An ordinal analysis of Π_N-Collection

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Abstract

In this paper we give an ordinal analysis of a set theory with Π_N-Collection.

1 Introduction

Throughout in this paper N denotes a fixed positive integer. In this paper we give an ordinal analysis of a Kripke-Platek set theory with the axiom of Infinity and one of Π_N-Collection, denoted by $\mathsf{KP}\omega + \Pi_N$-Collection. Our proof is an extension of [4, 5]. Since [5] has not yet appeared, some proofs are duplicated for the readers' conveniences.

In [5] we analyzed proof-theoretically a set theory $\mathsf{KP}\ell^r + (M \prec_{\Sigma_1} V)$ extending $\mathsf{KP}\ell^r$ with an axiom stating that 'there exists a transitive set M such that $M \prec_{\Sigma_1} V$'. An ordinal analysis of an extension $\mathsf{KP}i + (M \prec_{\Sigma_1} V)$ is given in M. Rathjen[14]. Our proof is an extension of [2, 5]. In [2], a set theory $\mathsf{KP}\Pi_N$ of Π_N-reflection is analyzed, which is an extension of M. Rathjen's analysis for Π_3-reflection in [13].

Σ^1_{N+2}-DC+BI [Σ^1_{N+2}-AC+BI] denotes a second order arithmetic obtained from ACA_0+BI by adding the axiom of Σ^1_{N+2}-Dependent Choice [Σ^1_{N+2}-Axiom of Choice], resp. It is easy to see that Σ^1_{N+2}-DC+BI is interpreted canonically to the set theory $\mathsf{KP}\omega + \Pi_N$-Collection $+ (V = L)$ with the axiom $V = L$ of constructibility. It is well known that Σ^1_{N+2}-DC_0 implies Σ^1_{N+2}-AC, which yields Δ^1_{N+2}-CA, a fortiori Σ^1_{N+1}-CA, cf. Lemma VII.6.6 of [15]. Moreover it is known that Σ^1_{N+2}-DC+BI is Π^1_4-conservative over Σ^1_{N+2}-AC+BI [over Δ^1_{N+2}-CA+BI], resp., cf. Exercise VII.5.13 and Theorem VII.6.16 of [15].

Let n be a positive integer. We say that an ordinal α is *n-stable* if $L_\alpha \prec_{\Sigma_n} L$ for the constructible universe $L = \bigcup_\alpha L_\alpha$. In general, a transitive and non-empty set M is *n-stable* if $M \prec_{\Sigma_n} V$ for the universe V. We see that $(V, \in) \models \mathsf{KP}\omega + \Pi_N$-Collection if V enjoys the $\Delta_0(\{st_i\}_{0<i\leq N})$-collection, where st_i denotes the predicate for the class $\{M \in V : M \prec_{\Sigma_i} V\}$ of i-stable sets in V.

We introduce an extension $S_{\mathbb{I}_N}$ of $\mathsf{KP}\omega + \Pi_N$-Collection in the language $\{\in\} \cup \{st_i\}_{0<i\leq N}$, which codifies $\Sigma(\{st_i\}_{0<i\leq N})$-reflection. We aim to give an ordinal analysis of the theory $S_{\mathbb{I}_N}$.

In the following theorems, Ω denotes the least recursively regular ordinal ω_1^{CK}, and ψ_Ω a collapsing function such that $\psi_\Omega(\alpha) < \Omega$. $\mathbb{I}_N$ is an ordinal term denoting an ordinal such that $L_{\mathbb{I}_N} \models \mathsf{KP}\omega + \Pi_N\text{-Collection} + (V = L)$.

First we show the following Theorem 1.1.

Theorem 1.1 *Suppose* $S_{\mathbb{I}_N} \vdash \theta^{L_\Omega}$ *for a* Σ_1*-sentence* θ *in the language* $\{\in\}$ *of set theory. Then* $L_{\psi_\Omega(\varepsilon_{\mathbb{I}_N+1})} \models \theta$ *holds.*

It is not hard to see that the ordinal $\psi_\Omega(\varepsilon_{\mathbb{I}_N+1})$ is computable. Let $<$ denote a computable well-ordering of type $\psi_\Omega(\varepsilon_{\mathbb{I}_N+1})$ on the set of natural numbers. Conversely we show that Σ^1_{N+2}-DC+BI proves that each initial segment of $\psi_\Omega(\varepsilon_{\mathbb{I}_N+1})$ is well-founded.

Theorem 1.2 $\Sigma^1_{N+2}\text{-DC+BI} \vdash Wo[\alpha]$ *for* each $\alpha < \psi_\Omega(\varepsilon_{\mathbb{I}_N+1})$.

For $T \supset \mathrm{ACA}_0$, $|T|$ denotes the proof-theoretic ordinal of T, i.e., the supremum of order types of computable well-orderings $\prec$ on the set of natural numbers for which T proves the fact that $\prec$ is a well-ordering. Also let $|\mathsf{KP}\omega + \Pi_N\text{-Collection}|_{\Sigma_1^\Omega}$ denote the Σ_1^Ω-ordinal of $\mathsf{KP}\omega + \Pi_N$-Collection, i.e., the ordinal $\min\{\alpha \leq \omega_1^{CK} : \forall \theta \in \Sigma_1 \left(\mathsf{KP}\omega + \Pi_N\text{-Collection} \vdash \theta^{L_\Omega} \Rightarrow L_\alpha \models \theta\right)\}$. For more on ordinal analysis see [3]. We conclude the following Theorem 1.3, where $\psi_\Omega(\varepsilon_{\mathbb{I}_N+1})$ denotes the order type of the initial segment $OT(\mathbb{I}_N)\cap\Omega$ of a notation system $OT(\mathbb{I}_N)$ of ordinals.

Theorem 1.3 $|\Delta^1_{N+2}\text{-CA+BI}| = |\Sigma^1_{N+2}\text{-AC+BI}| = |\Sigma^1_{N+2}\text{-DC+BI}| = |\mathsf{KP}\omega + \Pi_N\text{-Collection}|_{\Sigma_1^\Omega} = \psi_\Omega(\varepsilon_{\mathbb{I}_N+1})$.

Let $\mathbf{Z}_2 = \Sigma^1_\infty$-DC be the full second order arithmetic with the Dependent Choice schema, and $\mathsf{ZFC} - \mathsf{Power}$ denote the set theory ZFC minus the power set axiom. $\mathbf{Z}_2$ proves the (Π^1_1-)soundness of $\Sigma^1_{N+2}\text{-DC} + \text{BI}$, and hence $\mathbf{Z}_2$ proves that $(OT(\mathbb{I}_N), <)$ is a well ordering for *each* N. $\mathbf{Z}_2$ is canonically interpreted in $(\mathsf{ZFC} - \mathsf{Power}) + (V = L)$, which is Π^1_1-conservative over $\mathsf{ZFC} - \mathsf{Power}$.

Assume $\mathsf{ZFC} - \mathsf{Power} \vdash \theta$ for a sentence θ. Since $S_{\mathbb{I}_N}$ subsumes Π_N-Collection and Σ_N-Separation, there is an N such that $S_{\mathbb{I}_N} \vdash \theta$. Therefore we conclude the following.

Theorem 1.4 $\psi_\Omega(\mathbb{I}_\omega) := \sup\{\psi_\Omega(\mathbb{I}_N) : 0 < N < \omega\} = |\mathbf{Z}_2| = |\mathsf{ZFC} - \mathsf{Power}|_{\Sigma_1^\Omega}$.

Let us mention the contents of this paper. In the next section 2 a second order arithmetic Σ^1_{N+2}-DC+BI is interpreted to a set theory $\mathsf{KP}\omega + \Pi_N\text{-Collection} + (V = L)$, and $\mathsf{KP}\omega + \Pi_N$-Collection is shown to be a subtheory of a set theory $S_{\mathbb{I}_N}$. In section 3 ordinals for our analysis of Π_N-Collection are introduced, and a computable notation system $OT(\mathbb{I}_N)$ is extracted.

Theorem 1.1 is proved in sections 4 and 5. In section 4 operator controlled derivations are introduced. In section 5, stable ordinals are removed from derivations. Although our proof of Theorem 1.1 is based on operator controlled derivations introduced by W. Buchholz[9], it is hard for us to give its sketch here. See subsection 4.2 for an outline of the proof.

Theorem 1.2 is proved in sections 6 and 7. For $0 \leq i \leq N$, we introduce *i-maximal distinguished sets*, which are Σ^1_{2+i}-definable. A 0-maximal distinguished set is Σ^1_2-definable as in [4]. Σ^1_{N+2}-(Dependent) Choice is needed to handle limits of N-stable ordinals. Our proof of Theorem 1.2 is based on maximal distinguished class introduced again by Buchholz[7]. A sketch of the well-foundedness proof is outlined in subsection 6.1.

In the final section 8 let us conclude some standard outcomes of an ordinal analysis of the theory $\mathbf{Z}_2$.

IH denotes the Induction Hypothesis, MIH the Main IH, SIH the Subsidiary IH, and SSIH the Sub-Subsidiary IH.

2 Π_N-Collection

In this section a second order arithmetic Σ^1_{N+2}-DC+BI is interpreted canonically to a set theory $\mathsf{KP}\omega + \Pi_N$-Collection $+ (V = L)$, and $\mathsf{KP}\omega + \Pi_N$-Collection is shown to be a subtheory of a set theory $S_{\mathbb{I}_N}$.

For subsystems of second order arithmetic, we follow largely Simpson's monograph[15]. The schema Bar Induction, BI is denoted by TI in [15]. BI allows the transfinite induction schema for well-founded relations.

Σ^1_{N+2}-AC+BI denotes a second order arithmetic obtained from Π^1_1-CA$_0$ + BI by adding the axiom Σ^1_{N+2}-AC, $\forall n \exists X F(n, X) \to \exists Y \forall n F(n, Y_n)$ for each Π^1_{N+1}-formula $F(n, X)$, where $m \in Y_n \Leftrightarrow (n, m) \in Y$ for a bijective pairing function $(\cdot, \cdot)$. Σ^1_{N+2}-DC+BI denotes a second order arithmetic obtained from Π^1_1-CA$_0$+BI by adding the axiom Σ^1_{N+2}-DC for each Π^1_{N+1}-formula $F(n, X, Y)$, $\forall n \forall X \exists Y F(n, X, Y) \to \forall X_0 \exists Y \forall n[Y_0 = X_0 \land F(n, Y_n, Y_{n+1})]$. It is easy to see that the formulas F can be Σ^1_{N+2} in the axioms.

The axioms of the set theory $\mathsf{KP}\omega + \Pi_N$-Collection consists of those of $\mathsf{KP}\omega$ (Kripke-Platek set theory with the Axiom of Infinity, cf.[6, 12]) plus Π_N-Collection: for each Π_N-formula $A(x, y)$ in the language of set theory, $\forall x \in a \exists y A(x, y) \to \exists b \forall x \in a \exists y \in b A(x, y)$.

Σ_N-Separation denotes the axiom $\exists y \forall x(x \in y \leftrightarrow x \in a \land \varphi(x))$ for each Σ_N-formula $\varphi(x)$. Δ_{N+1}-Separation denotes the axiom $\forall x \in a(\varphi(x) \leftrightarrow \neg\psi(x)) \to \exists y \forall x(x \in y \leftrightarrow x \in a \land \varphi(x))$ for each Σ_{N+1}-formulas $\varphi(x)$ and $\psi(x)$.

Σ_{N+1}-Replacement denotes the axiom stating that if $\forall x \in a \exists! y \varphi(x, y)$, then there exists a function f with its domain $dom(f) = a$ such that $\forall x \in a\, \varphi(x, f(x))$ for each Σ_{N+1}-formula $\varphi(x, y)$.

Lemma 2.1 $\mathsf{KP}\omega + \Pi_N$-Collection *proves each of* Σ_N-Separation*,* $\Delta_{N+1}-$ Separation *and* Σ_{N+1}-Replacement*.*

Proof. We show that $\{x \in a : \varphi(x)\}$ exists as a set for each Σ_i-formula φ by (meta)induction on $i \leq N$. The case $i = 0$ follows from Δ_0-Separation. Let $\varphi \equiv \exists y\, \theta(x, y)$ with a Π_{i-1}-matrix θ. We have by logic $\forall x \in a \exists y(\exists z \theta(x, z) \to \theta(x, y))$. By Π_i-Collection pick a set b so that $\forall x \in a \exists y \in b(\varphi(x) \to \theta(x, y))$. In other words, $\{x \in a : \varphi(x)\} = \{x \in a : \exists y \in b\, \theta(x, y)\}$. If $i = 1$, then $\exists c[c = \{x \in a : \exists y \in b\, \theta(x, y)\}]$ by Δ_0-Separation. Let $2 \leq i \leq N$. By Π_{i-2}-Collection we obtain a Π_{i-1}-formula σ such that $\exists y \in b\, \theta(x, y) \leftrightarrow \sigma(x)$. By IH we obtain $\exists c[c = \{x \in a : \sigma(x)\}]$.

Δ_{N+1}-Separation follows from Σ_N-Separation as in [6], p.17, Theorem 4.5(Δ Separation), and Σ_{N+1}-Replacement follows from Δ_{N+1}-Separation as in [6], p.17, Theorem 4.6(Σ Replacement). □

For a formula A in the language of second order arithmetic let A^{set} denote the formula obtained from A by interpreting the first order variable x as $x \in \omega$ and the second order variable X as $X \subset \omega$.

The following is the Quantifier Theorem in p.125 of [12], in which $\mathsf{KP}l^r$ is defined as a set theory for limits of admissible sets with restricted induction. $\mathsf{KP}l^r$ is a subtheory of $\mathsf{KP}\omega + \Pi_N$-Collection. $Ad(x)$ designates that x is an admissible set.

Lemma 2.2 *For each Σ^1_{N+1}-formula $F(n, a, Y)$, there exists a Σ_N-formula $A_\Sigma(d, n, a, Y)$ in the language of set theory so that for $F_\Sigma(n, a, Y) :\Leftrightarrow \exists d[Ad(d) \land Y \in d \land A_\Sigma(d, n, a, Y)]$,*

$$\mathsf{KP}l^r \vdash n, a \in \omega \land Y \subset \omega \to \{F^{set}(n, a, Y) \leftrightarrow F_\Sigma(n, a, Y)\}.$$

For an ordinal α, L_α denotes the initial segment of Gödel's constructible universe $L = \bigcup_\alpha L_\alpha$. $x \in L$ is a Σ_1-formula. $<_L$ denotes a canonical Δ_1 well ordering of L such that if $y <_L x \in L_\alpha$, then $y \in L_\alpha$, cf. p.162 of [6]. $V = L$ denotes the axiom of Constructibility.

Lemma 2.3 *For each sentence A in the language of second order arithmetic,*

$$\Sigma^1_{N+2}\text{-DC} + \text{BI} \vdash A \Rightarrow \mathsf{KP}\omega + \Pi_N\text{-Collection} + (V = L) \vdash A^{set}.$$

Proof. By the Quantifier Theorem 2.2 $F^{set}(n, X, Y)$ is equivalent to a Π_N-formula $\varphi(n, X, Y)$ for a Π^1_{N+1}-formula $F(n, X, Y)$, $n \in \omega$ and $X \subset \omega$. It suffices to show for a Π_N-formula $\varphi(n, X, Y)$ that assuming $\forall n \in \omega \forall X \subset \omega \exists Y \subset \omega \varphi(n, X, Y)$ and $X_0 \subset \omega$, there exists a function f with its domain $dom(f) = \omega$ such that $\forall n \in \omega[f(0) = X_0 \land \varphi(n, f(n), f(n+1))]$. By induction on $k \in \omega$ using $V = L$ we see that there exists a unique family $(Y_n)_{n<k}$ of subsets of ω such that $\forall n < k[\varphi(n, Y_n, Y_{n+1}) \land \forall Z <_L Y_{n+1} \neg\varphi(n, Y_n, Z)]$, where $\forall Z <_L Y \neg\varphi(n, Y, Z)$ is equivalent to a Σ_N-formula under Π_{N-1}-Collection. By Σ_{N+1}-Replacement pick a function g with $dom(g) = \omega$ and $rng(g) \subset {}^{<\omega}\mathcal{P}(\omega)$ so that for any $k \in \omega$ $g(k)$ is the unique sequence $(Y_n)_{n<k} \in {}^k\mathcal{P}(\omega)$ with $Y_0 = X_0$. Then the function $f(n) = (g(n+1))\,(n)$ is a desired one. □

It is easy to see that $\mathsf{KP}\omega{+}\Pi_N$-Collection$+(V=L)\vdash A \Rightarrow \mathsf{KP}\omega{+}\Pi_N$-Collection $\vdash A^L$ for any A, and each Π^1_1-sentence B on ω is absolute for L, $\mathsf{KP}\omega{+}\Pi_N$-Collection $\vdash B \leftrightarrow B^L$.

Next we show that $\mathsf{KP}\omega + \Pi_N$-Collection is contained in a set theory $S_{\mathbb{I}_N}$. The language of the theory $S_{\mathbb{I}_N}$ is $\{\in, M_0\} \cup \{st_i\}_{0<i\leq N}$ with unary predicate constants st_i and an individual constant M_0. $st_i(a)$ is intended to denote the fact that a is an i-stable set and M_0 is intended to denote the least admissible set $L_{\omega_1^{CK}}$ above L_ω. The axioms of $S_{\mathbb{I}_N}$ are obtained from those[1] of $\mathsf{KP}\omega$ by adding the following axioms. By a $\Delta_0(\{st_i\}_{0<i<k})$-formula we mean a bounded formula in the language $\mathcal{L}_k = \{\in, M_0\} \cup \{st_i\}_{i<k}$.

1. The axioms for the admissible set M_0: $M_0 \neq \emptyset$, $\forall x \in M_0 \forall y \in x(y \in M_0)$, and the axioms stating that $(M_0, \in) \models \mathsf{KP}\omega$.

2. $\Delta_0(\{st_i\}_{0<i\leq N})$-collection:
$$\forall x \in a \exists y\, \theta(x,y) \to \exists b \forall x \in a \exists y \in b\, \theta(x,y)$$
for each $\Delta_0(\{st_i\}_{0<i\leq N})$-formula θ in which the predicates st_i may occur. Note that $\Sigma_1(\{st_i\}_{0<i\leq N})$-collection follows from this.

3. $$\forall a \exists b[a \in b \wedge st_N(b)] \tag{1}$$

4. For each $i+1 \leq N$:
$$st_{i+1}(a) \to M_0 \in a \wedge \forall y \in a \forall z \in y(z \in a) \wedge lst_i(a) \tag{2}$$
where $lst_i(a) :\Leftrightarrow st_i(a) \wedge \forall b \in a \exists c \in a\,(b \in c \wedge st_i(c))$ and $st_0(c) :\Leftrightarrow (0 = 0)$.

5. For $0 < i \leq N$:
$$st_i(a) \wedge \varphi(u) \wedge u \in a \to \varphi^a(u) \tag{3}$$
for each $\Sigma_1(\{st_j\}_{j<i})$-formula $\varphi \equiv (\exists x\, \theta)$ in the language $\mathcal{L}_i = \{\in, M_0\} \cup \{st_j\}_{j<i}$, where $\varphi^a \equiv (\exists x \in a\, \theta)$.

Note that if $lst_{i+1}(a)$ for a transitive set a, then $lst_i(a)$ holds.

Lemma 2.4 $S_{\mathbb{I}_N} \vdash st_i(M) \wedge u \in M \to [\varphi^M(u) \leftrightarrow \varphi(u)]$ *for set-theoretic* Σ_i-*formulas* φ.

Proof. Argue in $S_{\mathbb{I}_N}$. The case $i=1$ follows from the axiom (3). We show
$$st_k(a) \wedge u \in a \to \left[\theta^a(u) \leftrightarrow \exists b \in a\{st_i(b) \wedge u \in b \wedge \theta^b(u)\}\right] \tag{4}$$

[1] In the axiom schemata Δ_0-Separation and Δ_0-Collection, Δ_0-formulas remain to mean a Δ_0-formula in which st_i does not occur, while the axiom of foundation may be applied to a formula in which st_i may occur.

for $0 \leq i < k \leq N+1$ and $\Pi_1(\{st_j\}_{j<i-1})$-formula $\theta(u)$, where $a = V$, $st_{N+1}(V) :\Leftrightarrow (0=0)$ and $\theta^V(u) :\Leftrightarrow \theta$ when $k = N+1$.

Assume $st_k(a)$ and $\theta^a(u)$ with $u \in a$. By the axioms (1) and (2) there exists a set $b \in a$ such that $st_i(b)$ and $u \in b$. $\theta^b(u)$ follows logically. Conversely assume $\theta^b(u)$ for $b \in a$ such that $st_i(b)$ and $u \in b$. (3) yields $\theta(u)$, a fortiori $\theta^a(u)$. Thus (4) is shown.

Let $\varphi(u) \in \Sigma_{1+n}(\{st_j\}_{j<i})$ and $st_{i+n}(a)$ with $u \in a$. From (4) we see by (meta-)induction on n that there exists a $\Sigma_1(\{st_j\}_{j<i+n})$-formula θ such that $\varphi^a \leftrightarrow \theta^a$ and $\varphi \leftrightarrow \theta$.

Now we show $\varphi^M(u) \leftrightarrow \varphi(u)$, where $0 \leq n < N$, $st_{1+n}(M)$, $\varphi \in \Sigma_{1+n}$ and $u \in M$. Suppose $\varphi^M(u)$. Pick a $\Sigma_1(\{st_j\}_{j<n})$-formula θ such that $\varphi^M(u) \leftrightarrow \theta^M(u)$ and $\varphi(u) \leftrightarrow \theta(u)$. $\theta(u)$ follows logically, and $\varphi(u)$ follows. Conversely assume $\varphi(u)$. Then we obtain $\theta(u)$, and (3) yields $\theta^M(u)$, and hence $\varphi^M(u)$. $\Box$

Lemma 2.5 *$S_{\mathbb{I}_N}$ is an extension of* $\mathsf{KP}\omega + \Pi_N$-Collection. *Namely $S_{\mathbb{I}_N}$ proves Π_N-Collection.*

Proof. Argue in $S_{\mathbb{I}_N}$. Let $A(x,y)$ be a Π_N-formula in the language of set theory. We obtain by the axiom (1) and Lemma 2.4

$$A(x,y) \leftrightarrow \exists b(st_N(b) \land x, y \in b \land A^b(x,y)) \tag{5}$$

Assume $\forall x \in a \exists y A(x,y)$. Then we obtain $\forall x \in a \exists y \exists b(st_N(b) \land x, y \in b \land A^b(x,y))$ by (5). Since $st_N(b) \land x, y \in b \land A^b(x,y)$ is a $\Delta_0(\{st_i\}_{0<i\leq N})$-formula, pick a set c such that $\forall x \in a \exists y \in c \exists b \in c(st_N(b) \land x, y \in b \land A^b(x,y))$ by $\Delta_0(\{st_i\}_{0<i\leq N})$-Collection. Again by (5) we obtain $\forall x \in a \exists y \in c A(x,y)$. $\Box$

3 Ordinals for Π_N-Collection

In this section up to subsection 3.2 we work in a set theory $\mathsf{ZFC}(\{St_i\}_{0<i\leq N})$, where each St_i is a unary predicate symbol. Let St_0 denote the set of uncountable cardinals below $\mathbb{I}_N$. Ω and $\mathbb{I}_N$ are strongly critical numbers with $\Omega < \mathbb{I}_N$, i.e., non-zero ordinals closed under the binary Veblen function $\varphi\alpha\beta = \varphi_\alpha(\beta)$. We assume that $St_{i+1} \subset St_i$ for $i < N$, each St_i is an unbounded class of ordinals below $\mathbb{I}_N$ such that the least element of St_i is larger than Ω, $\Omega < \min(\bigcup_{0<i\leq N} St_i)$. The predicate St_i is identified with the class $\{\alpha \in ON : \alpha \in St_i\}$. $\alpha^{\dagger i}$ denotes the least ordinal$> \alpha$ in the class St_i when $\alpha < \mathbb{I}_N$. $\alpha^{\dagger i} := \mathbb{I}_N$ if $\alpha \geq \mathbb{I}_N$. Put $\alpha^\dagger := \alpha^{\dagger 1}$. Let $SSt_i := \{\alpha^{\dagger i} : \alpha \in ON\}$ and $LSt_i = St_i \setminus SSt_i$.

Γ_a denotes the a-th strongly critical number. For ordinals α, $\varepsilon(\alpha)$ denotes the least epsilon number above α, and $\Gamma(\alpha)$ the least strongly critical number above α. For ordinals α, β, and γ, $\gamma = \alpha - \beta$ designates that $\alpha = \beta + \gamma$. $\alpha \dot{+} \beta$ denotes the sum $\alpha + \beta$ when $\alpha + \beta$ equals to the commutative (natural) sum $\alpha \# \beta$, i.e., when either $\alpha = 0$ or $\alpha = \alpha_0 + \omega^{\alpha_1}$ with $\omega^{\alpha_1+1} > \beta$.

$u, v, w, x, y, z, \ldots$ range over sets in the universe, $a, b, c, \alpha, \beta, \gamma, \delta, \ldots$ range over ordinals$< \varepsilon(\mathbb{I}_N)$, and $\xi, \zeta, \eta, \ldots$ range over ordinals$< \Gamma(\mathbb{I}_N)$, and ordinals$\leq \mathbb{I}_N$ are denoted by $\pi, \kappa, \rho, \sigma, \tau, \lambda, \ldots$.

Let $\mathbb{S} \in St_i$ with $i > 0$. A 'Mahlo degree' $m(\pi)$ of ordinals $\pi < \mathbb{S}$ with higher reflections is defined to be a finite function $f : \mathbb{I}_N \to \varphi_{\mathbb{I}_N}(0)$. Let $\Lambda \leq \mathbb{I}_N$ be a strongly critical number. To denote ordinals$< \varphi_\Lambda(0)$, it is convenient for us to introduce an ordinal function $\tilde{\theta}_b(\xi; \Lambda) < \varphi_\Lambda(0)$ for $\xi < \varphi_\Lambda(0)$ and $b < \Lambda$ as in [4, 5], which is a b-th iterate of the exponential $\tilde{\theta}_1(\xi; \Lambda) = \Lambda^\xi$ with the base Λ.

Definition 3.1 Let $\Lambda \leq \mathbb{I}_N$ be a strongly critical number. $\varphi_b(\xi)$ denotes the binary Veblen function on $(\mathbb{I}_N)^{\dagger 0} = \omega_{\mathbb{I}_N+1}$ with $\varphi_0(\xi) = \omega^\xi$, and $\tilde{\varphi}_b(\xi; \Lambda) := \varphi_b(\Lambda \cdot \xi)$.

Let $b, \xi < (\mathbb{I}_N)^{\dagger 0}$. $\theta_b(\xi)$ $[\tilde{\theta}_b(\xi; \Lambda)]$ denotes a b-th iterate of $\varphi_0(\xi) = \omega^\xi$ [of $\tilde{\varphi}_0(\xi; \Lambda) = \Lambda^\xi$], resp. Specifically ordinals $\theta_b(\xi), \tilde{\theta}_b(\xi; \Lambda) < (\mathbb{I}_N)^{\dagger 0}$ are defined by recursion on b as follows. $\theta_0(\xi) = \tilde{\theta}_0(\xi; \Lambda) = \xi$, $\theta_{\omega^b}(\xi) = \varphi_b(\xi)$, $\tilde{\theta}_{\omega^b}(\xi; \Lambda) = \tilde{\varphi}_b(\xi; \Lambda)$, and $\theta_{c \dot{+} \omega^b}(\xi) = \theta_c(\theta_{\omega^b}(\xi))$, $\tilde{\theta}_{c \dot{+} \omega^b}(\xi; \Lambda) = \tilde{\theta}_c(\tilde{\theta}_{\omega^b}(\xi; \Lambda); \Lambda)$.

A finite set $SC(a)$ of strongly critical numbers is defined recursively as follows. $SC(0) = \emptyset$, $SC(a) = \bigcup_{i \leq m} SC(a_i)$ for $a = \omega^{a_m} \dot{+} \cdots \dot{+} \omega^{a_0}$, and $SC(a) = SC(b) \cup SC(c)$ for $a = \varphi_b(c)$ if a is not strongly critical. $SC(a) = \{a\}$ if a is strongly critical.

Let $\Lambda \leq \mathbb{I}_N$ be a strongly critical number. Let us define a normal form of non-zero ordinals $\xi < \varphi_\Lambda(0)$. Let $\xi = \Lambda^\zeta$. If $\zeta < \Lambda^\zeta$, then $\tilde{\theta}_1(\zeta; \Lambda)$ is the normal form of ξ, denoted by $\xi =_{NF} \tilde{\theta}_1(\zeta; \Lambda)$. Assume $\zeta = \Lambda^\zeta$, and let $b > 0$ be the maximal ordinal such that there exists an ordinal η with $\zeta = \tilde{\varphi}_b(\eta; \Lambda) > \eta$. Then $\xi = \tilde{\varphi}_b(\eta; \Lambda) =_{NF} \tilde{\theta}_{\omega^b}(\eta; \Lambda)$.

Let $\xi = \Lambda^{\zeta_m} a_m + \cdots + \Lambda^{\zeta_0} a_0$, where $\zeta_m > \cdots > \zeta_0$ and $0 < a_0, \ldots, a_m < \Lambda$. Let $\Lambda^{\zeta_i} =_{NF} \tilde{\theta}_{b_i}(\eta_i; \Lambda)$ with $b_i = \omega^{c_i}$ for each i. Then $\xi =_{NF} \tilde{\theta}_{b_m}(\eta_m; \Lambda) \cdot a_m + \cdots + \tilde{\theta}_{b_0}(\eta_0; \Lambda) \cdot a_0$.

Definition 3.2 Let $\xi < \varphi_\Lambda(0)$ be a non-zero ordinal with its normal form:

$$\xi = \sum_{i \leq m} \tilde{\theta}_{b_i}(\xi_i; \Lambda) \cdot a_i =_{NF} \tilde{\theta}_{b_m}(\xi_m; \Lambda) \cdot a_m + \cdots + \tilde{\theta}_{b_0}(\xi_0; \Lambda) \cdot a_0 \tag{6}$$

where $\tilde{\theta}_{b_i}(\xi_i; \Lambda) > \xi_i$, $\tilde{\theta}_{b_m}(\xi_m; \Lambda) > \cdots > \tilde{\theta}_{b_0}(\xi_0; \Lambda)$, $b_i = \omega^{c_i} < \Lambda$, and $0 < a_0, \ldots, a_m < \Lambda$. $\tilde{\theta}_{b_0}(\xi_0; \Lambda)$ is said to be the *tail* of ξ, denoted $\tilde{\theta}_{b_0}(\xi_0; \Lambda) = tl(\xi)$, and $\tilde{\theta}_{b_m}(\xi_m; \Lambda)$ the *head* of ξ, denoted $\tilde{\theta}_{b_m}(\xi_m; \Lambda) = hd(\xi)$.

1. ζ is a *segment* of ξ iff there exists an $n\, (0 \leq n \leq m+1)$ such that $\zeta =_{NF} \sum_{i \geq n} \tilde{\theta}_{b_i}(\xi_i; \Lambda) \cdot a_i = \tilde{\theta}_{b_m}(\xi_m; \Lambda) \cdot a_m + \cdots + \tilde{\theta}_{b_n}(\xi_n; \Lambda) \cdot a_n$ for ξ in (6).

2. Let $\zeta =_{NF} \tilde{\theta}_b(\xi; \Lambda)$ with $\tilde{\theta}_b(\xi; \Lambda) > \xi$ and $b = \omega^{b_0}$, and c be an ordinal. An ordinal $\tilde{\theta}_{-c}(\zeta; \Lambda)$ is defined recursively as follows. If $b \geq c$, then $\tilde{\theta}_{-c}(\zeta; \Lambda) = \tilde{\theta}_{b-c}(\xi; \Lambda)$. Let $c > b$. If $\xi > 0$, then $\tilde{\theta}_{-c}(\zeta; \Lambda) = \tilde{\theta}_{-(c-b)}(\tilde{\theta}_{b_m}(\xi_m; \Lambda); \Lambda)$ for the head term $hd(\xi) = \tilde{\theta}_{b_m}(\xi_m; \Lambda)$ of ξ in (6). If $\xi = 0$, then let $\tilde{\theta}_{-c}(\zeta; \Lambda) = 0$.

3. Let $\xi < \varphi_{\mathbb{I}_N}(0)$ be such that $SC(\xi) \subset \Lambda$ for a strongly critical number $\Lambda < \mathbb{I}_N$. Then $\xi[\Lambda : \mathbb{I}_N]$ denotes an ordinal$< \varphi_\Lambda(0)$ obtained from ξ by

changing the base $\mathbb{I}_N$ into Λ. This means that $\xi[\Lambda:\mathbb{I}_N]$ is obtained from ξ in (6) by replacing $\theta_{b_i}(\xi_i;\mathbb{I}_N)\cdot a_i$ by $\theta_{b_i}(\xi_i[\Lambda:\mathbb{I}_N];\Lambda)\cdot a_i$.

Proposition 3.3 *Let $\xi,\zeta<\varphi_{\mathbb{I}_N}(0)$ be such that $SC(\xi,\zeta)\subset\Lambda$ for a strongly critical number $\Lambda<\mathbb{I}_N$. Then $\xi<\zeta$ iff $\xi[\Lambda:\mathbb{I}_N]<\zeta[\Lambda:\mathbb{I}_N]$.*

Definition 3.4

1. A function $f:\Lambda\to\varphi_\Lambda(0)$ with a *finite* support $\mathrm{supp}(f)=\{c<\Lambda: f(c)\neq 0\}\subset\Lambda$ is said to be a *finite function* with *base* Λ if $\forall i>0(a_i=1)$ and $a_0=1$ when $b_0>1$ in $f(c)=_{NF}\tilde{\theta}_{b_m}(\xi_m;\Lambda)\cdot a_m+\cdots+\tilde{\theta}_{b_0}(\xi_0;\Lambda)\cdot a_0$ for any $c\in\mathrm{supp}(f)$.

   It is identified with the finite function $f\restriction\mathrm{supp}(f)$. When $c\not\in\mathrm{supp}(f)$, let $f(c):=0$. $f,g,h,\ldots$ range over finite functions.

   Let $SC(f):=\bigcup\{SC(c)\cup SC(f(c)): c\in\mathrm{supp}(f)\}$.

   For an ordinal c, f_c and f^c are restrictions of f to the domains $\mathrm{supp}(f_c)=\{d\in\mathrm{supp}(f): d<c\}$ and $\mathrm{supp}(f^c)=\{d\in\mathrm{supp}(f): d\geq c\}$. g_c*f^c denotes the concatenated function such that $\mathrm{supp}(g_c*f^c)=\mathrm{supp}(g_c)\cup\mathrm{supp}(f^c)$, $(g_c*f^c)(a)=g(a)$ for $a<c$, and $(g_c*f^c)(a)=f(a)$ for $a\geq c$.

2. Let f be a finite function and $c\leq\Lambda$, $\xi<\Gamma(\Lambda)$ ordinals. A relation $f<^c_\Lambda\xi$ is defined by induction on the cardinality of the finite set $\{d\in\mathrm{supp}(f): d>c\}$ as follows. If $f^c=\emptyset$, then $f<^c_\Lambda\xi$ holds. Let $f^c\neq\emptyset$. If $f^{c+1}=\emptyset$, then $f<^c_\Lambda\xi$ iff $f(c)<\xi$. Otherwise for $d=\min\{d>0: c+d\in\mathrm{supp}(f)\}$, $f<^c_\Lambda\xi$ iff there exists a segment μ of ξ such that $f(c)<\mu$ and $f<^{c+d}_\Lambda\tilde{\theta}_{-d}(tl(\mu);\Lambda)$, where $tl(\mu)$ is the tail of μ with base Λ.

The following Proposition 3.5 is shown in [4].

Proposition 3.5

1. *$\zeta\leq\xi<\varphi_\Lambda(0)\Rightarrow\tilde{\theta}_{-c}(\zeta;\Lambda)\leq\tilde{\theta}_{-c}(\xi;\Lambda)$.*
2. *$\tilde{\theta}_c(\tilde{\theta}_{-c}(\zeta;\Lambda);\Lambda)\leq\zeta$ for $\zeta<\varphi_\Lambda(0)$.*

Although the following Proposition 3.6 is shown in [5], let us reproduce its proof.

Proposition 3.6 *$f<^c_\Lambda\xi\leq\zeta\Rightarrow f<^c_\Lambda\zeta$.*

Proof. By induction on the cardinality n of the finite set $\{d\in\mathrm{supp}(f): d>c\}=\{c+d_1<\cdots<c+d_n\}$ with $c<c+d_1$. If $n=0$, then there is nothing to prove. Let $n>0$. We have $f(c)<\mu$, and $f<^{c+d_1}_\Lambda\tilde{\theta}_{-d_1}(tl(\mu);\Lambda)$ for a segment μ of ξ. We show the existence of a segment λ of ζ such that $\mu\leq\lambda$, and $\tilde{\theta}_{-d_1}(tl(\mu);\Lambda)\leq\tilde{\theta}_{-d_1}(tl(\lambda);\Lambda)$. Then IH yields $f<^{c+d_1}_\Lambda\tilde{\theta}_{-d_1}(tl(\lambda);\Lambda)$, and $f<^c_\Lambda\zeta$ follows.

If μ is a segment of ζ, then $\lambda=\mu$ works. Otherwise $\xi<\zeta$ and there exists a segment λ of ζ such that $\mu<\lambda$, and $tl(\mu)<tl(\lambda)$. We obtain $\tilde{\theta}_{-d_1}(tl(\mu);\Lambda)\leq\tilde{\theta}_{-d_1}(tl(\lambda);\Lambda)$ by Proposition 3.5.1. $\Box$

3.1 Skolem hulls and Mahlo classes

In this subsection Skolem hulls $\mathcal{H}_a(X)$, collapsing functions ψ and Mahlo classes $Mh^a_{i,c}(\xi)$ are introduced. ψ-functions are introduced in Buchholz[8].

Definition 3.7 Let $A \subset \mathbb{I}_N$ be a set, and $\alpha \leq \mathbb{I}_N$ a limit ordinal.

$$\alpha \in M(A) :\Leftrightarrow A \cap \alpha \text{ is stationary in } \alpha \Leftrightarrow \text{ every club subset of } \alpha \text{ meets } A.$$

In the following Definition 3.8, $\varphi\alpha\beta = \varphi_\alpha(\beta)$ denotes the binary Veblen function on $(\mathbb{I}_N)^{\dagger 0}$. For $a < \varepsilon(\mathbb{I}_N)$, $c < \mathbb{I}_N$, $\xi < \Gamma(\mathbb{I}_N)$, and $X \subset \mathbb{I}_N$, define simultaneously classes $\mathcal{H}_a(X) \subset \Gamma(\mathbb{I}_N)$, $Mh^a_{i,c}(\xi) \subset (\mathbb{I}_N{+}1)$ $(i > 0)$, and ordinals $\psi_{\mathbb{I}_N}(a) \leq \mathbb{I}_N$ and $\psi^f_\kappa(a) \leq \kappa$ by recursion on ordinals a as follows.

Definition 3.8 Let $a < \varepsilon(\mathbb{I}_N)$, $c < \mathbb{I}_N$, $\xi < \Gamma(\mathbb{I}_N)$, and $X \subset \mathbb{I}_N$.

1. (Inductive definition of $\mathcal{H}_a(X)$)

   (a) $\{0, \Omega, \mathbb{I}_N\} \cup X \subset \mathcal{H}_a(X)$, where $\Omega \in SSt_0$.

