Essential Betti Numbers of Surface-knot Diagrams

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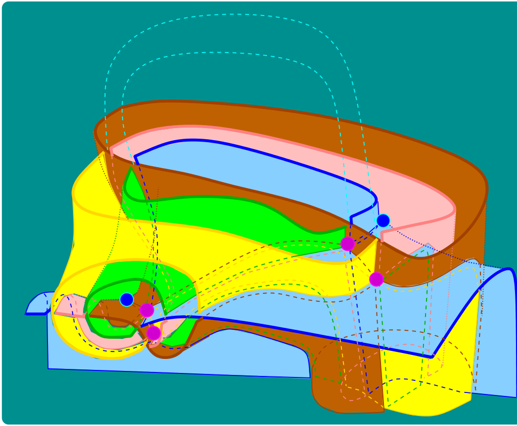
Contents

Motivation

A **surface-knot** F is a closed oriented surface smoothly embedded in $\mathbb{R}^4$ The orthogonal projection $\mathrm{proj} : \mathbb{R}^4 \to \mathbb{R}^3$ maps F onto $\mathrm{proj}(F)$ with double point set, isolated triple points, and isolated branch points. The image $\mathrm{proj}(F)$ with crossing information is called a **surface-knot diagram** and denoted by Δ_F.



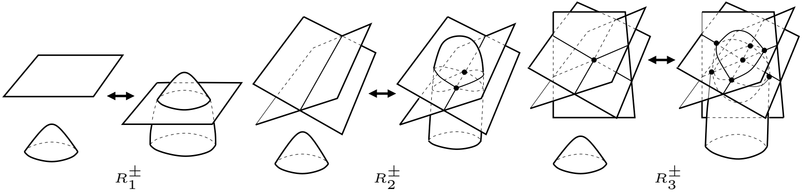
Motivation

There is a geometric surface-knot invariant, called a **triple point number** of a surface-knot. This is defined as the minimal number of triple points for all possible surface-knot diagrams. Only few triple point numbers have been determined (Satoh, Shima et.al.)

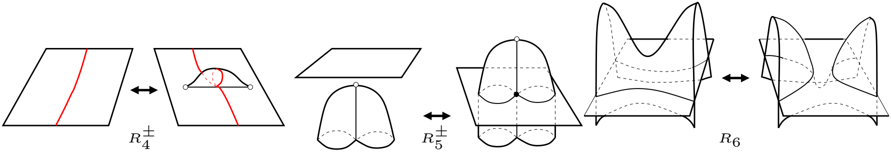
To investigate the triple point numbers, we need to construct diagrams. We would like to know any relationship between the topology of F and the number of triple points in Δ_F.

Roseman moves

Two surface-knot diagrams are **equivalent** if they have projected images of the same type of a surface-knot. Two equivalent surface-knot diagrams are modified from one to the other by a finite sequence of seven types of local moves called **Roseman moves** (D. Roseman 1998). Roseman's seven moves are expressed by six moves (TY 2005, K. Kawamura 2015).



Roseman moves



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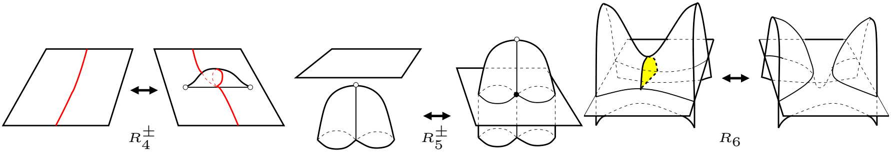
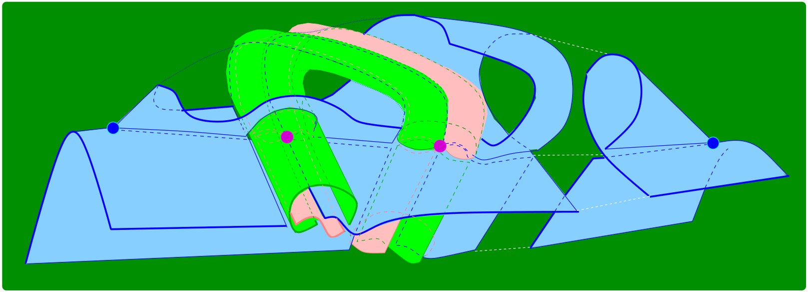


Deforming diagrams

The $2k$-twist-spun trefoil knot is deformed into the t-minimal surface-knot diagram by deforming each single twisting part.



