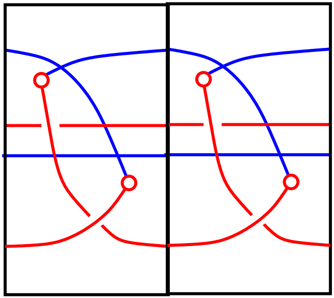
Theorem

For a surface-knot F coloured by a dihedral quandle X of order n,

$$2\mu(F, X) \leq t(F)$$

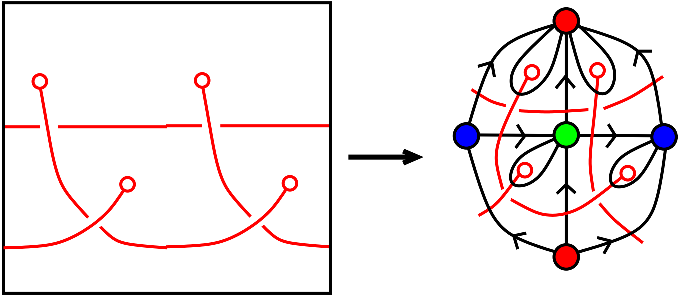
Applications

We can construct a double decker set obtained by pasting copies of the following primitive diagram and the resulting double decker set is constructible as a tri-colourable surface-knot diagram.



Applications

A surface-knot diagram of the $2k$-twist spun trefoil can be constructed by pasting copies of the primitive diagram. The lower decker set of the primitive diagram induces a pseudo-cycle:



Applications

We have the following:

Theorem

For the $2k$-twist spun trefoil $\widetilde{F}$,

$$t(\widetilde{F}) = 4k \quad (k = 1, 2, \dots n)$$

Proof The $2k$-twist spun trefoil $\widetilde{F}$ has a covering diagram $D_{\widetilde{F}}$ over a surface diagram of the double twist spun trefoil D_F with degree k. Both are tri-coloured.

Let $\widetilde{c}$ be a pseudo-Cole for $K_{D_{\widetilde{F}}}$. Then $\mathrm{Col}_{\#}(\widetilde{c})$ is a quandle cycle in $C_3^Q(X)$. By Satoh's result, for every cycle $c \in C_3^Q(X)$,

$$4 \leq |c|$$

Applications

Therefore,

$$4 \leq |\mathrm{Col}_{\#}(\widetilde{c})|$$

This implies that

$$4\mu(\widetilde{F}, X) \leq 4\deg(p)\mu(F, X) \leq t(\widetilde{F}).$$

Therefore, $4k \leq t(F)$. There is a surface-knot diagram of $\widetilde{F}$ with $4k$ triple points. Therefore, $t(\widetilde{F}) = 4k$.

Thank You