   (b) If $x, y \in \mathcal{H}_a(X)$, then $x + y \in \mathcal{H}_a(X)$ and $\varphi xy \in \mathcal{H}_a(X)$.

   (c) Let $\alpha = \psi_\pi(b)$ with $\pi \in \mathcal{H}_a(X) \cap SSt_0 \cap \mathbb{I}_N$, $b \in \mathcal{H}_a(X) \cap a$ such that $\{\pi, b\} \subset \mathcal{H}_b(\alpha)$. Then $\alpha \in \mathcal{H}_a(X)$.

   (d) Let $\alpha = \psi_{\mathbb{I}_N}(b)$ with $b \in \mathcal{H}_a(X) \cap a$. Then $\alpha \in \mathcal{H}_a(X) \cap (LSt_N \cup \{\mathbb{I}_N\})$.

   (e) Let $\alpha \in \mathcal{H}_a(X) \cap \mathbb{I}_N$. Then $\alpha^{\dagger i} \in \mathcal{H}_a(X) \cap SSt_i$ for each $0 < i \leq N$.

   (f) Let $\alpha = \psi^f_\pi(b)$ with $b < a$, and a finite function $f : \mathbb{I}_N \to \varphi_{\mathbb{I}_N}(0)$ such that $\{\pi, b\} \cup SC(f) \subset \mathcal{H}_a(X) \cap \mathcal{H}_b(\alpha)$. Then $\alpha \in \mathcal{H}_a(X)$.

2. (Definitions of $Mh^a_{i,c}(\xi)$ and $Mh^a_{i,c}(f)$ for $0 < i \leq N$)
   The classes $Mh^a_{i,c}(\xi)$ are defined for $c < \mathbb{I}_N$, $a < \varepsilon(\mathbb{I}_N)$ and $\xi < \varphi_{\mathbb{I}_N}(0)$. By main induction on ordinals $\pi < \mathbb{I}_N$ with subsidiary induction on $c < \mathbb{I}_N$ we define $\pi \in Mh^a_{i,c}(\xi)$ iff $\pi \in LSt_{i-1}$, $\{a, c, \xi\} \subset \mathcal{H}_a(\pi)$ and the following condition is met for any finite functions $f, g : \mathbb{I}_N \to \varphi_{\mathbb{I}_N}(0)$ such that $f <^c_{\mathbb{I}_N} \xi$:

   $$SC(f, g) \subset \mathcal{H}_a(\pi) \,\&\, \pi \in Mh^a_{i,0}(g_c) \Rightarrow \pi \in M(Mh^a_{i,0}(g_c * f^c))$$

   where $SC(f, g) = SC(f) \cup SC(g)$ and

   $$\begin{aligned} Mh^a_{i,c}(f) &:= \bigcap\{Mh^a_{i,d}(f(d)) : d \in \mathrm{supp}(f^c)\} \\ &= \bigcap\{Mh^a_{i,d}(f(d)) : c \leq d \in \mathrm{supp}(f)\}. \end{aligned}$$

   $Mh^a_{i,0}(g_c) = \bigcap\{Mh^a_{i,d}(g(d)) : d \in \mathrm{supp}(g_c)\} = \bigcap\{Mh^a_{i,d}(g(d)) : c > d \in \mathrm{supp}(g)\}$. When $f = \emptyset$ or $f^c = \emptyset$, let $Mh^a_{i,c}(\emptyset) := LSt_{i-1}$.

3. (Definition of $\psi^f_\pi(a)$)
   Let a, π be ordinals, and $f : \mathbb{I}_N \to \varphi_{\mathbb{I}_N}(0)$ a finite function. Then $\psi^f_{i,\pi}(a)$ denotes the least ordinal $\kappa < \pi$ such that

$$\kappa \in Mh^a_{i,0}(f) \,\&\, \mathcal{H}_a(\kappa) \cap \pi \subset \kappa \,\&\, \{\pi, a\} \cup SC(f) \subset \mathcal{H}_a(\kappa) \tag{7}$$

   if such a κ exists. Otherwise set $\psi^f_{i,\pi}(a) = \pi$.

4. $\psi_\Omega(a) := \min(\{\Omega\} \cup \{\beta : \mathcal{H}_a(\beta) \cap \Omega \subset \beta\})$ and

$$\psi_{\mathbb{I}_N}(a) := \min(\{\mathbb{I}_N\} \cup \{\kappa \in LSt_N : \mathcal{H}_a(\kappa) \cap \mathbb{I}_N \subset \kappa\}) \tag{8}$$

5. For classes $A \subset \mathbb{I}_N$, let $\alpha \in M^a_{i,c}(A)$ iff $\alpha \in A$ and for any finite functions $g : \mathbb{I}_N \to \varphi_{\mathbb{I}_N}(0)$

$$\alpha \in Mh^a_{i,0}(g_c) \,\&\, SC(g_c) \subset \mathcal{H}_a(\alpha) \Rightarrow \alpha \in M\left(Mh^a_{i,0}(g_c) \cap A\right) \tag{9}$$

The following Propositions 3.9, 3.10 and 3.11 are seen as in [5].

Proposition 3.9 *Assume* $\pi \in Mh^a_{i,c}(\zeta)$ *and* $\xi < \zeta$ *with* $SC(\xi) \subset \mathcal{H}_a(\pi)$. *Then* $\pi \in Mh^a_{i,c}(\xi) \cap M^a_{i,c}(Mh^a_{i,c}(\xi))$.

Proof. Proposition 3.6 yields $\pi \in Mh^a_{i,c}(\xi)$. $\pi \in M^a_{i,c}(Mh^a_{i,c}(\xi))$ is seen from the function f such that $f <^c_{\mathbb{I}_N} \zeta$ with $\mathrm{supp}(f) = \{c\}$ and $f(c) = \xi$. □

Proposition 3.10 *Suppose* $\pi \in Mh^a_{i,c}(\xi)$.

1. *Let* $f <^c_{\mathbb{I}_N} \xi$ *with* $SC(f) \subset \mathcal{H}_a(\pi)$. *Then* $\pi \in M^a_{i,c}(Mh^a_{i,c}(f^c))$.
2. *Let* $\pi \in M^a_{i,d}(A)$ *for* $d > c$ *and* $A \subset \mathbb{I}_N$. *Then* $\pi \in M^a_{i,c}(Mh^a_{i,c}(\xi) \cap A)$.

Proof. 3.10.1. Let g be a function such that $\pi \in Mh^a_{i,0}(g_c)$ with $SC(g_c) \subset \mathcal{H}_a(\pi)$. We obtain $\pi \in M\left(Mh^a_{i,0}(g_c) \cap Mh^a_{i,c}(f^c)\right)$ by Definition 3.8.2 of $\pi \in Mh^a_{i,c}(\xi)$.
3.10.2. Let $\pi \in M^a_{i,d}(A)$ for $d > c$. Then $\pi \in Mh^a_{i,c}(\xi) \cap A$. Let g be a function such that $\pi \in Mh^a_{i,0}(g_c)$ with $SC(g_c) \subset \mathcal{H}_a(\pi)$. We obtain by (9) and $d > c$ with the function $g_c * h$, $\pi \in M\left(Mh^a_{i,0}(g_c) \cap Mh^a_{i,c}(\xi) \cap A\right)$, where $\mathrm{supp}(h) = \{c\}$ and $h(c) = \xi$. □

Proposition 3.11 *Each of* $x \in \mathcal{H}_a(y)$, $x \in Mh^a_{i,c}(f)$ *and* $x = \psi^f_\kappa(a)$ *is a* $\Delta_1(\{St_i\}_{0<i\leq N})$*-predicate in* $\mathsf{ZFC}(\{St_i\}_{0<i\leq N})$.

Proof. An inspection of Definition 3.8 shows that $x \in \mathcal{H}_a(y)$, $\psi^f_\kappa(a)$ and $x \in Mh^a_{i,c}(f)$ are simultaneously defined by recursion on $a < \varepsilon(\mathbb{I}_N)$, in which $x \in Mh^a_{i,c}(f)$ is defined by recursion on ordinals $x < \mathbb{I}_N$ with subsidiary recursion on $c < \mathbb{I}_N$. □

3.2 A small large cardinal hypothesis

It is convenient for us to assume the existence of a small large cardinal in justification of Definition 3.8. *Shrewd cardinals* as well as $\mathcal{A}$-*shrewd cardinals* are introduced by M. Rathjen[14].

Definition 3.12 (Rathjen[14])
Let $\eta > 0$. A cardinal κ is η-*shrewd* iff for any $P \subset V_\kappa$, and a set-theoretic formula $\varphi(x, y)$ if $V_{\kappa+\eta} \models \varphi[P, \kappa]$, then there are $0 < \kappa_0, \eta_0 < \kappa$ such that $V_{\kappa_0+\eta_0} \models \varphi[P \cap V_{\kappa_0}, \kappa_0]$. For classes $\mathcal{A}$, κ is $\mathcal{A}$-η-*shrewd* iff for any $P \subset V_\kappa$, and a formula $\varphi(x, y)$ in the language $\{\in, R\}$ with a unary predicate R if $(V_{\kappa+\eta}; \mathcal{A}) \models \varphi[P, \kappa]$, then there are $0 < \kappa_0, \eta_0 < \kappa$ such that $(V_{\kappa_0+\eta_0}; \mathcal{A}) \models \varphi[P \cap V_{\kappa_0}, \kappa_0]$, where $(V_\alpha; \mathcal{A})$ denotes the structure $(V_\alpha, \in; \mathcal{A} \cap V_\alpha)$, and for the formulas φ in the language $\{\in, R\}$, $R(t)$ is interpreted as $t \in \mathcal{A} \cap V_\alpha$ in $(V_\alpha; \mathcal{A}) \models \varphi$.

Obviously each $\mathcal{A}$-η-shrewd cardinal is η-shrewd. We see easily that each η-shrewd cardinal is regular. A cardinal κ is said to be $(< \eta)$-*shrewd* [$\mathcal{A}$-$(< \eta)$-*shrewd*] if κ is δ-shrewd [$\mathcal{A}$-δ-shrewd] for every $\delta < \eta$, resp.

On the other side subtle cardinals are introduced by R. Jensen and K. Kunen. The following Lemma 3.13 is shown in [14] by Rathjen.

Lemma 3.13 (Lemma 2.7 of [14])
Let π be a subtle cardinal. The set $\{\kappa \in V_\pi : (V_\pi; \mathcal{A}) \models$ 'κ is $\mathcal{A}$-shrewd'$\}$ of $\mathcal{A}$-shrewd cardinals in $(V_\pi; \mathcal{A})$ is stationary in π for each class $\mathcal{A}$.

Definition 3.14 Let π be a cardinal. The classes $\mathcal{B}_n$ and $\mathcal{A}_n$ are defined recursively for $n < \omega$. Let

$$\begin{array}{rcl} \mathcal{B}_0 & = & \{\kappa \in V_\pi : V_\pi \models \text{`}\kappa \text{ is an uncountable cardinal'}\} \\ \mathcal{A}_n & = & \{\langle i, \sigma \rangle : i \leq n, \sigma \in \mathcal{B}_i\} \\ \mathcal{B}_{n+1} & = & \{\kappa \in V_\pi : (V_\pi; \mathcal{A}_n) \models \text{`}\kappa \text{ is an } \mathcal{A}_n\text{-shrewd cardinal'}\}. \end{array}$$

We say that a cardinal $\kappa \in V_\pi$ is n-*shrewd* in π iff $\kappa \in \mathcal{B}_n$. An n-shrewd carinal is an n-*shrewd limit* iff the set of n-shrewd cardinals is cofinal in it.

$\mathcal{B}_1$ is the set of shrewd cardinals in V_π, and a 1-shrewd cardinal is a shrewd cardinal in π. Each $\mathcal{A}_{n+1}$-shrewd cardinal is $\mathcal{A}_n$-shrewd, and each $(n + 1)$-shrewd cardinal is n-shrewd.

Lemma 3.15 *Let π be a subtle cardinal.*

1. *The set of n-shrewd cardinals in π is stationary in π for each $n < \omega$.*
2. *Let κ be an $(n+1)$-shrewd cardinal in π. If $(V_{\kappa+\eta}; \mathcal{A}_n) \models \varphi[P, \kappa]$ for $0 < \eta < \pi$, $P \subset V_\kappa$ and a formula $\varphi(x, y)$ in $\{\in, R\}$, then there are an n-shrewd limit $\kappa_0 < \kappa$ and $0 < \eta_0 < \kappa$ such that $(V_{\kappa_0+\eta_0}; \mathcal{A}_n) \models \varphi[P \cap V_{\kappa_0}, \kappa_0]$.*

Proof. 3.15.1. From Lemma 3.13 we see that the set of $\mathcal{A}_{n-1}$-shrewd cardinals is stationary in a subtle cardinal π.
3.15.2. Let κ be an $(n+1)$-shrewd cardinal in π. Then κ is n-shrewd, and hence $(V_{\kappa+\eta};\mathcal{A}_n)\models \exists x(x\in P)\land R(\langle n,\kappa\rangle)$ for each $P=\{\alpha\}\subset V_\kappa$ with $\kappa<\kappa+\eta<\pi$. Since κ is $\mathcal{A}_n$-shrewd, there are $0<\kappa_0,\eta_0<\kappa$ such that $(V_{\kappa_0+\eta_0};\mathcal{A}_n)\models \exists x(x\in P\cap V_{\kappa_0})\land R(\langle n,\kappa_0\rangle)$. This means that $\alpha<\kappa_0$ is n-shrewd. Therefore κ is an n-shrewd limit.

Suppose $(V_{\kappa+\eta};\mathcal{A}_n)\models\varphi[P,\kappa]$ for $0<\eta<\pi$, $P\subset V_\kappa$ and a formula $\varphi(x,y)$ in $\{\in,R\}$. Then $(V_{\kappa+\eta};\mathcal{A}_n)\models\varphi[P,\kappa]\land R(\langle n,\kappa\rangle)\land\forall\alpha<\kappa\exists\sigma<\kappa(\sigma>\alpha\land R(\langle n,\sigma\rangle))$. Since κ is $\mathcal{A}_n$-shrewd, there are an n-shrewd limit $\kappa_0<\kappa$ and $0<\eta_0<\kappa$ such that $(V_{\kappa_0+\eta_0};\mathcal{A}_n)\models\varphi[P\cap V_{\kappa_0},\kappa_0]$. □

In this subsection we work in an extension T of ZFC by adding the axiom stating that there exists a regular cardinal $\mathbb{I}_N$ in which the set of N-shrewd cardinals is stationary. Ω denotes the least uncountable ordinal ω_1, For $0<i\leq N$, $St_i=\mathcal{B}_i$ the class of i-shrewd cardinals in $V_{\mathbb{I}_N}$. LSt_i denotes the class of i-shrewd limits in $V_{\mathbb{I}_N}$. Let $St_{N+1}=SSt_{N+1}=\{\mathbb{I}_N\}$ with $\mathbb{I}_N=\Omega^{\dagger(N+1)}$. Also St_0 denotes the class of uncountable cardinals in $V_{\mathbb{I}_N}$, and LSt_0 the class of limit cardinals in $V_{\mathbb{I}_N}$. A *successor n-shrewd cardinal* is an n-shrewd cardinal in $V_{\mathbb{I}_N}$, but not in LSt_n.

Lemma 3.16 $T\vdash\forall a<\Gamma(\mathbb{I}_N)[\psi_{\mathbb{I}_N}(a)<\mathbb{I}_N]$.

Proof. We see that the set $C=\{\kappa<\mathbb{I}_N:\mathcal{H}_a(\kappa)\cap\mathbb{I}_N\subset\kappa\}$ is a club subset of the regular cardinal $\mathbb{I}_N$. This shows the existence of a $\kappa\in LSt_N\cap C$, and hence $\psi_{\mathbb{I}_N}(a)<\mathbb{I}_N$ by the definition (8). □

$\alpha^{\dagger i^{(k)}}$ is defined by recursion on $k<\omega$ by $\alpha^{\dagger i^{(0)}}=\alpha$ and $\alpha^{\dagger i^{(k+1)}}=(\alpha^{\dagger i^{(k)}})^{\dagger i}$.

Proposition 3.17 *Let $a\in\mathcal{H}_a(\psi_{\mathbb{I}_N}(a))$, $b\in\mathcal{H}_b(\psi_{\mathbb{I}_N}(b))$, $c\in\mathcal{H}_c(\psi_\Omega(c))$ and $d\in\mathcal{H}_d(\psi_\Omega(d))$.*

1. *$\psi_{\mathbb{I}_N}(a)<\psi_{\mathbb{I}_N}(b)$ iff $a<b$.*
2. *$\Omega^{\dagger N^{(k)}}<\psi_{\mathbb{I}_N}(b)$ for every $k<\omega$.*
3. *Let $\alpha=\psi_{\mathbb{I}_N}(a)$ and $0<k<\omega$. Then $\alpha^{\dagger N^{(k)}}<\psi_{\mathbb{I}_N}(b)$ iff $\alpha<\psi_{\mathbb{I}_N}(b)$. $\psi_{\mathbb{I}_N}(b)<\alpha^{\dagger N^{(k)}}$ iff $\psi_{\mathbb{I}_N}(b)\leq\alpha$.*
4. *$\psi_\Omega(c)<\psi_\Omega(d)$ iff $c<d$.*
5. *If $x<y$, then $\psi_{\mathbb{I}_N}(x)\leq\psi_{\mathbb{I}_N}(y)$.*

Proof. 3.17.2 and 3.17.3. Let $\beta=\psi_{\mathbb{I}_N}(b)$. By the definition (8) and $\Omega\in\mathcal{H}_b(\beta)\cap\mathbb{I}_N\subset\beta$ we obtain $\Omega<\beta$. Let $\alpha\in\{\Omega,\psi_{\mathbb{I}_N}(a)\}$. If $\alpha<\beta$, then $\beta\in LSt_N$ yields $\alpha^{\dagger N^{(k)}}<\beta$.
3.17.5. We obtain $\psi_{\mathbb{I}_N}(y)\in LSt_N$ and $\mathcal{H}_x(\psi_{\mathbb{I}_N}(y))\cap\mathbb{I}_N\subset\mathcal{H}_y(\psi_{\mathbb{I}_N}(y))\cap\mathbb{I}_N\subset\psi_{\mathbb{I}_N}(y)$ by $x<y$ and Lemma 3.16. Hence $\psi_{\mathbb{I}_N}(x)\leq\psi_{\mathbb{I}_N}(y)$. □

3.3 ψ-functions

In this subsection we work in $\mathsf{ZFC}(\{St_i\}_{0<i\leq N})$ with $St_i=\mathcal{B}_i$, and show that $\psi^f_{i,\mathbb{S}}(a)<\mathbb{S}$ for i-shrewd cardinal $\mathbb{S}$ in Lemma 3.19, and introduce an *irreducibility* of finite functions in Definition 3.24 using Lemma 3.21, which is needed to define a normal form in ordinal notations.

Lemma 3.18 *Let $\mathbb{S}$ be an i-shrewd cardinal with $0<i\leq N$, $a<\varepsilon(\mathbb{I}_N)$, $h:\mathbb{I}_N\to\varphi_{\mathbb{I}_N}(0)$ a finite function with $\{a\}\cup SC(h)\subset\mathcal{H}_a(\mathbb{S})$. Then $\mathbb{S}\in Mh^a_{i,0}(h)\cap M(Mh^a_{i,0}(h))$.*

Proof. By induction on $\xi<\varphi_{\mathbb{I}_N}(0)$ we show $\mathbb{S}\in Mh^a_{i,c}(\xi)$ for $\{a,c,\xi\}\subset\mathcal{H}_a(\mathbb{S})$.

Let $\{a,c,\xi\}\cup SC(f)\subset\mathcal{H}_a(\mathbb{S})$ with $f<^c_{\mathbb{I}_N}\xi$ and $a<\varepsilon(\mathbb{I}_N)$. We show $\mathbb{S}\in M^a_{i,c}(Mh^a_{i,c}(f^c))$, which yields $\mathbb{S}\in Mh^a_{i,c}(\xi)$. IH yields $\mathbb{S}\in Mh^a_{i,c}(f^c)$ by Proposition 3.5.2, $\tilde{\theta}_{-e}(\zeta;\mathbb{I}_N)\leq\zeta$. By the definition (9) it suffices to show that

$$\forall g[\mathbb{S}\in Mh^a_{i,0}(g_c)\,\&\,SC(g_c)\subset\mathcal{H}_a(\mathbb{S})\Rightarrow\mathbb{S}\in M\left(Mh^a_{i,0}(g_c)\cap Mh^a_{i,c}(f^c)\right)].$$

Let $g:\mathbb{I}_N\to\varphi_{\mathbb{I}_N}(0)$ be a finite function such that $SC(g_c)\subset\mathcal{H}_a(\mathbb{S})$ and $\mathbb{S}\in Mh^a_{i,0}(g_c)$. We have to show $\mathbb{S}\in M(A\cap B)$ for $A=Mh^a_{i,0}(g_c)\cap\mathbb{S}$ and $B=Mh^a_{i,c}(f^c)\cap\mathbb{S}$. Let C be a club subset of $\mathbb{S}$.

We have $\mathbb{S}\in Mh^a_{i,0}(g_c)\cap Mh^a_{i,c}(f^c)$, and $\{a\}\cup SC(g_c,f^c)\subset\mathcal{H}_a(\mathbb{S})$. Pick a $b<\mathbb{S}$ so that $\{a\}\cup SC(g_c,f^c)\subset\mathcal{H}_a(b)$. Since the cardinality of the set $\mathcal{H}_a(\mathbb{S})$ is equal to $\mathbb{S}$, pick a bijection $F:\mathbb{S}\to\mathcal{H}_a(\mathbb{S})$. Each $\alpha<\Gamma(\mathbb{I}_N)$ with $\alpha\in\mathcal{H}_a(\mathbb{S})$ is identified with its code, denoted by $F^{-1}(\alpha)<\mathbb{S}$. Let P be the class $P=\{(\pi,d,\alpha)\in\mathbb{S}^3:\pi\in Mh^{F(\alpha)}_{i,F(d)}(F(\xi))\}$, where $F(d)\in\mathcal{H}_a(\mathbb{S})\cap(c+1)$ and $F(\alpha)<\Gamma(\mathbb{I}_N)$ with $\{F(d),F(\alpha)\}\subset\mathcal{H}_a(\pi)$. For fixed i, a and c, the set $\{(d,\zeta)\in(\mathcal{H}_a(\mathbb{S})\cap(c+1))\times\Gamma(\mathbb{I}_N):\mathbb{S}\in Mh^a_{i,d}(\zeta)\}$ is defined from the classes P and $\{St_j\}_{j<i}$ by recursion on ordinals $d\leq c$.

Let φ be a formula in $\{\in\}\cup\{St_j\}_{j<i}$ such that $(V_{\mathbb{S}+c^{\dagger i}};\{St_j\}_{j<i})\models\varphi[P,C,\mathbb{S},b]$ iff $\mathbb{S}\in Mh^a_{i,0}(g_c)\cap Mh^a_{i,c}(f^c)$ and C is a club subset of $\mathbb{S}$, where $\{St_j\}_{j<i}=\mathcal{A}_{i-1}$. Since $\mathbb{S}$ is i-shrewd in $V_{\mathbb{I}_N}$, pick $b<\mathbb{S}_0<\eta<\mathbb{S}$ such that $(V_{\mathbb{S}_0+\eta};\{St_j\}_{j<i})\models\varphi[P\cap\mathbb{S}_0,C\cap\mathbb{S}_0,\mathbb{S}_0,b]$. We obtain $\mathbb{S}_0\in A\cap B\cap C$.

Therefore $\mathbb{S}\in Mh^a_{i,c}(\xi)$ is shown for every $\{c,\xi\}\subset\mathcal{H}_a(\mathbb{S})$. This yields $\mathbb{S}\in Mh^a_{i,0}(h)$ for $SC(h)\subset\mathcal{H}_a(\mathbb{S})$. $\mathbb{S}\in M(Mh^a_{i,0}(h))$ follows from the i-shrewdness of $\mathbb{S}$. □

Lemma 3.19 *Let $\mathbb{S}$ be an i-shrewd cardinal, a an ordinal, and $f:\mathbb{I}_N\to\varphi_{\mathbb{I}_N}(0)$ a finite function such that $\{a,\mathbb{S}\}\cup SC(f)\subset\mathcal{H}_a(\mathbb{S})$. Then $\psi^f_{i,\mathbb{S}}(a)<\mathbb{S}$ holds.*

Proof. Suppose $\{a,\mathbb{S}\}\cup SC(f)\subset\mathcal{H}_a(\mathbb{S})$. By Lemma 3.18 we obtain $\mathbb{S}\in M(Mh^a_{i,0}(f))$. The set $C=\{\kappa<\mathbb{S}:\mathcal{H}_a(\kappa)\cap\mathbb{S}\subset\kappa,\{a,\mathbb{S}\}\cup SC(f)\subset\mathcal{H}_a(\kappa)\}$ is a club subset of the regular cardinal $\mathbb{S}$, and $Mh^a_{i,0}(f)$ is stationary in $\mathbb{S}$. This shows the existence of a $\kappa\in Mh^a_{i,0}(f)\cap C\cap\mathbb{S}$, and hence $\psi^f_{i,\mathbb{S}}(a)<\mathbb{S}$ by the definition (7). □

Proposition 3.20 *Let α be either Ω or an i-shrewd cardinal for $0 < i \leq N$ and $\mathbb{S} = \alpha^{\dagger i}$. Assume $\{a, \mathbb{S}\} \cup SC(f) \subset \mathcal{H}_a(\mathbb{S})$ for an ordinal a and a finite function $f : \mathbb{I}_N \to \varphi_{\mathbb{I}_N}(0)$. Then $\alpha^{\dagger j} < \psi^f_{i,\mathbb{S}}(a)$ for every $j < i$, and $\psi^f_{i,\mathbb{S}}(a) \in LSt_{i-1} \backslash St_i$.*

Proof. Let $\kappa = \psi^f_{i,\mathbb{S}}(a) < \mathbb{S}$. We obtain $\alpha \in \mathcal{H}_a(\kappa)$ by $\alpha^{\dagger i} = \mathbb{S} \in \mathcal{H}_a(\kappa)$, and $\alpha^{\dagger j} \in \mathcal{H}_a(\kappa) \cap \mathbb{S}$ for $\mathbb{S} \in LSt_j$. $\alpha < \kappa$ is seen from $\alpha^{\dagger j} \in \mathcal{H}_a(\kappa) \cap \mathbb{S} \subset \kappa$ in the definition (7). □

The following Lemma 3.21 and Corollary 3.23 are seen as in [5].

Lemma 3.21 *Assume $\mathbb{I}_N > \pi \in Mh^a_{i,d}(\xi) \cap Mh^a_{i,c}(\xi_0)$, $\xi_0 \neq 0$, and $d < c$. Moreover let $\xi_1 \in \mathcal{H}_a(\pi)$ for $\xi_1 \leq \tilde{\theta}_{c-d}(\xi_0; \mathbb{I}_N)$, and $tl(\xi) \geq \xi_1$ when $\xi \neq 0$. Then $\pi \in Mh^a_{i,d}(\xi + \xi_1) \cap M^a_{i,d}(Mh^a_{i,d}(\xi + \xi_1))$.*

Proof. $\pi \in M^a_{i,d}(Mh^a_{i,d}(\xi + \xi_1))$ follows from $\pi \in Mh^a_{i,d}(\xi + \xi_1)$ and $\pi \in Mh^a_{i,c}(\xi_0) \subset M^a_{i,c}(Mh^a_{i,c}(\emptyset))$ by Proposition 3.10.1.

Let f be a finite function such that $SC(f) \subset \mathcal{H}_a(\pi)$, and $f <^d_{\mathbb{I}_N} \xi + \xi_1$. We show $\pi \in M^a_{i,d}(Mh^a_{i,d}(f^d))$ by main induction on the cardinality of the finite set $\{e \in \text{supp}(f) : e > d\}$ with subsidiary induction on ξ_1.

First let $f <^d_{\mathbb{I}_N} \mu$ for a segment μ of ξ. By Proposition 3.9 we obtain $\pi \in Mh^a_{i,d}(\mu)$ and $\pi \in M^a_{i,d}(Mh^a_{i,d}(f^d))$.

In what follows let $f(d) = \xi + \zeta$ with $\zeta < \xi_1$. By SIH we obtain $\pi \in Mh^a_{i,d}(f(d)) \cap M^a_{i,d}(Mh^a_{i,d}(f(d)))$. If $\{e \in \text{supp}(f) : e > d\} = \emptyset$, then $Mh^a_{i,d}(f^d) = Mh^a_{i,d}(f(d))$, and we are done. Otherwise let $e = \min\{e \in \text{supp}(f) : e > d\}$. By SIH we can assume $f <^e_{\mathbb{I}_N} \tilde{\theta}_{-(e-d)}(tl(\xi_1); \mathbb{I}_N)$. We obtain $f <^e_{\mathbb{I}_N} \tilde{\theta}_{-(e-d)}(\tilde{\theta}_{c-d}(\xi_0; \mathbb{I}_N); \mathbb{I}_N) = \tilde{\theta}_{-e}(\tilde{\theta}_c(\xi_0; \mathbb{I}_N); \mathbb{I}_N)$ by $\xi_1 \leq \tilde{\theta}_{c-d}(\xi_0; \mathbb{I}_N)$, Propositions 3.6 and 3.5.1. We claim that $\pi \in M^a_{i,c_0}(Mh^a_{i,c_0}(f^{c_0}))$ for $c_0 = \min\{c, e\}$. If $c = e$, then the claim follows from the assumption $\pi \in Mh^a_{i,c}(\xi_0)$ and $f <^e_{\mathbb{I}_N} \xi_0$. Let $e = c + e_0 > c$. Then $\tilde{\theta}_{-e}(\tilde{\theta}_c(\xi_0; \mathbb{I}_N); \mathbb{I}_N) = \tilde{\theta}_{-e_0}(hd(\xi_0); \mathbb{I}_N)$, and $f <^c_{\mathbb{I}_N} \xi_0$ with $f(c) = 0$ yields the claim. Let $c = e + c_1 > e$. Then $\tilde{\theta}_{-e}(\tilde{\theta}_c(\xi_0; \mathbb{I}_N); \mathbb{I}_N) = \tilde{\theta}_{c_1}(\xi_0; \mathbb{I}_N)$. MIH yields the claim.

On the other hand we have $Mh^a_{i,d}(f^d) = Mh^a_{i,d}(f(d)) \cap Mh^a_{i,c_0}(f^{c_0})$. $\pi \in Mh^a_{i,d}(f(d)) \cap M^a_{i,c_0}(Mh^a_{i,c_0}(f^{c_0}))$ with $d < c_0$ yields by Proposition 3.10.2, $\pi \in M^a_{i,d}(Mh^a_{i,d}(f(d)) \cap Mh^a_{i,c_0}(f^{c_0}))$, i.e., $\pi \in M^a_{i,d}(Mh^a_{i,d}(f^d))$. □

Definition 3.22 For finite functions $f, g : \mathbb{I}_N \to \varphi_{\mathbb{I}_N}(0)$, $Mh^a_{i,0}(g) \prec Mh^a_{i,0}(f)$ iff the following holds:

$$\forall \pi \in Mh^a_{i,0}(f) \left(SC(g) \subset \mathcal{H}_a(\pi) \Rightarrow \pi \in M(Mh^a_{i,0}(g)) \right).$$

Corollary 3.23 *Let $f, g : \mathbb{I}_N \to \varphi_{\mathbb{I}_N}(0)$ be finite functions and $c \in \text{supp}(f)$. Assume that there exists an ordinal $d < c$ such that $(d, c) \cap \text{supp}(f) = (d, c) \cap \text{supp}(g) = \emptyset$, $g_d = f_d$, $g(d) < f(d) + \tilde{\theta}_{c-d}(f(c); \mathbb{I}_N) \cdot \omega$, and $g <^c_{\mathbb{I}_N} f(c)$.*

Then $Mh^a_{i,0}(g) \prec Mh^a_{i,0}(f)$ holds. In particular if $\pi \in Mh^a_{i,0}(f)$ and $SC(g) \subset \mathcal{H}_a(\pi)$, then $\psi^g_{i,\pi}(a) < \pi$.

Proof. Let $\pi \in Mh^a_{i,0}(f) = \bigcap\{Mh^a_{i,e}(f(e)) : e \in \mathrm{supp}(f)\}$ and $SC(g) \subset \mathcal{H}_a(\pi)$. Lemma 3.21 with $\pi \in Mh^a_{i,d}(f(d)) \cap Mh^a_{i,c}(f(c))$ yields $\pi \in Mh^a_{i,d}(g(d)) \cap M^a_{i,c}(Mh^a_{i,c}(g^c))$. On the other hand we have $\pi \in Mh^a_{i,0}(g_d) = \bigcap\{Mh^a_{i,e}(f(e)) : e \in \mathrm{supp}(f) \cap d\}$. Hence $\pi \in M(Mh^a_{i,0}(g))$.

Now suppose $SC(g) \subset \mathcal{H}_a(\pi)$. The set $C = \{\kappa < \pi : \mathcal{H}_a(\kappa) \cap \pi \subset \kappa, \{\pi, a\} \cup SC(g) \subset \mathcal{H}_a(\kappa)\}$ is a club subset of the regular cardinal π, and $Mh^a_{i,0}(g)$ is stationary in π. This shows the existence of a $\kappa \in Mh^a_{i,0}(g) \cap C \cap \pi$, and hence $\psi^g_{i,\pi}(a) < \pi$ by the definition (7). □

Definition 3.24 An *irreducibility* of finite functions $f : \mathbb{I}_N \to \varphi_{\mathbb{I}_N}(0)$ is defined by induction on the cardinality n of the finite set $\mathrm{supp}(f)$. If $n \leq 1$, f is defined to be irreducible. Let $n \geq 2$ and $c < c+d$ be the largest two elements in $\mathrm{supp}(f)$, and let g be a finite function such that $\mathrm{supp}(g) = \mathrm{supp}(f_c) \cup \{c\}$, $g_c = f_c$ and $g(c) = f(c) + \tilde{\theta}_d(f(c+d); \mathbb{I}_N)$.

Then f is irreducible iff $tl(f(c)) > \tilde{\theta}_d(f(c+d); \mathbb{I}_N)$ and g is irreducible.

Definition 3.25 Let $f, g : \mathbb{I}_N \to \varphi_{\mathbb{I}_N}(0)$ be irreducible finite functions, and b an ordinal. Let us define a relation $f <^b_{lx} g$ by induction on the cardinality $\#\{e \in \mathrm{supp}(f) \cup \mathrm{supp}(g) : e \geq b\}$ as follows. $f <^b_{lx} g$ holds iff $f^b \neq g^b$ and for the ordinal $c = \min\{c \geq b : f(c) \neq g(c)\}$, one of the following conditions is met:

1. $f(c) < g(c)$ and let μ be the shortest segment of $g(c)$ such that $f(c) < \mu$. Then for any $c < c+d \in \mathrm{supp}(f)$, if $tl(\mu) \leq \tilde{\theta}_d(f(c+d); \mathbb{I}_N)$, then $f <^{c+d}_{lx} g$ holds.
2. $f(c) > g(c)$ and let ν be the shortest segment of $f(c)$ such that $\nu > g(c)$. Then there exist a $c < c+d \in \mathrm{supp}(g)$ such that $f <^{c+d}_{lx} g$ and $tl(\nu) \leq \tilde{\theta}_d(g(c+d); \mathbb{I}_N)$.

In [4] the following Proposition 3.26 is shown.

Proposition 3.26 *Let $f, g : \mathbb{I}_N \to \varphi_{\mathbb{I}_N}(0)$. If $f <^0_{lx} g$, then $Mh^a_{i,0}(f) \prec Mh^a_{i,0}(g)$.*

Proposition 3.27 *Let $f, g : \mathbb{I}_N \to \varphi_{\mathbb{I}_N}(0)$ be irreducible functions, and assume that $\psi^f_{i,\pi}(b) < \pi$ and $\psi^g_{i,\kappa}(a) < \kappa$.*

Then $\psi^f_{i,\pi}(b) < \psi^g_{i,\kappa}(a)$ iff one of the following cases holds:

1. $\pi \leq \psi^g_{i,\kappa}(a)$.
2. $b < a$, $\psi^f_{i,\pi}(b) < \kappa$, *and* $SC(f) \cup \{\pi, b\} \subset \mathcal{H}_a(\psi^g_{i,\kappa}(a))$.
3. $b > a$, *and* $SC(g) \cup \{\kappa, a\} \not\subset \mathcal{H}_b(\psi^f_{i,\pi}(b))$.
4. $b = a$, $\kappa < \pi$, *and* $\kappa \notin \mathcal{H}_b(\psi^f_{i,\pi}(b))$.
5. $b = a$, $\pi = \kappa$, $SC(f) \subset \mathcal{H}_a(\psi^g_{i,\kappa}(a))$, *and* $f <^0_{lx} g$.
6. $b = a$, $\pi = \kappa$, $SC(g) \not\subset \mathcal{H}_b(\psi^f_{i,\pi}(b))$.

Proof. This is seen from Proposition 3.26 as in [2]. □

3.4 A computable notation system for Π_N-collection

Although Propositions 3.17, 3.20, and 3.27 suffice for us to define a computable notation system for $\mathcal{H}_{\varepsilon(\mathbb{I}_N)}(0)$, we need a notation system closed under Mostowski collapsings to remove stable ordinals from derivations as in [5], cf. section 5. Two new constructors $\mathbb{I}_N[\cdot]$ and $\mathbb{S}^{\dagger\vec{i}}[\rho/\mathbb{S}]$ are used to generate terms in $OT(\mathbb{I}_N)$.

Definition 3.28 $\rho \prec \sigma$ denotes the transitive closure of the relation $\{(\rho, \sigma) : \exists f, a(\rho = \psi^f_\sigma(a))\}$. Let $\rho \preceq \sigma :\Leftrightarrow \rho \prec \sigma \lor \rho = \sigma$.

Let $\mathbb{S} \in SSt_i$ and $\rho \prec \mathbb{S}$. We define a set $M_\rho = \mathcal{H}_b(\rho)$ from ρ in (10) in such a way that $\mathcal{H}_b(\rho) \cap \mathbb{S} \subset \rho$. Then a Mostowski collapsing $M_\rho \ni \alpha \mapsto \alpha[\rho/\mathbb{S}]$ in Definition 3.33 maps ordinal terms $\alpha \in M_\rho$ to $\alpha[\rho/\mathbb{S}] < \mathbb{S}$ isomorphically. The transitive collapse $(M_\rho)^{[\rho/\mathbb{S}]} = \{\alpha[\rho/\mathbb{S}] : \alpha \in M_\rho\}$ is an initial segment in $OT(\mathbb{I}_N)$ such that $(M_\rho)^{[\rho/\mathbb{S}]} < \kappa$ if $\rho < \kappa \prec \mathbb{S}$. Note that both ρ and κ can be interpreted as uncountable cardinals, and the cardinality of the set M_ρ is equal to ρ.

Let us define simultaneously the followings: A set $OT(\mathbb{I}_N)$ of terms over constants $0, \Omega, \mathbb{I}_N$ and constructors $+, \varphi, \psi$, $\mathbb{I}_N[*]$, $*^{\dagger i}$ $(0 < i \leq N)$, and $*_0[*_1/*_2]$. Its subsets SSt_i, LSt_i with $St_i = SSt_i \cup LSt_i$, and sets M_ρ $(\rho \in \Psi)$, finite sets $K_X(\alpha)$ of subterms of α for $X \subset OT(\mathbb{I}_N)$. Let $SSt = \bigcup_{0<i\leq N} SSt_i$ and $LSt = \bigcup_{0<i\leq N} LSt_i$. For each $\mathbb{S} \in SSt$, there exists a unique i such that $\mathbb{S} \in SSt_i$.