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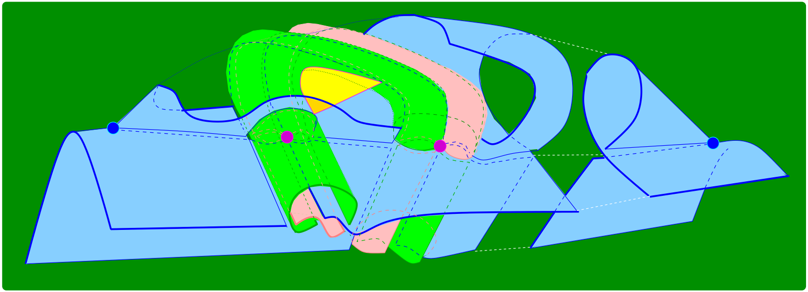

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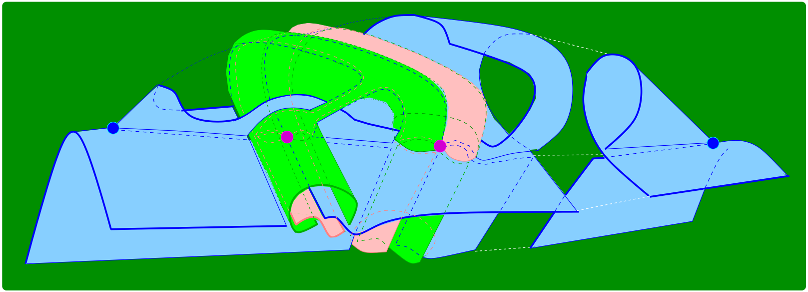

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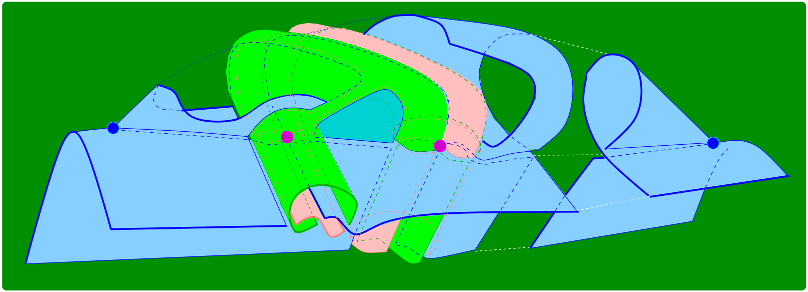

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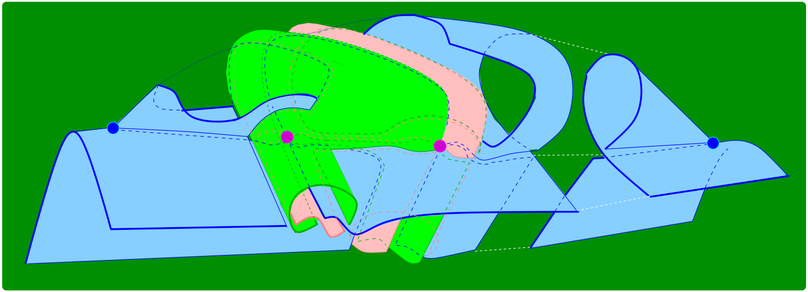