For $i > 0$, $\kappa \in St_i$ is intended to designate that κ is an i-shrewd cardinal, or κ is an i-stable ordinal. $\kappa \in SSt_i$ $[\kappa \in LSt_i]$ is intended to designate that κ is a successor i-stable ordinal [κ is a limit of i-stable ordinals], resp. $\kappa \in St_0$ is intended to designate that κ is an uncountable cardinal, or κ is either a recursively regular ordinal or their limit. We have $St_i = SSt_i \cup LSt_i$ with $SSt_i \cap LSt_i = \emptyset$, and $St_{i+1} \subset LSt_i$. If $\mathbb{S} \in SSt_i$, then the ordinal term $\psi^f_\mathbb{S}(a)$ in Definition 3.31.5 denotes the ordinal $\psi^f_{i,\mathbb{S}}(a)$ in (7) of Definition 3.8.3.

$\alpha =_{NF} \alpha_m + \cdots + \alpha_0$ means that $\alpha = \alpha_m + \cdots + \alpha_0$ with $\alpha_m \geq \cdots \geq \alpha_0$ and each α_i is a non-zero additive principal number. $\alpha =_{NF} \varphi\beta\gamma$ means that $\alpha = \varphi\beta\gamma$ and $\beta, \gamma < \alpha$.

Sets $SC(\alpha)$ of strongly critical numbers are slightly modified as $SC(\Omega) = SC(\mathbb{I}_N) = \emptyset$. Specifically $SC(0) = \emptyset$, $SC(\alpha) = \bigcup_{i\leq m} SC(\alpha_i)$ for $\alpha =_{NF} \alpha_m + \cdots + \alpha_0$, and $SC(a) = SC(b) \cup SC(c)$ for $a =_{NF} \varphi_b(c)$. $SC(\Omega) = SC(\mathbb{I}_N) = \emptyset$. $SC(a) = \{a\}$ if $a \notin \{\Omega, \mathbb{I}_N\}$ is strongly critical.

For $\alpha = \psi^f_\pi(a)$, let $m(\alpha) = f$. $SC(f) = \bigcup\{SC(c) \cup SC(f(c)) : c \in \mathrm{supp}(f)\}$. Immediate subterms of terms are defined as follows. $k(\alpha_m + \cdots + \alpha_0) = \{\alpha_0, \ldots, \alpha_m\}$, $k(\varphi\alpha\beta) = \{\alpha, \beta\}$, $k(\psi_{\mathbb{I}_N}(a)) = \{\mathbb{I}_N, a\}$, and $k(\psi^f_\sigma(\alpha)) = \{\sigma, \alpha\} \cup SC(f)$.

Note that in the following Definition 3.31, e.g., there is no clause for constructing $\kappa = \psi_\mathbb{S}(a)$ from a for $\mathbb{S} \notin SSt$.

Definition 3.29 1. $\alpha \in \Psi :\Leftrightarrow \exists\kappa, f, a(\alpha = \psi^f_\kappa(a))$ and $\alpha \in \Psi_\mathbb{S} :\Leftrightarrow \exists\kappa \preceq \mathbb{S}\exists f, a(\alpha = \psi^f_\kappa(a))$.

2. For a sequence $\vec{i} = (i_0, i_1, \ldots, i_n)$ of numbers, let $\alpha^{\dagger\vec{i}} = (\cdots((\alpha^{\dagger i_0})^{\dagger i_1})\cdots)^{\dagger i_n}$.

3. By $\vec{i} \leq i$ let us understand that $\vec{i} = (i_0, i_1, \ldots, i_n)$ is a non-empty and non-increasing sequence of numbers such that $0 < i_n \leq \cdots \leq i_1 \leq i_0 \leq i$.

Definition 3.30

1. Let $\alpha \preceq \psi^g_{\mathbb{S}}(b)$ for an $\mathbb{S} \in SSt$ and a g with $b = \mathtt{p}_0(\alpha)$. Then let
$$M_\alpha := \mathcal{H}_b(\alpha) \tag{10}$$

2. For $\alpha \in \Psi$, an ordinal $\mathtt{p}_0(\alpha)$ is defined.
   (a) If $\alpha \preceq \psi^g_{\mathbb{S}}(b)$, then $\mathtt{p}_0(\alpha) = b$.
   (b) If there are ρ and $\beta \in M_\rho$ such that $LSt_i \ni \rho \prec \mathbb{S} \in SSt_{i+1}$ and $\alpha = \beta[\rho/\mathbb{S}] \neq \beta$, then $\mathtt{p}_0(\alpha) = \mathtt{p}_0(\beta)$.
   (c) $\mathtt{p}_0(\alpha) = 0$ otherwise.

3. $\alpha^\dagger := \alpha^{\dagger 1}$.

Definition 3.31 (Definitions of $OT(\mathbb{I}_N)$ and $K_X(\alpha)$)
Let $St_i = SSt_i \cup LSt_i \subset OT(\mathbb{I}_N)$ with $SSt_i \cap LSt_i = \emptyset$ and $St_{i+1} \subset LSt_i$. For $\delta, \alpha \in OT(\mathbb{I}_N)$, $K_\delta(\alpha) = K_X(\alpha)$, where $X = \{\beta \in OT(\mathbb{I}_N) : \beta < \delta\}$.

1. $\{0, \Omega, \mathbb{I}_N\} \subset OT(\mathbb{I}_N)$ and $\Omega^{\dagger i} \in SSt_i$ for $0 < i \leq N$. Let $St_{N+1} = \{\mathbb{I}_N\}$. $m(\alpha) = K_X(\alpha) = \emptyset$ for $\alpha \in \{0, \mathbb{I}_N, \Omega\} \cup \{\Omega^{\dagger i} : 0 < i \leq N\}$.

2. If $\alpha =_{NF} \alpha_m + \cdots + \alpha_0\,(m > 0)$ with $\{\alpha_i : i \leq m\} \subset OT(\mathbb{I}_N)$, then $\alpha \in OT(\mathbb{I}_N)$, and $m(\alpha) = \emptyset$.
   Let $\alpha =_{NF} \varphi\beta\gamma < \varepsilon(\mathbb{I}_N)$ with $\{\beta, \gamma\} \subset OT(\mathbb{I}_N)$. Then $\alpha \in OT(\mathbb{I}_N)$ and $m(\alpha) = \emptyset$.
   In each case $K_X(\alpha) = K_X(k(\alpha))$.

3. Let $\alpha = \psi_\Omega(a)$ with $a \in OT(\mathbb{I}_N)$ and $K_\alpha(a) < a$. Then $\alpha \in OT(\mathbb{I}_N)$.
   Let $m(\alpha) = \emptyset$. $K_X(\alpha) = \emptyset$ if $\alpha \in X$. $K_X(\alpha) = \{a\} \cup K_X(a)$ if $\alpha \notin X$.

4. Let $\alpha = \psi_{\mathbb{I}_N}(a)$ with $a \in OT(\mathbb{I}_N)$ such that $K_\alpha(a) < a$. Then $\alpha \in LSt_N$ and $\alpha^{\dagger i} \in SSt_i$ for $0 < i \leq N$. For $\beta \in \{\alpha, \alpha^{\dagger i}\}$, $m(\beta) = \emptyset$. Also $K_X(\alpha^{\dagger i}) = \emptyset$ if $\alpha^{\dagger i} \in X$. $K_X(\alpha^{\dagger i}) = K_X(\alpha)$ if $\alpha^{\dagger i} \notin X$. $K_X(\alpha) = \emptyset$ if $\alpha \in X$. $K_X(\alpha) = \{a\} \cup K_X(a)$ if $\alpha \notin X$.

5. Let $\mathbb{T} \in LSt_k \cup \{\Omega\}$ and $\mathbb{S} = \mathbb{T}^{\dagger\vec{i}} \in SSt_{i+1}$ for a non-empty and non-increasing sequence of numbers $\vec{i} = (i_0 \geq i_1 \geq \cdots \geq i_n)$ such that $i_0 \leq k$ and $i_n = i + 1$, cf. Proposition 3.32. Let $\alpha = \psi^f_{\mathbb{S}}(a)$, where $\{a, \mathbb{S}\} \subset OT(\mathbb{I}_N)$, and if $f \neq \emptyset$, then there are $\{d, \xi\} \subset OT(\mathbb{I}_N)$ such that $\mathrm{supp}(f) = \{d\}$, $0 < f(d) = \xi < (\mathbb{I}_N)^2$, $d < \mathbb{I}_N$. If $K_{\mathbb{S}}(\{\mathbb{S}, a\} \cup SC(f)) < a$ for $SC(f) = SC(\{d, \xi\})$, and
$$SC(f) \subset \mathcal{H}_a(SC(a)) \tag{11}$$

then $\alpha \in LSt_i$ and $\alpha^{\dagger j} \in SSt_j$ for $0 < j \leq i$.

Let $a = \mathtt{p}_0(\alpha)$, $m(\alpha) = f$. $K_X(\alpha) = \emptyset$ if $\alpha \in X$. $K_X(\alpha) = \{a\} \cup K_X(\{a, \mathbb{S}\} \cup SC(f))$ if $\alpha \notin X$.

$m(\alpha^{\dagger j}) = \emptyset$. $K_X(\alpha^{\dagger j}) = \emptyset$ if $\alpha^{\dagger j} \in X$. $K_X(\alpha^{\dagger j}) = K_X(\alpha)$ if $\alpha^{\dagger j} \notin X$.

6. Let $\{\pi, a, d\} \subset OT(\mathbb{I}_N)$ with $\pi \prec \mathbb{S} \in SSt_{i+1}$, $m(\pi) = f$, $d < c \in \mathrm{supp}(f)$, and $(d, c) \cap \mathrm{supp}(f) = \emptyset$.

   When $g \neq \emptyset$, let g be an irreducible finite function such that $SC(g) \subset OT(\mathbb{I}_N)$, $g_d = f_d$, $(d, c) \cap \mathrm{supp}(g) = \emptyset$, $g(d) < f(d) + \tilde{\theta}_{c-d}(f(c); \mathbb{I}_N) \cdot \omega$, and $g <^c_{\mathbb{I}_N} f(c)$.

   Then $\alpha = \psi^g_\pi(a) \in LSt_i$ and $\alpha^{\dagger j} \in SSt_j$ for $0 < j \leq i$ if $K_\pi(k(\alpha)) < a$, and

$$SC(g) \cup \{\mathtt{p}_0(\alpha)\} \subset M_\alpha \tag{12}$$

   Let $m(\alpha) = g$. $K_X(\alpha) = \emptyset$ if $\alpha \in X$. $K_X(\alpha) = \{a\} \cup K_X(k(\alpha))$ if $\alpha \notin X$.

   $m(\alpha^{\dagger j}) = \emptyset$. $K_X(\alpha^{\dagger j}) = \emptyset$ if $\alpha^{\dagger j} \in X$. $K_X(\alpha^{\dagger j}) = K_X(\alpha)$ if $\alpha^{\dagger j} \notin X$.

7. Let $\mathbb{S} \in SSt_i$ and $0 < k \leq i$. Then $\mathbb{S}^{\dagger k} \in SSt_k$.

   $m(\mathbb{S}^{\dagger k}) = \emptyset$. $K_X(\mathbb{S}^{\dagger k}) = \emptyset$ if $\mathbb{S}^{\dagger k} \in X$. $K_X(\mathbb{S}^{\dagger k}) = K_X(\mathbb{S})$ if $\mathbb{S}^{\dagger k} \notin X$.

8. Let $SSt^M_i = SSt_i \cup \{\alpha[\rho/\mathbb{S}] : \rho \prec \mathbb{S} \in SSt^M, \alpha \in M_\rho \cap SSt^M_i\}$ and $SSt^M = \bigcup_{0<i\leq N} SSt^M_i$. Also let $LSt^M_i = LSt_i \cup \{\alpha[\rho/\mathbb{S}] : \rho \prec \mathbb{S} \in SSt^M, \alpha \in M_\rho \cap LSt^M_i\}$ and $LSt^M = \bigcup_{0<i\leq N} LSt^M_i$.

   Let $\rho \prec \mathbb{S} \in SSt^M_{i+1}$ and $\vec{i} = (i_0 \geq i_1 \geq \cdots \geq i_n)\, (n \geq 0)$ with $0 < i_n \leq i_0 \leq i + 1$. Then $(\mathbb{S}^{\dagger \vec{i}}[\rho/\mathbb{S}]) \in SSt^M_{i_n} \subset OT(\mathbb{I}_N)$, where a term $\mathbb{S}^{\dagger \vec{i}}[\rho/\mathbb{S}]$ is built from terms $\mathbb{S}^{\dagger \vec{i}}$, ρ and $\mathbb{S}$ by the constructor $*_0[*_1/*_2]$.

9. Let $\alpha = \beta[\rho/\mathbb{S}]$ with $\mathbb{S} < \beta \in M_\rho$, $\rho \prec \mathbb{S}$, and $\mathbb{S} \in SSt^M$. Then $\alpha \in OT(\mathbb{I}_N) \setminus St$.

Note that in Definition 3.31.5,

$$K_\alpha(k(\mathbb{T})) \cup \{b\} < a \tag{13}$$

follows from $\mathbb{S} = \mathbb{T}^{\dagger \vec{i}} \in \mathcal{H}_a(\alpha)$ if $\mathbb{T} = \psi^g_\sigma(b) \in LSt_k$ with $k(\mathbb{T}) = \{\sigma, b\} \cup SC(g)$, and $\alpha = \psi^f_{\mathbb{S}}(a)$.

Proposition 3.32 *Let* $\alpha \in OT(\mathbb{I}_N)$.

1. $\alpha \in LSt_N$ *iff* $\alpha = \psi_{\mathbb{I}_N}(a)$ *for an* a. *For* $0 < i < N$, $\alpha \in LSt_i \cap \Psi$ *iff there exists an* $\mathbb{S} \in SSt_{i+1}$ *such that* $\alpha \prec \mathbb{S}$.

2. $\beta \in SSt_k$ *iff there exists an* $\alpha \in \{\Omega\} \cup (LSt_i \cap \Psi)$ *for an* $k \leq i \leq N$ *and a non-empty and non-increasing sequence* $\vec{i} = (i_0 \geq i_1 \geq \cdots \geq i_n)$ *of numbers such that* $k = i_n > 0$, $\alpha \in LSt_i \Rightarrow i_0 \leq i$ *and* $\beta = \alpha^{\dagger \vec{i}}$.

3. *Let $\psi^f_{\mathbb{S}}(a) \in OT(\mathbb{I}_N)$ with $\mathbb{S} \in SSt$. Suppose that there exists a sequence $\{(\mathbb{T}_m, \mathbb{S}_m, \vec{i}_m)\}_{m\leq n}$ of $\mathbb{T}_m \in LSt \cap \Psi$, $\mathbb{S}_m \in SSt$ and sequences $\vec{i}_m$ of numbers such that $\mathbb{T}_0 = \psi_{\mathbb{I}_N}(b)$, $\mathbb{S}_m = \mathbb{T}_m^{\dagger\vec{i}_m}$ and $\mathbb{T}_{m+1} \prec \mathbb{S}_m$ $(m < n)$, and $\mathbb{S} = \mathbb{S}_n$. Then $b < a$ holds.*

4. *$\alpha \in SSt^M$ iff there exists a ρ and an $\vec{i}$ such that $\alpha \in \{\rho^{\dagger\vec{i}}, \mathbb{S}^{\dagger\vec{i}}[\rho/\mathbb{S}]\}$.*

Proof. 3.32.1 and 3.32.2. We see these from Definitions 3.31.1, 3.31.4, 3.31.5, 3.31.6 and 3.31.7.
3.32.3. Let $\mathbb{T}_m = \psi^{g_m}_{\sigma_m}(b_m)$ and $\mathbb{T}_m \preceq \psi^{f_{m-1}}_{\mathbb{S}_{m-1}}(a_{m-1})$ for $\psi^{f_{m-1}}_{\mathbb{S}_{m-1}} = \psi_{\mathbb{I}_N}$ and $a_{-1} = b$. In general, if $\sigma = \psi^f_\tau(c) \in \mathcal{H}_b(\psi^g_\sigma(b))$ with $\psi^g_\sigma(b) < \sigma$, then $c < b$. Hence $a_{m-1} \leq b_m$. On the other we obtain $b_m < a_m$ by (13), where $a_n = a$. Therefore $b = a_{-1} < a_n = a$. □

Sets $\mathcal{H}_\gamma(X)$ are defined for $\{\gamma\}\cup X \subset OT(\mathbb{I}_N)$ in such a way that $\alpha \in \mathcal{H}_\gamma(X)$ iff $K_X(\alpha) < \gamma$ for $\alpha, \gamma \in OT(\mathbb{I}_N)$ and $X \subset OT(\mathbb{I}_N)$. In particular $OT(\mathbb{I}_N) = \mathcal{H}_{\varepsilon(\mathbb{I}_N)}(0)$, and $\mathcal{H}_\gamma(X)$ is closed under Mostowski collapsing $\alpha \mapsto \alpha[\rho/\mathbb{S}]$ if $\gamma \geq \mathbb{I}_N$, and differs from sets defined in Definition 3.8.

We define terms $\alpha[\rho/\mathbb{S}]$, sets $K_X(\alpha[\rho/\mathbb{S}])$ and a relation $\beta < \gamma$ on $OT(\mathbb{I}_N)$ recursively as follows.

Definition 3.33 (Definitions of $\alpha[\rho/\mathbb{S}]$ and $K_X(\alpha[\rho/\mathbb{S}])$)
Let $\rho \prec \mathbb{S} \in SSt^M_{i+1}$. We define a term $\alpha[\rho/\mathbb{S}] \in OT(\mathbb{I}_N)$ for $\alpha \in M_\rho$ in such a way that $\alpha[\rho/\mathbb{S}] = \alpha$ iff $\alpha < \rho$. Moreover $\alpha[\rho/\mathbb{S}] \in St$ iff either $\alpha[\rho/\mathbb{S}] = \alpha \in St$ or $\alpha[\rho/\mathbb{S}] = \rho \in SSt$.

Also $K_X(\alpha[\rho/\mathbb{S}])$ is defined recursively as follows. The map $\alpha \mapsto \alpha[\rho/\mathbb{S}]$ commutes with ψ, φ, $\mathbb{I}_N[\cdot]$, and $+$. $K_X(\alpha[\rho/\mathbb{S}]) = \emptyset$ if $\alpha[\rho/\mathbb{S}] \in X$.

1. $\alpha[\rho/\mathbb{S}] := \alpha$ when $\alpha < \mathbb{S}$.

   In what follows assume $\alpha \geq \mathbb{S}$, $\alpha[\rho/\mathbb{S}] \geq \rho$ and $\alpha[\rho/\mathbb{S}] \notin X$.

2. $(\mathbb{S})[\rho/\mathbb{S}] := \rho$ and $(\mathbb{I}_N)[\rho/\mathbb{S}] := \mathbb{I}_N[\rho]$.

   For $\vec{i} = (i_0 \geq i_1 \geq \cdots \geq i_n) \leq i+1$, $(\mathbb{S}^{\dagger\vec{i}})[\rho/\mathbb{S}] := (\mathbb{S}^{\dagger\vec{i}}[\rho/\mathbb{S}]) \in SSt^M_{i_n}$, cf. Definition 3.31.8. Here $\mathbb{S}^{\dagger\vec{i}}[\rho/\mathbb{S}] \neq \rho^{\dagger\vec{i}}$.

   $K_X(\alpha[\rho/\mathbb{S}]) = K_X(\rho)$ if $\alpha[\rho/\mathbb{S}] \in \{\mathbb{I}_N[\rho], \mathbb{S}^{\dagger\vec{i}}[\rho/\mathbb{S}]\}$.

3. Let $\alpha = \psi_{\mathbb{I}_N}(a)$. Then $\alpha[\rho/\mathbb{S}] = \psi_{\mathbb{I}_N[\rho]}(a[\rho/\mathbb{S}])$.

   $K_X(\alpha[\rho/\mathbb{S}]) = K_X(\{\rho, a[\rho/\mathbb{S}]\}) \cup \{a[\rho/\mathbb{S}]\}$.

4. Let $\alpha = \psi^f_\kappa(a)$. Then $\alpha[\rho/\mathbb{S}] = \psi^{f[\rho/\mathbb{S}]}_{\kappa[\rho/\mathbb{S}]}(a[\rho/\mathbb{S}])$, where $(f[\rho/\mathbb{S}]) : \mathbb{I}_N[\rho] \to \varphi_{\mathbb{I}_N[\rho]}(0)$, $\mathrm{supp}(f[\rho/\mathbb{S}]) = (\mathrm{supp}(f))[\rho/\mathbb{S}] = \{c[\rho/\mathbb{S}] : c \in \mathrm{supp}(f)\}$ and $(f[\rho/\mathbb{S}])(c[\rho/\mathbb{S}]) = (f(c))[\rho/\mathbb{S}]$ for $f : \mathbb{I}_N[\rho] \to \varphi_{\mathbb{I}_N[\rho]}(0)$ and $c \in \mathrm{supp}(f)$.

   $K_X(\alpha[\rho/\mathbb{S}]) = K_X(\{\kappa[\rho/\mathbb{S}], a[\rho/\mathbb{S}]\} \cup SC(f[\rho/\mathbb{S}])) \cup \{a[\rho/\mathbb{S}]\}$.

   $M_{\alpha[\rho/\mathbb{S}]} = \mathcal{H}_{b[\rho/\mathbb{S}]}(\alpha[\rho/\mathbb{S}])$ for $b = \mathtt{p}_0(\alpha)$ and $b[\rho/\mathbb{S}] = \mathtt{p}_0(\alpha[\rho/\mathbb{S}])$.

5. Let $\alpha = \mathbb{I}_N[\tau] \neq \mathbb{I}_N$. Then $\alpha[\rho/\mathbb{S}] = \mathbb{I}_N[\tau[\rho/\mathbb{S}]]$, where $\mathbb{I}_N[\tau] \in M_\rho$ iff $\tau \in M_\rho$. $K_X(\alpha[\rho/\mathbb{S}]) = K_X(\tau[\rho/\mathbb{S}])$.

6. Let $\alpha = \psi_{\mathbb{I}_N[\tau]}(a)$ for $\mathbb{I}_N[\tau] \neq \mathbb{I}_N$. Then $\alpha[\rho/\mathbb{S}] = \psi_{\mathbb{I}_N[\tau[\rho/\mathbb{S}]]}(a[\rho/\mathbb{S}])$. $K_X(\alpha[\rho/\mathbb{S}]) = K_X(\{\tau[\rho/\mathbb{S}], a[\rho/\mathbb{S}]\}) \cup \{a[\rho/\mathbb{S}]\}$.

7. Let $\alpha = \tau^{\dagger\vec{j}}$ with $\mathbb{S} < \tau \in LSt^M$. Then $\alpha[\rho/\mathbb{S}] = (\tau[\rho/\mathbb{S}])^{\dagger\vec{j}}$, where $\tau^{\dagger\vec{j}} \in M_\rho$ iff $\tau \in M_\rho$. $K_X(\alpha[\rho/\mathbb{S}]) = K_X(\tau[\rho/\mathbb{S}])$.

8. Let $\alpha = \mathbb{T}^{\dagger\vec{j}}[\tau/\mathbb{T}]$, where $\tau \prec \mathbb{T} \in SSt^M$. Then $\alpha[\rho/\mathbb{S}] = \mathbb{T}_1^{\dagger\vec{j}}[\tau_1/\mathbb{T}_1]$, where $\tau_1 = \tau[\rho/\mathbb{S}] \prec \mathbb{T}_1 = \mathbb{T}[\rho/\mathbb{S}] \in SSt^M$ and $\mathbb{T}_1^{\dagger\vec{j}} = (\mathbb{T}_1)^{\dagger\vec{j}}$. $K_X(\alpha[\rho/\mathbb{S}]) = K_X(\tau[\rho/\mathbb{S}])$.

9. Let $\alpha = \varphi\beta\gamma$. Then $\alpha[\rho/\mathbb{S}] = \varphi(\beta[\rho/\mathbb{S}])(\gamma[\rho/\mathbb{S}])$. $K_X(\alpha[\rho/\mathbb{S}]) = K_X(\beta[\rho/\mathbb{S}], \gamma[\rho/\mathbb{S}])$.

10. For $\alpha = \alpha_m + \cdots + \alpha_0\,(m > 0)$, $\alpha[\rho/\mathbb{S}] = (\alpha_m[\rho/\mathbb{S}]) + \cdots + (\alpha_0[\rho/\mathbb{S}])$. $K_X(\alpha[\rho/\mathbb{S}]) = \bigcup\{K_X(\alpha_i[\rho/\mathbb{S}])) : i \leq m\}$.

A relation $\alpha < \beta$ for $\alpha, \beta \in OT(\mathbb{I}_N)$ is defined according to Lemmas 3.16 and 3.19, Propositions 3.17, 3.20, and 3.27, and Corollary 3.23, provided that $\alpha \in \mathcal{H}_\gamma(X)$ is replaced by $K_X(\alpha) < \gamma$. The relation enjoys $\psi_\kappa^f(a) < \kappa$ according to Lemma 3.19 and Corollary 3.23. Moreover we obtain $\mathbb{S}^{\dagger i} < \psi^{g_0}_{\mathbb{S}^{\dagger(i+1)}}(b_0) < \mathbb{S}^{\dagger(i+1)}$ for $i+1 \leq N$, and $LSt_N \ni \tau_0 = \psi_{\mathbb{I}_N}(c_0) < \psi^{h_0}_{\tau_0^\dagger}(d_0) < \tau_0^\dagger < \mathbb{I}_N$ by Proposition 3.20 and Lemma 3.16. Hence if $\mathbb{S} < \psi_{\mathbb{I}_N}(c_0)$, then $\mathbb{S} < \mathbb{S}^{\dagger i} < \psi^{g_0}_{\mathbb{S}^{\dagger(i+1)}}(b_0) < \mathbb{S}^{\dagger(i+1)} < \tau_0 = \psi_{\mathbb{I}_N}(c_0) < \psi^{h_0}_{\tau_0^\dagger}(d_0) < \tau_0^\dagger < \mathbb{I}_N$. The Mostowski collapsing $\cdot[\rho/\mathbb{S}]$ maps these inequalities isomorphically to $\rho < \mathbb{S}^{\dagger i}[\rho/\mathbb{S}] < \psi^g_{\mathbb{S}^{\dagger(i+1)}[\rho/\mathbb{S}]}(b) < \mathbb{S}^{\dagger(i+1)}[\rho/\mathbb{S}] < \tau = \psi_{\mathbb{I}_N[\rho]}(c) < \psi^h_{\tau^\dagger}(d) < \tau^\dagger < \mathbb{I}_N[\rho] < \rho^{\dagger 0}$, where $b = b_0[\rho/\mathbb{S}]$, etc.

Definition 3.34 For terms $\pi, \kappa \in OT(\mathbb{I}_N)$, a relation $\pi \prec^R \kappa$ is defined recursively as follows.

1. Let $\pi \prec \kappa \preceq \mathbb{S} \in SSt^M_{i+1}$, and $\vec{i} \leq i+1$. Then each of $\pi \prec^R \kappa$, $\mathbb{S}^{\dagger\vec{i}}[\pi/\mathbb{S}] \prec^R \kappa$ and $\mathbb{I}_N[\pi] \prec^R \kappa$ holds. Moreover $\pi^{\dagger\vec{i}} \prec^R \kappa$ holds provided that $\pi^{\dagger\vec{i}} \notin SSt$.

2. $\tau \prec^R \pi \prec^R \kappa \Rightarrow \tau \prec^R \kappa$.

Let $\pi \preceq^R \kappa :\Leftrightarrow \pi \prec^R \kappa \vee \pi = \kappa$. For $\mathbb{S} \in SSt$, let

$$L(\mathbb{S}) := \{\alpha \in OT(\mathbb{I}_N) : \alpha \prec^R \mathbb{S}\}.$$

Note that $L(\mathbb{S}) \cap SSt = \emptyset$, and $SSt \ni \rho^{\vec{i}} \not\prec^R \mathbb{S}$ for $LSt_i \ni \rho \prec \mathbb{S} \in SSt_{i+1}$ and $\vec{i} \leq i$. For each strongly critical number $\Omega < \alpha \notin \{\mathbb{I}_N\} \cup St$, there exists a unique $\mathbb{S} \in SSt$ such that $\alpha \prec^R \mathbb{S}$. If $\beta \prec^R \mathbb{T}$ and $\alpha \prec^R \mathbb{S}$ with $\mathbb{T} < \mathbb{S}$, then $\beta < \alpha$. In other words, $L(\mathbb{T}) < L(\mathbb{S})$ for layers $L(\mathbb{S})$. Moreover if $\eta \notin \bigcup_{\mathbb{S} \in SSt} L(\mathbb{S}) \cup SSt$ and $\eta \in \Psi$, then either $\eta \prec \Omega$ or $\eta \prec \mathbb{I}_N$.

Definition 3.35 Let $\beta, \alpha \in OT(\mathbb{I}_N) \cap \mathbb{I}_N$ be strongly critical numbers. $\beta < \alpha$ iff one of the following cases holds:

1. $\beta = \psi_\Omega(b)$, $\alpha = \psi_\Omega(a)$ and $b < a$.
2. $\beta = \psi_\pi(b)$, $\alpha = \psi_\kappa(a)$, $\pi = \kappa \in \{\mathbb{I}_N\} \cup \{\sigma \in OT(\mathbb{I}_N) : \exists\rho(\sigma = \mathbb{I}_N[\rho])\}$, and $b < a$.
3. $\beta = \Omega$ and $\Omega \neq \alpha \neq \psi_\Omega(a)$.
4. $\mathbb{S}^{\dagger\vec{i}} < \mathbb{T}^{\dagger\vec{j}}$ iff $(\mathbb{S}) * \vec{i} <_{lx} (\mathbb{T}) * \vec{j}$ for $\mathbb{S}, \mathbb{T} \in \{\Omega\} \cup (LSt \cap \Psi)$, where $\vec{i} = (i_0, i_1, \ldots, i_n) <_{lx} (j_0, j_1, \ldots, j_m) = \vec{j}$ iff either $\exists k \leq \min\{n, m\}(\forall p < k(i_p = j_p) \,\&\, i_k < j_k)$ or $n < m \,\&\, \forall p \leq n(i_p = j_p)$.
5. (a) There is an $\mathbb{S} \in SSt$ such that $\alpha \prec^R \mathbb{S} > \beta \in LSt_N$.
   (b) There is a $\mathbb{T} \in SSt$ such that $\beta \prec^R \mathbb{T} < \alpha$.
6. There are $\mathbb{T}, \mathbb{S} \in SSt$ such that $\beta \prec^R \mathbb{T}$ and $\alpha \prec^R \mathbb{S}$ with $\mathbb{T} < \mathbb{S}$.
7. There is an $\mathbb{S} \in SSt$ such that $\beta, \alpha \prec^R \mathbb{S}$ and one of the following holds:
   (a) $\beta = \psi_\pi^f(b)$, $\alpha = \psi_\kappa^g(a)$, and there is a $\rho \preceq^R \mathbb{S}$ such that $\kappa, \pi \preceq \rho$ and one of the following holds:
      i. $\pi \leq \alpha$.
      ii. $b < a$, $\beta < \kappa$, and $K_\alpha(SC(f) \cup \{\pi, b\}) < a$
      iii. $b > a$ and $b \leq K_\beta(SC(g) \cup \{\kappa, a\})$.
      iv. $b = a$, $\kappa < \pi$, and $b \leq K_\beta(\kappa)$.
      v. $b = a$, $\pi = \kappa$, $K_\alpha(SC(f)) < a$, and $f <_{lx}^0 g$.
      vi. $b = a$, $\pi = \kappa$, and $b \leq K_\beta(SC(g))$.
   (b) There are $\mathbb{I}_N[\rho] \prec^R \mathbb{S}$, c, d and $\vec{i}, \vec{j}$ such that $\beta \preceq^R (\psi_{\mathbb{I}_N[\rho]}(d))^{\dagger\vec{i}}$, $\alpha \preceq^R (\psi_{\mathbb{I}_N[\rho]}(c))^{\dagger\vec{j}}$ and $\psi_{\mathbb{I}_N[\rho]}(d) < \psi_{\mathbb{I}_N[\rho]}(c)$.
   (c) There are $\vec{i}$, $\mathbb{I}_N[\rho]$ such that $\rho \prec \mathbb{T} \preceq^R \mathbb{S}$, $\beta \preceq^R \mathbb{T}^{\dagger\vec{i}}[\rho/\mathbb{T}]$ and $\alpha \preceq^R \mathbb{I}_N[\rho]$.
   (d) There are $\rho \prec \mathbb{T} \preceq^R \mathbb{S}$, $\sigma, \tau \prec \mathbb{U} = \mathbb{T}^{\dagger\vec{i}}[\rho/\mathbb{T}]$, $\vec{k}$ and $\vec{l}$ such that $\tau < \sigma$, $(\tau, \sigma) \neq (\beta, \alpha)$, $\alpha = \sigma \vee \alpha \preceq^R \mathbb{I}_N[\sigma] \vee \alpha \preceq^R \mathbb{U}^{\dagger\vec{k}}[\sigma/\mathbb{U}]$, and $\beta = \tau \vee \beta \preceq^R \mathbb{I}_N[\tau] \vee \beta \preceq^R \mathbb{U}^{\dagger\vec{l}}[\tau/\mathbb{U}]$.

Lemma 3.36 $(OT(\mathbb{I}_N), <)$ *is a computable linear order. Specifically each of* $\alpha < \beta$ *and* $\alpha = \beta$ *is decidable for* $\alpha, \beta \in OT(\mathbb{I}_N)$*, and* $\alpha \in OT(\mathbb{I}_N)$ *is decidable for terms* α *over symbols* $\{0, \Omega, \mathbb{I}_N, +, \varphi, \psi\}$*,* $\{{}^{\dagger i} : 0 < i \leq N\}$*,* $\mathbb{I}_N[*]$ *and* $*_0[*_1/*_2]$.

In particular the order type of the initial segment $\{\alpha \in OT(\mathbb{I}_N) : \alpha < \Omega\}$ *is less than* ω_1^{CK} *if it is well-founded.*

In what follows by ordinals we mean ordinal terms in $OT(\mathbb{I}_N)$. $\ell\alpha$ denotes the length of ordinal terms α, which means the number of occurrences of symbols in α.

Proposition 3.37 *If $\mathbb{S} \in St_{i+1} = SSt_{i+1} \cup LSt_{i+1}$ and $\alpha < \mathbb{S}$, then $\alpha^{\dagger i} < \mathbb{S}$.*

Proof. This is seen from Proposition 3.32 and Definition 3.35. □

Proposition 3.38 $\{\mathbb{S}\} \cup SC(m(\rho)) \cup \{\mathtt{p}_0(\rho)\} \subset M_\rho$ *for $\rho \in \Psi_{\mathbb{S}}$.*

Proof. If $\rho = \psi^f_{\mathbb{S}}(a)$ with an $\mathbb{S} \in SSt$, then we obtain $f = m(\rho)$, $a = \mathtt{p}_0(\rho)$, $\{\mathbb{S}\} \cup SC(f) \cup \{\mathtt{p}_0(\rho)\} \subset \mathcal{H}_a(\alpha) = M_\rho$ by Definition 3.31.5. Otherwise $\{\mathbb{S}\} \cup SC(m(\rho)) \cup \{\mathtt{p}_0(\rho)\} \subset M_\rho$ follows from (12) in Definition 3.31.6. □

An ordinal term $\sigma \in OT(\mathbb{I}_N)$ is said to be *regular* if either $\sigma \in \{\Omega, \mathbb{I}_N\} \cup \{\sigma \in OT(\mathbb{I}_N) : \exists\rho(\sigma = \mathbb{I}_N[\rho])\}$ or $\psi^f_\sigma(a)$ is in $OT(\mathbb{I}_N)$ for some f and a. Reg denotes the set of regular terms. Then $Reg = SSt^M \cup \{\mathbb{I}_N[\rho] : \exists \mathbb{S} \in SSt^M(\rho \prec \mathbb{S})\} \cup \{\Omega, \mathbb{I}_N\}$. We see that for each $\alpha \in \Psi$, there exists a $\kappa \in Reg_0 := (Reg \setminus \Psi)$ such that $\alpha \prec \kappa$. Such a κ is either in $\{\Omega, \mathbb{I}_N\}$ or one of the form $\mathbb{I}_N[\rho]$, $\rho^{\dagger\vec{i}}$ or $\mathbb{S}^{\dagger\vec{i}}[\rho/\mathbb{S}]$ with a non-empty $\vec{i}$.

Proposition 3.39 *Let $\psi^f_\pi(a) < \psi^g_\kappa(b) < \pi < \kappa$ and $\pi \preceq \rho$ and $\kappa \preceq \tau$ with $\{\rho, \tau\} \subset Reg_0$. Then $\rho = \tau$.*

Proof. From Definition 3.35 we see that the only possible case is Definition 3.35(7a). □

Lemma 3.40 *For $\rho \prec \mathbb{S}$ and $\mathbb{S} \in SSt$, $\{\alpha[\rho/\mathbb{S}] : \alpha \in M_\rho\}$ is a transitive collapse of M_ρ in the following sense. Let $\{\alpha, \beta, \gamma\} \subset M_\rho$.*

1. $\beta < \alpha \Leftrightarrow \beta[\rho/\mathbb{S}] < \alpha[\rho/\mathbb{S}]$.
2. $\beta \prec^R \alpha \Leftrightarrow \beta[\rho/\mathbb{S}] \prec^R \alpha[\rho/\mathbb{S}]$.
3. $\mathbb{S} < \gamma \Rightarrow \big(K_\gamma(\beta) < \alpha \Leftrightarrow K_{\gamma[\rho/\mathbb{S}]}(\beta[\rho/\mathbb{S}]) < \alpha[\rho/\mathbb{S}]\big)$.
4. $OT(\mathbb{I}_N) \cap \alpha[\rho/\mathbb{S}] = \{\gamma[\rho/\mathbb{S}] : \gamma \in M_\rho \cap \alpha\}$.

Proof. We show Lemmas 3.40.1- 3.40.3 simultaneously by induction on the sum $2^{\ell\alpha} + 2^{\ell\beta}$ for $\alpha, \beta \in M_\rho$. We see easily that $\mathbb{S} > \Gamma(\mathbb{I}_N[\rho]) > \alpha[\rho/\mathbb{S}] > \rho$ when $\alpha > \mathbb{S}$. Also $\alpha[\rho/\mathbb{S}] \leq \alpha$.
3.40.2 and 3.40.3 are seen from IH.
3.40.1. Let $k(\psi^g_\kappa(a)) = SC(g) \cup \{\kappa, a\}$. Let $\mathbb{S} < \beta = \psi^f_\pi(b) < \psi^g_\kappa(a) = \alpha$ with $k(\beta, \alpha) \subset M_\rho$. From IH with Definition 3.35 we see that $\beta[\rho/\mathbb{S}] = \psi^{f[\rho/\mathbb{S}]}_{\pi[\rho/\mathbb{S}]}(b[\rho/\mathbb{S}]) < \psi^{g[\rho/\mathbb{S}]}_{\kappa[\rho/\mathbb{S}]}(a[\rho/\mathbb{S}]) = \alpha[\rho/\mathbb{S}]$. Other cases are seen from IH.
3.40.3. Suppose $K_\gamma(\beta) < \alpha$ for $\mathbb{S} < \gamma$. Then $K_{\gamma[\rho/\mathbb{S}]}(\beta[\rho/\mathbb{S}]) < \alpha[\rho/\mathbb{S}]$ is seen from IH and Lemma 3.40.1 using the fact $\gamma[\rho/\mathbb{S}] > \rho$.