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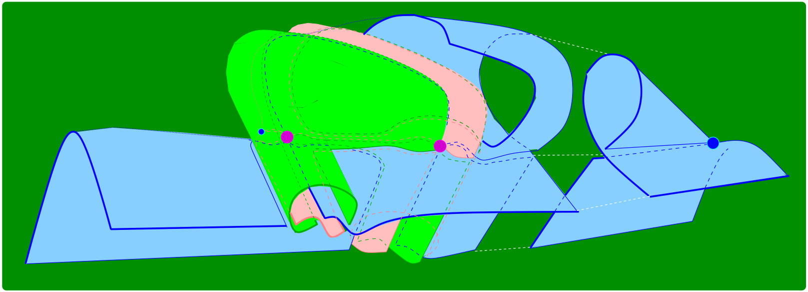

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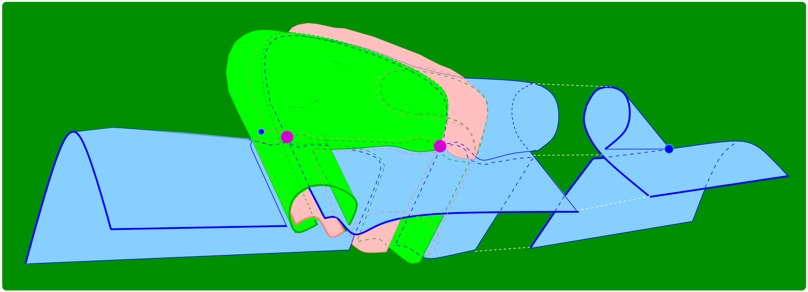

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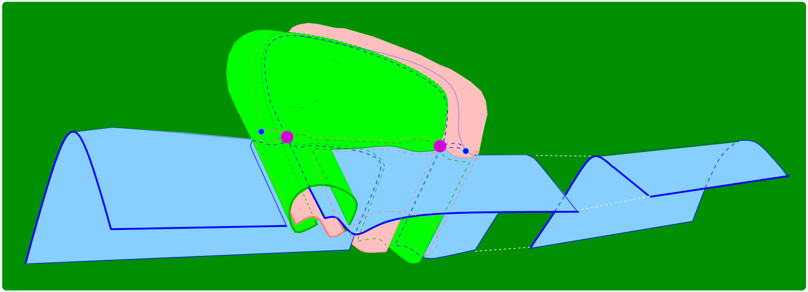


Double Decker Sets

The pre-image of singularities of the projection proj is:

$$S = \{x \in F \mid \#((\mathrm{proj}|_F)^{-1}(\mathrm{proj}(x)) > 1\}$$

is the union of two families of immersed circles and immersed open intervals:

$$\begin{aligned} \mathcal{S}_a &= \{s_{a1}, s_{a2}, \ldots, s_{al}\} \\ \mathcal{S}_b &= \{s_{b1}, s_{b2}, \ldots, s_{bl}\} \end{aligned}$$

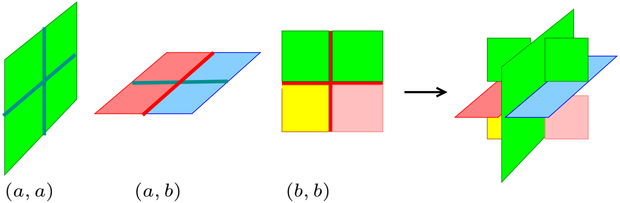
where for $x \in s_{ai}$, $y \in s_{bi}$ $(i = 1, 2, \ldots, l)$, if $\mathrm{proj}(x) = \mathrm{proj}(y)$, then $h(x) > h(y)$.
The closure $\mathrm{cl}(S)$ is called the **double decker set** (Carter-Saito).
The closures $S_a = \mathrm{cl}(\bigcup \mathcal{S}_a)$ and $S_b = \mathrm{cl}(\bigcup \mathcal{S}_b)$ are called **upper decker set**, and **lower decker set** respectively.