3.40.4. Let $\beta \in OT(\mathbb{I}_N) \cap \alpha[\rho/\mathbb{S}]$ for $\alpha \in M_\rho$. We show by induction on $\ell\beta$ that there exists a $\gamma \in M_\rho$ such that $\beta = \gamma[\rho/\mathbb{S}]$. If $\beta < \rho$, then $\beta[\rho/\mathbb{S}] = \beta$. Also $\rho = \mathbb{S}[\rho/\mathbb{S}]$ and $\mathbb{I}_N[\rho] = (\mathbb{I}_N)[\rho/\mathbb{S}]$. Let $\Gamma(\mathbb{I}_N[\rho]) > \alpha[\rho/\mathbb{S}] > \beta > \rho$. We may assume $\mathbb{I}_N[\rho] > \beta > \rho$ by IH.

If $\beta = \mathbb{I}_N[\tau]$, then $\mathbb{I}_N[\tau] > \tau$. Pick a $\kappa \in M_\rho$ such that $\kappa[\rho/\mathbb{S}] = \tau$. Then $\beta = (\mathbb{I}_N[\kappa])[\rho/\mathbb{S}]$.

If $\beta = \tau^{\dagger\vec{i}}$, then $\tau^{\dagger\vec{i}} > \tau$. Pick a $\kappa \in M_\rho$ such that $\kappa[\rho/\mathbb{S}] = \tau$. Then $\beta = (\kappa^{\dagger\vec{i}})[\rho/\mathbb{S}]$.

If $\beta = \mathbb{T}_1^{\dagger\vec{j}}[\tau_1/\mathbb{T}_1]$, then $\mathbb{T}_1^{\dagger\vec{j}}[\tau_1/\mathbb{T}_1] > \tau_1$. Pick a $\tau \in M_\rho$ such that $\tau[\rho/\mathbb{S}] = \tau_1$. Then for $\tau \prec \mathbb{T} \in SSt^M$, we obtain $\beta = (\mathbb{T}^{\dagger\vec{j}}[\tau/\mathbb{T}])[\rho/\mathbb{S}]$.

Finally let $\beta = \psi_\pi^f(b)$ with $k(\beta) \subset \mathcal{H}_b(\beta)$, $b < \Gamma(\mathbb{I}_N[\rho])$ and $f : \Lambda \to \varphi_\Lambda(0)$ for $\pi \preceq \sigma^{\dagger\vec{k}}$ with a $\vec{k} \neq \emptyset$. We have $\beta \prec \sigma^{\dagger\vec{k}}$, $\rho < \beta < \mathbb{I}_N[\rho]$, and $\rho \prec \mathbb{S}$. By Definition 3.35 we obtain $\sigma \neq \mathbb{S}$. Suppose $\beta < \mathbb{S} < \sigma^{\dagger\vec{k}}$. Then $\alpha < \rho$ by Definition 3.35. Hence we may assume $\sigma^{\dagger\vec{k}} < \mathbb{S}$. Then we obtain $\rho < \sigma^{\dagger k} < \mathbb{I}_N[\rho]$. Hence $\sigma \prec^R \mathbb{I}_N[\rho]$ or $\sigma \prec^R \mathbb{S}^{\dagger\vec{i}}[\rho/\mathbb{S}]$ for an $\vec{i}$. By IH with $\pi \leq \sigma^{\dagger\vec{k}}$ there are $\{c, \kappa, \lambda\} \subset M_\rho$ and $g : \lambda \to \varphi_\lambda(0)$ such that $c[\rho/\mathbb{S}] = b$, $\kappa[\rho/\mathbb{S}] = \pi$, $\lambda[\rho/\mathbb{S}] = \Lambda$, $SC(g) \subset M_\rho$, $g[\rho/\mathbb{S}] = f$ in the sense that $(\mathrm{supp}(g))[\rho/\mathbb{S}] = \mathrm{supp}(f)$ and $(g(d))[\rho/\mathbb{S}] = f(d[\rho/\mathbb{S}])$ for every $d \in \mathrm{supp}(g)$. Let $\gamma = \psi_\kappa^g(c) \in M_\rho$. Then $\gamma[\rho/\mathbb{S}] = \psi_\pi^f(b) = \beta$ and $k(\gamma) \subset \mathcal{H}_c(\gamma)$.

Other cases are seen from IH. □

Lemma 3.41 *1. Let $\alpha = \psi_\Omega(a)$ with $a \in \mathcal{H}_a(\alpha)$. Then $\mathcal{H}_a(\alpha) \cap \Omega \subset \alpha$.*

2. Let $\alpha = \psi_{\mathbb{I}_N}(a)$ with $a \in \mathcal{H}_a(\alpha)$. Then $\mathcal{H}_a(\alpha) \cap \mathbb{I}_N \subset \alpha$.

3. Let $\mathbb{S} \in SSt$, and $\alpha = \psi_\kappa^f(a) < \kappa$ with $\kappa \preceq \mathbb{S}$ and $\{\kappa, a\} \cup SC(f) \subset \mathcal{H}_a(\alpha)$. Then $\mathcal{H}_a(\alpha) \cap \kappa \subset \alpha$.

Proof. We see $\beta \in \mathcal{H}_a(\alpha) \cap \Omega \Rightarrow \beta < \alpha = \psi_\Omega(a)$ by induction on the lengths $\ell\beta$ of β. Lemmas 3.41.2 and 3.41.3 are seen similarly using the fact $\rho < \alpha \Rightarrow \mathbb{I}_N[\rho] < \alpha$ for $\alpha \in \{\psi_{\mathbb{I}_N}(a), \psi_\kappa^f(a)\}$. □

Proposition 3.42 *Let $\mathbb{S} \in SSt$, and $\rho = \psi_\kappa^f(a) < \kappa$ with $\kappa \preceq \mathbb{S}$ and $\mathcal{H}_\gamma(\kappa) \cap \mathbb{S} \subset \kappa$ for $\gamma \leq a$. Then $\mathcal{H}_\gamma(\rho) \cap \mathbb{S} \subset \rho$.*

Proof. If $\kappa = \mathbb{S}$, then $\mathcal{H}_\gamma(\rho) \cap \mathbb{S} \subset \mathcal{H}_a(\rho) \cap \mathbb{S} \subset \rho$ by $\gamma \leq a$ and Lemma 3.41.3. Let $\kappa = \psi_\pi^g(b) < \mathbb{S}$. We have $\kappa \in \mathcal{H}_a(\rho)$ by (7), and hence $b < a$ by $\mathbb{S} > \kappa > \rho$. We obtain $\mathcal{H}_\gamma(\rho) \cap \mathbb{S} \subset \mathcal{H}_\gamma(\kappa) \cap \mathbb{S} \subset \kappa$. $\gamma \leq a$ with Lemma 3.41.3 yields $\mathcal{H}_\gamma(\rho) \cap \mathbb{S} \subset \mathcal{H}_\gamma(\rho) \cap \kappa \subset \mathcal{H}_a(\rho) \cap \kappa \subset \rho$. □

Lemma 3.43 *Let $\rho \in \Psi_{\mathbb{S}}$ for an $\mathbb{S} \in SSt$.*

1. *$\mathcal{H}_\gamma(M_\rho) \subset M_\rho$ if $\gamma \leq \mathtt{p}_0(\rho)$.*
2. *$M_\rho \cap \mathbb{S} = \rho$ and $\rho \notin M_\rho$.*
3. *If $\sigma < \rho$ and $\mathtt{p}_0(\sigma) \leq \mathtt{p}_0(\rho)$, then $M_\sigma \subset M_\rho$.*

Proof. Lemmas 3.43.2 and 3.43.3 are seen readily.
3.43.1. Let $\gamma \leq b = \mathsf{p}_0(\rho)$. We show $\alpha \in M_\rho = \mathcal{H}_b(\rho)$ by induction on $\ell\alpha$ for $\alpha \in \mathcal{H}_\gamma(M_\rho)$. Let $k(\alpha) \subset \mathcal{H}_\gamma(M_\rho) \cap \mathcal{H}_a(\alpha)$ be such that $a < \gamma \leq b$ and $\alpha = \psi^g_\kappa(a) \in \mathcal{H}_\gamma(M_\rho)$. IH yields $k(\alpha) \subset M_\rho$. We obtain $\alpha \in \mathcal{H}_b(\rho)$.

Other cases are seen from IH. □

Definition 3.44 (Mostowski uncollapsing)
Let α be an ordinal term and $\rho \prec \mathbb{S}$ with $\mathbb{S} \in SSt$. If there exists a $\beta \in M_\rho$ such that $\alpha = \beta[\rho/\mathbb{S}]$, then $\alpha[\rho/\mathbb{S}]^{-1} := \beta$. Otherwise $\alpha[\rho/\mathbb{S}]^{-1} := 0$. Let $X[\rho/\mathbb{S}]^{-1} := \{\alpha[\rho/\mathbb{S}]^{-1} : \alpha \in X\}$ for a set X of ordinal terms.

We see that ordinal terms ρ and $\beta \in M_\rho$ with $\rho \leq \alpha = \beta[\rho/\mathbb{S}] < \Gamma(\mathbb{I}_N[\rho])$ are uniquely determined from α, when such β and ρ exist.

4 Operator controlled derivations

We prove Theorem 1.1 assuming that the notation system $(OT(\mathbb{I}_N), <)$ is a well ordering. Operator controlled derivations are introduced by W. Buchholz[9], which we follow. In this section except otherwise stated, $\alpha, \beta, \gamma, \ldots, a, b, c, d, \ldots$ and $\xi, \zeta, \nu, \mu, \ldots$ range over ordinal terms in $OT(\mathbb{I}_N)$, $f, g, h, \ldots$ range over finite functions.

4.1 Classes of sentences

Following Buchholz[9] let us introduce a language of ramified set theory RS.

Definition 4.1 *RS-terms* and their *levels* are inductively defined.

1. For each $\alpha \in OT(\mathbb{I}_N) \cap \mathbb{I}_N$, L_α is an RS-term of level α.
2. Let $\phi(x, y_1, \ldots, y_n)$ be a set-theoretic formula in the language $\{\in\}$, and $a_1, \ldots, a_n$ RS-terms of levels$< \alpha \in OT(\mathbb{I}_N) \cap \mathbb{I}_N$.

   Then $[x \in \mathsf{L}_\alpha : \phi^{\mathsf{L}_\alpha}(x, a_1, \ldots, a_n)]$ is an RS-term of level α.

Let us identify the individual constant M_0 in the language of $S_{\mathbb{I}_N}$ with the RS-term L_Ω.

Definition 4.2 1. $|u|$ denotes the level of RS-terms u, and $Tm(\alpha)$ the set of RS-terms of level$< \alpha \in OT(\mathbb{I}_N) \cap (\mathbb{I}_N + 1)$. $Tm = Tm(\mathbb{I}_N)$ is then the set of RS-terms, which are denoted by $u, v, w, \ldots$

2. *RS-formulas* are constructed from *literals* $u \in v, u \notin v$ and $st_i(u), \neg st_i(u)$ for $0 < i \leq N$ by propositional connectives $\vee, \wedge$, bounded quantifiers $\exists x \in u, \forall x \in u$ and unbounded quantifiers $\exists x, \forall x$. Unbounded quantifiers $\exists x, \forall x$ are denoted by $\exists x \in \mathsf{L}_{\mathbb{I}_N}, \forall x \in \mathsf{L}_{\mathbb{I}_N}$, resp.

   It is convenient for us not to restrict propositional connectives $\vee, \wedge$ to binary ones. Specifically when A_i are RS-formulas for $i < n < \omega$, $A_0 \vee \cdots \vee$

A_{n-1} and $A_0\land\cdots\land A_{n-1}$ are RS-formulas. Even when $n=1$, $A_0\lor\cdots\lor A_0$ is understood to be different from the formula A_0. For $\Gamma=\{A_i : i<n\}$ we write $\bigvee\Gamma\equiv(A_0\lor\cdots\lor A_{n-1})$ and $\bigwedge\Gamma\equiv(A_0\land\cdots\land A_{n-1})$.

3. For RS-terms and RS-formulas ι, $\mathsf{k}(\iota)$ denotes the set of ordinal terms α such that the constant L_α occurs in ι, and $|\iota|=\max(\mathsf{k}(\iota)\cup\{0\})$.

   Also let $\mathcal{B}(\mathsf{k}(\iota)) := \bigcup\{\mathcal{B}(\alpha) : \alpha\in\mathsf{k}(\iota)\}$, cf. Definition 4.10 and (19) in Definition 4.14.

   Let $\mathsf{k}(n)=\mathcal{B}(\mathsf{k}(n))=\emptyset$ and $|n|=0$ for natural numbers n.

4. $\mathcal{L}_i=\{\in\}\cup\{st_j : 0<j<i\}$.

5. $\Delta_0(\mathcal{L}_i)$-formulas, $\Sigma_1(\mathcal{L}_i)$-formulas and $\Sigma(\mathcal{L}_i)$-formulas are defined as in [6]. Specifically if ψ is a $\Sigma(\mathcal{L}_i)$-formula, then so is the formula $\forall y\in z\,\psi$. $\theta^{(u)}$ denotes a $\Delta_0(\mathcal{L}_i)$-formula obtained from a $\Sigma(\mathcal{L}_i)$-formula θ by restricting each unbounded existential quantifier to u.

6. For a $\Sigma_1(\mathcal{L}_i)$-formula $\psi(x_1,\ldots,x_m)$ and $u_1,\ldots,u_m\in Tm(\kappa)$ with $\kappa\leq\mathbb{I}_N$, $\psi^{(\mathsf{L}_\kappa)}(u_1,\ldots,u_m)$ is a $\Sigma_1(\mathcal{L}_i:\kappa)$*-formula.* $\Delta_0(\mathcal{L}_i:\kappa)$-formulas and $\Sigma(\mathcal{L}_i:\kappa)$-formulas are defined similarly

7. For $\theta\equiv\psi^{(\mathsf{L}_\kappa)}(u_1,\ldots,u_m)\in\Sigma(\mathcal{L}_i:\kappa)$ and $\lambda<\kappa$, with $u_1,\ldots,u_m\in Tm(\lambda)$, $\theta^{(\lambda,\kappa)}:\equiv\psi^{(\mathsf{L}_\lambda)}(u_1,\ldots,u_m)$.

In what follows we consider only *sentences* without free variables. Sentences are denoted A, C possibly with indices.

For each sentence A, either a disjunction is assigned as $A\simeq\bigvee(A_\iota)_{\iota\in J}$, or a conjunction is assigned as $A\simeq\bigwedge(A_\iota)_{\iota\in J}$. By $st_i(u)$ we understand that there is a *successor* i-stable ordinal $\mathbb{S}$ such that $L_{\mathbb{S}}=u$.

Definition 4.3

1. For $v,u\in Tm(\mathbb{I}_N)$ with $|v|<|u|$, let

$$(v\dot{\in}u):\equiv\begin{cases} A(v) & \text{if } u\equiv[x\in\mathsf{L}_\alpha : A(x)]\\ v\not\in\mathsf{L}_0 & \text{if } u\equiv\mathsf{L}_\alpha\end{cases}$$

   and $(u=v):\equiv(\forall x\in u(x\in v)\land\forall x\in v(x\in u))$.

2. When $A\simeq\bigvee(A_\iota)_{\iota\in J}$, let $\neg A\simeq\bigwedge(A_\iota)_{\iota\in J}$.

3. $(v\in u):\simeq\bigvee(A_w)_{w\in J}$ for $A_w:\equiv((w\dot{\in}u)\land(w=v))$ and $J=Tm(|u|)$.

4. $(A_0\lor\cdots\lor A_{n-1}):\simeq\bigvee(A_\iota)_{\iota\in J}$ for $J:=n$.

5. For $u\in Tm(\mathbb{I}_N)\cup\{\mathsf{L}_{\mathbb{I}_N}\}$, $\exists x\in u\,A(x):\simeq\bigvee(A_v)_{v\in J}$ for $A_v:\equiv((v\dot{\in}u)\land A(v))$ and $J=Tm(|u|)$, where $Tm(|\mathsf{L}_{\mathbb{I}_N}|)=Tm(\mathbb{I}_N)=Tm$ and $(v\dot{\in}\mathsf{L}_{\mathbb{I}_N}):\equiv(v\not\in\mathsf{L}_0)$.

6. $st_i(u):\simeq\bigvee(\mathsf{L}_{\mathbb{S}}=u)_{\mathsf{L}_{\mathbb{S}}\in J_i}$ with $J_i=\{\mathsf{L}_{\mathbb{S}} : |u|\geq\mathbb{S}\in SSt_i\}$, where st_i denotes the predicate symbol in the language $\mathcal{L}_{N+1}$, while $SSt_i\subset OT(\mathbb{I}_N)$ in the definition of J_i.

7. For $A \simeq \bigvee(A_\iota)_{\iota\in J}$ let $[\rho]J = \{\iota \in J : \mathsf{k}(\iota) \subset M_\rho\}$.

It is clear that $\mathsf{k}(A_\iota) \subset \mathcal{H}_0(\mathsf{k}(A) \cup \mathsf{k}(\iota))$.

The rank $\mathrm{rk}(\iota)$ of sentences or terms ι is defined slightly modified from [9] so that the following Proposition 4.5 holds.

Definition 4.4 1. $\mathrm{rk}(\neg A) := \mathrm{rk}(A)$.

2. $\mathrm{rk}(\mathsf{L}_\alpha) = \omega\alpha$.

3. $\mathrm{rk}([x \in \mathsf{L}_\alpha : A(x)]) = \max\{\omega\alpha, \mathrm{rk}(A(\mathsf{L}_0))\}$.

4. $\mathrm{rk}(v \in u) = \max\{\mathrm{rk}(v) + 4, \mathrm{rk}(u) + 1\}$.

5. $\mathrm{rk}(st_i(u)) = \mathrm{rk}(u) + 5$.

6. $\mathrm{rk}(A_0 \lor \cdots \lor A_{n-1}) = \max(\{0\} \cup \{\mathrm{rk}(A_i) + 1 : i < n\})$.

7. $\mathrm{rk}(\exists x \in u\, A(x)) = \max\{\mathrm{rk}(u), \mathrm{rk}(A(\mathsf{L}_0))\} + 2$ for $u \in Tm(\mathbb{I}_N) \cup \{\mathsf{L}_{\mathbb{I}_N}\}$.

For finite sets Δ of sentences, let $\mathrm{rk}(\Delta) = \max(\{0\} \cup \{\mathrm{rk}(\delta) : \delta \in \Delta\})$.

Proposition 4.5 *Let A be a sentence with $A \simeq \bigvee(A_\iota)_{\iota\in J}$ or $A \simeq \bigwedge(A_\iota)_{\iota\in J}$.*

1. $\mathrm{rk}(A) < \mathbb{I}_N + \omega$.

2. $\mathrm{rk}(\bigvee\Gamma) = \max(\{0\} \cup \{\mathrm{rk}(A) + 1 : A \in \Gamma\})$.

3. $\omega|u| \leq \mathrm{rk}(u) \in \{\omega|u| + i : i \in \omega\}$, *and* $\omega|A| \leq \mathrm{rk}(A) \in \{\omega|A| + i : i \in \omega\}$.

4. $\mathrm{rk}(st_i(u)) \in \{\mathrm{rk}(u) + i : i < \omega\}$.

5. *For* $v \in Tm(|u|)$, $\mathrm{rk}(v \dot{\in} u) \leq \mathrm{rk}(u)$.

6. $\forall\iota \in J(\mathrm{rk}(A_\iota) < \mathrm{rk}(A))$.

Proof. 4.5.5. Let $\alpha = |u|$. We obtain $\mathrm{rk}(v) < \omega(|v| + 1) \leq \omega\alpha$ by Proposition 4.5.3. First let u be L_α. Then $(v \dot{\in} u) \equiv (v \notin \mathsf{L}_0)$, and $\mathrm{rk}(v \notin \mathsf{L}_0) = \max\{\mathrm{rk}(v) + 4, 1\} < \omega\alpha = \mathrm{rk}(u)$.

Next let u be an RS-term $[x \in \mathsf{L}_\alpha : A(x)]$ with $A(x) \equiv (\phi^{\mathsf{L}_\alpha}(x, u_1, \ldots, u_n))$ for a set-theoretic formula $\phi(x, y_1, \ldots, y_n)$, and RS-terms $u_1, \ldots, u_n \in Tm(\alpha)$. Then $(v \dot{\in} u) \equiv (A(v))$. If ϕ is a bounded formula, then we see from Proposition 4.5.3 that $\mathrm{rk}(A(v)) < \omega\alpha$. Otherwise $\mathrm{rk}(A(v)) = \omega\alpha + i$ for an $i < \omega$. Hence $\mathrm{rk}(A(v)) = \mathrm{rk}(A(\mathsf{L}_0)) = \mathrm{rk}(u)$.

4.5.6. First let A be a formula $v \in u$, and $w \in Tm(\alpha)$ with $\alpha = |u| > 0$. Then $\mathrm{rk}(w \dot{\in} u) \leq \mathrm{rk}(u)$ by Proposition 4.5.5. Moreover $\max\{\mathrm{rk}(\forall x \in w(x \in v)), \mathrm{rk}(\forall x \in v(x \in w))\} = \max\{\mathrm{rk}(w), \mathrm{rk}(v), \mathrm{rk}(\mathsf{L}_0 \in v), \mathrm{rk}(\mathsf{L}_0 \in w)\} + 2$. We have $\max\{\mathrm{rk}(w), \mathrm{rk}(\mathsf{L}_0 \in w)\} + 2 < \omega\alpha \leq \mathrm{rk}(u)$, and $\mathrm{rk}(\mathsf{L}_0 \in v) = \max\{4, \mathrm{rk}(v) + 1\}$. Hence $\max\{\mathrm{rk}(w \dot{\in} u), \mathrm{rk}(\forall x \in w(x \in v)), \mathrm{rk}(\forall x \in v(x \in w))\} + 2 \leq \max\{\mathrm{rk}(v) + 3, \mathrm{rk}(u)\}$. Therefore $\mathrm{rk}(A_w) < \mathrm{rk}(A)$.

Next let A be a formula $\exists x \in u\, B(x)$, and $v \in Tm(\alpha)$ with $\alpha = |u|$. Then $\mathrm{rk}(v\dot{\in}u) \leq \mathrm{rk}(u)$ by Proposition 4.5.5. Moreover either $\mathrm{rk}(B(v)) < \omega(|v|+1) \leq \omega\alpha \leq \mathrm{rk}(u)$ or $\mathrm{rk}(B(v)) = \mathrm{rk}(B(\mathsf{L}_0))$. This shows $\mathrm{rk}(A_v) < \mathrm{rk}(A)$.

Finally let A be a formula $st_i(u)$, and $A_\alpha \equiv (\mathsf{L}_\alpha = u)$ with $\alpha \leq |u|$ and $\alpha \in SSt_i$. In particular $0 < \alpha \leq |u|$. We obtain $\max\{\mathrm{rk}(\forall x \in \mathsf{L}_\alpha(x \in u)), \mathrm{rk}(\forall x \in u(x \in \mathsf{L}_\alpha))\} = \max\{\mathrm{rk}(\mathsf{L}_\alpha), \mathrm{rk}(u), 3\} + 3$, where $\mathrm{rk}(\mathsf{L}_\alpha) = \omega\alpha \leq \mathrm{rk}(u)$. Hence $\mathrm{rk}(A_\alpha) = \mathrm{rk}(u) + 4 < \mathrm{rk}(A)$. □

Definition 4.6 Let $\rho \prec \mathbb{S} \in SSt_i$ for an $0 < i \leq N$, and $\mathsf{k}(\iota) \subset M_\rho$ for RS-terms and RS-formulas ι. Then $\iota^{[\rho/\mathbb{S}]}$ denotes the result of replacing each unbounded quantifier Qx by $Qx \in \mathsf{L}_{\mathbb{I}_N[\rho]}$, and each ordinal term $\alpha \in \mathsf{k}(\iota)$ by $\alpha[\rho/\mathbb{S}]$ for the Mostowski collapse in Definition 3.33. $\iota^{[\rho/\mathbb{S}]}$ is defined recursively as follows.

1. $(\mathsf{L}_\alpha)^{[\rho/\mathbb{S}]} \equiv \mathsf{L}_{\alpha[\rho/\mathbb{S}]}$ with $\alpha \in M_\rho$. When $\{\alpha\} \cup \bigcup\{\mathsf{k}(u_i) : i \leq n\} \subset M_\rho$, $\left([x \in \mathsf{L}_\alpha : \phi^{\mathsf{L}_\alpha}(x, u_1, \ldots, u_n)]\right)^{[\rho/\mathbb{S}]}$ is defined to be the RS-term $[x \in \mathsf{L}_{\alpha[\rho/\mathbb{S}]} : \phi^{L_{\alpha[\rho/\mathbb{S}]}}(x, (u_1)^{[\rho/\mathbb{S}]}, \ldots, (u_n)^{[\rho/\mathbb{S}]})]$.
2. $(\neg A)^{[\rho/\mathbb{S}]} \equiv \neg A^{[\rho/\mathbb{S}]}$. $(u \in v)^{[\rho/\mathbb{S}]} \equiv \left(u^{[\rho/\mathbb{S}]} \in v^{[\rho/\mathbb{S}]}\right)$. $(A_0 \vee \cdots \vee A_{n-1})^{[\rho/\mathbb{S}]} \equiv \left((A_0)^{[\rho/\mathbb{S}]} \vee \cdots \vee (A_{n-1})^{[\rho/\mathbb{S}]}\right)$. $(\exists x \in u\, A)^{[\rho/\mathbb{S}]} \equiv (\exists x \in u^{[\rho/\mathbb{S}]} A^{[\rho/\mathbb{S}]})$. $(\exists x A)^{[\rho/\mathbb{S}]} \equiv (\exists x \in \mathsf{L}_{\mathbb{I}_N[\rho]} A^{[\rho/\mathbb{S}]})$.

The following Propositions 4.7, 4.8 and 4.9 are seen from Lemma 3.40.

Proposition 4.7 *Let $\rho \prec \mathbb{S}$.*

1. *Let v be an RS-term with $\mathsf{k}(v) \subset M_\rho$, and $\alpha = |v|$. Then $v^{[\rho/\mathbb{S}]}$ is an RS-term of level $\alpha[\rho/\mathbb{S}]$, $\left|v^{[\rho/\mathbb{S}]}\right| = \alpha[\rho/\mathbb{S}]$ and $\mathsf{k}(v^{[\rho/\mathbb{S}]}) = (\mathsf{k}(v))^{[\rho/\mathbb{S}]}$.*
2. *Let $\alpha \leq \mathbb{I}_N$ be such that $\alpha \in M_\rho$. Then $(Tm(\alpha))^{[\rho/\mathbb{S}]} := \{v^{[\rho/\mathbb{S}]} : v \in Tm(\alpha), \mathsf{k}(v) \subset M_\rho\} = Tm(\alpha[\rho/\mathbb{S}])$.*
3. *Let A be an RS-formula with $\mathsf{k}(A) \subset M_\rho$. Then $A^{[\rho/\mathbb{S}]}$ is an RS-formula such that $\mathsf{k}(A^{[\rho/\mathbb{S}]}) \subset \{\alpha[\rho/\mathbb{S}] : \alpha \in \mathsf{k}(A)\} \cup \{\mathbb{I}_N[\rho]\} \cap \mathcal{H}_{\mathbb{S}}(\mathsf{k}(A) \cup \{\rho\})$.*

Proof. 4.7.1. We see easily that $v^{[\rho/\mathbb{S}]}$ is an RS-term of level $\alpha[\rho/\mathbb{S}]$.
4.7.2. We see $(Tm(\alpha))^{[\rho/\mathbb{S}]} \subset Tm(\alpha[\rho/\mathbb{S}])$ from Proposition 4.7.1. Conversely let u be an RS-term with $\mathsf{k}(u) = \{\beta_i : i < n\}$ and $\max\{\beta_i : i < n\} = |u| < \alpha[\rho/\mathbb{S}]$. By Lemma 3.40 there are ordinal terms $\gamma_i \in OT(\mathbb{I}_N)$ such that $\gamma_i \in M_\rho$ and $\gamma_i[\rho/\mathbb{S}] = \beta_i$. Let v be an RS-term obtained from u by replacing each constant L_{β_i} by L_{γ_i}. We obtain $v^{[\rho/\mathbb{S}]} \equiv u$, $v \in Tm(\alpha)$, and $\mathsf{k}(v) = \{\gamma_i : i < n\} \subset M_\rho$. This means $v \in (Tm(\alpha))^{[\rho/\mathbb{S}]}$.
4.7.3. We see readily that $\mathsf{k}(A^{[\rho/\mathbb{S}]}) \subset \{\alpha[\rho/\mathbb{S}] : \alpha \in \mathsf{k}(A)\} \cup \{\mathbb{I}_N[\rho]\}$. From this and Proposition 4.11.2, $\mathsf{k}(A^{[\rho/\mathbb{S}]}) \subset \mathcal{H}_{\mathbb{S}}(\mathsf{k}(A) \cup \{\rho\})$ follows. □

Proposition 4.8 *For RS-formulas A, let $A \simeq \bigvee(A_\iota)_{\iota \in J}$ and assume $\mathsf{k}(A) \subset M_\rho$ with $\rho \prec \mathbb{S}$. Then $A^{[\rho/\mathbb{S}]} \simeq \bigvee\left((A_\iota)^{[\rho/\mathbb{S}]}\right)_{\iota \in [\rho]J}$ for $[\rho]J = \{\iota \in J : \mathsf{k}(\iota) \subset M_\rho\}$.*

Proof. This is seen from Proposition 4.7.2. □

Proposition 4.9 *Let* $\mathsf{k}(\iota) \subset M_\rho$ *with* $\rho \prec \mathbb{S}$. *Then* $\mathrm{rk}(\iota^{[\rho/\mathbb{S}]}) = (\mathrm{rk}(\iota))\,[\rho/\mathbb{S}]$.

Proof. We see that $\mathrm{rk}(\iota) \in M_\rho$ from Proposition 4.5.3. The proposition is seen from the facts $(\omega\alpha)[\rho/\mathbb{S}] = \omega(\alpha[\rho/\mathbb{S}])$ and $(\alpha+1)[\rho/\mathbb{S}] = \alpha[\rho/\mathbb{S}] + 1$ when $\alpha \in M_\rho$. □

4.2 A preview of elimination procedures of stable ordinals

Let us explain briefly our elimination procedures of stable ordinals in this section and section 5. In the previous paper [5], we analyzed an axiom $L_\mathbb{S} \prec_{\Sigma_1} L$ proof-theoretically. The axiom is a schema $\exists x\, B(x,v) \land v \in L_\mathbb{S} \to \exists x \in L_\mathbb{S}\, B(x,v)$ for Δ_0-formulas B. The schema says that $\mathbb{S}$ 'reflects' $\Pi_{\mathbb{S}^+}$-formulas in transfinite levels for a bigger ordinal $\mathbb{S}^+ > \mathbb{S}$ such that $L = L_{\mathbb{S}^+}$. In order to analyze the reflections, Mahlo classes $Mh^a_{i,c}(\xi)$ are introduced in Definition 3.8.2. $\pi \in Mh^a_{i,c}(\xi)$ reflects every fact $\pi \in Mh^a_{i,0}(g_c) = \bigcap\{Mh^a_{i,d}(g(d)) : c > d \in \mathrm{supp}(g)\}$ on the ordinals $\pi \in Mh^a_{i,c}(\xi)$ in lower level, down to 'smaller' Mahlo classes $Mh^a_{i,c}(f) = \bigcap\{Mh^a_{i,d}(f(d)) : c \le d \in \mathrm{supp}(f)\}$.

This apparatus would suffice to analyze reflections in transfinite levels. We need another for the axiom $L_\mathbb{S} \prec_{\Sigma_1} L$, i.e., a (formal) *Mostowski collapsing*: Assume that $B(u,v)$ with $v \in L_\mathbb{S}$ for a Δ_0-formula B. We need to find a substitute $u' \in L_\mathbb{S}$ for $u \in L$ such that $B(u',v)$. For simplicity let us assume that $v = \beta < \mathbb{S}$ and $u = \alpha$ are ordinals. We may assume that $\alpha \ge \mathbb{S}$. Let $\rho < \mathbb{S}$ be an ordinal, which is bigger than every ordinal$< \mathbb{S}$ occurring in the 'context' of $B(\alpha,\beta)$. This means that $\delta < \rho$ holds for every ordinal $\delta < \mathbb{S}$ occurring in a 'relevant' branch of a derivation of $B(\alpha,\beta)$. Then we can define a Mostwosiki collapsing $\alpha \mapsto \alpha[\rho/\mathbb{S}]$ for ordinal terms α such that $\beta[\rho/\mathbb{S}] = \beta$ for each relevant $\beta < \mathbb{S}$ and $\mathbb{S}[\rho/\mathbb{S}] = \rho$, cf. Definition 3.33. Then we see that $B(\alpha[\rho/\mathbb{S}],\beta)$ holds.

Let M_ρ denote a set of ordinal terms α such that every subterm $\beta < \mathbb{S}$ of α is smaller than ρ. It is shown in Lemma 3.43.1 that $\mathcal{H}_\gamma(M_\rho) \subset M_\rho$ if $\mathcal{H}_\gamma(\rho) \cap \mathbb{S} \subset \rho$. Let $\mathcal{H}_\gamma[\Theta] \vdash^a_c \Gamma$, and assume that $\{\gamma, a, c\} \cup \mathsf{k}(\Gamma) \subset \mathcal{H}_\gamma[\Theta]$. Moreover let us assume that $\Theta \subset M_\rho$ holds. Then we obtain $\{\gamma, a, c\} \cup \mathsf{k}(\Gamma) \subset \mathcal{H}_\gamma[\Theta] \subset \mathcal{H}_\gamma(M_\rho) \subset M_\rho$. This means that $\mathsf{k}(\Gamma) \subset M_\rho$ holds as long as $\Theta \subset M_\rho$ holds, i.e., as long as we are concerned with branches for $\mathsf{k}(\iota) \subset M_\rho$ in, e.g., inferences $(\bigwedge)$: $A \simeq \bigwedge(A_\iota)_{\iota\in J}$

$$\frac{\{\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_c \Gamma, A, A_\iota\}_{\iota\in J}}{\mathcal{H}_\gamma[\Theta] \vdash^a_c \Gamma, A}\,(\bigwedge) \leadsto \frac{\{\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_c \Gamma, A, A_\iota\}_{\iota\in J,\, \mathsf{k}(\iota)\subset M_\rho}}{\mathcal{H}_\gamma[\Theta] \vdash^a_c \Gamma, A}\,(\bigwedge) \tag{14}$$

and dually $\mathsf{k}(\iota) \subset M_\rho$ for a minor formula A_ι of a $(\bigvee)$ with the main formula $A \simeq \bigvee(A_\iota)_{\iota\in J}$, *provided that* $\mathcal{H}_\gamma(\rho) \cap \mathbb{S} \subset \rho$. The proviso means that $\gamma_1 \ge \gamma$ when $\rho = \psi^f_\mathbb{S}(\gamma_1)$. Such a ρ is in $\mathcal{H}_\gamma[\Theta]$ only when $\rho \in \Theta$. Let us try to replace the inferences for the stability of $\mathbb{S}$

$$\frac{\mathcal{H}_\gamma[\Theta] \vdash \Gamma, B(u) \quad \{\mathcal{H}_\gamma[\Theta \cup \{\sigma\}] \vdash \Gamma, \neg B(u)^{[\sigma/\mathbb{S}]}\}_{\Theta\subset M_\sigma}}{\mathcal{H}_\gamma[\Theta] \vdash \Gamma}\,(\mathrm{stbl})$$

by inferences for reflection of ρ with $\Theta \subset M_\rho$: If $B(u)^{[\rho/\mathbb{S}]}$ holds, then $B(u)^{[\sigma/\mathbb{S}]}$ holds for some $\sigma < \rho$.

$$\frac{\mathcal{H}_\gamma[\Theta \cup \{\rho\}] \vdash \Gamma^{[\rho/\mathbb{S}]}, B(u)^{[\rho/\mathbb{S}]} \quad \{\mathcal{H}_\gamma[\Theta \cup \{\rho, \sigma\}] \vdash \Gamma^{[\rho/\mathbb{S}]}, \neg B(u)^{[\sigma/\mathbb{S}]}\}_{\Theta \subset M_\sigma, \sigma < \rho}}{\mathcal{H}_\gamma[\Theta \cup \{\rho\}] \vdash \Gamma^{[\rho/\mathbb{S}]}} \text{ (rfl)}$$

In analyzing the inferences for reflections in transfinite levels, formulas $\Gamma^{[\rho/\mathbb{S}]}$ are replaced by $\Gamma^{[\sigma/\mathbb{S}]}$. This means that $\alpha[\sigma/\mathbb{S}]$ is substituted for each $\alpha[\rho/\mathbb{S}]$. Namely a composition of uncollapsing and collapsing $\alpha[\rho/\mathbb{S}] \mapsto \alpha \mapsto \alpha[\sigma/\mathbb{S}]$ arises. Hence we need $\alpha \in M_\sigma \subsetneq M_\rho$ for $\sigma < \rho$. However we have $\sigma \notin M_\sigma$ although $\sigma \in M_\rho$, and we cannot replace $[\rho/\mathbb{S}]$ by $[\sigma/\mathbb{S}]$ in the upper part of $\Gamma^{[\rho/\mathbb{S}]}, B(u)^{[\rho/\mathbb{S}]}$. The schema seems to be broken.

Instead of an explicit collapsing ${}^{[\rho/\mathbb{S}]}$, formulas could put on *caps* $\rho, \sigma, \ldots$ in such a way that $\mathsf{k}(A^{(\sigma)}) = \mathsf{k}(A)$. This means that the cap σ does not 'occur' in a capped formula $A^{(\sigma)}$. If we choose an ordinal γ_0 big enough (depending on a given finite proof figure), every ordinal 'occurring' in derivations (including the subscript $\gamma \leq \gamma_0$ in the operators $\mathcal{H}_\gamma$) is in $\mathcal{H}_{\gamma_0}(\emptyset)$ for the ordinal γ_0, while each cap ρ exceeds the *threshold* γ_0 in the sense that $\rho \notin \mathcal{H}_{\gamma_0}(\rho) \cap \mathbb{S} \subset \rho$. Then every ordinal 'occurring' in derivations is in the domain M_ρ of the Mostowski collapsing $\alpha \mapsto \alpha[\rho/\mathbb{S}]$.

The ordinal γ_0 is a threshold, which means that every ordinal occurring in derivations is in $\mathcal{H}_{\gamma_0}(0)$ and the subscript $\gamma \leq \gamma_0$ in $\mathcal{H}_\gamma$, while each $\rho \in \mathtt{Q}$ for a finite set $\mathtt{Q}$ of ordinals, exceeds γ_0 in such a way that $\mathsf{p}_0(\rho) \geq \gamma_0$ for the ordinal $\mathsf{p}_0(\rho)$ in Definition 3.30.2. This ensures us that $\mathcal{H}_\gamma(M_\rho) \subset M_\rho$. In the end, inferences for reflections are removed in [5] by moving outside $\mathcal{H}_{\gamma_0}(0)$.

Now we have several (successor) stable ordinals $\mathbb{S}, \mathbb{T}, \ldots \in dom(\mathtt{Q})$ for a *finite* collection $dom(\mathtt{Q})$ of successor stable ordinals, cf. Definition 4.22.1. Inferences for stability and their children for reflections are eliminated first for bigger $\mathbb{S} > \mathbb{T}$, and then smaller ones $\mathbb{T}$. Therefore we need an assignment $dom(\mathtt{Q}) \ni \mathbb{S} \mapsto \gamma^{\mathtt{Q}}_{\mathbb{S}}$ for thresholds so that $\gamma^{\mathtt{Q}}_{\mathbb{S}} < \gamma^{\mathtt{Q}}_{\mathbb{T}}$ if $\mathbb{S} > \mathbb{T}$ in Definition 4.36.4.

We define two derivability relations $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}) \vdash^{*a}_c \Gamma; \Pi^{\{\cdot\}}$ and $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_{c,d,e,\beta} \Gamma$ in subsections 4.4 and 4.5, resp. In the former relation, c is a bound of ranks of the inference rules for stability and of cut formulas as well as successor stable ordinals collected in $dom(\mathtt{Q})$. In each an operator $\mathcal{H}_\gamma$ together with a finite set Θ of ordinals and a finite family $\mathtt{Q} \subset \coprod_{\mathbb{S}} \Psi_{\mathbb{S}}$ controls ordinals occurring in derivations, where $dom(\mathtt{Q})$ is a finite set of successor stable ordinals $\mathbb{S}$ and $\mathtt{Q}(\mathbb{S})$ is a finite set of ordinals $\rho \in \Psi_{\mathbb{S}}$ for each $\mathbb{S} \in dom(\mathtt{Q})$. Furthermore in the latter relation, $\mathtt{Q}$ carries thresholds.

The rôle of the former calculus $\vdash^{*a}_c$ is twofold: first finite proof figures are embedded in the calculus, and second the cut rank c in $\vdash^{*a}_c$ is lowered to $\mathbb{I}_N$. Then the derivation is collapsed down to a $\beta < \mathbb{I}_N$ using the collapsing function $\psi_{\mathbb{I}_N}(\alpha)$.

The standard requirement $\mathsf{k}(\Gamma) \subset \mathcal{H}_\gamma[\Theta]$ in operator controlled derivations is weakened to (22) and (28) in Definitions 4.23 and 4.39. These say the following:

Assume that, e.g., $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}) \vdash_c^{*a} \Gamma; \Pi^{\{\cdot\}}$ holds, and an ordinal α occurs in a formula $A \in \Gamma$. Then α is in the set $\mathcal{H}_\gamma[\Theta(\mathtt{Q})]$, where $\Theta(\mathtt{Q}) = \Theta \cup \bigcup_{\mathbb{S}} \mathtt{Q}(\mathbb{S})$.

The weakened condition comes from a proof of Tautology lemma 4.24.2 as follows. Let $\sigma \in \Psi_{\mathbb{S}}$, $A \simeq \bigvee(A_\iota)_{\iota \in J}$ and $I = \{\iota^{[\sigma/\mathbb{S}]} : \iota \in [\sigma]J\}$, where $\iota \in [\sigma]J$ iff $\iota \in J$ and $\mathsf{k}(\iota) \subset M_\sigma$. Let $\mathrm{rk}(A) \geq \mathbb{S}$. Otherwise we don't need to collapse the formula A. Then $A^{[\sigma/\mathbb{S}]} \simeq \bigvee(B_\nu)_{\nu \in I}$ with $B_\nu \equiv A_\iota^{[\sigma/\mathbb{S}]}$ for $\nu = \iota^{[\sigma/\mathbb{S}]}$, $\mathrm{rk}(A^{[\sigma/\mathbb{S}]}) = \mathrm{rk}(A)[\sigma/\mathbb{S}]$ and $\mathsf{k}(\iota^{[\sigma/\mathbb{S}]}) = \mathsf{k}(\iota)^{[\sigma/\mathbb{S}]}$ by Proposition 4.8. A standard proof of the tautology $\neg A^{[\sigma/\mathbb{S}]}, A^{[\sigma/\mathbb{S}]}$ runs as follows:

$$\dfrac{\dfrac{\mathcal{H}_\gamma[\mathsf{k}(A^{[\sigma/\mathbb{S}]}) \cup \mathsf{k}(\iota^{[\sigma/\mathbb{S}]})] \vdash_0^{2d_\iota[\sigma/\mathbb{S}]} \neg A_\iota^{[\sigma/\mathbb{S}]}, A_\iota^{[\sigma/\mathbb{S}]}}{\mathcal{H}_\gamma[\mathsf{k}(A^{[\sigma/\mathbb{S}]}) \cup \mathsf{k}(\iota^{[\sigma/\mathbb{S}]})] \vdash_0^{2d_\iota[\sigma/\mathbb{S}]+1} \neg A_\iota^{[\sigma/\mathbb{S}]}, A^{[\sigma/\mathbb{S}]}}\,(\bigvee)}{\mathcal{H}_\gamma[\mathsf{k}(A^{[\sigma/\mathbb{S}]})] \vdash_0^{2d[\sigma/\mathbb{S}]} \neg A^{[\sigma/\mathbb{S}]}, A^{[\sigma/\mathbb{S}]}}\,(\bigwedge) \tag{15}$$

where $d = \mathrm{rk}(A)$ and $d_\iota = \mathrm{rk}(A_\iota)$ with $\iota \in [\sigma]J$, and $\mathbb{S} \in dom(\mathtt{Q})$ with $\sigma \in \mathtt{Q}(\mathbb{S})$. Here $\mathsf{k}(A^{[\sigma/\mathbb{S}]}) \not\subset M_\sigma$.

We obtain $\mathsf{k}(A^{[\sigma/\mathbb{S}]}) \subset \mathcal{H}_{\mathbb{S}}[\mathsf{k}(A) \cup \{\sigma\}]$ and $\mathsf{k}(\iota^{[\sigma/\mathbb{S}]}) \subset \mathcal{H}_{\mathbb{S}}[\mathsf{k}(\iota) \cup \{\sigma\}]$ by Proposition 4.11. For every ordinal $\alpha[\sigma/\mathbb{S}]$ occurring in $A^{[\sigma/\mathbb{S}]}$, either $\alpha \in \mathcal{H}_{\mathbb{S}}[\mathsf{k}(A)]$ or there exists a $\beta \in \mathcal{H}_{\mathbb{S}}[\mathsf{k}(A)]$ such that $\alpha = \beta[\sigma/\mathbb{S}]$. Thus we arrive at the weakened condition (22), and obtain $\mathsf{k}(A^{[\sigma/\mathbb{S}]}) \subset \mathcal{H}_{\mathbb{S}}[\mathsf{k}(A) \cup \mathtt{Q}(\mathbb{S})]$. In Definition 4.23 of the $*$-calculus, the operator $\mathcal{H}_\gamma$ controls ordinals occurring in derivations of $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}) \vdash_c^{*a} \Gamma; \Pi^{(\cdot)}$ using ordinals in Θ with the help of the family $\mathtt{Q}$. Instead of a standard one, we prove the tautology $\neg A^{[\sigma/\mathbb{S}]}, A^{[\sigma/\mathbb{S}]}$ as follows:

$$\dfrac{\dfrac{(\mathcal{H}_{\mathbb{I}_N}, \mathsf{k}(A) \cup \mathsf{k}(\iota), \mathtt{Q}) \vdash_{\mathbb{I}_N}^{2d_\iota} \neg A_\iota^{[\sigma/\mathbb{S}]}, A_\iota^{[\sigma/\mathbb{S}]}}{(\mathcal{H}_{\mathbb{I}_N}, \mathsf{k}(A) \cup \mathsf{k}(\iota), \mathtt{Q}) \vdash_{\mathbb{I}_N}^{2d_\iota+1} \neg A_\iota^{[\sigma/\mathbb{S}]}, A^{[\sigma/\mathbb{S}]}}\,(\bigvee)}{(\mathcal{H}_{\mathbb{I}_N}, \mathsf{k}(A), \mathtt{Q}) \vdash_{\mathbb{I}_N}^{2d} \neg A^{[\sigma/\mathbb{S}]}, A^{[\sigma/\mathbb{S}]}}\,(\bigwedge) \tag{16}$$

where $2d \in \mathcal{H}_0[\mathsf{k}(A)] \subset \mathcal{H}_{\mathbb{I}_N}[\mathsf{k}(A)]$ for (21). Observe that the derivation in (16) is obtained from the standard one in (15) by uncollapsing $\alpha[\sigma/\mathbb{S}] \mapsto \alpha$.

Let $B(\mathsf{L}_0)$ be a formula with $\mathrm{rk}(B(\mathsf{L}_0)) < \mathbb{S}$ and u an RS-term such that $\mathsf{k}(B(u)) \subset M_\sigma$. We have $B(u)^{[\sigma/\mathbb{S}]} \equiv B(u^{[\sigma/\mathbb{S}]})$. From the derivation of the tautology $\neg B(u)^{[\sigma/\mathbb{S}]}, B(u)^{[\sigma/\mathbb{S}]}$, the axiom $\neg \exists x\, B(x), \exists x \in L_{\mathbb{S}} B(x)$ is derived in Lemma 4.26 using an inference (stbl) for the stability of a successor stable ordinal $\mathbb{S}$ as follows.

$$\dfrac{\dfrac{\neg B(u), B(u) \qquad \dfrac{\neg B(u)^{[\sigma/\mathbb{S}]}, B(u^{[\sigma/\mathbb{S}]})}{\{\neg B(u)^{[\sigma/\mathbb{S}]}, \exists x \in L_{\mathbb{S}} B(x)\}_{\mathsf{k}(B(u)) \subset M_\sigma}}\,(\bigvee)}{\neg B(u), \exists x \in L_{\mathbb{S}} B(x)}\,(\mathrm{stbl})}{\neg \exists x\, B(x), \exists x \in L_{\mathbb{S}} B(x)}\,(\bigwedge)$$

where $u^{[\sigma/\mathbb{S}]} \in Tm(\mathbb{S})$ and σ ranges over ordinals such that $\mathsf{k}(B(u)) \subset M_\sigma$. The inference says that 'if $B(u)$, then there exists an ordinal σ such that $B(u)^{[\sigma/\mathbb{S}]}$'.

In Capping lemma 5.1 of subsection 4.5 the relation $\vdash_c^{*a}$ is embedded in another derivability relation $\vdash_{c,d,e,\beta}^a$ by putting caps ρ on formulas. Let $\sigma < \rho$.

Then $\mathsf{k}(B(u^{[\sigma/\mathbb{S}]}))\subset M_\rho$. In the above derivation each formula puts on the cap ρ except $\neg B(u)^{[\sigma/\mathbb{S}]}$. An inference (rfl) for reflection says that 'if $B(u)^{(\rho)}$, then there exists an ordinal σ such that $B(u)^{(\sigma)}$'. Therefore the above derivation turns to the following.

$$\cfrac{\neg B(u)^{(\rho)}, B(u)^{(\rho)} \quad \cfrac{\neg B(u)^{[\sigma/\mathbb{S}]}, B(u^{[\sigma/\mathbb{S}]})^{(\rho)}}{\{\neg B(u)^{[\sigma/\mathbb{S}]}, (\exists x\in L_{\mathbb{S}}B(x))^{(\rho)}\}_{\mathsf{k}(B(u))\subset M_\sigma,\sigma<\rho}}(\bigvee)}{\cfrac{\neg B(u)^{(\rho)}, (\exists x\in L_{\mathbb{S}}B(x))^{(\rho)}}{(\neg\exists x\, B(x))^{(\rho)}, (\exists x\in L_{\mathbb{S}}B(x))^{(\rho)}}(\bigwedge)}\text{(rfl)} \tag{17}$$

In doing so, it is better to distinguish $\neg B(u)^{[\sigma/\mathbb{S}]}$ from $B(u^{[\sigma/\mathbb{S}]})$ formally. The latter $B(u^{[\sigma/\mathbb{S}]})$ puts on a bigger cap ρ as $B(u^{[\sigma/\mathbb{S}]})^{(\rho)}$, while the former $\neg B(u)^{[\sigma/\mathbb{S}]}$ changes to $\neg B(u)^{(\sigma)}$ with a smaller cap $\sigma<\rho$. Let us replace the collapsed formula $\neg B(u)^{[\sigma/\mathbb{S}]}$ by an uncollapsed $\neg B(u)^{\{\sigma\}}$, and collect uncollapsed formulas to the right of the semicolon as $;\Pi^{\{\cdot\}}$. This results in the $*$-calculus $(\mathcal{H}_\gamma,\Theta;\mathtt{Q})\vdash^{*a}_{c}\Gamma;\Pi^{\{\cdot\}}$, and a derivation of $\neg A^{[\sigma/\mathbb{S}]}; A^{\{\sigma\}}$ runs as follows.

$$\cfrac{\cfrac{(\mathcal{H}_{\mathbb{I}_N},\mathsf{k}(A)\cup\mathsf{k}(\iota);\mathtt{Q})\vdash^{2d_\iota}_{\mathbb{I}_N}\neg A_\iota^{[\sigma/\mathbb{S}]};A_\iota^{\{\sigma\}}}{(\mathcal{H}_{\mathbb{I}_N},\mathsf{k}(A)\cup\mathsf{k}(\iota);\mathtt{Q})\vdash^{2d_\iota+1}_{\mathbb{I}_N}\neg A_\iota^{[\sigma/\mathbb{S}]};A^{\{\sigma\}}}(\bigvee)}{(\mathcal{H}_{\mathbb{I}_N},\mathsf{k}(A);\mathtt{Q})\vdash^{2d}_{\mathbb{I}_N}\neg A^{[\sigma/\mathbb{S}]};A^{\{\sigma\}}}(\bigwedge)$$

The derivation (17) turns to the following:

$$\cfrac{\neg B(u)^{(\rho)}, B(u)^{(\rho)};\emptyset \quad \cfrac{B(u^{[\sigma/\mathbb{S}]})^{(\rho)};\neg B(u)^{(\sigma)}}{\{(\exists x\in L_{\mathbb{S}}B(x))^{(\rho)};\neg B(u)^{(\sigma)}\}_{\mathsf{k}(B(u))\subset M_\sigma,\sigma<\rho}}(\bigvee)}{\cfrac{\neg B(u)^{(\rho)}, (\exists x\in L_{\mathbb{S}}B(x))^{(\rho)};\emptyset}{(\neg\exists x\, B(x))^{(\rho)}, (\exists x\in L_{\mathbb{S}}B(x))^{(\rho)};\emptyset}(\bigwedge)}\text{(rfl)} \tag{18}$$

$\mathsf{k}(A)\subset M_\rho$ should be satisfied for each capped formula $A^{(\rho)}$, and this would follow from $\mathsf{k}(A)\subset\mathcal{H}_\gamma[\Theta(\mathtt{Q})]$ and $\Theta(\mathtt{Q})\subset M_\rho$. However $\rho\not\in M_\rho$ for $\rho\in\mathtt{Q}(\mathbb{S})$. Looking back the derivation (16) and $\mathsf{k}(A^{[\sigma/\mathbb{S}]})\subset\mathcal{H}_{\mathbb{S}}[\mathsf{k}(A)\cup\mathtt{Q}(\mathbb{S})]$, we see that the extra part $\bigcup_{\mathbb{S}}\mathtt{Q}(\mathbb{S})$ in $\Theta(\mathtt{Q})$ is needed to capture the ordinals $\sigma<\rho$ in the derivation (18). Thus we arrive at a classification of ordinals in the set $\bigcup_{\mathbb{S}}\mathtt{Q}(\mathbb{S})$: The *temporary part* denoted by $\partial\mathtt{Q}$ and the *fixed part* by $\mathtt{Q}^\circ$ in Definition 4.36.2. Ordinals ρ in $\partial\mathtt{Q}$ are caps on which formulas $B(u)$ put, while the formulas $\neg B(u)^{(\sigma)}$ in derivations (18) puts on caps σ in $\mathtt{Q}^\circ$, cf. Capping lemma 5.1. Ordinals in $\bigcup_{\mathbb{S}}\mathtt{Q}(\mathbb{S})$ might occur actually in derivations only when these are in $\mathtt{Q}^\circ$. See the conditions (27) and (28) in Definition 4.39.

(27) says that $\Theta(\mathtt{Q}^\circ)=\Theta\cup\bigcup_{\mathbb{S}}\mathtt{Q}^\circ(\mathbb{S})\subset M_{\partial\mathtt{Q}}=\bigcap_{\rho\in\partial\mathtt{Q}}M_\rho$, while $\mathsf{k}(\Gamma)\subset\mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)]$ is imposed in (28). One of the reasons for the constraint (27) is to ensure the condition (12) in Definition 3.31.6, which says that every ordinal occurring in the finite function $m(\rho)$ has to be in M_ρ. A cap $\rho\in\partial\mathtt{Q}$ of the capped formula $A^{(\rho)}$ is replaced by another cap κ to $A^{(\kappa)}$ in the main lemma of Recapping 5.4, and the rank of the reflected formulas $B(u)$ in inferences (rfl)

is lowered. In doing so, a new ordinal $\kappa = \psi^h_\rho(\alpha)$ 'enters' in derivation. Here a finite function $h = m(\kappa)$ is constructed from the function $m(\rho)$ and some ordinals b, d, a, where ordinals b and d are ranks of formulas in derivations, and a the ordinal height of the derivation. Two constraints yield $\{b, d, a\} \subset M_\rho$, and the ordinal κ is chosen so that a specified finite subset of M_ρ is a subset of M_κ, cf. Definition 4.38.

The ordinals in the temporary part $\mathtt{Q}^\circ$ are finally removed from $\Theta(\mathtt{Q}^\circ)$ in Lemma 5.11 as follows. For this we need another constraint (29), which says that $\mathtt{Q}(\mathbb{S}) \subset \mathcal{H}_{\gamma^{\mathtt{Q}}_{\mathbb{S}}+\mathbb{I}_N}[\Theta(\mathtt{Q}^\circ \restriction \mathbb{S})]$, where $\mathtt{Q}^\circ \restriction \mathbb{S}$ denotes the restriction of $\mathtt{Q}^\circ$ to $\mathbb{S}$.

In Lemma 5.7 we show that the largest successor stable ordinal $\mathbb{S}$ in $dom(\mathtt{Q}) \cap \mathbb{S}^\dagger$ as well as caps $\rho \in \mathtt{Q}(\mathbb{S})$ can be removed from derivations in the following way: Let $\mathrm{rk}(\Xi) < \mathbb{S}$ and each cap ρ in Ξ is in $\mathtt{Q} \restriction \mathbb{S}$. If $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_{\mathbb{S}^\dagger,\mathbb{S}^\dagger,\mathbb{S}^\dagger,\beta} \Xi$, then $(\mathcal{H}_{\gamma_1}, \Theta, \mathtt{Q} \restriction \mathbb{S}) \vdash^{\tilde{a}}_{\mathbb{S},\mathbb{S},\mathbb{S},\beta} \Xi$ holds for an ordinal $\tilde{a}$ and $\gamma_1 = \gamma^{\mathtt{Q}}_{\mathbb{S}} + \mathbb{I}_N$ if $\mathbb{S} \in dom(\mathtt{Q})$. This is done as follows. First Recapping 5.4 yields $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{\mathbb{S}+\omega a}_{\mathbb{S}^\dagger,\mathbb{S},\mathbb{S}^\dagger,\beta} \Xi$, and we obtain a derivation in which the rank of each reflected formula A in inferences (rfl) is less than $\mathbb{S}$. Then we obtain $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{\tilde{a}}_{\mathbb{S},\mathbb{S},\mathbb{S}^\dagger,\beta} \Xi$ for $\tilde{a} = \varphi_{\mathbb{S}^\dagger}(\mathbb{S}+\omega a)$ by Cut-elimination 4.44. Thus we obtain a derivation in which the rank of every formula is less than $\mathbb{S}$. Then the formula $A^{(\rho)}$ takes off the cap $\rho \in \mathtt{Q}(\mathbb{S})$, and the set $\mathtt{Q}(\mathbb{S})$ no longer helps operators $\mathcal{H}_\gamma$. Now we have $\mathtt{Q}(\mathbb{S}) \subset \mathcal{H}_{\gamma_1}[\Theta(\mathtt{Q}^\circ \restriction \mathbb{S})]$ for $\gamma_1 = \gamma^{\mathtt{Q}}_{\mathbb{S}} + \mathbb{I}_N$ by (29). By lifting the threshold $\gamma_0 \le \gamma^{\mathtt{Q}}_{\mathbb{S}}$ to a larger one γ_1, we obtain $\mathtt{Q}(\mathbb{S}) \subset \mathcal{H}_{\gamma_1}[\Theta(\mathtt{Q}^\circ \restriction \mathbb{S})]$ and $\mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ \restriction \mathbb{S}^\dagger)] \subset \mathcal{H}_{\gamma_1}[\Theta(\mathtt{Q}^\circ \restriction \mathbb{S})]$. This explains the constraint (29).

The reason of the introduction of trail and the set $\mathcal{B}_0(\alpha)$ of ordinals α in Definition 4.14 are two fold. For a stable ordinal $\mathbb{S}$ and its next stable ordinal $\mathbb{S}^\dagger$, we see that if $\mathcal{B}_0(\mathbb{S}^\dagger) \subset \mathcal{H}_\gamma[\Theta]$, then $\mathbb{S} \in \mathcal{B}_0(\mathbb{S}) \subset \mathcal{H}_\gamma[\Theta]$ since the set $\mathcal{H}_\gamma[\Theta]$ is closed under $\mathbb{T} \mapsto \mathbb{T}^\dagger$. The fact is used in Lemma 5.7. On the other side, in proving the axiom (2) in Lemma 4.26 we need the fact that if both of a limit i-stable ordinal $\mathbb{T}$ and an ordinal $\alpha < \mathbb{T}$ are 'captured' in $\mathcal{H}_\gamma[\Theta]$, then so is a successor i-stable ordinal $\mathbb{S}$ such that $\alpha < \mathbb{S} < \mathbb{T}$. Or in other words, such an $\mathbb{S}$ should be constructed from data included in ordinals $\mathbb{T}$ and α. The data we need are trails, cf. Proposition 4.16. Then the finite sets Θ should satisfy $\mathcal{B}(\Theta) \subset \Theta$, cf. Propositions 4.15.2, 4.17.3, 4.15.6 and 4.15.9. As we said above, the addition of $E_{\mathbb{S}}(\alpha)$ to $\mathcal{B}(\alpha)$ is to construct the collapsed ordinals $\alpha[\rho/\mathbb{S}]$ from $E_{\mathbb{S}}(\alpha)$ and ρ.

Now details follow.

4.3 Sets M_ρ, trails and stepping-down

In this subsection some facts on sets M_ρ, ordinal terms and finite functions are established. These facts are needed in this and next secrtion 5.

Definition 4.10 For $\alpha \in OT(\mathbb{I}_N)$ and $\mathbb{S} \in SSt$, a finite set $E_{\mathbb{S}}(\alpha) \subset \mathbb{S}$ of subterms of α is defined recursively as follows.

1. $E_{\mathbb{S}}(\alpha) = E_{\mathbb{S}}(SC(\alpha)) := \bigcup\{E_{\mathbb{S}}(\beta) : \beta \in SC(\alpha)\}$ if $\alpha \notin SC(\alpha)$.

In what follows let $SC(\alpha) = \{\alpha\}$.

2. $E_{\mathbb{S}}(\alpha) = \{\alpha\}$ if $SC(\alpha) \ni \alpha < \mathbb{S}$.

   In what follows let $SC(\alpha) \ni \alpha \geq \mathbb{S}$.

3. $E_{\mathbb{S}}(\mathbb{S}) = \emptyset$.

4. $E_{\mathbb{S}}(\alpha) = E_{\mathbb{S}}(\{\sigma, a\} \cup SC(f)) = \bigcup\{E_{\mathbb{S}}(\beta) : \beta \in \{\sigma, a\} \cup SC(f)\}$ if $\alpha = \psi_{\sigma}^{f}(a)$.

5. $E_{\mathbb{S}}(\alpha) = E_{\mathbb{S}}(\mathbb{T})$ if $\alpha = \mathbb{T}^{\dagger i}$.

6. $E_{\mathbb{S}}(\alpha) = E_{\mathbb{S}}(\tau)$ if $\alpha = \mathbb{I}_N[\tau]$.

7. $E_{\mathbb{S}}(\alpha) = E_{\mathbb{S}}(\{\tau, \mathbb{T}\})$ if $\alpha = \mathbb{T}^{\dagger\vec{i}}[\tau/\mathbb{T}]$.

Let $E(\alpha) = \bigcup\{E_{\mathbb{S}}(\alpha) : \mathbb{S} \in SSt\}$.

Proposition 4.11 *1. $SC(\alpha) \subset E(\alpha) = E(E(\alpha))$, where $E(X) = \bigcup\{E(\beta) : \beta \in X\}$ for stes X of ordinals.*

2. Let $\alpha \in M_\rho$ with $\rho \in \Psi_{\mathbb{S}}$. Then $\alpha[\rho/\mathbb{S}] \in \mathcal{H}_{\mathbb{S}}(E_{\mathbb{S}}(\alpha) \cup \{\rho\})$ and $E(\alpha[\rho/\mathbb{S}]) \subset E(\alpha) \cup E(\rho) \cup SC(\alpha[\rho/\mathbb{S}])$.

3. $\forall\beta \in E(\alpha)\exists\gamma \in SC(\alpha)(\beta \leq \gamma)$.

Proof. 4.11.1. Let $\beta \in E_{\mathbb{S}}(\alpha)$. By induction on $\ell\alpha$ we show $E_{\mathbb{T}}(\beta) \subset E_{\mathbb{T}}(\alpha) \cup E_{\mathbb{S}}(\alpha)$. By IH we may assume $SC(\alpha) \ni \alpha < \mathbb{S}$, $E_{\mathbb{S}}(\alpha) = \{\alpha\}$ and $\beta = \alpha$. If $\alpha < \mathbb{T}$, then $E_{\mathbb{T}}(\alpha) = \{\alpha\} \subset E_{\mathbb{S}}(\alpha)$. Let $\mathbb{T} \leq \alpha$. Then $E_{\mathbb{T}}(\beta) = E_{\mathbb{T}}(\alpha)$. Hence $E(E(\alpha)) \subset E(\alpha)$.

Conversely let $\beta \in SC(\alpha)$ and $\beta < \mathbb{S} \in SSt$. Then $E_{\mathbb{S}}(\beta) = \{\beta\}$, and $SC(\alpha) \subset E(\alpha)$. Hence $E(\alpha) = E(SC(\alpha)) \subset E(E(\alpha))$.

4.11.2. By induction on $\ell\alpha$. $\alpha[\rho/\mathbb{S}] \in \mathcal{H}_{\mathbb{S}}(E_{\mathbb{S}}(\alpha) \cup \{\rho\})$ follows from the facts $M_\rho \cap \mathbb{S} = \rho$ and $\alpha[\rho/\mathbb{S}] < \mathbb{S}$. For each $\mathbb{T}$ we show $E_{\mathbb{T}}(\alpha[\rho/\mathbb{S}]) \subset E_{\mathbb{T}}(\alpha) \cup E_{\mathbb{T}}(\rho) \cup SC(\alpha[\rho/\mathbb{S}])$. If $\mathbb{T} \geq \mathbb{S}$, then $E_{\mathbb{T}}(\alpha[\rho/\mathbb{S}]) \subset SC(\alpha[\rho/\mathbb{S}])$. Let $\mathbb{T} < \mathbb{S} \leq \alpha$. Then $\mathbb{T} < \alpha[\rho/\mathbb{S}]$. $E_{\mathbb{T}}(\alpha[\rho/\mathbb{S}]) \subset E_{\mathbb{T}}(\alpha) \cup E_{\mathbb{T}}(\rho)$ is seen by induction on $\ell\alpha$.

4.11.3. By induction on $\ell\alpha$. By IH we may assume that $\alpha \in SC(\alpha)$. Let $\beta \in E_{\mathbb{T}}(\alpha)$. If $\alpha < \mathbb{T}$, then $\beta = \alpha$. Let $\mathbb{T} \leq \alpha$. Then $\beta < \mathbb{T} \leq \alpha$. □

Proposition 4.12 *Let α be a strongly critical number such that $\Omega < \alpha < \mathbb{I}_N$. There exists a unique sequence $(\alpha_n)_{n \leq m}$ such that $\alpha_0 = \psi_{\mathbb{I}_N}(a)$ for an a, $\alpha_m = \alpha$ and each α_{n+1} is one of the forms $\psi_{\alpha_n}^{f}(b)$, $\alpha_n^{\dagger\vec{i}}$, $\mathbb{I}_N[\alpha_n]$, $\mathbb{S}^{\dagger\vec{i}}[\alpha_n/\mathbb{S}]$ for some $f, b, \vec{i}$ and $\mathbb{S}$. The sequence $(\alpha_n)_{n \leq m}$ is said to be the* trail *to α, and denoted by* trail(α).

For a term α_n in the trail to α, if $\alpha_n < \alpha$, then $\alpha_n < \alpha_k$ for $n < k \leq m$, and $E_{\mathbb{T}}(\alpha_n) \subset E_{\mathbb{T}}(\alpha)$ for every $SSt \ni \mathbb{T} \leq \alpha_n$.

Furthermore $\alpha_0 \leq \alpha$, and $E_{\mathbb{T}}(\alpha_0) \subset E_{\mathbb{T}}(\alpha)$ holds for every $SSt \ni \mathbb{T} \leq \alpha_0$.

Proof. This is seen by inspection of Definitions 3.31 and 3.33. If $\alpha_n > \alpha_{n+1}$, then we would have $\alpha_{n+1} \prec \alpha_n$ and $\alpha < \alpha_n$ by Definition 3.35. □

Proposition 4.13 *Let $\rho \in \Psi_{\mathbb{S}}$ with a successor stable ordinal $\mathbb{S}$. Assume $\mathbb{S} < \psi_{\mathbb{I}_N}(\gamma)$, $\mathbb{I}_N \leq \gamma \leq \mathtt{p}_0(\rho)$, $\alpha \in \mathcal{H}_\gamma(\psi_{\mathbb{I}_N}(\gamma))$ and $E_{\mathbb{S}}(\alpha) \subset \rho \in \Psi_{\mathbb{S}}$. Then $\alpha \in M_\rho = \mathcal{H}_{\mathtt{p}_0(\rho)}(\rho)$.*

Proof. By induction on $\ell\alpha$. By IH we may assume that $\mathbb{S} < \alpha < \mathbb{I}_N$. Let $\alpha = \psi_{\mathbb{I}_N}(a)$. Then $a \in \mathcal{H}_\gamma(\psi_{\mathbb{I}_N}(\gamma)) \cap \gamma$ and $E_{\mathbb{S}}(\alpha) = E_{\mathbb{S}}(a)$. IH yields $a \in M_\rho$, and $\alpha \in M_\rho$ by $a < \gamma \leq \mathtt{p}_0(\rho)$. □

Definition 4.14 For $\alpha \in OT(\mathbb{I}_N)$, a finite set $\mathcal{B}_0(\alpha)$ is defined recursively as follows.

1. $\mathcal{B}_0(\alpha) = \mathcal{B}_0(SC(\alpha)) := \bigcup\{\mathcal{B}_0(\beta) : \beta \in SC(\alpha)\}$ if $\alpha \not\in SC(\alpha)$.
2. $\mathcal{B}_0(\alpha) = \{\alpha\}$ if $SC(\alpha) \ni \alpha < \Omega$.
3. $\mathcal{B}_0(\alpha) = \{\alpha\} \cup (\mathrm{trail}(\alpha) \cap St \cap \alpha)$ if $\Omega < \alpha \in SC(\alpha)$.

Let $\mathcal{B}_0(X) = \bigcup\{\mathcal{B}_0(\beta) : \beta \in X\}$ for sets X of ordinals, and

$$\mathcal{B}(\alpha) = \mathcal{B}_0(E(\alpha)) \tag{19}$$

Proposition 4.15

1. *$SC(\alpha) \subset \mathcal{B}_0(\alpha)$ and $E(\alpha) \subset \mathcal{B}(\alpha)$.*
2. *$SC(\alpha) \subset \mathcal{B}(\alpha)$ and $\mathcal{B}(\alpha) = \mathcal{B}(SC(\alpha))$.*
3. *$\mathcal{B}_0(\mathcal{B}_0(\alpha)) \subset \mathcal{B}_0(\alpha)$.*
4. *$\forall\beta \in \mathcal{B}_0(\alpha)\exists\gamma \in SC(\alpha)(\beta \leq \gamma)$ and $\forall\beta \in \mathcal{B}(\alpha)\exists\gamma \in SC(\alpha)(\beta \leq \gamma)$.*
5. *$E(\mathcal{B}_0(\alpha)) \subset E(\alpha) \cup \mathcal{B}_0(\alpha)$.*
6. *$\mathcal{B}(\mathcal{B}(\alpha)) = \mathcal{B}(\alpha)$.*
7. *Let $\rho \in \Psi_{\mathbb{S}}$ with $\mathbb{S} \in SSt$. Then $\mathcal{B}(\alpha[\rho/\mathbb{S}]) \subset \mathcal{B}(\{\alpha, \rho, \mathbb{S}\}) \cup SC(\alpha[\rho/\mathbb{S}])$.*
8. *For $\alpha = \psi^f_\sigma(a)$, $\mathcal{B}(\alpha) \subset \{\alpha\} \cup \mathcal{B}(\{\sigma, a\} \cup SC(f))$.*
9. *Let $\mathcal{B}(\Theta) \subset \Theta$ for a finite set Θ of ordinals, and $\alpha \in \mathcal{H}_\gamma[\Theta]$ with $\gamma \geq \mathbb{I}_N$. Then $\mathcal{B}(\alpha) \subset \mathcal{H}_\gamma[\Theta]$.*

Proof. 4.15.1. We have $SC(\alpha) \subset \mathcal{B}_0(\alpha)$. Hence $E(\alpha) \subset \mathcal{B}_0(E(\alpha)) = \mathcal{B}(\alpha)$.
4.15.2. By Proposition 4.11.1 we have $SC(\alpha) \subset E(\alpha)$, and hence $SC(\alpha) \subset \mathcal{B}(\alpha)$ by Proposition 4.15.1.
4.15.3. This is seen by induction on $\ell\alpha$ using the fact that $\mathrm{trail}(\mathbb{S}) \cap \mathbb{S} \subset \mathrm{trail}(\alpha)$ for $\mathbb{S} \in \mathrm{trail}(\alpha) \cap St \cap \alpha$.
4.15.4. By induction on $\ell\alpha$ we show $\forall\beta \in \mathcal{B}_0(\alpha)\exists\gamma \in SC(\alpha)(\beta \leq \gamma)$. $\forall\beta \in \mathcal{B}(\alpha)\exists\gamma \in SC(\alpha)(\beta \leq \gamma)$ follows from this and Proposition 4.11.3. By IH we may assume that $\Omega < \alpha \in SC(\alpha)$. For $\beta \in \mathcal{B}_0(\alpha)$ we see $\beta \leq \alpha$.
4.15.5. By induction on $\ell\alpha$. By IH we may assume that $\Omega < \alpha \in SC(\alpha)$. For $\beta \in \mathcal{B}_0(\alpha)$, we show $E(\beta) \subset E(\alpha) \cup \mathcal{B}_0(\alpha)$. Let $\beta \in \mathrm{trail}(\alpha) \cap St \cap \alpha$. Then we obtain $E(\beta) \subset E(\alpha)$ by Proposition 4.12.

4.15.6. By Propositions 4.11.1 and 4.15.1 we obtain $E(\alpha) = E(E(\alpha)) \subset E(\mathcal{B}(\alpha))$, and $\mathcal{B}(\alpha) = \mathcal{B}_0(E(\alpha)) \subset \mathcal{B}_0(E(\mathcal{B}(\alpha))) = \mathcal{B}(\mathcal{B}(\alpha))$. Conversely we obtain $E(\mathcal{B}_0(\alpha)) \subset E(\alpha) \cup \mathcal{B}_0(\alpha)$ by Proposition 4.15.5. Hence $E(\mathcal{B}_0(E(\alpha))) \subset E(\alpha) \cup \mathcal{B}_0(E(\alpha))$ by Proposition 4.11.1. Therefore $\mathcal{B}(\mathcal{B}(\alpha)) = \mathcal{B}_0(E(\mathcal{B}_0(E(\alpha)))) \subset \mathcal{B}_0(E(\alpha)) \cup \mathcal{B}_0(\mathcal{B}_0(E(\alpha))) \subset \mathcal{B}(\alpha)$ by Proposition 4.15.3.
4.15.7. By Proposition 4.11.2 we have $E(\alpha[\rho/\mathbb{S}]) \subset E(\alpha) \cup E(\rho) \cup SC(\alpha[\rho/\mathbb{S}])$. On the other side, we see $\mathcal{B}_0(\alpha[\rho/\mathbb{S}]) \subset \mathcal{B}_0(\alpha) \cup \mathcal{B}_0(\mathbb{S}) \cup SC(\alpha[\rho/\mathbb{S}])$ by induction on $\ell\alpha$. When $\mathbb{S} \leq \alpha \in SC(\alpha)$, we obtain $\mathrm{trail}(\alpha[\rho/\mathbb{S}]) \cap St \cap (\alpha[\rho/\mathbb{S}]) \subset \mathrm{trail}(\mathbb{S}) \cap \mathbb{S}$.
4.15.8. We have $E(\alpha) \subset \{\alpha\} \cup E(\{\sigma, a\} \cup SC(f))$, and $\mathcal{B}(\alpha) \subset \mathcal{B}_0(\alpha) \cup \mathcal{B}(\{\sigma, a\} \cup SC(f))$. On the other hand we have $\mathcal{B}_0(\alpha) \subset \{\alpha\} \cup \mathcal{B}_0(\sigma)$. Hence $\mathcal{B}(\alpha) \subset \{\alpha\} \cup \mathcal{B}(\{\sigma, a\} \cup SC(f))$.
4.15.9. By induction on $\ell\alpha$. □

Proposition 4.16 *Let $\mathbb{T} \in SSt_{i+1}$ be a successor $(i+1)$-stable ordinal, and $\alpha < \mathbb{T}$ an ordinal. Then there exists a successor i-stable ordinal $\alpha < \mathbb{S} < \mathbb{T}$ such that $\mathcal{B}(\mathbb{S}) \subset \mathcal{H}_0(\mathcal{B}(\alpha, \mathbb{T}))$ for $\mathcal{B}(\alpha, \mathbb{T}) = \mathcal{B}(\alpha) \cup \mathcal{B}(\mathbb{T})$.*

Proof. By induction on the lengths $\ell\alpha$ of ordinal terms α. By IH we may assume that $\Omega < \alpha < \mathbb{I}_N$ and $\alpha \in SC(\alpha)$. Let $\mathbb{T} = \mathbb{U}^{\dagger(i+1)}$ with $\mathbb{U} \in St \cup \{\Omega\}$. Then $\mathrm{trail}(\mathbb{U}) \subset \mathrm{trail}(\mathbb{T})$ and $\mathcal{B}(\mathbb{U}) \subset \mathcal{H}_0(\mathcal{B}(\mathbb{T}))$.
Case 1. There exists a $k > 0$ such that $\alpha < \mathbb{U}^{\dagger i^{(k)}}$, where $\mathbb{U}^{\dagger i^{(0)}} = \mathbb{U}$ and $\mathbb{U}^{\dagger i^{(k+1)}} = (\mathbb{U}^{\dagger i^{(k)}})^{\dagger i}$: Pick a $k > 0$ such that $\alpha < \mathbb{S} = \mathbb{U}^{\dagger i^{(k)}}$. We obtain $\mathcal{B}(\mathbb{S}) \subset \mathcal{H}_0(\mathcal{B}(\mathbb{T}))$, and $\mathbb{S} < \mathbb{T}$ is seen from $\mathbb{T} \in LSt_i$.
Case 2. Otherwise: Then we see from Definition 3.35 that there exists a $\rho \in \mathcal{B}_0(\alpha)$ such that $\rho \prec \mathbb{T}$. We obtain $\rho \leq \alpha$ and $\mathrm{trail}(\rho) \subset \mathrm{trail}(\alpha)$. Pick a $k > 0$ such that $\alpha < \mathbb{S} = \rho^{\dagger i^{(k)}} < \mathbb{T}$. We obtain $\mathrm{trail}(\mathbb{S}) = \{\mathbb{S}\} \cup \mathrm{trail}(\rho)$ and $\mathcal{B}(\mathbb{S}) \subset \mathcal{H}_0(\mathcal{B}(\alpha))$. □

Proposition 4.17 *Let $\alpha \in OT(\mathbb{I}_N)$ and $\rho \in \Psi_{\mathbb{S}}$ with $\mathbb{S} \in SSt$.*

1. *If $\alpha \in M_\rho$, then $E(\alpha) \subset M_\rho$.*
2. *If $\alpha \in M_\rho$, then $\mathcal{B}_0(\alpha) \subset M_\rho$.*
3. *If $\alpha \in M_\rho$, then $\mathcal{B}(\alpha) \subset M_\rho$.*

Proof. Proposition 4.17.3 follows from Propositions 4.17.1 and 4.17.2, each of which is shown by induction on $\ell\alpha$. By IH we may assume that $\Omega < \alpha < \mathbb{I}_N$ and $\alpha \in SC(\alpha)$. Let $M_\rho = \mathcal{H}_b(\rho)$ with $b = \mathtt{p}_0(\rho)$.
4.17.1. Let $\beta \in E_{\mathbb{T}}(\alpha)$. If $\mathbb{T} < \mathbb{S}$, then $\beta < \mathbb{T} < \rho$. If $\alpha < \mathbb{T}$, then $\beta = \alpha$. We may assume that $\rho < \mathbb{S} \leq \mathbb{T} \leq \alpha$. For example let $\alpha = \psi_\sigma^f(a) \in M_\rho = \mathcal{H}_b(\rho)$. Then $E_{\mathbb{T}}(\alpha) = E_{\mathbb{T}}(\{\sigma, a\} \cup SC(f))$ and $\{\sigma, a\} \cup SC(f) \subset M_\rho$. IH yields $E_{\mathbb{T}}(\alpha) \subset M_\rho$. Other cases are seen similarly.
4.17.2. Let $(\alpha_n)_{n \leq m}$ be the trail to α. First let $\alpha_n \in \mathrm{trail}(\alpha) \cap \alpha$. If $\alpha < \mathbb{S}$, then $\alpha_n < \alpha < \rho$ by Proposition 4.15.4. Let $\rho < \mathbb{S} \leq \alpha_m = \alpha \in M_\rho$. Let $k = \min\{k : n \leq k \leq m, \alpha_k \geq \rho\}$. If $n < k$, then $\alpha_n < \rho$ and $\alpha \in M_\rho$.