Double Decker Sets

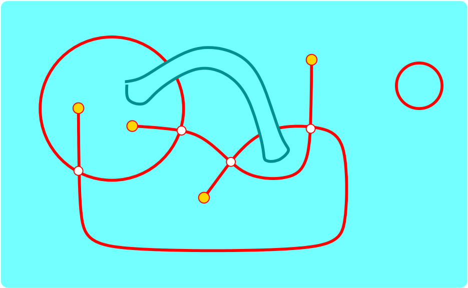
Lemma 2.1 (Carter-Saito (1998))

Let F be a closed orientable surface. Let $f : F \to \mathbb{R}^3$ be a generic map. Then there is an embedding $g : F \to \mathbb{R}^4$ such that $\mathrm{proj} \circ g = f$ *if and only if*

1. $S(f) = \bigcup \mathcal{S}_a \cup \bigcup \mathcal{S}_b$.
2. *For each triple point, the pre-images are crossings of types* (a, a), (a, b) *and* (b, b).

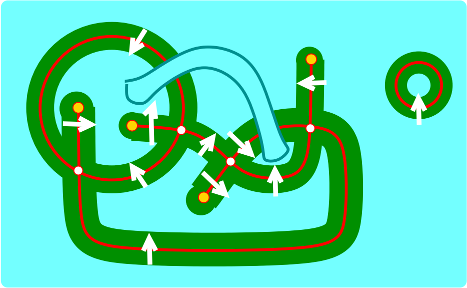


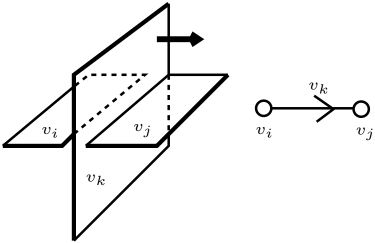
Rectangular cell-complexes



$F \setminus S_b = \{R_0, \ldots, R_n\}$. Let $N(S_b)$ be a small neighbourhood of S_b in F. $F \setminus N(S_b) = \{V_0, \ldots, V_n\}$; $V_i \subset R_i$ $(i = 0, \ldots, n)$.

Rectangular cell-complexes





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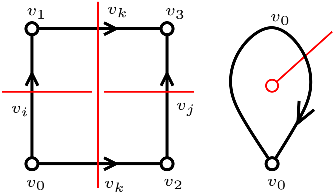
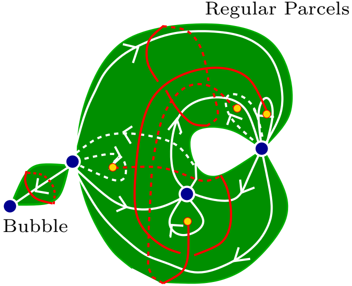
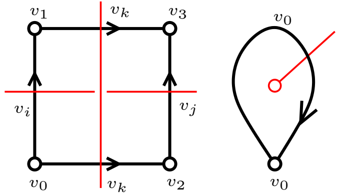
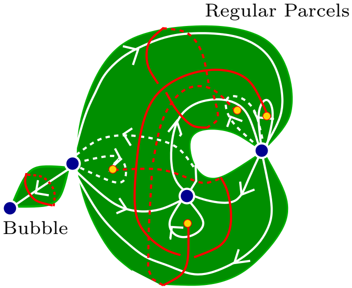




The quotient map $q : F \to F/_\sim$ is defined by $q(V_i) = v_i$, $(i = 0, \ldots, n)$. $F/_\sim$ admits a 2-dimensional complex structure, and it is denoted by K_{Δ_F}.

Rectangular cell-complexes

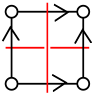
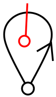
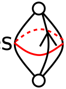




A subcomplex of K_{Δ_F} induced from a connected lower decker set is called a **parcel** A parcel induced from a simple closed curve in S_b is called a **bubble**. A parcel including the crossings is called a **regular parcel**.

Topology of K_{Δ_F}

We use the folloiwng notations.

- V, E, and f denote the sets of vertices, edges, and faces of K_{Δ_F} respectively.
- ρ is the number of rectangles  $= t(\Delta_F)$
- λ is the number of loop discs  $= b(\Delta_F)$
- ζ is the number of bubbles  $=$ the number of simple closed lower decker curves.
- ψ is the number of rectangular parcels.

Topology of K_{Δ_F}

Then the Euler characteristic of $|K_{\Delta_F}|$ is:

$$\begin{aligned}
\chi(|K_{\Delta_F}| &= |V| - |E| + |f| \\
&= |V| - \left(\frac{\lambda + 4\rho}{2}\right) - \zeta + (\rho + \lambda + \zeta) \\
&= |V| + \frac{\lambda}{2} - \rho \\
&= 1 - \beta_1 + \psi + \zeta \ (= \beta_0 - \beta_1 + \beta_2)
\end{aligned}$$

where β_i is the i-th Betti number of $|K_{\Delta_F}|$ for $i = 0, 1, 2$. Since $2 \leq |V|$,

$$2 \leq \rho - \frac{\lambda}{2} + 1 - \beta_1 + \psi + \zeta$$

$$\beta_1 \leq t(\Delta_F) - \frac{\lambda}{2} - 1 + \psi + \zeta$$

Topology of K_{Δ_F}

Theorem 3.1

Let F be a surface-knot. Let K be the cell-complex of a surface-knot diagram Δ_F of F. The underlying space $|K|$ has ψ regular parcels, ζ bubbles, λ loop discs, and ρ rectangles. Then the following holds.

$$\beta_1 \leq t(\Delta_F) - \frac{\lambda}{2} - 1 + \psi + \zeta,$$

where β_1 is the first Betti number of $|K|$.

Reduced Complex

Let K_Δ be the rectangular complex of a diagram Δ. Two vertices u, v in K are equivalent if there is a path from u to v such that all edges in the path are of bubbles. The quotient complex $K_\Delta/_\sim$ is called a **reduced rectangular complex** denoted by $\widetilde{K_\Delta}$. The first Betti number $\widetilde{\beta}_1 = \beta_1(\widetilde{K}_\Delta$ is called an **essential Betti number** of Δ. Then the following holds.

$$\widetilde{\beta}_1 \leq \rho - \frac{\lambda}{2} - 1 + \psi$$

t-minimal diagrams

A surface-knot diagram Δ_F of F is **t-minimal** if $t(\Delta_F) = t(\Delta_F)$.

We assume that a t-minimal diagram Δ induces only one regular parcel. Then $\widetilde{K}_\Delta$ consists of the combinations of loop discs and rectangular cells, and there exists at least 2 branch points. Therefore, the following holds.

$$\begin{aligned}\widetilde{\beta}_1 &\leq t(\Delta) - \frac{\lambda}{2} - 1 + \psi \\ &\leq t(F) - 1\end{aligned}$$

t-minimal diagrams

Lemma 3.1 (AK-TY (2016))

Let F be a surface-knot of genus 1. Then

$$3 \leq t(F)$$

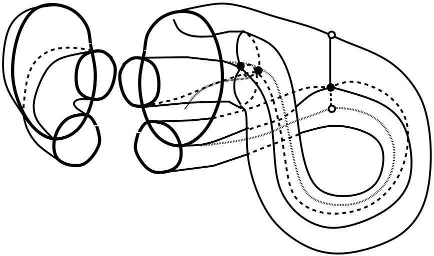
We try to construct a t-minimal surface-knot diagram Δ with only triple points and branch points ($t(\Delta) = 3$). Then

$$\widetilde{\beta}_1 \leq 2$$

The genus of F is at most 1 but Satoh proved that it is not 0. Thus it is 1.

t-minimal diagrams

We found the following diagram (only one) with three triple points and two branch points.



Its K has $|V| = 2$ thus $\pi F \cong \mathbb{Z}$ and also $\widetilde{\beta}_1 = 2$. Also, it is trivial.

Thank You for listening!