Otherwise we obtain $\rho \leq \alpha_n < \alpha_k$ for every k with $n < k \leq m$ by Proposition 4.12. Hence $\alpha_n \in M_\rho = \mathcal{H}_b(\rho)$. □

The following Definition 4.18 is needed in subsection 5.2.

Definition 4.18 Let $s(f) = \max(\{0\} \cup \mathrm{supp}(f))$ for finite function f, and $s(\rho) = s(m(\rho))$.

Let $\Lambda < \mathbb{I}_N$ be a strongly critical number, which is a base for $\tilde{\theta}$-function. Let $f : \Lambda \to \varphi_\Lambda(0)$ be a non-empty and irreducible finite function. Then f is said to be *special* if there exists an ordinal α such that $f(s(f)) = \alpha + \Lambda$. For a special finite function f, f' denotes a finite function such that $\mathrm{supp}(f') = \mathrm{supp}(f)$, $f'(c) = f(c)$ for $c \neq s(f)$, and $f'(s(f)) = \alpha$ with $f(s(f)) = \alpha + \Lambda$.

A special function $h^b(g; a)$ is defined from ordinals a, b and a finite function g as in [5].

Definition 4.19 Let $\Lambda < \mathbb{I}_N$ be a strongly critical number, which is a base for $\tilde{\theta}$-function. Let f, g be special finite functions.

1. For ordinals $a \leq \Lambda$, $b < s(g)$, let us define a special finite function $h = h^b(g; a)$ as follows. $s(h) = b$, and $h_b = g_b$. To define $h(b)$, let $\{b = b_0 < b_1 < \cdots < b_n = s(g)\} = \{b, s(g)\} \cup ((b, s(g)) \cap \mathrm{supp}(g))$. Define recursively ordinals α_i by $\alpha_n = \alpha + a$ with $g(s(g)) = \alpha + \Lambda$. $\alpha_i = g(b_i) + \tilde{\theta}_{c_i}(\alpha_{i+1}; \Lambda)$ for $c_i = b_{i+1} - b_i$. Finally let $h(b) = \alpha_0 + \Lambda$.

2. $f_b * g^b$ denotes a special function h such that $\mathrm{supp}(h) = \mathrm{supp}(f_b) \cup \mathrm{supp}(g^b)$, $h'(c) = f'(c)$ for $c < b$, and $h'(c) = g'(c)$ for $c \geq b$.

The following Proposition 4.20 is seen as in [5].

Proposition 4.20 *Let k be a finite function, f, g special finite functions such that $f_d = g_d$ and $f <^d g'(d)$ for a $d \in \mathrm{supp}(g)$, and $\rho \in \Psi_{\mathbb{S}}$ with $g = m(\rho)$. $\tilde{\theta}$ denotes the function $\tilde{\theta}_b(\xi; \Lambda)$ in Definition 3.1 with base Λ.*

1. *For $b < d$ and $a < \Lambda$, $f_b = (h^b(g; a))_b$ and $f <^b (h^b(g; a))'(b)$.*

2. *Let $b \leq e < d$, $a_0 < a < \Lambda$, and $h = (h^e(g; a_0)) * f^{e+1}$. Then $h_b = (h^b(g; a))_b$ and $h <^b (h^b(g; a))'(b)$.*

Proof. 4.20.1. Let $h = h^b(g; a)$. We have $h_b = g_b = f_b$. Let $b + x \in \mathrm{supp}(f) \cap d \subset \mathrm{supp}(g) \cap d$. Then $f(b + x) = g(b + x) < \tilde{\theta}_{-x}(h'(b))$ and $g'(d) < \tilde{\theta}_{-(d-b)}(h'(b))$. Proposition 3.6 yields the proposition.

4.20.2. Note that $h = (h^e(g; a_0))' * f^{e+1}$. We have $h_b = g_b = (h^b(g; a))_b$. For $b+x \in \mathrm{supp}(g) \cap e$, $h(b+x) = (h^e(g; a_0))(b+x) = g(b+x) < \tilde{\theta}_{-x}((h^b(g; a))'(b))$, and $h(e) = (h^e(g; a_0))(e) < \tilde{\theta}_{-(e-b)}((h^b(g; a))'(b))$ by $a_0 < a$. For $e < e + x \in \mathrm{supp}(f) \cap d$, we obtain $h(e+x) = f(e+x) = g(e+x) < \tilde{\theta}_{-(e+x-b)}((h^b(g; a))'(b))$. For $d+x \in \mathrm{supp}(f)$, we obtain $h(d+x) = f(d+x) < \tilde{\theta}_{-x}(g'(d)) \leq \tilde{\theta}_{-(d+x-b)}((h^b(g; a))'(b))$. Therefore $h <^b (h^b(g; a))'(b)$. □

4.4 Operator controlled $*$-derivations

Let $\mathcal{H}_\gamma[\Theta] := \mathcal{H}_\gamma(\Theta)$ and $\mathcal{H}_\gamma := \mathcal{H}_\gamma(0)$. By a *successor stable ordinal* we mean ordinals in $SSt = \bigcup_{0<i\leq N} SSt_i$, and $\mathbb{S}^\dagger := \mathbb{S}^{\dagger 1}$. In this section and the next section 5 let us fix an ordinal $\mathbb{I}_N \leq \gamma_0 \in \mathcal{H}_0$. The ordinal γ_0 depends on a given finite proof figure in $S_{\mathbb{I}_N}$, and is specified in the end of section 5.

Definition 4.21 By an *uncollapsed formula* we mean a pair $\{A, \rho\}$ of RS-sentence A and an ordinal $\rho \prec \mathbb{S}$ for a successor stable ordinal $\mathbb{S}$ such that $\mathsf{k}(A) \subset M_\rho$. Such a pair is denoted by $A^{\{\rho\}}$. When we write $\Gamma^{\{\rho\}}$, we tacitly assume that $\mathsf{k}(\Gamma) \subset M_\rho$.

$\mathcal{B}(\alpha)$ denotes the set defined in (19) of Definition 4.14. For ordinals α, we see $\mathcal{B}(\alpha) \subset M_\rho$ iff $\alpha \in M_\rho$ from Propositions 4.15.2 and 4.17.3. Hence $\mathcal{B}(\mathsf{k}(\iota)) \subset M_\rho$ iff $\mathsf{k}(\iota) \subset M_\rho$ for RS-terms and RS-formulas ι. On the other hand we have $\max(\{0\} \cup \mathcal{B}(\alpha)) \leq \max(\{0\} \cup SC(\alpha))$ by Proposition 4.15.4.

Definition 4.22

1. A *finite family* for an ordinal γ_0 is a finite function $\mathtt{Q} \subset \coprod_{\mathbb{S}} \Psi_{\mathbb{S}}$ such that its domain $dom(\mathtt{Q})$ is a finite set of successor stable ordinals and $\mathtt{Q}(\mathbb{S})$ is a finite set of ordinals κ in $\Psi_{\mathbb{S}}$ for each $\mathbb{S} \in dom(\mathtt{Q})$ with a special finite function $m(\kappa)$, and $\gamma_0 \leq \mathsf{p}_0(\kappa)$, where $M_{\mathtt{Q}} = \bigcap_{\mathbb{S}\in dom(\mathtt{Q})} M_{\mathtt{Q}(\mathbb{S})}$ with $M_{\mathtt{Q}(\mathbb{S})} = \bigcap_{\sigma\in\mathtt{Q}(\mathbb{S})} M_\sigma$ and $M_\emptyset = OT(\mathbb{I}_N)$. Let $\mathtt{Q}(\mathbb{T}) = \emptyset$ for $\mathbb{T} \not\in dom(\mathtt{Q})$.

2. Let Θ be a finite set of ordinals and $\mathtt{Q}$ a finite family. Let

$$\Theta(\mathtt{Q}) := \Theta \cup \mathcal{B}\left(\bigcup\{\mathtt{Q}(\mathbb{S}) : \mathbb{S} \in dom(\mathtt{Q})\}\right) \tag{20}$$

We define a derivability relation $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_c \Gamma; \Pi^{\{\cdot\}}$ where c is a bound of ranks of the inference rules $(i\text{-stbl}(\mathbb{S}))$, one of ranks of cut formulas, and of $dom(\mathtt{Q}_\Pi)$. The relation depends on an ordinal γ_0, and should be written as $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_{c,\gamma_0} \Gamma; \Pi^{\{\cdot\}}$. However the ordinal γ_0 will be fixed. So let us omit it. Note that if $\gamma_0 \leq \mathsf{p}_0(\sigma)$ for $\sigma \prec \mathbb{S}$, then $\mathcal{H}_{\gamma_0}(\sigma) \cap \mathbb{S} \subset \sigma$ by Proposition 3.42.

Definition 4.23 Let Θ be a finite set of ordinals such that $\mathcal{B}(\Theta) \subset \Theta$, a, c ordinals, and $\mathtt{Q}_\Pi$ a finite family for γ_0 such that $dom(\mathtt{Q}_\Pi) \subset c$. Let $\Pi = \bigcup_{(\mathbb{S},\sigma)\in\mathtt{Q}_\Pi} \Pi_\sigma$ be a set of formulas such that $\Pi \subset \Delta_0(\mathcal{L}_{N+1})$, $\mathsf{k}(\Pi_\sigma) \subset M_\sigma$ for each $(\mathbb{S}, \sigma) \in \mathtt{Q}_\Pi$. Let $\Pi^{\{\cdot\}} = \bigcup_{(\mathbb{S},\sigma)\in\mathtt{Q}_\Pi} \Pi_\sigma^{\{\sigma\}}$.

$(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_{c,\gamma_0} \Gamma; \Pi^{\{\cdot\}}$ holds for a set Γ of formulas if $\mathbb{I}_N \leq \gamma \leq \gamma_0$,

$$\{\gamma, a, c, \gamma_0\} \cup dom(\mathtt{Q}_\Pi) \subset \mathcal{H}_\gamma[\Theta] \tag{21}$$

$$\mathsf{k}(\Gamma \cup \Pi) \subset \mathcal{H}_\gamma[\Theta(\mathtt{Q}_\Pi)] \tag{22}$$

and one of the following cases holds:

($\bigvee$) There exist $A \simeq \bigvee(A_\iota)_{\iota\in J}$, $\iota \in J$, and an ordinal $a(\iota) < a$ such that $A \in \Gamma$ and $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a(\iota)}_c \Gamma, A_\iota; \Pi^{\{\cdot\}}$.

($\bigvee$)$^{\{\cdot\}}$ There exist $A^{\{\sigma\}} \in \Pi^{\{\cdot\}}$, $A \simeq \bigvee(A_\iota)_{\iota\in J}$, $\iota \in [\sigma]J$, and an ordinal $a(\iota) < a$ such that $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a(\iota)}_c \Gamma; \Pi^{\{\cdot\}}, A_\iota^{\{\sigma\}}$.

($\bigwedge$) There exist an $A \simeq \bigwedge(A_\iota)_{\iota\in J}$ such that $A \in \Gamma$. For each $\iota \in J$, $(\mathcal{H}_\gamma, \Theta \cup \mathcal{B}(\mathsf{k}(\iota)); \mathtt{Q}_\Pi) \vdash^{*a(\iota)}_c \Gamma, A_\iota; \Pi^{\{\cdot\}}$ holds for an ordinal $a(\iota) < a$.

($\bigwedge$)$^{\{\cdot\}}$ There exist $A^{\{\sigma\}} \in \Pi^{\{\cdot\}}$ such that $A \simeq \bigwedge(A_\iota)_{\iota\in J}$. For each $\iota \in [\sigma]J$, $(\mathcal{H}_\gamma, \Theta \cup \mathcal{B}(\mathsf{k}(\iota)); \mathtt{Q}_\Pi \vdash^{*a(\iota)}_c \Gamma; A_\iota^{\{\sigma\}}, \Pi^{\{\cdot\}}$ holds for an ordinal $a(\iota) < a$.

(cut) There exist an ordinal $a_0 < a$ and a formula C such that $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a_0}_c \Gamma, \neg C; \Pi^{\{\cdot\}}$ and $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a_0}_c C, \Gamma; \Pi^{\{\cdot\}}$ with $\mathrm{rk}(C) < c$.

($\Sigma(St)$-rfl) There exist ordinals $a_\ell, a_r < a$ and a formula $C \in \Sigma(\mathcal{L}_{N+1})$ such that $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a_\ell}_c \Gamma, C; \Pi^{\{\cdot\}}$ and $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a_r}_c \neg\exists x\, C^{(x,\mathbb{I}_N)}, \Gamma; \Pi^{\{\cdot\}}$, where $c \geq \mathbb{I}_N$.

($\Sigma(\Omega)$-rfl) There exist ordinals $a_\ell, a_r < a$ and a formula $C \in \Sigma(\mathcal{L}_0 : \Omega)$ such that $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a_\ell}_c \Gamma, C; \Pi^{\{\cdot\}}$ and $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a_r}_c \neg\exists x < \Omega\, C^{(x,\Omega)}, \Gamma; \Pi^{\{\cdot\}}$, where $c \geq \Omega$.

($i-\mathrm{stbl}(\mathbb{S})$) Let $0 < i \leq N$. There exist an ordinal $a_0 < a$, a successor i-stable ordinal $\mathbb{S} \in SSt_i \cap c$, a formula $B(\mathsf{L}_0) \in \Delta_0(\mathcal{L}_i)$ with $\mathrm{rk}(B(\mathsf{L}_0)) < \mathbb{S}$, and a $u \in Tm(\mathbb{I}_N)$ such that $\mathbb{S} \leq \mathrm{rk}(B(u)) < c$ for which the following hold:

$$\mathbb{S} \in \mathcal{H}_\gamma[\Theta] \tag{23}$$

and $(\mathcal{H}_\gamma, \Theta; \mathtt{R}_\Pi) \vdash^{*a_0}_c \Gamma, B(u); \Pi^{\{\cdot\}}$ for $dom(\mathtt{R}_\Pi) = dom(\mathtt{Q}_\Pi) \cup \{\mathbb{S}\}$ and $\mathtt{R}_\Pi(\mathbb{S}) = \mathtt{Q}_\Pi(\mathbb{S})$.

For every $\sigma \in \Psi_{\mathbb{S}}$ such that $\mathtt{R}^\sigma_\Pi = \mathtt{R}_\Pi \cup \{(\mathbb{S}, \sigma)\}$ is a finite family for γ_0 and

$$\Theta(\mathtt{Q}_\Pi) \subset M_\sigma \tag{24}$$

$(\mathcal{H}_\gamma, \Theta; \mathtt{R}^\sigma_\Pi) \vdash^{*a_0}_c \Gamma; \neg B(u)^{\{\sigma\}}, \Pi^{\{\cdot\}}$ holds, where $dom(\mathtt{R}^\sigma_\Pi) = dom(\mathtt{R}_\Pi)$ and $(\mathtt{R}^\sigma_\Pi)(\mathbb{S}) = \mathtt{R}_\Pi(\mathbb{S}) \cup \{\sigma\}$.

$$\frac{(\mathcal{H}_\gamma, \Theta; \mathtt{R}_\Pi) \vdash^{*a_0}_c \Gamma, B(u); \Pi^{\{\cdot\}} \quad \{(\mathcal{H}_\gamma, \Theta; \mathtt{R}^\sigma_\Pi) \vdash^{*a_0}_c \Gamma; \neg B(u)^{\{\sigma\}}, \Pi^{\{\cdot\}}\}_\sigma}{(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_c \Gamma; \Pi^{\{\cdot\}}}$$

Note that in (24) we have $\mathbb{S} \in M_\sigma$ by Proposition 3.38. Let $\mathcal{B}(\Theta) \subset \Theta$. By Propositions 4.15.6 and 4.15.9 we have $\mathcal{B}(\alpha) \subset \mathcal{H}_\gamma[\Theta]$ if $\alpha \in \mathcal{H}_\gamma[\Theta]$. In particular $\mathcal{B}(\mathsf{k}(\iota)) \subset \mathcal{H}_\gamma[\Theta]$ holds when $\mathsf{k}(\iota) \subset \mathcal{H}_\gamma[\Theta]$.

We will state some lemmas for the operator controlled derivations. These can be shown as in [9].

Lemma 4.24 (Tautology) *Let* $d = \mathrm{rk}(A)$*,* $\mathbb{I}_N \leq \gamma \leq \gamma_0$ *and* $\{\gamma, \gamma_0\} \subset \mathcal{H}_\gamma[\mathsf{k}(A)]$.

1. $(\mathcal{H}_\gamma, \mathcal{B}(\mathsf{k}(A)); \emptyset) \vdash^{*2d}_{\mathbb{I}_N,\gamma_0} \neg A, A; \emptyset$.

2. $(\mathcal{H}_\gamma, \mathcal{B}(\mathsf{k}(A) \cup \{\mathbb{S}\}); \{(\mathbb{S}, \sigma)\}) \vdash^{*2d}_{\mathbb{I}_N,\gamma_0} \neg A^{[\sigma/\mathbb{S}]}; A^{\{\sigma\}}$ *if* $\sigma \in \Psi_{\mathbb{S}}$*,* $\mathsf{k}(A) \subset M_\sigma$ *and* $A \in \Delta_0(\mathcal{L}_{N+1})$*.*

Proof. Each is seen by induction on $d = \mathrm{rk}(A)$. Let us consider Lemma 4.24.2. Let $\Theta = \mathcal{B}(\mathsf{k}(A) \cup \{\mathbb{S}\})$, $\mathsf{Q}_\Pi = \{(\mathbb{S}, \sigma)\}$ and $B \equiv A^{[\sigma/\mathbb{S}]}$. Then $\Theta(\mathsf{Q}_\Pi) = \mathcal{B}(\mathsf{k}(A) \cup \{\mathbb{S}, \sigma\})$. We have $\{\gamma, 2d, \mathbb{I}_N, \gamma_0\} \cup \mathsf{k}(A) \subset \mathcal{H}_\gamma[\Theta]$. For (22), we obtain by Proposition 4.7.3, $\mathsf{k}(A^{[\sigma/\mathbb{S}]}) \subset \mathcal{H}_{\mathbb{S}}(\mathsf{k}(A) \cup \{\sigma\}) \subset \mathcal{H}_\gamma[\Theta(\mathsf{Q}_\Pi)]$ with $\mathbb{S} < \mathbb{I}_N \leq \gamma$ if $B \not\equiv A$, and $\mathsf{k}(A^{[\sigma/\mathbb{S}]}) \subset \mathcal{H}_\gamma[\mathsf{k}(A)]$ else. Moreover $\mathbb{S} \in \mathcal{H}_\gamma[\Theta]$ for (21).

Let $A \simeq \bigvee(A_\iota)_{\iota \in J}$. We obtain $B \simeq \bigvee(A_\iota^{[\sigma/\mathbb{S}]})_{\iota \in [\sigma]J}$ by Proposition 4.8.

Let $\Theta_\iota = \Theta \cup \mathcal{B}(\mathsf{k}(\iota))$ and $\iota \in [\sigma]J$. For $d > d_\iota = \mathrm{rk}(A_\iota)$ with $d_\iota \subset \mathcal{H}_\gamma[\Theta_\iota]$ we obtain $(\mathcal{H}_\gamma, \Theta_\iota; \mathsf{Q}_\Pi) \vdash^{*2d_\iota}_{\mathbb{I}_N} \neg A_\iota^{[\sigma/\mathbb{S}]}; A_\iota^{\{\sigma\}}$ by IH.

$$
\dfrac{\dfrac{\dfrac{}{(\mathcal{H}_\gamma, \Theta_\iota; \mathsf{Q}_\Pi) \vdash^{*2d_\iota}_{\mathbb{I}_N} \neg A_\iota^{[\sigma/\mathbb{S}]}; A_\iota^{\{\sigma\}}}\ \text{IH}}{\{(\mathcal{H}_\gamma, \Theta_\iota; \mathsf{Q}_\Pi) \vdash^{*2d_\iota+1}_{\mathbb{I}_N} \neg A_\iota^{[\sigma/\mathbb{S}]}; A^{\{\sigma\}}\}_{\iota \in [\sigma]J}}\ (\bigvee)^{\{\cdot\}}}{(\mathcal{H}_\gamma, \Theta; \mathsf{Q}_\Pi) \vdash^{*2d}_{\mathbb{I}_N} \neg A^{[\sigma/\mathbb{S}]}; A^{\{\sigma\}}}\ (\bigwedge)
$$

and

$$
\dfrac{\dfrac{\dfrac{}{(\mathcal{H}_\gamma, \Theta_\iota; \mathsf{Q}_\Pi) \vdash^{*2d_\iota}_{\mathbb{I}_N} A_\iota^{[\sigma/\mathbb{S}]}; \neg A_\iota^{\{\sigma\}}}\ \text{IH}}{\{(\mathcal{H}_\gamma, \Theta_\iota; \mathsf{Q}_\Pi) \vdash^{*2d_\iota+1}_{\mathbb{I}_N} A^{[\sigma/\mathbb{S}]}; \neg A_\iota^{\{\sigma\}}\}_{\iota \in [\sigma]J}}\ (\bigvee)}{(\mathcal{H}_\gamma, \Theta; \mathsf{Q}_\Pi) \vdash^{*2d}_{\mathbb{I}_N} A^{[\sigma/\mathbb{S}]}; \neg A^{\{\sigma\}}}\ (\bigwedge)^{\{\cdot\}}
$$

□

Lemma 4.25 (Equality) *Let* $d = \mathrm{rk}(A(\mathsf{L}_0))$*,* $\gamma \geq \mathbb{I}_N$*,* $\mathcal{B}(\mathsf{k}(A, u, v)) = \mathcal{B}(\mathsf{k}(A)) \cup \mathcal{B}(\mathsf{k}(u)) \cup \mathcal{B}(\mathsf{k}(v))$ *and* $\{\gamma, \gamma_0\} \subset \mathcal{H}_\gamma[\mathcal{B}(\mathsf{k}(A, u, v))]$*.*

Then $(\mathcal{H}_\gamma, \mathcal{B}(\mathsf{k}(A, u, v)); \emptyset) \vdash^{*\omega(|u|\#|v|)\#2d}_{\mathbb{I}_N,\gamma_0} u \neq v, \neg A(u), A(v); \emptyset$*.*

Proof. This is seen by induction on $d = \mathrm{rk}(A)$ as in [9, 3].

First show that $(\mathcal{H}_\gamma, \mathcal{B}(\mathsf{k}(u, v, w)); \emptyset) \vdash^{*\alpha}_{\mathbb{I}_N,\gamma_0} u \neq v, u \notin w, v \in w; \emptyset$, $(\mathcal{H}_\gamma, \mathcal{B}(\mathsf{k}(u, v, w)); \emptyset) \vdash^{*\alpha}_{0,\gamma_0} u \neq v, u \neq w, v = w; \emptyset$ and $(\mathcal{H}_\gamma, \mathcal{B}(\mathsf{k}(u, v, w)); \emptyset) \vdash^{*\alpha}_{\mathbb{I}_N,\gamma_0} u \neq v, w \notin u, w \in v; \emptyset$ simultaneously by induction on the natural sum $|u|\#|v|\#|w|$, where $\alpha = \omega(|u|\#|v|\#|w|)$. Then the lemma is seen by induction on $d = \mathrm{rk}(A(\mathsf{L}_0))$. □

Lemma 4.26 (Embedding of Axioms) *For each axiom* A *in* $S_{\mathbb{I}_N}$ *there is an* $m < \omega$ *such that* $(\mathcal{H}_{\mathbb{I}_N}, \emptyset; \emptyset) \vdash^{*\mathbb{I}_N \cdot 2+m}_{\mathbb{I}_N+m,\gamma_0} A; \emptyset$ *holds.*

Proof. In the proof, let us suppress the operator $\mathcal{H}_{\mathbb{I}_N}$, the second subscript γ_0, and write $\vdash^*$ for $\vdash^{*\mathbb{I}_N+m}_{\mathbb{I}_N+m,\gamma_0}$ for an $m < \omega$.

We show first that the axiom (3) follows from an inference ($i-$stbl($\mathbb{S}$)). Let $\varphi(y) \equiv (\exists x\, \theta(x,y))$ be a $\Sigma_1(\{st_j\}_{j<i})$-formula such that $\mathrm{rk}(\theta(\mathsf{L}_0,\mathsf{L}_0)) < \omega$. Also let u, w be RS-terms, $\mathbb{S}$ a successor i-stable ordinal, and $B(x) \equiv \theta(x,w)$.

Let $\mathsf{k}_w = \mathsf{k}(B(\mathsf{L}_0)) = \mathsf{k}(w)$, $\mathsf{k}_u = \mathsf{k}(u)$, and $\Theta := \mathcal{B}(\mathsf{k}_w \cup \mathsf{k}_u \cup \{\mathbb{S}\})$, where $\mathbb{S} \in \mathcal{H}_\gamma[\Theta]$ for (23). We show

$$\mathcal{B}(\mathsf{k}_w) \cup \mathcal{B}(\mathbb{S}); \emptyset \vdash^* w \not\in \mathsf{L}_{\mathbb{S}}, \neg\exists x\, B(x), \exists x \in \mathsf{L}_{\mathbb{S}} B(x); \tag{25}$$

First assume $|w| < \mathbb{S}$. Then $\mathrm{rk}(B(\mathsf{L}_0)) = \mathrm{rk}(\theta(\mathsf{L}_0, w)) < \mathbb{S}$. We obtain by Tautology 4.24.1, $\Theta; \mathsf{Q} \vdash^{*2d}_{\mathbb{I}_N} \neg B(u), B(u); \emptyset$, where d=rk(B(u)), $dom(\mathsf{Q}) = \{\mathbb{S}\}$ and $\mathsf{Q}(\mathbb{S}) = \emptyset$. We may assume that $\mathbb{I}_N > d \geq \mathbb{S}$ with $|u| \geq \mathbb{S}$.

Let $\sigma \in \Psi_{\mathbb{S}}$ be an ordinal such that $\Theta \subset M_\sigma$ and $\gamma_0 \leq \mathsf{p}_0(\sigma)$. Tautology 4.24.2 yields $\Theta; \{(\mathbb{S},\sigma)\} \vdash^{*2d}_{\mathbb{I}_N} B(u)^{[\sigma/\mathbb{S}]}; \neg B(u)^{\{\sigma\}}$. Then for $\exists x \in \mathsf{L}_{\mathbb{S}} B(x) \simeq \bigvee(B(v))_{v\in J}$ we obtain $u^{[\sigma/\mathbb{S}]} \in Tm(\mathbb{S}) = J$ with $B(u^{[\sigma/\mathbb{S}]}) \equiv B(u)^{[\sigma/\mathbb{S}]}$. When $|w| < \mathbb{S}$, (25) is seen as follows:

$$\dfrac{\dfrac{\Theta; \mathsf{Q} \vdash^{*2d}_{\mathbb{I}_N} \neg B(u), B(u); \quad \left\{\dfrac{\Theta; \{(\mathbb{S},\sigma)\} \vdash^{*2d}_{\mathbb{I}_N} B(u^{[\sigma/\mathbb{S}]}); \neg B(u)^{\{\sigma\}}}{\Theta; \{(\mathbb{S},\sigma)\} \vdash^{*2d+1}_{\mathbb{I}_N} \exists x \in \mathsf{L}_{\mathbb{S}} B(x); \neg B(u)^{\{\sigma\}}}\,(\bigvee)\right\}_\sigma}{\Theta; \vdash^{*2d+1}_{\mathbb{I}_N} \neg B(u), \exists x \in \mathsf{L}_{\mathbb{S}} B(x);}\,(i-\mathrm{stbl}(\mathbb{S}))}{\mathcal{B}(\mathsf{k}_w) \cup \mathcal{B}(\mathbb{S}); \vdash^{*2d+2}_{\mathbb{I}_N} \neg\exists x\, B(x), \exists x \in \mathsf{L}_{\mathbb{S}} B(x);}\,(\bigwedge)$$

Assume $|w| \geq \mathbb{S}$, and let $v \in Tm(\mathbb{S})$. Then $|v| < \mathbb{S}$ and $(v \dot{\in} \mathsf{L}_{\mathbb{S}}) \equiv (v \not\in \mathsf{L}_0)$. We obtain by (25)

$$\mathcal{B}(\mathsf{k}(v)) \cup \mathcal{B}(\mathbb{S}); \emptyset \vdash^* \neg\exists x\, \theta(x,v), \exists x \in \mathsf{L}_{\mathbb{S}} \theta(x,v);$$

We obtain

$$\mathcal{B}(\mathsf{k}(w,v)) \cup \mathcal{B}(\mathbb{S}); \emptyset \vdash^* \neg(v \dot{\in} \mathsf{L}_{\mathbb{S}}), w \neq v, \neg\exists x\, \theta(x,w), \exists x \in \mathsf{L}_{\mathbb{S}} \theta(x,w);$$

by Equality 4.25 followed by (cut)'s with $|v|, |w| < \mathbb{I}_N$ and $\mathrm{rk}(\exists x\, \theta(x,w)) = \mathbb{I}_N + 2$. Then a $(\bigvee)$ followed by a $(\bigwedge)$ yields (25), where $(w \not\in \mathsf{L}_{\mathbb{S}}) \simeq \bigwedge(\neg(v \dot{\in} \mathsf{L}_{\mathbb{S}}) \vee w \neq v)_{v \in Tm(\mathbb{S})}$.

Let v be an RS-term with $|v| \geq \mathbb{S}$. We obtain by (25) and Equality 4.25

$$\mathcal{B}(\mathsf{k}(w,v)) \cup \mathcal{B}(\mathbb{S}); \vdash^* \mathsf{L}_{\mathbb{S}} \neq v, w \not\in v, \neg\exists x\, \theta(x,w), \exists x \in v\, \theta(x,w);$$

We have $\neg st_i(v) \simeq \bigwedge(\mathsf{L}_{\mathbb{S}} \neq v)_J$ with $J = \{\mathsf{L}_{\mathbb{S}} : |v| \geq \mathbb{S} \in SSt_i\}$. A $(\bigwedge)$ yields the axiom (3)

$$\mathcal{B}(\mathsf{k}(w,v)); \vdash^* \neg st_i(v), \neg\varphi(w), w \not\in v, \varphi^v(w);$$

Next we show the axiom (1). Let u be an RS-term and $\beta = \alpha^{\dagger N}$ for $\alpha = |u|$. Then $\beta \in \mathcal{H}_0[\mathsf{k}(u)]$. We obtain $\mathcal{B}(\mathsf{k}(u)); \emptyset \vdash^* u = u; \emptyset$ and $\mathcal{B}(\mathsf{k}(u)); \emptyset \vdash^* \mathsf{L}_\beta = \mathsf{L}_\beta; \emptyset$. Hence

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{B}(\mathsf{k}(u)); \emptyset \vdash^* u = u; \emptyset}{\mathcal{B}(\mathsf{k}(u)); \emptyset \vdash^* u \in \mathsf{L}_\beta; \emptyset}\,(\bigvee) \quad \dfrac{\mathcal{B}(\mathsf{k}(u)); \emptyset \vdash^* \mathsf{L}_\beta = \mathsf{L}_\beta; \emptyset}{\mathcal{B}(\mathsf{k}(u)); \emptyset \vdash^* st_N(\mathsf{L}_\beta); \emptyset}\,(\bigvee)}{\mathcal{B}(\mathsf{k}(u)); \emptyset \vdash^* u \in \mathsf{L}_\beta \wedge st_N(\mathsf{L}_\beta); \emptyset}\,(\bigwedge)}{\mathcal{B}(\mathsf{k}(u)); \emptyset \vdash^* \exists y\,(u \in y \wedge st_N(y)); \emptyset}\,(\bigvee)}{\emptyset; \emptyset \vdash^* \forall x \exists y\,(x \in y \wedge st_N(y)); \emptyset}\,(\bigwedge)$$

Third we show the axiom (2). Let $\mathbb{T} \in SSt_{i+1}$ be a successor $(i+1)$-stable ordinal. We obtain $\mathcal{B}(\mathbb{T}); \emptyset \vdash^* \theta(\mathsf{L}_{\mathbb{T}})$ for $\theta(x) \equiv (st_i(x) \land \mathsf{L}_\Omega \in x \land \forall y \in x \forall z \in y(z \in x))$ with $\mathsf{L}_\Omega \equiv M_0$.

For a given $\alpha < \mathbb{T}$ pick a successor i-stable ordinal $\alpha < \mathbb{S} < \mathbb{T}$ such that $\mathbb{S} \in \mathcal{H}_0[\mathcal{B}(\alpha, \mathbb{T})]$ by Proposition 4.16.

Let $|v| = \alpha < \mathbb{T}$. We obtain $(\mathsf{L}_{\mathbb{S}} \dot{\in} \mathsf{L}_{\mathbb{T}}) \equiv (\mathsf{L}_{\mathbb{S}} \not\in \mathsf{L}_0)$, $\mathcal{B}(v); \emptyset \vdash^* v = v; \emptyset$, and $\mathcal{B}(\mathsf{k}(v) \cup \{\mathbb{T}\}); \emptyset \vdash^* \mathsf{L}_{\mathbb{S}} = \mathsf{L}_{\mathbb{S}}; \emptyset$. Hence $\mathcal{B}(\mathsf{k}(v) \cup \{\mathbb{T}\}); \emptyset \vdash^* v \in \mathsf{L}_{\mathbb{S}} \land st_i(\mathsf{L}_{\mathbb{S}}); \emptyset$, and $\mathcal{B}(\mathsf{k}(v) \cup \{\mathbb{T}\}); \emptyset \vdash^* \exists z \in \mathsf{L}_{\mathbb{T}}(v \in z \land st_i(z)); \emptyset$. Let w and u be RS-terms. Equality 4.25 yields $\mathcal{B}(\mathsf{k}(w) \cup \{\mathbb{T}\}); \emptyset \vdash^* w \not\in \mathsf{L}_{\mathbb{T}}, \exists z \in \mathsf{L}_{\mathbb{T}}(w \in z \land st_i(z)); \emptyset$, and $\mathcal{B}(\mathsf{k}(w, u) \cup \{\mathbb{T}\}); \emptyset \vdash^* u \neq \mathsf{L}_{\mathbb{T}}, w \not\in u, \exists z \in u(w \in z \land st_i(z)); \emptyset$. A $(\bigwedge)$ yields $\mathcal{B}(\mathsf{k}(w, u)); \emptyset \vdash^* \neg st_{i+1}(u), w \not\in u, \exists z \in u(w \in z \land st_i(z)); \emptyset$.

$\Delta_0(\mathcal{L}_{N+1})$-Collection follows from an inference $(\Sigma(St){-}\mathrm{rfl})$, and the Δ_0-collection for the set $M_0 = \mathsf{L}_\Omega$ follows from an inference $(\Sigma(\Omega){-}\mathrm{rfl})$. Other axioms in $\mathsf{KP}\omega$, i.e., axioms for pair, union, Δ_0-Separation and foundation are seen as in [9, 3]. □

Lemma 4.27 (Embedding) *If $S_{\mathbb{I}_N} \vdash \Gamma$ for sets Γ of sentences, there are $m, k < \omega$ such that $(\mathcal{H}_{\mathbb{I}_N}, \emptyset; \emptyset) \vdash^{*\mathbb{I}_N \cdot 2 + k}_{\mathbb{I}_N + m, \gamma_0} \Gamma; \emptyset$ holds.*

Proof. This follows from Lemma 4.26 as in [9, 3]. □

Lemma 4.28 *Let $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_c \Gamma; \Pi^{\{\cdot\}}$, $\gamma \leq \gamma_1 \leq \gamma_0$ with $\gamma_1 \in \mathcal{H}_{\gamma_1}[\Theta_1]$ and $\Theta \subset \mathcal{B}(\Theta_1) \subset \Theta_1$. Then $(\mathcal{H}_{\gamma_1}, \Theta_1; \mathtt{Q}_\Pi) \vdash^{*a}_c \Gamma; \Pi^{\{\cdot\}}$ holds.*

Proof. By induction on a. We need to prune some branches at inferences $(i{-}\mathrm{stbl}(\mathbb{S}))$ for (24) with $\Theta(\mathtt{Q}_\Pi) \subset \Theta_1(\mathtt{Q}_\Pi)$. □

Lemma 4.29 *Let $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_c \Gamma; \Pi^{\{\cdot\}}$, and $\mathbb{S} < \mathbb{I}_N \leq c$ be a successor stable ordinal and $\sigma \in \Psi_{\mathbb{S}}$. Assume $\mathtt{P} = \mathtt{Q}_\Pi \cup \{(\mathbb{S}, \sigma)\}$ is a finite family for γ_0, and $\mathbb{S} \in \mathcal{H}_\gamma[\Theta]$. Then $(\mathcal{H}_\gamma, \Theta; \mathtt{P}) \vdash^{*a}_c \Gamma; \Pi^{\{\cdot\}}$ holds.*

Proof. By induction on a. By the assumption (21) is enjoyed in $(\mathcal{H}_\gamma, \Theta; \mathtt{P}) \vdash^{*a}_c \Gamma; \Pi^{\{\cdot\}}$. We need to prune some branches at inferences $(i{-}\mathrm{stbl}(\mathbb{S}))$ for (24) with $\Theta(\mathtt{Q}_\Pi) \subset \Theta(\mathtt{P})$. □

Lemma 4.30 (Inversion) *Let $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_c \Gamma; \Pi^{\{\cdot\}}$ with $A \simeq \bigwedge(A_\iota)_{\iota \in J}$, $A \in \Gamma$, and $\iota \in J$.*

Then $(\mathcal{H}_\gamma, \Theta_\iota; \mathtt{Q}_\Pi) \vdash^{*a}_c \Gamma, A_\iota; \Pi^{\{\cdot\}}$ holds for $\Theta_\iota = \Theta \cup \mathcal{B}(\mathsf{k}(\iota))$.

Proof. By induction on a. We obtain $(\mathcal{H}_\gamma, \Theta_\iota; \mathtt{Q}_\Pi) \vdash^{*a}_c \Gamma; \Pi^{\{\cdot\}}$ by Lemma 4.28. □

Lemma 4.31 (Reduction) *Let $C \simeq \bigvee(C_\iota)_{\iota \in J}$ and $\neg(\Omega \leq c < \mathbb{I}_N)$ with $\mathrm{rk}(C) \leq c$. Assume $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_c \Gamma_0, \neg C; \Pi^{\{\cdot\}}$ and $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*b}_c C, \Gamma_1; \Pi^{\{\cdot\}}$. Then $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a+b}_c \Gamma_0, \Gamma_1; \Pi^{\{\cdot\}}$.*

Proof. By induction on b.
Case 1. Consider first the case when $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*b}_c C, \Gamma_1; \Pi^{\{\cdot\}}$ follows from a $(\bigvee)$ with its major formula C. We have $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*b(\iota)}_c C_\iota, C, \Gamma_1; \Pi^{\{\cdot\}}$ for an $\iota \in J$. IH yields $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a+b(\iota)}_c C_\iota, \Gamma_0, \Gamma_1; \Pi^{\{\cdot\}}$.

Let $\Theta_\iota = \Theta \cup \mathcal{B}(\mathsf{k}(\iota))$. We obtain $(\mathcal{H}_\gamma, \Theta_\iota; \mathtt{Q}_\Pi) \vdash^{*a}_c \Gamma_0, \neg C_\iota; \Pi^{\{\cdot\}}$ by Inversion 4.30. On the other hand we have $\mathcal{B}(\mathsf{k}(C_\iota)) \subset \mathcal{H}_\gamma[\Theta(\mathtt{Q}_\Pi)]$ by (22) and Propositions 4.15.6 and 4.15.9. $\mathcal{H}_\gamma[(\Theta_\iota)(\mathtt{Q}_\Pi)] = \mathcal{H}_\gamma[\Theta(\mathtt{Q}_\Pi)]$ follows provided that $\mathsf{k}(\iota) \subset \mathsf{k}(C_\iota)$. Hence $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_c \Gamma_0, \neg C_\iota; \Pi^{\{\cdot\}}$.

A (*cut*) with the cut formula C_ι yields $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a+b}_c \Gamma_0, \Gamma_1; \Pi^{\{\cdot\}}$ for $\mathrm{rk}(C_\iota) < \mathrm{rk}(C) \leq c$.
Case 2. Second assume that $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*b}_c C, \Gamma_1; \Pi^{\{\cdot\}}$ follows from an $(i\text{-}\mathrm{stbl}(\mathbb{S}))$. We have an ordinal $b_0 < b$ and a formula $B(u)$ such that for $dom(\mathtt{R}_\Pi) = dom(\mathtt{Q}_\Pi) \cup \{\mathbb{S}\}$ and $\mathtt{R}^\sigma_\Pi = \mathtt{R}_\Pi \cup \{(\mathbb{S}, \sigma)\}$

$$\frac{(\mathcal{H}_\gamma, \Theta; \mathtt{R}_\Pi) \vdash^{*b_0}_c C, \Gamma_1, B(u); \Pi^{\{\cdot\}} \quad \{(\mathcal{H}_\gamma, \Theta; \mathtt{R}^\sigma_\Pi) \vdash^{*b_0}_c C, \Gamma_1; \neg B(u)^{\{\sigma\}}, \Pi^{\{\cdot\}}\}_\sigma}{(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*b}_c C, \Gamma_1; \Pi^{\{\cdot\}}}$$

where $\mathbb{S} \in \mathcal{H}_\gamma[\Theta]$ and $\Theta(\mathtt{Q}_\Pi) \subset M_\sigma$ by (24). By Lemma 4.29 we obtain $(\mathcal{H}_\gamma, \Theta; \mathtt{R}^\sigma_\Pi) \vdash^{*a}_c \Gamma_0, \neg C; \Pi^{\{\cdot\}}$ for each σ. IH followed by an $(i\text{-}\mathrm{stbl}(\mathbb{S}))$ yields

$$\frac{(\mathcal{H}_\gamma, \Theta; \mathtt{R}_\Pi) \vdash^{*a+b_0}_c \Gamma_0, \Gamma_1, B(u); \Pi^{\{\cdot\}} \quad \{(\mathcal{H}_\gamma, \Theta; \mathtt{R}^\sigma_\Pi) \vdash^{*a+b_0}_c \Gamma_0, \Gamma_1; \neg B(u)^{\{\sigma\}}, \Pi^{\{\cdot\}}\}_\sigma}{(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a+b}_c \Gamma_0, \Gamma_1; \Pi^{\{\cdot\}}}$$

Other cases are seen from IH. □

Lemma 4.32 (Cut-elimination) *Let* $c \in \mathcal{H}_\gamma[\Theta]$ *and* $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_{c+b} \Gamma; \Pi^{\{\cdot\}}$, *where either* $c \geq \mathbb{I}_N$ *or* $\neg(c < \Omega < c + b)$. *Then* $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*\varphi_b(a)}_c \Gamma; \Pi^{\{\cdot\}}$.

Proof. By main induction on b with subsidiary induction on a using Reduction 4.31. □

Lemma 4.33 (Σ-persistency) *Let* $A \in \Sigma(\mathcal{L}_{N+1})$ *with* $\mathrm{rk}(A) \leq \mathbb{I}_N$, $dom(\mathtt{Q}_\Pi) \subset \alpha < \beta$, $\beta \in \mathcal{H}_\gamma[\Theta] \cap \mathbb{I}_N$, *and* $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_c \Gamma, A^{(\alpha, \mathbb{I}_N)}; \Pi^{\{\cdot\}}$.

Then $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_c \Gamma, A^{(\beta, \mathbb{I}_N)}; \Pi^{\{\cdot\}}$.

Proof. This is seen by induction on a. (22) follows from $\beta \in \mathcal{H}_\gamma[\Theta]$. □

Lemma 4.34 (Collapsing) *Assume* $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_{\mathbb{I}_N, \gamma_0} \Gamma; \Pi^{\{\cdot\}}$ *for* $\Gamma \subset \Sigma(\mathcal{L}_{N+1})$. *Assume* $\Theta \subset \mathcal{H}_\gamma(\psi_{\mathbb{I}_N}(\gamma))$ *and* $\hat{a} := \gamma + \omega^a < \gamma_0$.

Then $(\mathcal{H}_{\hat{a}+1}, \Theta; \mathtt{Q}_\Pi) \vdash^{*\beta}_{\beta, \gamma_0} \Gamma^{(\beta, \mathbb{I}_N)}; \Pi^{\{\cdot\}}$ *holds for* $\beta = \psi_{\mathbb{I}_N}(\hat{a})$.

Proof. By induction on a as in [9]. Let us omit the second subscript γ_0 in the proof.

We have $\{\gamma, a\} \cup dom(\mathtt{Q}_\Pi) \subset \mathcal{H}_\gamma[\Theta]$ by (21). We obtain $\beta \in \mathcal{H}_{\hat{a}+1}[\Theta]$, and $dom(\mathtt{Q}_\Pi) \subset \mathcal{H}_\gamma(\psi_{\mathbb{I}_N}(\gamma)) \cap \mathbb{I}_N = \psi_{\mathbb{I}_N}(\gamma) \subset \beta$ by the assumption. This yields $\mathtt{Q}_\Pi(\mathbb{S}) \subset \mathbb{S} \subset \psi_{\mathbb{I}_N}(\gamma)$ for every $\mathbb{S} \in dom(\mathtt{Q}_\Pi)$, and $\Theta(\mathtt{Q}_\Pi) \subset \mathcal{H}_\gamma(\psi_{\mathbb{I}_N}(\gamma))$.

$\beta=\psi_{\mathbb{I}_N}(\hat{a})$ needs to be in LSt_N due to the axiom (1). On the other hand we have $\mathsf{k}(\Gamma\cup\Pi)\subset\mathcal{H}_\gamma[\Theta(\mathtt{Q}_\Pi)]$ by (22). We obtain

$$\mathsf{k}(\Gamma\cup\Pi)\subset\psi_{\mathbb{I}_N}(\gamma)\subset\beta \tag{26}$$

Case 1. The last inference is an $(i-\mathrm{stbl}(\mathbb{S}))$: We have $\mathbb{S}\in\mathcal{H}_\gamma[\Theta]$. Let $B(\mathsf{L}_0)\in\Delta_0(\mathcal{L}_{N+1})$ be a $\bigwedge$-formula with $\mathrm{rk}(B(\mathsf{L}_0))<\mathbb{S}$ and a term $u\in Tm(\mathbb{I}_N)$ such that $(\mathcal{H}_\gamma,\Theta;\mathtt{R}_\Pi)\vdash^{*a_0}_{\mathbb{I}_N}\Gamma,B(u);\Pi^{\{\cdot\}}$ for an ordinal $a_0\in\mathcal{H}_\gamma[\Theta]\cap a$ and $dom(\mathtt{R}_\Pi)=dom(\mathtt{Q}_\Pi)\cup\{\mathbb{S}\}$. $(\mathcal{H}_{\hat{a}+1},\Theta;\mathtt{R}_\Pi)\vdash^{*\beta_0}_{\mathbb{I}_N}\Gamma^{(\beta,\mathbb{I}_N)},B(u);\Pi^{\{\cdot\}}$ follows from IH with Σ-persistency 4.33, where $\beta_0=\psi_{\mathbb{I}_N}(\widehat{a_0})$ with $\widehat{a_0}=\gamma+\omega^{a_0}$.

We obtain $\mathsf{k}(B(u))\subset\mathcal{H}_\gamma(\beta)$ by (26), and $\mathrm{rk}(B(u))<\beta$ for $\mathrm{rk}(B(u))<\mathbb{I}_N$ by Proposition 4.5.3.

On the other hand we have $(\mathcal{H}_\gamma,\Theta;\mathtt{R}^\sigma_\Pi)\vdash^{*a_0}_{\mathbb{I}_N}\Gamma;\neg B(u)^{\{\sigma\}},\Pi^{\{\cdot\}}$ for every $\sigma\in\Psi_{\mathbb{S}}$ such that $\Theta(\mathtt{Q}_\Pi)\subset M_\sigma$. IH with Σ-persistency 4.33 yields $(\mathcal{H}_{\hat{a}+1},\Theta;\mathtt{R}^\sigma_\Pi)\vdash^{*\beta_0}_{\mathbb{I}_N}\Gamma^{(\beta,\mathbb{I}_N)};\neg B(u)^{\{\sigma\}},\Pi^{\{\cdot\}}$. $(\mathcal{H}_{\hat{a}+1},\Theta;\mathtt{Q}_\Pi)\vdash^{*\beta}_{\beta,\gamma_0}\Gamma^{(\beta,\mathbb{I}_N)};\Pi^{\{\cdot\}}$. follows from an $(i-\mathrm{stbl}(\mathbb{S}))$.

Case 2. The case when the last inference is a $(\Sigma(St)\text{-rfl})$ on $\mathbb{I}_N$: We have ordinals $a_\ell,a_r<a$ and a formula $C\in\Sigma(\mathcal{L}_{N+1})$ such that $(\mathcal{H}_\gamma,\Theta;\mathtt{Q}_\Pi)\vdash^{*a_\ell}_{\mathbb{I}_N}\Gamma,C;\Pi^{\{\cdot\}}$ and $(\mathcal{H}_\gamma,\Theta;\mathtt{Q}_\Pi)\vdash^{*a_r}_{\mathbb{I}_N}\neg\exists x\, C^{(x,\mathbb{I}_N)},\Gamma;\Pi^{\{\cdot\}}$.

Let $\beta_\ell=\psi_{\mathbb{I}_N}(\widehat{a_\ell})\in\mathcal{H}_{\hat{a}+1}[\Theta(\mathtt{Q}_\Pi)]\cap\beta$ with $\widehat{a_\ell}=\gamma+\omega^{a_\ell}$. $\beta_\ell<\beta$ follows from $a_\ell\in\mathcal{H}_\gamma[\Theta(\mathtt{Q}_\Pi)]\subset\mathcal{H}_\gamma(\beta)$ and Proposition 3.17.1. IH with Σ-persistency 4.33 yields $(\mathcal{H}_{\hat{a}+1},\Theta;\mathtt{Q}_\Pi)\vdash^{*\beta_\ell}_{\beta}\Gamma^{(\beta,\mathbb{I}_N)},C^{(\beta_\ell,\mathbb{I}_N)};\Pi^{\{\cdot\}}$.

Inversion 4.30 yields $(\mathcal{H}_{\widehat{a_\ell}+1},\Theta;\mathtt{Q}_\Pi)\vdash^{*a_r}_{\mathbb{I}_N}\neg C^{(\beta_\ell,\mathbb{I}_N)},\Gamma;\Pi^{\{\cdot\}}$.

For $\beta_r=\psi_{\mathbb{I}_N}(\widehat{a_r})\in\mathcal{H}_{\hat{a}+1}[\Theta(\mathtt{Q}_\Pi)]\cap\beta$ with $\widehat{a_r}=\widehat{a_\ell}+1+\omega^{a_r}$, we obtain $\widehat{a_r}<\hat{a}$ by $a_\ell,a_r<a$, and $\beta_r<\beta$ follows from $\{a_\ell,a_r\}\subset\mathcal{H}_\gamma[\Theta(\mathtt{Q}_\Pi)]\subset\mathcal{H}_\gamma(\beta)$ and Proposition 3.17.1. IH with Σ-persistency 4.33 yields $(\mathcal{H}_{\hat{a}+1},\Theta;\mathtt{Q}_\Pi)\vdash^{*\beta_r}_{\beta}\neg C^{(\beta_\ell,\mathbb{I}_N)},\Gamma^{(\beta,\mathbb{I}_N)};\Pi^{\{\cdot\}}$. We obtain $(\mathcal{H}_{\hat{a}+1},\Theta;\mathtt{Q}_\Pi)\vdash^{*\beta}_{\beta}\Gamma^{(\beta,\mathbb{I}_N)};\Pi^{\{\cdot\}}$ by a (cut).

Case 3. The last inference is a $(\bigwedge)$: We have an $A\simeq\bigwedge(A_\iota)_{\iota\in J}$ such that $A\in\Gamma\subset\Sigma(\mathcal{L}_{N+1})$ and $(\mathcal{H}_\gamma,\Theta_\iota;\mathtt{Q}_\Pi)\vdash^{*a(\iota)}_{\mathbb{I}_N}\Gamma,A_\iota;\Pi^{\{\cdot\}}$ with $a(\iota)<a$ and $\Theta_\iota=\Theta\cup\mathcal{B}(\mathsf{k}(\iota))$ for each $\iota\in J$.

We obtain $\mathsf{k}(A)\subset\psi_{\mathbb{I}_N}(\gamma)$ by (26). Let $\iota\in J$. Since $A\in\Sigma(\mathcal{L}_{N+1})$, we obtain $\mathsf{k}(\iota)\subset\psi_{\mathbb{I}_N}(\gamma)$, and $\mathcal{B}(\mathsf{k}(\iota))\subset\psi_{\mathbb{I}_N}(\gamma)$ by Proposition 4.15.4. Let $\widehat{a_\iota}=\gamma+\omega^{a(\iota)}<\hat{a}$ by $a(\iota)<a$. Then $a(\iota)\in\mathcal{H}_\gamma[(\Theta_\iota)(\mathtt{Q}_\Pi)]\subset\mathcal{H}_\gamma(\beta)$ and $\beta_\iota=\psi_{\mathbb{I}_N}(\widehat{a_\iota})<\beta$. IH with Σ-persistency 4.33 yields $(\mathcal{H}_{\hat{a}+1},\Theta_\iota;\mathtt{Q}_\Pi)\vdash^{*\beta_\iota}_{\beta}\Gamma^{(\beta,\mathbb{I}_N)},(A_\iota)^{(\beta,\mathbb{I}_N)};\Pi^{\{\cdot\}}$. $(\mathcal{H}_{\hat{a}+1},\Theta;\mathtt{Q}_\Pi)\vdash^{*\beta}_{\beta}\Gamma^{(\beta,\mathbb{I}_N)};\Pi^{\{\cdot\}}$ follows by a $(\bigwedge)$.

Case 4. The last inference is a $(\bigvee)$: We have an $A\simeq\bigvee(A_\iota)_{\iota\in J}$ such that $A\in\Gamma$ and $(\mathcal{H}_\gamma,\Theta;\mathtt{Q}_\Pi)\vdash^{*a(\iota)}_{\mathbb{I}_N}\Gamma,A_\iota;\Pi^{\{\cdot\}}$ with $a(\iota)<a$ and an $\iota\in J$. Assuming $\mathsf{k}(\iota)\subset\mathsf{k}(A_\iota)$, we obtain $\mathsf{k}(\iota)\subset\beta$ by (26). IH followed by a $(\bigvee)$ yields the lemma.

Other cases are seen from IH as in [9]. □

Lemma 4.35 *Let $\Gamma\subset\Sigma(\mathcal{L}_0:\Omega)$ be a set of formulas. Suppose $\Theta\subset\mathcal{H}_\gamma(\psi_\Omega(\gamma))$ and $(\mathcal{H}_\gamma,\Theta;\emptyset)\vdash^{*a}_{\Omega,\gamma_1}\Gamma;\emptyset$. Let $\beta=\psi_\Omega(\hat{a})$ with $\hat{a}=\gamma+\omega^a<\gamma_1$. Then $(\mathcal{H}_{\hat{a}+1},\Theta;\emptyset)\vdash^{*\beta}_{\beta,\gamma_1}\Gamma^{(\beta,\Omega)};\emptyset$ holds.*

Proof. By induction on a as in [9]. □

4.5 Operator controlled derivations with caps

Let β be the ordinal in Collapsing 4.34, and $\Lambda := \Gamma(\beta)$. Λ is the base of the $\tilde{\theta}$-function $\tilde{\theta}_b(\xi) = \tilde{\theta}_b(\xi; \Lambda)$ in Definition 3.1. Definitions 4.36.4, 4.38 and 4.39 depend on the ordinals γ_0, Λ.

Definition 4.36 1. For a finite set Γ of formulas let $\mathrm{rk}(\Gamma) = \max(\{0\} \cup \{\mathrm{rk}(A) : A \in \Gamma\})$ and $\mathrm{rk}(\bigvee \Gamma) = \max(\{0\} \cup \{\mathrm{rk}(A) + 1 : A \in \Gamma\})$.

2. For a finite family $\mathtt{Q} \subset \coprod_{\mathbb{S}} \Psi_{\mathbb{S}}$ in the sense of Definition 4.22.1 let

$$\partial\mathtt{Q} := \{(\mathbb{S}, \max(\mathtt{Q}(\mathbb{S}))) : \mathbb{S} \in dom(\mathtt{Q}), \mathtt{Q}(\mathbb{S}) \neq \emptyset\}$$

and

$$\mathtt{Q}^{\circ} := \mathtt{Q} \setminus \partial\mathtt{Q} = \{(\mathbb{S}, \sigma) \in \mathtt{Q} : \sigma < \max(\mathtt{Q}(\mathbb{S}))\}.$$

Let $M_{\partial\mathtt{Q}} := \bigcap_{(\mathbb{S},\rho)\in\partial\mathtt{Q}} M_\rho$, and $\iota \in [\partial\mathtt{Q}]J :\Leftrightarrow \mathsf{k}(\iota) \subset M_{\partial\mathtt{Q}}$ for $\iota \in J$.

3. By a *capped formula* we mean a pair (A, ρ) of *RS*-sentence A and an ordinal $\rho \prec \mathbb{S}$ with a successor stable ordinal $\mathbb{S}$ such that $\mathsf{k}(A) \subset M_\rho$. Such a pair is denoted by $A^{(\rho)}$. It is convenient for us to regard *uncapped formulas* A as capped formulas $A^{(\mathtt{u})}$ with its cap $\mathtt{u}$, where $[\mathtt{u}]J = J$ with $M_{\mathtt{u}} = OT(\mathbb{I}_N) \cap \mathbb{I}_N$.

   A *sequent* is a finite set of capped or uncapped formulas, denoted by $\Gamma_0^{(\rho_0)}, \ldots, \Gamma_n^{(\rho_n)}, \Pi^{(\mathtt{u})}$, where each formula in the set $\Gamma_i^{(\rho_i)}$ puts on the cap ρ_i. When we write $\Gamma^{(\rho)}$, we tacitly assume that $\mathsf{k}(\Gamma) \subset M_\rho$.

   A capped formula $A^{(\rho)}$ is said to be a $\Sigma(\mathcal{L}_i : \pi)$-formula if $A \in \Sigma(\mathcal{L}_i : \pi)$. Let $\mathsf{k}(A^{(\rho)}) := \mathsf{k}(A)$.

4. A pair $\mathtt{Q} = ((\mathtt{Q})_0, \gamma_{\cdot}^{\mathtt{Q}})$ is said to be a *finite family for* γ_0 *with thresholds* if $(\mathtt{Q})_0$ is a finite family in the sense of Definition 4.22 and the following conditions are met. Let $dom(\mathtt{Q}) = dom((\mathtt{Q})_0)$, $\mathtt{Q}(\mathbb{S}) = (\mathtt{Q})_0(\mathbb{S})$, and $\bigcup \mathtt{Q} = \bigcup\{\mathtt{Q}(\mathbb{S}) : \mathbb{S} \in dom(\mathtt{Q})\}$.

   (a) $\gamma_{\cdot}^{\mathtt{Q}}$ is a map $dom(\mathtt{Q}) \ni \mathbb{S} \mapsto \gamma_{\mathbb{S}}^{\mathtt{Q}}$ such that $\gamma_0 + (\mathbb{I}_N)^2 > \gamma_{\mathbb{S}}^{\mathtt{Q}} \geq \gamma_0 + \mathbb{I}_N$, $\gamma_{\mathbb{S}}^{\mathtt{Q}} \geq \gamma_{\mathbb{T}}^{\mathtt{Q}} + \mathbb{I}_N$ for $\{\mathbb{S} < \mathbb{T}\} \subset dom(\mathtt{Q})$.
   $\mathtt{Q}$ is said to have *gaps* η if $\gamma_{\mathbb{S}}^{\mathtt{Q}} \geq \gamma_{\mathbb{T}}^{\mathtt{Q}} + \mathbb{I}_N \cdot \eta$ holds for $\{\mathbb{S} < \mathbb{T}\} \subset dom(\mathtt{Q})$, and $\gamma_{\mathbb{S}}^{\mathtt{Q}} \geq \gamma_0 + \mathbb{I}_N \cdot \eta$ for $\mathbb{S} \in dom(\mathtt{Q})$.

   (b) For each $\rho \in \mathtt{Q}(\mathbb{S})$, $m(\rho) : \Lambda \to \varphi_\Lambda(0)$ is special, and $\gamma_{\mathbb{S}}^{\mathtt{Q}} \leq \mathtt{p}_0(\rho)$.

   The thresholds function $\gamma_{\cdot}^{\mathtt{Q}}$ is uniquely extended for $\mathbb{S} \in \{\Omega\} \cup St$ by $\gamma_{\mathbb{S}}^{\mathtt{Q}} := \gamma_{\mathbb{T}}^{\mathtt{Q}}$ for $\mathbb{T} = \min\{\mathbb{T} \in dom(\mathtt{Q}) : \mathbb{T} \geq \mathbb{S}\}$ if such a $\mathbb{T}$ exists. Otherwise let $\gamma_{\mathbb{S}}^{\mathtt{Q}} = \gamma_0$.

   For an ordinal e, let $\mathtt{Q}\restriction e$ denote the restriction of $\mathtt{Q}$ to e. Namely $dom(\mathtt{Q}\restriction e) = \{\mathbb{S} \in dom(\mathtt{Q}) : \mathbb{S} < e\}$ and $\gamma_{\mathbb{S}}^{\mathtt{Q}\restriction e} = \gamma_{\mathbb{S}}^{\mathtt{Q}}$ for every $\mathbb{S} \in dom(\mathtt{Q}\restriction e)$.

5. For a finite family $\mathtt{Q}$ for γ_0 with thresholds and a pair $(\mathbb{S},\rho)$ such that $(\mathtt{Q})_0\cup\{(\mathbb{S},\rho)\}$ is a finite family for γ_0, $\mathtt{Q}\cup\{(\mathbb{S},\rho)\}=\mathtt{R}=((\mathtt{R})_0,\gamma^{\mathtt{R}}_{\cdot})$ denotes a finite family for γ_0 with thresholds enjoying the following:
   (a) $dom(\mathtt{R})=dom(\mathtt{Q})\cup\{\mathbb{S}\}$, $\mathtt{R}(\mathbb{T})=\mathtt{Q}(\mathbb{T})$ for $\mathbb{T}\neq\mathbb{S}$ and $\mathtt{R}(\mathbb{S})=\mathtt{Q}(\mathbb{S})\cup\{\rho\}$.
   (b) $\gamma^{\mathtt{R}}_{\cdot}$ extends $\gamma^{\mathtt{Q}}_{\cdot}$ in such a way that $\gamma^{\mathtt{R}}_{\mathbb{T}}=\gamma^{\mathtt{Q}}_{\mathbb{T}}$ for $\mathbb{T}\in dom(\mathtt{Q})$, $\gamma^{\mathtt{R}}_{\mathbb{S}}\geq\gamma^{\mathtt{Q}}_{\mathbb{T}}+\mathbb{I}_N$ for every $\mathbb{S}<\mathbb{T}\in dom(\mathtt{Q})$, $\gamma^{\mathtt{Q}}_{\mathbb{U}}\geq\gamma^{\mathtt{R}}_{\mathbb{S}}+\mathbb{I}_N$ for every $\mathbb{S}>\mathbb{U}\in dom(\mathtt{Q})$, and $\gamma^{\mathtt{R}}_{\mathbb{S}}\geq\gamma_0+\mathbb{I}_N$.

   A pair $\mathtt{Q}=((\mathtt{Q})_0,\gamma^{\mathtt{Q}}_{\cdot})$ is simply denoted by $\mathtt{Q}$ when $\gamma^{\mathtt{Q}}_{\cdot}$ is irrelevant.

Lemma 4.37 *Let $\rho\in\partial\mathtt{Q}(\mathbb{S})$ for a finite family $\mathtt{Q}$ for γ_0 with thresholds function $\gamma^{\mathtt{Q}}_{\cdot}$. Assume $\Theta\cup dom(\mathtt{Q})\subset M_\rho$ and $\forall\mathbb{T}\in dom(\mathtt{Q})\left(\mathtt{Q}^{\circ}(\mathbb{T})\subset\mathcal{H}_{\gamma^{\mathtt{Q}}_{\mathbb{S}}+\mathbb{I}_N}[\Theta(\mathtt{Q}^{\circ}\restriction\mathbb{T})]\right)$ for a finite set Θ of ordinals, cf. (29). Then $\bigcup_{\mathbb{T}\in dom(\mathtt{Q})}\mathtt{Q}^{\circ}(\mathbb{T})\subset M_\rho$ holds.*

Proof. Let $\mathbb{S},\mathbb{T}\in dom(\mathtt{Q})$ with $\rho\in\partial\mathtt{Q}(\mathbb{S})$. We show $\mathtt{Q}^{\circ}(\mathbb{T})\subset M_\rho$ by induction on the cardinality of the finite set $\{\mathbb{U}\in dom(\mathtt{Q}):\mathbb{U}<\mathbb{T}\}$. First let $\mathbb{S}\geq\mathbb{T}$ and $\sigma\in\mathtt{Q}^{\circ}(\mathbb{T})$. If $\mathbb{S}>\mathbb{T}$, then $\sigma<\mathbb{T}<\rho\in\Psi_{\mathbb{S}}$. $\sigma\in M_\rho$ follows. Otherwise $\sigma\in\mathtt{Q}^{\circ}(\mathbb{S})\subset\rho$ follows from $\rho\in\partial\mathtt{Q}(\mathbb{S})$. Next let $\mathbb{S}<\mathbb{T}$. We have $\Theta\subset M_\rho$ and $\mathtt{Q}^{\circ}(\mathbb{T})\subset\mathcal{H}_{\gamma^{\mathtt{Q}}_{\mathbb{T}}+\mathbb{I}_N}[\Theta(\mathtt{Q}^{\circ}\restriction\mathbb{T})]$ by the assumption. For $dom(\mathtt{Q})\ni\mathbb{U}<\mathbb{T}$ we have $\mathtt{Q}^{\circ}(\mathbb{U})\subset M_\rho$ by IH, and hence $\Theta(\mathtt{Q}^{\circ}\restriction\mathbb{T})\subset M_\rho$. On the other hand we have $\gamma^{\mathtt{Q}}_{\mathbb{T}}+\mathbb{I}_N<\gamma^{\mathtt{Q}}_{\mathbb{S}}\leq\mathtt{p}_0(\rho)$. Lemma 3.43.1 yields $\mathtt{Q}^{\circ}(\mathbb{T})\subset\mathcal{H}_{\mathtt{p}_0(\rho)}(M_\rho)\subset M_\rho$. □

Definition 4.38 Let $\rho\in\Psi_{\mathbb{S}}$ and Θ,Θ_1 be finite sets of ordinals.

1. $\kappa\in L^{\mathtt{Q}}_{\rho}(\Theta,\Theta_1)$ iff $\kappa\in\Psi_{\mathbb{S}}\cap\rho$, $\Theta\cup\Theta_1\cup\{\mathtt{p}_0(\rho)\}\cup SC(m(\rho))\cup\mathtt{Q}^{\circ}(\mathbb{S})\subset M_\kappa$, $\gamma^{\mathtt{Q}}_{\mathbb{S}}\leq\mathtt{p}_0(\kappa)\leq\mathtt{p}_0(\rho)$, $\kappa\in\mathcal{H}_{\gamma^{\mathtt{Q}}_{\mathbb{S}}+\mathbb{I}_N}[\Theta(\mathtt{Q}^{\circ}\restriction\mathbb{S})]$, and $m(\kappa)$ is special.

2. $H^{\mathtt{Q}}_{\rho}(f,\Theta,\Theta_1)$ denotes the *resolvent class* defined by $\kappa\in H^{\mathtt{Q}}_{\rho}(f,\Theta,\Theta_1)$ iff $\kappa\in L^{\mathtt{Q}}_{\rho}(\Theta,\Theta_1)$ and $f\leq m(\kappa)$, where $f\leq g:\Leftrightarrow\forall i(f'(i)\leq g'(i))$ for special finite functions f,g.

Let Γ be a sequent, Θ a finite set of ordinals$<\mathbb{I}_N$, $\{\gamma,a,c,d,e\}\subset OT(\mathbb{I}_N)$, and $\mathtt{Q}$ a finite family for γ_0 with thresholds.

We define another derivability relation $(\mathcal{H}_\gamma,\Theta,\mathtt{Q})\vdash^{a}_{c,d,e,\gamma_0}\Gamma$, where c is a bound of ranks of cut formulas, d a bound of ranks in the inference rules $(i\text{−rfl}_{\mathbb{S}}(\rho,f,\Theta_1))$, and e a bound of ordinals $\mathbb{S}$. The relation depends on ordinals β,γ_0, and should be written as $(\mathcal{H}_\gamma,\Theta;\mathtt{Q})\vdash^{a}_{c,d,e,\beta,\gamma_0}\Gamma$. However the ordinals β,γ_0 will be fixed. So let us omit it.

Definition 4.39 Let $\mathtt{Q}=((\mathtt{Q})_0,\gamma^{\mathtt{Q}}_{\cdot})$ be a finite family for γ_0 with thresholds, Θ a finite set of ordinals such that $\mathcal{B}(\Theta)\subset\Theta$, and a,c,d,e ordinals such that $dom(\mathtt{Q})\subset e$. Let $\beta<\psi_{\mathbb{I}_N}(\gamma_0)$ be a fixed ordinal in Collapsing 4.34 and $\Lambda=\Gamma(\beta)$.

Let $\Gamma=\bigcup\{\Gamma^{(\rho)}_{\rho}:\rho\in\{\mathtt{u}\}\cup\bigcup\mathtt{Q}\}$ be a set of formulas such that $\mathsf{k}(\Gamma_\rho)\subset M_\rho\cap M_{\partial\mathtt{Q}}$ for each cap $\rho\in\{\mathtt{u}\}\cup\bigcup\mathtt{Q}$.

$(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_{c,d,e} \Gamma$ holds if $\gamma_0 \leq \gamma$, each of the following (27), (28) and (29) holds, cf. (21) and (22), and one of the following cases $(\bigvee)$, $(\bigwedge)$, (cut), $(\Sigma(\Omega)\text{-rfl})$ and $(i\text{−rfl}_{\mathbb{S}}(\rho, f, \Theta_1))$ holds:

$$\Theta(\mathtt{Q}^\circ) \subset M_{\partial \mathtt{Q}} \tag{27}$$

$$\{\mathbb{S}, \gamma^{\mathtt{Q}}_{\mathbb{S}} : \mathbb{S} \in dom(\mathtt{Q})\} \cup \{\gamma, a, c, d, e, \beta, \gamma_0\} \cup \mathsf{k}(\Gamma) \subset \mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)] \tag{28}$$

$$\forall \mathbb{S} \in dom(\mathtt{Q}) \left(\mathtt{Q}(\mathbb{S}) \subset \mathcal{H}_{\gamma^{\mathtt{Q}}_{\mathbb{S}} + \mathbb{I}_N}[\Theta(\mathtt{Q}^\circ \restriction \mathbb{S})] \right) \tag{29}$$

$(\bigvee)$ There exist an $A \simeq \bigvee(A_\iota)_{\iota \in J}$, a cap $\rho \in \{\mathtt{u}\} \cup \bigcup \mathtt{Q}$, $\iota \in [\rho]J$, and an ordinal $a(\iota) < a$ such that $A^{(\rho)} \in \Gamma$ and $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a(\iota)}_{c,d,e} \Gamma, (A_\iota)^{(\rho)}$.

$(\bigwedge)$ There exist an $A \simeq \bigwedge(A_\iota)_{\iota \in J}$ and a cap $\rho \in \{\mathtt{u}\} \cup \bigcup \mathtt{Q}$ such that $A^{(\rho)} \in \Gamma$. For $\iota \in [\rho]J \cap [\partial \mathtt{Q}]J$, there is an ordinal $a(\iota) < a$ such that $(\mathcal{H}_\gamma, \Theta_\iota, \mathtt{Q}) \vdash^{a(\iota)}_{c,d,e} \Gamma, (A_\iota)^{(\rho)}$ holds for $\Theta_\iota = \Theta \cup \mathcal{B}(\mathsf{k}(\iota))$.

(cut) There exist $\rho \in \{\mathtt{u}\} \cup \bigcup \mathtt{Q}$ an ordinal $a_0 < a$, and a formula C with $\mathrm{rk}(C) < c$, for which $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a_0}_{c,d,e} \Gamma, \neg C^{(\rho)}$ and $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a_0}_{c,d,e} C^{(\rho)}, \Gamma$ hold.

$(\Sigma(\Omega)\text{-rfl})$ There exist ordinals $a_\ell, a_r < a$ and an uncapped formula $C \in \Sigma(\mathcal{L}_0 : \Omega)$ such that $c \geq \Omega$, $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a_\ell}_{c,d,e} \Gamma, C$ and $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a_r}_{c,d,e} \neg \exists x < \pi\, C^{(x,\Omega)}, \Gamma$.

$(i\text{−rfl}_{\mathbb{S}}(\rho, f, \Theta_1))$ There exists a successor i-stable ordinal $\mathbb{S} < e$ such that

$$\mathbb{S} \in \mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)] \tag{30}$$

$\rho \in \Psi_{\mathbb{S}}$ is an ordinal such that $\rho = \max(\mathtt{Q}^\rho(\mathbb{S}))$, i.e., $\rho \in \partial \mathtt{Q}^\rho(\mathbb{S})$ and

$$\Theta \subset M_\rho \,\&\, SC(\rho) \cup \{\mathtt{p}_0(\rho)\} \subset M_{\partial \mathtt{Q}} \,\&\, \rho \in \mathcal{H}_{\gamma^{\mathtt{Q}^\rho}_{\mathbb{S}} + \mathbb{I}_N}[\Theta(\mathtt{Q}^\circ \restriction \mathbb{S})] \tag{31}$$

where $\mathtt{Q}^\rho = \mathtt{Q} \cup \{(\mathbb{S}, \rho)\}$, cf. (27) and (29), and $s \in \mathrm{supp}(m(\rho))$ is an ordinal, f is a special function, $a_0 < a$ is an ordinal, D is an $\mathcal{L}_i$-formula, which is a finite conjunction with $D \equiv \bigwedge(D_n)_{n<m}$, and Θ_1 is a finite set of ordinals such that $\Theta_1 \subset M_{\partial \mathtt{Q}^\rho}$ enjoying the following conditions (r1), (r2), (r3) and (r4).

(r1) $\mathrm{rk}(D) < \min\{s, d\}$.

(r2) For $g = m(\rho)$, $SC(f) \subset \mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)]$ and $f_s = g_s \,\&\, f^s <^s_\Lambda g'(s)$, cf. Definition 3.31.6.

(r3) For each $n < m$, $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}^\rho) \vdash^{a_0}_{c,d,e} \Gamma, D_n^{(\rho)}$ holds.

(r4) $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}^{\rho\sigma}) \vdash^{a_0}_{c,d,e} \Gamma, \neg D^{(\sigma)}$ holds for every $\sigma \in H^{\mathtt{Q}^\rho}_\rho(f, \Theta, \Theta_1)$, where $(\mathtt{Q}^{\rho\sigma})_0 = (\mathtt{Q}^\rho)_0 \cup \{(\mathbb{S}, \sigma)\}$ and $\gamma^{\mathtt{Q}^{\rho\sigma}}_{\cdot} = \gamma^{\mathtt{Q}^\rho}_{\cdot}$.

In this subsection the ordinals β and γ_0 will be fixed, and we write $\vdash^a_{c,d,e}$ for $\vdash^a_{c,d,e,\beta,\gamma_0}$. Note that $\mathtt{Q}(\mathbb{S}) \subset \mathcal{H}_\gamma[\Theta]$ need not to hold.

Lemma 4.40 (Tautology) *Let $\mathtt{Q}$ be a finite family for γ_0 with thresholds $\gamma^{\mathtt{Q}}_{\cdot}$, b, e, γ be ordinals, and $\rho \in \{\mathtt{u}\} \cup \bigcup \mathtt{Q}$ such that $\mathsf{k}(A) \subset M_\rho$ for a formula A.*

Assume that $\mathcal{B}(\Theta) \subset \Theta$, and $\Theta, \mathtt{Q}, b, e, \beta, \gamma_0, \gamma, \gamma^{\mathtt{Q}}_{\cdot}$ and A enjoy (27), (28), and (29).

Then $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{2d}_{0,b,e,\beta,\gamma_0} \neg A^{(\rho)}, A^{(\rho)}$ holds for $d = \mathrm{rk}(A)$.

Proof. By induction on $d = \mathrm{rk}(A)$. By (28) we have $\mathsf{k}(A) \subset \mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)]$. Let $A \simeq \bigvee(A_\iota)_{\iota \in J}$ and $\iota \in [\partial\mathtt{Q}]J \cap [\rho]J$. Then $\mathsf{k}(\iota) \subset M_{\partial\mathtt{Q}} \cap M_\rho$ for (27). On the other hand we have $d_\iota = \mathrm{rk}(A_\iota) < \mathrm{rk}(A) = d$ and $d_\iota \in \mathcal{H}_0[\mathsf{k}(A_\iota)][\subset \mathcal{H}_0[\mathsf{k}(A, \iota)] \subset \mathcal{H}_\gamma[\Theta_\iota(\mathtt{Q}^\circ)]$ for $\Theta_\iota = \Theta \cup \mathcal{B}(\mathsf{k}(\iota))$. Hence (28) is enjoyed in $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{2d_\iota}_{0,b,e,\beta,\gamma_0} \neg A^{(\rho)}_\iota, A^{(\rho)}_\iota$. □

Lemma 4.41 *Let $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_{c,d,e} \Gamma$. Let $\rho \in \Psi_{\mathbb{S}}$ be an ordinal such that $\mathtt{R} = \mathtt{Q} \cup \{(\mathbb{S}, \rho)\}$ is a finite family for γ_0 with thresholds, $\{\mathbb{S}, \gamma^{\mathtt{R}}_{\mathbb{S}}\} \subset \mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)]$, $\Theta \cup dom(\mathtt{Q}) \subset M_\rho$, $SC(\rho) \cup \{\mathtt{p}_0(\rho)\} \subset M_{\partial\mathtt{Q}}$ and $\rho \in \mathcal{H}_{\gamma^{\mathtt{R}}_{\mathbb{S}}+\mathbb{I}_N}[\Theta(\mathtt{Q}^\circ \restriction \mathbb{S})]$, cf. (31).*

Then $(\mathcal{H}_\gamma, \Theta, \mathtt{R}) \vdash^a_{c,d,e} \Gamma$ holds.

Proof. By induction on a as in Lemma 4.29. Let $\rho \in \partial\mathtt{R}(\mathbb{S})$. By $\Theta \subset M_\rho$ and Lemma 4.37, (27) holds in $(\mathcal{H}_\gamma, \Theta, \mathtt{R}) \vdash^a_{c,d,e} \Gamma$. Also we have $\mathtt{Q}^\circ \subset \mathtt{R}^\circ$ for (28). □

Lemma 4.42 (Reduction) *Let $C \simeq \bigvee(C_\iota)_{\iota \in J}$, and $\mathrm{rk}(C) \le c$ with $\Omega \le c < \mathbb{I}_N$. Assume $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_{c,d,e,\beta,\gamma_0} \Gamma_0, \neg C^{(\tau)}$ and $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^b_{c,d,e,\beta,\gamma_0} C^{(\tau)}, \Gamma_1$.*

Then $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a\#b}_{c,d,e,\beta} \Gamma_0, \Gamma_1$ holds for the natural sum $a\#b$ of ordinals a and b.

Proof. By induction on $a\#b$. In the proof let us write $\vdash^a_c$ for $\vdash^a_{c,d,e,\beta,\gamma_0}$.

Case 1. The last inference in $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_c \Gamma_0, \neg C^{(\tau)}$ is a $(\bigwedge)$ with its major formula $\neg C^{(\tau)}$, and one in $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^b_c C^{(\tau)}, \Gamma_1$ is a $(\bigvee)$ with its major formula $C^{(\tau)}$: We have $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{b(\iota)}_c (C_\iota)^{(\tau)}, C^{(\tau)}, \Gamma_1$ for an $\iota \in [\tau]J$ and a $b(\iota) < b$. We obtain $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a\#b(\iota)}_c (C_\iota)^{(\tau)}, \Gamma_0, \Gamma_1$ by IH.

We obtain $\mathsf{k}(C_\iota) \subset \mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)]$ by (28). On the other hand we have $\Theta(\mathtt{Q}^\circ) \subset M_{\partial\mathtt{Q}}$ by (27). Hence $\mathsf{k}(\iota) \subset M_{\partial\mathtt{Q}}$, i.e., $\iota \in [\partial\mathtt{Q}]J$ provided that $\mathsf{k}(\iota) \subset \mathsf{k}(C_\iota)$. $\mathcal{H}_\gamma[\Theta_\iota(\mathtt{Q}^\circ)] = \mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)]$ follows by Propositions 4.15.6 and 4.15.9 for $\Theta_\iota = \Theta \cup \mathcal{B}(\mathsf{k}(\iota))$. Moreover $\mathtt{Q}(\mathbb{S}) \subset \mathcal{H}_{\gamma^{\mathtt{Q}}_{\mathbb{S}}+\mathbb{I}_N}[\Theta(\mathtt{Q}^\circ \restriction \mathbb{S})]$ for every $\mathbb{S} \in dom(\mathtt{Q})$ by (29). On the other hand we have $(\mathcal{H}_\gamma, \Theta_\iota, \mathtt{Q}) \vdash^{a(\iota)}_c \Gamma_0, \neg C^{(\tau)}, \neg(C_\iota)^{(\tau)}$ for an $a(\iota) < a$. $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a(\iota)}_c \Gamma_0, \neg C^{(\tau)}, \neg(C_\iota)^{(\tau)}$ follows. IH yields $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a(\iota)\#b}_c \Gamma_0, \Gamma_1, \neg(C_\iota)^{(\tau)}$. We obtain $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a\#b}_c \Gamma_0, \Gamma_1$ by a (cut) with $\mathrm{rk}(C_\iota) < \mathrm{rk}(C) = c$. Suppressing the part $(\mathcal{H}_\gamma, \Theta, \mathtt{Q})$, let us depict it as follows.

$$\dfrac{\dfrac{\vdash^{a(\iota)}_c \Gamma_0, \neg C^{(\tau)}, \neg(C_\iota)^{(\tau)} \quad \vdash^b_c C^{(\tau)}, \Gamma_1}{\vdash^{a(\iota)\#b}_c \Gamma_0, \Gamma_1, \neg(C_\iota)^{(\tau)}}\,\mathrm{IH} \quad \dfrac{\vdash^a_c \Gamma_0, \neg C^{(\tau)} \quad \vdash^{b(\iota)}_c (C_\iota)^{(\tau)}, C^{(\tau)}, \Gamma_1}{\vdash^{a\#b(\iota)}_c (C_\iota)^{(\tau)}, \Gamma_0, \Gamma_1}\,\mathrm{IH}}{(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a\#b}_c \Gamma_0, \Gamma_1}\,(cut)$$

Case 2. One of $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_c \Gamma_0, \neg C^{(\tau)}$ and $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^b_c C^{(\tau)}, \Gamma_1$ follows from a (cut): For example let for $\mathrm{rk}(D) < c$ and $b_0 < b$

$$\frac{(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{b_0}_c C^{(\tau)}, \Gamma_1, \neg D^{(\rho)} \quad (\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{b_0}_c C^{(\tau)}, \Gamma_1, D^{(\rho)}}{(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^b_c C^{(\tau)}, \Gamma_1} \ (cut)$$

We obtain $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a\#b_0}_c \Gamma_0, \Gamma_1, \neg D^{(\rho)}$ and $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a\#b_0}_c \Gamma_0, \Gamma_1, D^{(\rho)}$ by IH. $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a\#b}_c \Gamma_0, \Gamma_1$ follows by a (cut).
Case 3. Otherwise: Consider the case when the last inference in $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_c \Gamma_0, \neg C^{(\tau)}$ is an $(i - \mathrm{rfl}_{\mathbb{S}}(\rho, f, \Theta_1))$ with an ordinal $\mathbb{S} < e$. We have $\Theta \subset M_\rho$ by (31) and D is a finite conjunction $D \simeq \bigwedge(D_n)_{n<m}$. For $n < m$ and each $\sigma \in H^{\mathtt{Q}^\rho}_\rho(f, \Theta, \Theta_1)$ we have

$$(\mathcal{H}_\gamma, \Theta, \mathtt{Q}^\rho) \vdash^{a_0}_c \Gamma_0, \neg C^{(\tau)}, D^{(\rho)}_n$$

and

$$(\mathcal{H}_\gamma, \Theta, \mathtt{Q}^{\rho\sigma}) \vdash^{a_0}_c \Gamma_0, \neg C^{(\tau)}, \neg D^{(\sigma)}$$

Lemma 4.41 yields $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}^\rho) \vdash^b_c C^{(\tau)}, \Gamma_1$ and $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}^{\rho\sigma}) \vdash^b_c C^{(\tau)}, \Gamma_1$. By IH we obtain $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}^\rho) \vdash^{a_0\#b}_c \Gamma_0, \Gamma_1, D^{(\rho)}_n$, and $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}^{\rho\sigma}) \vdash^{a_0\#b}_c \Gamma_0, \Gamma_1, \neg D^{(\sigma)}$ for each σ. An $(i - \mathrm{rfl}_{\mathbb{S}}(\rho, f, \Theta_1))$ yields $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a\#b}_c \Gamma_0, \Gamma_1$.

Other cases are seen similarly. □

Remark 4.43 In the **Case 3** of the proof of Reduction 4.42, e.g., when $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}^{\rho\sigma}) \vdash^{a_0}_{c,d,e,\beta} \Gamma_0, \neg C^{(\tau)}, \neg D^{(\sigma)}$ is derived from a $(\bigwedge)$ with $\Theta_2 \supset \Theta$

$$\frac{\{(\mathcal{H}_\gamma, \Theta_\iota, \mathtt{Q}^{\rho\sigma}) \vdash^{a_0(k,\iota)}_{c,d,e,\beta} \Gamma_0, \neg C^{(\tau)}, \neg(C_\iota)^{(\tau)}, \neg D^{(\sigma)}\}_{\iota\in[\partial\mathtt{Q}^{\rho\sigma}]J\cap[\tau]J}}{(\mathcal{H}_\gamma, \Theta, \mathtt{Q}^{\rho\sigma}) \vdash^{a_0}_{c,d,e,\beta} \Gamma_0, \neg C^{(\tau)}, \neg D^{(\sigma)}} \ (\bigwedge)$$

it is not possible to exchange the inference $(\bigwedge)$ with $(i - \mathrm{rfl}_{\mathbb{S}}(\rho, f, \Theta_1))$ since there may exist a $\sigma \in H^{\mathtt{Q}^\rho}_\rho(f, \Theta_\iota, \Theta_1)$ such that $\sigma \notin H^{\mathtt{Q}^\rho}_\rho(f, \Theta, \Theta_1)$. Specifically $\mathcal{H}_{\gamma^{\mathtt{Q}^\rho}_{\mathbb{S}}+\mathbb{I}_N}[\Theta(\mathtt{Q}^\circ \restriction \mathbb{S})] \subsetneq \mathcal{H}_{\gamma^{\mathtt{Q}^\rho}_{\mathbb{S}}+\mathbb{I}_N}[\Theta_\iota(\mathtt{Q}^\circ \restriction \mathbb{S})]$, cf. Definition 4.38. This means that an Inversion lemma does not hold for the derivability relation $\vdash$.

Lemma 4.44 (Cut-elimination) *If $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_{c+c_1,d,e,\beta,\gamma_0} \Gamma$ with $\Omega \le c \in \mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)]$ and $c + c_1 < \mathbb{I}_N$, then $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{\varphi_{c_1}(a)}_{c,d,e,\beta,\gamma_0} \Gamma$.*

Proof. By main induction on c_1 with subsidiary induction on a using Reduction 4.42. □

5 Elimination of stable ordinals

5.1 Capping and recapping

In this subsection the relation $\vdash^*$ is embedded in $\vdash$ by putting caps on formulas, and then caps are changed to smaller caps.

Lemma 5.1 (Capping) *Let $\Gamma\cup\Pi\subset\Delta_0(\mathcal{L}_{N+1})$ be a set of uncapped formulas with $\mathrm{rk}(\Gamma\cup\Pi)<\beta$, where $\beta<\psi_{\mathbb{I}_N}(\gamma_0)$ is a fixed limit ordinal in Collapsing 4.34 such that $a,\beta<\mathbb{I}_N$ and $dom(\mathtt{Q}_\Pi)\subset\beta$. Let $(\mathcal{H}_\gamma,\Theta;\mathtt{Q}_\Pi)\vdash^{*a}_{\beta,\gamma_0}\Gamma;\Pi^{\{\cdot\}}$, where $\mathbb{I}_N\leq\gamma\leq\gamma_0$, $\Gamma=\Gamma_{\mathtt{u}}\cup\bigcup_{\mathbb{S}\in dom(\mathtt{Q}_\Pi)}\Gamma_{\mathbb{S}}$, and $\Pi^{\{\cdot\}}=\bigcup_{(\mathbb{S},\sigma)\in\mathtt{Q}_\Pi}\Pi^{\{\sigma\}}_\sigma$. Let $\Lambda=\Gamma(\beta)$.*

For each $\mathbb{S}\in dom(\mathtt{Q}_\Pi)$, let $\rho_{\mathbb{S}}=\psi^{g_{\mathbb{S}}}_{\mathbb{S}}(\delta_{\mathbb{S}})$ be an ordinal with a $\delta_{\mathbb{S}}$ and a special finite function $g_{\mathbb{S}}=m(\rho_{\mathbb{S}}):\Lambda\to\varphi_\Lambda(0)$ such that $\mathrm{supp}(g_{\mathbb{S}})=\{\beta\}$ with $g_{\mathbb{S}}(\beta)=\alpha_{\mathbb{S}}+\Lambda$, $\Lambda(2a+1)\leq\alpha_{\mathbb{S}}+\Lambda$, $SC(g_{\mathbb{S}})=SC(\beta,\alpha_{\mathbb{S}})\subset\mathcal{H}_0(SC(\delta_{\mathbb{S}}))$, cf. (11), and $\{\alpha_{\mathbb{S}},\delta_{\mathbb{S}}\}\subset\mathcal{H}_\gamma[\Theta]$.

Let $\mathtt{Q}=((\mathtt{Q})_0,\gamma^{\mathtt{Q}}_{\cdot})$ be a finite family for γ_0 with thresholds such that the following holds.

1. *The thresholds function $\gamma^{\mathtt{Q}}_{\cdot}$ enjoys $\gamma^{\mathtt{Q}}_{\mathbb{S}}\leq\delta_{\mathbb{S}}<\gamma^{\mathtt{Q}}_{\mathbb{S}}+\mathbb{I}_N$ for each $\mathbb{S}\in dom(\mathtt{Q})$.*
2. *$\mathtt{Q}(\mathbb{S})=\mathtt{Q}_\Pi(\mathbb{S})\cup\{\rho_{\mathbb{S}}\}$ for $\mathbb{S}\in dom(\mathtt{Q})=dom(\mathtt{Q}_\Pi)$.*

Let $\widehat{\Gamma}=\Gamma_{\mathtt{u}}\cup\bigcup_{\mathbb{S}\in dom(\mathtt{Q}_\Pi)}\{A^{(\rho_{\mathbb{S}})}:A\in\Gamma_{\mathbb{S}}\}$, and $\Pi^{(\cdot)}=\bigcup_{(\mathbb{S},\sigma)\in\mathtt{Q}_\Pi}\Pi^{(\sigma)}_\sigma$. Assume the following:

1. *$\Theta\subset\mathcal{H}_{\gamma_0}(\psi_{\mathbb{I}_N}(\gamma_0))$.*
2. *$\gamma^{\mathtt{Q}}_{\mathbb{S}}\in\mathcal{H}_\gamma[\Theta]$, $\Theta\cup\mathtt{Q}_\Pi(\mathbb{S})\subset M_{\rho_{\mathbb{S}}}$ and $\mathtt{Q}_\Pi(\mathbb{S})\subset\mathcal{H}_{\gamma^{\mathtt{Q}}_{\mathbb{S}}+\mathbb{I}_N}[\Theta(\mathtt{Q}_\Pi\restriction\mathbb{S})]$ for every $\mathbb{S}\in dom(\mathtt{Q}_\Pi)$.*
3. *$\mathtt{p}_0(\sigma)\leq\mathtt{p}_0(\rho_{\mathbb{S}})=\delta_{\mathbb{S}}$ for each $(\mathbb{S},\sigma)\in\mathtt{Q}_\Pi$.*
4. *$\mathtt{Q}$ has gaps $(\varphi_{\beta+1}(\beta)+1)\cdot 2^a$.*

Then $(\mathcal{H}_{\gamma_0},\Theta,\mathtt{Q})\vdash^{2a}_{\beta,\beta,\beta,\beta,\gamma_0}\widehat{\Gamma},\Pi^{(\cdot)}$ holds.

Remark 5.2 We have $\{\gamma_0,a,\beta\}\subset\mathcal{H}_\gamma[\Theta]$ by (21). Let $\gamma^{\mathtt{Q}}_{\mathbb{S}}=\gamma_0+\mathbb{I}_N\cdot(\varphi_{\beta+1}(\beta)+1)\cdot 2^a\cdot k$ for $k=\#\{\mathbb{T}\in dom(\mathtt{Q}):\mathbb{T}\geq\mathbb{S}\}$. Then $\gamma^{\mathtt{Q}}_{\mathbb{S}}\in\mathcal{H}_\gamma[\Theta]$ for (28). $\gamma^{\mathtt{Q}}_{\cdot}$ is a threshold function in Definition 4.36.4a.

For the gap $\varphi_{\beta+1}(\beta)+1$, see Lemma 5.11.

Proof of Lemma 5.1. This is seen by induction on a. Let us write $\vdash^a_\beta$ for $\vdash^a_{\beta,\beta,\beta,\beta,\gamma_0}$ in the proof.

We have $dom(\mathtt{Q}_\Pi)\subset\beta$, $\{\gamma,a,\beta,\gamma_0\}\cup dom(\mathtt{Q}_\Pi)\subset\mathcal{H}_\gamma[\Theta]$ by (21), and for each $A\in\Gamma\cup\Pi$, $\mathsf{k}(A)\subset\mathcal{H}_\gamma[\Theta(\mathtt{Q}_\Pi)]$ by (22).

The assumption $\mathtt{Q}_\Pi(\mathbb{S})\subset M_{\rho_{\mathbb{S}}}$ means that $\rho_{\mathbb{S}}=\max(\mathtt{Q}(\mathbb{S}))$ and $\rho_{\mathbb{S}}\in\partial\mathtt{Q}(\mathbb{S})$. Hence $\mathtt{Q}^\circ=\mathtt{Q}_\Pi$. We have $\forall\mathbb{S}\in dom(\mathtt{Q}_\Pi)(\gamma^{\mathtt{Q}}_{\mathbb{S}}\in\mathcal{H}_\gamma[\Theta])$ by the assumption.

On the other hand we have $\forall\mathbb{S}\in dom(\mathtt{Q}_\Pi)(\mathtt{Q}_\Pi(\mathbb{S})\subset\mathcal{H}_{\gamma^{\mathtt{Q}}_{\mathbb{S}}+\mathbb{I}_N}[\Theta(\mathtt{Q}^\circ\restriction\mathbb{S})])$ by the assumption, and $\{\mathbb{S},\beta,\alpha_{\mathbb{S}},\delta_{\mathbb{S}}\}\subset\mathcal{H}_\gamma[\Theta]$ with $\delta_{\mathbb{S}}<\gamma^{\mathtt{Q}}_{\mathbb{S}}+\mathbb{I}_N$ by (21) and the assumptions. Hence by Proposition 4.15.8 we obtain $\rho_{\mathbb{S}}\in\mathcal{H}_{\gamma^{\mathtt{Q}}_{\mathbb{S}}+\mathbb{I}_N}[\Theta]$, and (29) is enjoyed. Therefore (28) and (29) are enjoyed in $(\mathcal{H}_{\gamma_0},\Theta,\mathtt{Q})\vdash^{2a}_{\beta,\beta,\beta,\beta,\gamma_0}\widehat{\Gamma},\Pi^{(\cdot)}$.

We have $\Theta(\mathtt{Q}^\circ) = \Theta(\mathtt{Q}_\Pi) = \Theta \cup \mathcal{B}\left(\bigcup\{\mathtt{Q}_\Pi(\mathbb{T}) : \mathbb{T} \in dom(\mathtt{Q}_\Pi)\}\right)$ with $\mathtt{Q}_\Pi(\mathbb{T}) = \mathtt{Q}^\circ(\mathbb{T})$ and $M_{\partial\mathtt{Q}} = \bigcap_{\mathbb{S}\in dom(\mathtt{Q})} M_{\rho_\mathbb{S}}$. We obtain $\Theta \subset M_{\rho_\mathbb{S}}$ and for $\mathbb{T} \in dom(\mathtt{Q})$, $\mathtt{Q}_\Pi(\mathbb{T}) \subset \mathcal{H}_{\gamma^{\mathtt{Q}}_{\mathbb{T}}+\mathbb{I}_N}[\Theta(\mathtt{Q}^\circ\restriction\mathbb{T})]$ by the assumption. Hence Lemma 4.37 yields

$$\Theta(\mathtt{Q}^\circ) \subset M_{\partial\mathtt{Q}} \tag{32}$$

and (27) is enjoyed.

We obtain $\mathsf{k}(\Gamma \cup \Pi) \subset M_{\partial\mathtt{Q}}$. Furthermore when $A \in \Pi_\sigma$, $\mathsf{k}(A) \subset M_\sigma$ is assumed. We obtain $\mathsf{k}(\Pi) \subset M_{\partial\mathtt{Q}} \cap M_\sigma$.

Case 1. First consider the case when the last inference is an $(i-\mathrm{stbl}(\mathbb{S}))$: We have a successor i-stable ordinal $\mathbb{S}$ such that $\mathbb{S} \in \mathcal{H}_\gamma[\Theta]$ by (23), a formula $B(0) \in \Delta_0(\mathcal{L}_i)$ with $\mathrm{rk}(B(0)) < \mathbb{S}$, an ordinal $a_0 < a$, and a term $u \in Tm(\mathbb{I}_N)$ with $\mathbb{S} \le \mathrm{rk}(B(u)) < \beta$.

For every ordinal $\sigma \in \Psi_\mathbb{S}$ such that $\Theta(\mathtt{Q}_\Pi) \subset M_\sigma$ and $\mathtt{p}_0(\sigma) \ge \gamma_0$, the following holds for $dom(\mathtt{R}_\Pi) = dom(\mathtt{Q}_\Pi) \cup \{\mathbb{S}\}$ and $\mathtt{R}^\sigma_\Pi = \mathtt{R}_\Pi \cup \{(\mathbb{S}, \sigma)\}$.

$$\frac{(\mathcal{H}_\gamma, \Theta; \mathtt{R}_\Pi) \vdash^{*a_0}_\beta \Gamma, B(u); \Pi^{\{\cdot\}} \quad \{(\mathcal{H}_\gamma, \Theta; \mathtt{R}^\sigma_\Pi) \vdash^{*a_0}_\beta \Gamma; \neg B(u)^{\{\sigma\}}, \Pi^{\{\cdot\}}\}_\sigma}{(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_\beta \Gamma; \Pi^{\{\cdot\}}}$$

When $\mathbb{S} \notin dom(\mathtt{Q}_\Pi)$, let $\mathtt{R} = \mathtt{Q} \cup \{(\mathbb{S}, \rho_\mathbb{S})\}$, and ordinals $\gamma^{\mathtt{R}}_\mathbb{S}$ and $\rho_\mathbb{S}$ are defined as follows. First let $\gamma^{\mathtt{R}}_\mathbb{T} = \gamma^{\mathtt{Q}}_\mathbb{T}$ for $\mathbb{T} \in dom(\mathtt{Q})$. If there is no $\mathbb{S} > \mathbb{T} \in dom(\mathtt{Q})$, then $\gamma^{\mathtt{R}}_\mathbb{S} = \gamma_0 + \mathbb{I}_N \cdot (\varphi_{\beta+1}(\beta) + 1) \cdot 2^{a_0}$. Assume there is a largest $\mathbb{S} > \mathbb{T} \in dom(\mathtt{Q})$. Then let $\gamma^{\mathtt{R}}_\mathbb{S} = \gamma^{\mathtt{Q}}_\mathbb{T} + \mathbb{I}_N \cdot (\varphi_{\beta+1}(\beta)+1) \cdot 2^{a_0}$. In each case we obtain $\gamma^{\mathtt{R}}_\mathbb{S} \in \mathcal{H}_\gamma[\Theta]$ by $\{\gamma_0, \beta, \gamma^{\mathtt{Q}}_\mathbb{T}, a_0\} \subset \mathcal{H}_\gamma[\Theta]$. Suppose that there is a least $\mathbb{S} < \mathbb{U} \in dom(\mathtt{Q})$. Since $\mathtt{Q}$ is assumed to have gaps $(\varphi_{\beta+1}(\beta)+1)\cdot 2^a$, we obtain $\gamma_0 + \mathbb{I}_N \cdot (\varphi_{\beta+1}(\beta)+1) \cdot 2^a \le \gamma^{\mathtt{Q}}_\mathbb{U}$ and $\gamma^{\mathtt{Q}}_\mathbb{T} + \mathbb{I}_N \cdot (\varphi_{\beta+1}(\beta) + 1) \cdot 2^a \le \gamma^{\mathtt{Q}}_\mathbb{U}$. We see from $a_0 < a$ that $\mathtt{R}$ has gaps $(\varphi_{\beta+1}(\beta) + 1) \cdot 2^{a_0}$.

Let $\alpha_\mathbb{S} = \Lambda(2a) > \Lambda(2a_0)$ and $\delta_\mathbb{S} = \gamma^{\mathtt{R}}_\mathbb{S} \# a \# \beta \# b$ for $b = \max(\{0\} \cup E_\mathbb{S}(\Theta))$ with the set $E_\mathbb{S}(\alpha)$ in Definition 4.10. We obtain $\alpha_\mathbb{S} < \mathbb{I}_N \le \gamma \le \gamma_0 \le \gamma^{\mathtt{R}}_\mathbb{S}$ and $\{\alpha_\mathbb{S}, \delta_\mathbb{S}\} \subset \mathcal{H}_0[\{a, \beta, \gamma^{\mathtt{R}}_\mathbb{S}\} \cup E_\mathbb{S}(\Theta)] \subset \mathcal{H}_\gamma[\Theta]$. Also $\delta_\mathbb{S} < \gamma^{\mathtt{R}}_\mathbb{S} + \mathbb{I}_N$ by $\max\{a, \beta, \mathbb{S}\} < \mathbb{I}_N$. Moreover $\{a, \beta\} \subset \mathcal{H}_0(SC(\delta_\mathbb{S}))$. Hence (11) is enjoyed for $\rho_\mathbb{S} = \psi^{g_\mathbb{S}}_\mathbb{S}(\delta_\mathbb{S})$, cf. Proposition 6.6.2.

Next we show $\Theta \cup \mathtt{Q}_\Pi(\mathbb{S}) \subset M_{\rho_\mathbb{S}}$. We have $\mathtt{Q}_\Pi(\mathbb{S}) = \emptyset$. We obtain $b = \max(\{0\} \cup E_\mathbb{S}(\Theta)) \in \mathcal{H}_{\delta_\mathbb{S}}(\rho_\mathbb{S}) \cap \mathbb{S} = \rho_\mathbb{S}$ by (7), and hence $E_\mathbb{S}(\Theta) \subset \rho_\mathbb{S}$. On the other hand we have $\Theta \subset \mathcal{H}_{\gamma_0}(\psi_{\mathbb{I}_N}(\gamma_0))$ by the assumption. Also $\gamma_0 \le \gamma^{\mathtt{R}}_\mathbb{S} \le \delta_\mathbb{S} = \mathtt{p}_0(\rho_\mathbb{S})$. Proposition 4.13 yields $\Theta \subset \mathcal{H}_{\mathtt{p}_0(\rho_\mathbb{S})}(\rho_\mathbb{S}) = M_{\rho_\mathbb{S}}$.

Let h be a special finite function such that $\mathrm{supp}(h) = \{\beta\}$ and $h(\beta) = \Lambda(2a_0 + 1)$. Then $h_\beta = (g_\mathbb{S})_\beta = \emptyset$ and $h^\beta <^\beta_\Lambda (g_\mathbb{S})'(\beta)$ by $h(\beta) = \Lambda(2a_0 + 1) < \Lambda(2a) \le \alpha_\mathbb{S} = (g_\mathbb{S})'(\beta)$. Let $\sigma \in H^{\mathtt{R}}_{\rho_\mathbb{S}}(h, \Theta, \emptyset)$. We have $\Theta \subset M_\sigma$ and $\sigma \in \mathcal{H}_{\gamma^{\mathtt{R}}_\mathbb{S}+\mathbb{I}_N}[\Theta(\mathtt{R}^\circ\restriction\mathbb{S})]$ by Definition 4.38.

For example let $\sigma = \psi^h_{\rho_\mathbb{S}}(\delta_\mathbb{S} + b + 1)$. We obtain, cf. (12), $SC(h) \cup \{\mathtt{p}_0(\sigma)\} \cup \Theta \cup \{\mathtt{p}_0(\rho_\mathbb{S})\} \cup SC(m(\rho_\mathbb{S})) = SC(\{a_0, a, \beta\}) \cup \Theta \cup \{\delta_\mathbb{S}, \alpha_\mathbb{S}\} \subset \mathcal{H}_\gamma[\Theta] \subset \mathcal{H}_{\delta_\mathbb{S}}(\sigma) = M_\sigma$ and $\mathtt{p}_0(\sigma) = \mathtt{p}_0(\rho_\mathbb{S})$. We see $\sigma \in \mathcal{H}_0[\{\rho_\mathbb{S}, \beta, a_0, \delta_\mathbb{S}\} \cup \Theta] \subset \mathcal{H}_{\gamma^{\mathtt{R}}_\mathbb{S}+\mathbb{I}_N}[\Theta]$ from Proposition 4.15.8. Therefore $\sigma \in H^{\mathtt{R}}_{\rho_\mathbb{S}}(h, \Theta, \emptyset)$.

Since $\mathtt{Q}$ is assumed to have gaps $(\varphi_{\beta+1}(\beta) + 1) \cdot 2^a$, we may assume that $\mathtt{R}$ as well as $\mathtt{R}^\sigma$ has gaps $(\varphi_{\beta+1}(\beta) + 1) \cdot 2^{a_0}$.

We obtain by IH for $\rho_{\mathbb{S}} > \sigma \in M_{\rho_{\mathbb{S}}}$ and $\mathrm{rk}(B(u)) < \beta$ for (r1), $(\mathcal{H}_{\gamma_0}, \Theta, \mathtt{R}) \vdash^{2a_0}_{\beta} \widehat{\Gamma}, B(u)^{(\rho_{\mathbb{S}})}, \Pi^{(\cdot)}$, and $(\mathcal{H}_{\gamma_0}, \Theta, \mathtt{R}^{\sigma}) \vdash^{2a_0}_{\beta} \widehat{\Gamma}, \Pi^{(\cdot)}, \neg B(u)^{(\sigma)}$.

Let $D \equiv \bigwedge(B(u))$ with $D \simeq \bigwedge(D_n)_{n<1}$ and $D_0 \equiv B(u)$. We obtain $\mathrm{rk}(D) = \mathrm{rk}(B(u)) + 1 < \beta$ and $(\mathcal{H}_{\gamma_0}, \Theta, \mathtt{R}) \vdash^{2a_0}_{\beta} \widehat{\Gamma}, D_0^{(\rho_{\mathbb{S}})}, \Pi^{(\cdot)}$ and $(\mathcal{H}_{\gamma_0}, \Theta, \mathtt{R}^{\sigma}) \vdash^{2a_0+1}_{\beta} \widehat{\Gamma}, \Pi^{(\cdot)}, \neg D^{(\sigma)}$ by a $(\bigvee)$. An $(i\text{–rfl}_{\mathbb{S}}(\rho_{\mathbb{S}}, h, \emptyset))$ yields $(\mathcal{H}_{\gamma_0}, \Theta, \mathtt{Q}) \vdash^{2a}_{\beta} \widehat{\Gamma}, \Pi^{(\cdot)}$.

$$\dfrac{(\mathcal{H}_{\gamma_0}, \Theta, \mathtt{R}) \vdash^{2a_0}_{\beta} \widehat{\Gamma}, D_0^{(\rho_{\mathbb{S}})}, \Pi^{(\cdot)} \quad (\mathcal{H}_{\gamma_0}, \Theta, \mathtt{R}^{\sigma}) \vdash^{2a_0+1}_{\beta} \widehat{\Gamma}, \Pi^{(\cdot)}, \neg D^{(\sigma)}}{(\mathcal{H}_{\gamma_0}, \Theta, \mathtt{Q}) \vdash^{2a}_{\beta} \widehat{\Gamma}, \Pi^{(\cdot)}}\ (i\text{–rfl}_{\mathbb{S}}(\rho_{\mathbb{S}}, h, \emptyset))$$

Case 2. When the last inference is a (cut): There exist $a_0 < a$ and C such that $\mathrm{rk}(C) < \beta$, $(\mathcal{H}_{\gamma}, \Theta; \mathtt{Q}_{\Pi}) \vdash^{*a_0}_{\beta} \Gamma, \neg C; \Pi^{\{\cdot\}}$ and $(\mathcal{H}_{\gamma}, \Theta; \mathtt{Q}_{\Pi}) \vdash^{*a_0}_{\beta} \Gamma, C; \Pi^{\{\cdot\}}$. IH followed by a (cut) with an uncapped cut formula $C^{(\mathtt{u})}$ yields the lemma.
Case 3. Third the last inference introduces a $\bigvee$-formula A.
Case 3.1. First let $A \in \Gamma_{\mathbb{S}}$ be introduced by a $(\bigvee)$, and $A \simeq \bigvee(A_{\iota})_{\iota \in J}$. Then $A^{(\rho_{\mathbb{S}})} \in \Gamma^{(\rho_{\mathbb{S}})}$. There are an $\iota \in J$ and an ordinal $a(\iota) < a$ such that $(\mathcal{H}_{\gamma}, \Theta; \mathtt{Q}_{\Pi}) \vdash^{*a(\iota)}_{\beta} \Gamma, A_{\iota}; \Pi^{\{\cdot\}}$. We obtain $\mathsf{k}(\iota) \subset \mathcal{H}_{\gamma}[\Theta(\mathtt{Q}^{\circ})] \subset M_{\partial\mathtt{Q}}$ by (22) and (32) provided that $\mathsf{k}(\iota) \subset \mathsf{k}(A_{\iota})$. Hence $\iota \in [\partial\mathtt{Q}]J \subset [\rho_{\mathbb{S}}]J$. IH yields $(\mathcal{H}_{\gamma_0}, \Theta, \mathtt{Q}) \vdash^{2a(\iota)}_{\beta} \widehat{\Gamma}, (A_{\iota})^{(\rho_{\mathbb{S}})}, \Pi^{(\cdot)}$. $(\mathcal{H}_{\gamma_0}, \Theta, \mathtt{Q}) \vdash^{2a}_{\beta} \widehat{\Gamma}, \Pi^{(\cdot)}$ follows from a $(\bigvee)$.
Case 3.2. Second $A^{\{\sigma\}} \in \Pi^{\{\sigma\}}_{\sigma}$ is introduced by a $(\bigvee)^{\{\cdot\}}$ with $A \simeq \bigvee(A_{\iota})_{\iota \in J}$. Then $A^{(\sigma)} \in \Pi^{(\cdot)}$. There are an $\iota \in [\sigma]J$ and an ordinal $a(\iota) < a$ such that $(\mathcal{H}_{\gamma}, \Theta; \mathtt{Q}_{\Pi}) \vdash^{*a(\iota)}_{\beta} \Gamma; A_{\iota}^{\{\sigma\}}, \Pi^{\{\cdot\}}$. IH yields $(\mathcal{H}_{\gamma_0}, \Theta, \mathtt{Q}) \vdash^{2a(\iota)}_{\beta} \widehat{\Gamma}, (A_{\iota})^{(\sigma)}, \Pi^{(\cdot)}$. We obtain $(\mathcal{H}_{\gamma_0}, \Theta, \mathtt{Q}) \vdash^{2a}_{\beta} \widehat{\Gamma}, \Pi^{(\cdot)}$ with $A^{(\sigma)} \in \Pi^{(\cdot)}$ by a $(\bigvee)$.
Case 3.3. Third the case when $A \in \Gamma_{\mathtt{u}}$ is introduced by a $(\bigvee)$ is seen from IH.
Case 4. Fourth the last inference introduces a $\bigwedge$-formula A.
Case 4.1. First let $A \in \Gamma_{\mathbb{S}}$ be introduced by a $(\bigwedge)$, and $A \simeq \bigwedge(A_{\iota})_{\iota \in J}$. For every $\iota \in J$, $(\mathcal{H}_{\gamma}, \Theta_{\iota}; \mathtt{Q}_{\Pi}) \vdash^{*a(\iota)}_{\beta} \Gamma, A_{\iota}; \Pi^{\{\cdot\}}$ holds for an $a(\iota) < a$ and $\Theta_{\iota} = \Theta \cup \mathcal{B}(\mathsf{k}(\iota))$. Let $\iota \in [\partial\mathtt{Q}]J$. We obtain $\Theta_{\iota} \subset M_{\partial\mathtt{Q}} \subset M_{\rho_{\mathbb{S}}}$ for every $\mathbb{S} \in dom(\mathtt{Q})$. On the other hand we have $\mathrm{rk}(A) < \beta < \psi_{\mathbb{I}_N}(\gamma_0)$. Hence $\Theta_{\iota} \subset \mathcal{H}_{\gamma_0}(\psi_{\mathbb{I}_N}(\gamma_0))$. IH yields $(\mathcal{H}_{\gamma_0}, \Theta_{\iota}, \mathtt{Q}) \vdash^{2a(\iota)}_{\beta} \widehat{\Gamma}, (A_{\iota})^{(\rho_{\mathbb{S}})}, \Pi^{(\cdot)}$. $(\mathcal{H}_{\gamma_0}, \Theta, \mathtt{Q}) \vdash^{2a}_{\beta} \widehat{\Gamma}, \Pi^{(\cdot)}$ follows by a $(\bigwedge)$.
Case 4.2. Second $A^{\{\sigma\}} \in \Pi^{\{\cdot\}}$ is introduced by a $(\bigwedge)^{\{\cdot\}}$ with $(\mathbb{S}, \sigma) \in \mathtt{Q}_{\Pi}$. Let $A \simeq \bigwedge(A_{\iota})_{\iota \in J}$. For each $\iota \in [\sigma]J$ there is an ordinal $a(\iota) < a$ such that $(\mathcal{H}_{\gamma}, \Theta_{\iota}; \mathtt{Q}_{\Pi}) \vdash^{*a(\iota)}_{\beta} \Gamma; A_{\iota}^{\{\sigma\}}, \Pi^{\{\cdot\}}$.

For each $\iota \in [\sigma]J \cap [\partial\mathtt{Q}]J$, IH yields $(\mathcal{H}_{\gamma_0}, \Theta_{\iota}, \mathtt{Q}) \vdash^{2a(k,\iota)}_{\beta} \widehat{\Gamma}, (A_{\iota})^{(\sigma)}, \Pi^{(\cdot)}$. $(\mathcal{H}_{\gamma_0}, \Theta, \mathtt{Q}) \vdash^{2a}_{\beta} \widehat{\Gamma}, \Pi^{(\cdot)}$ follows from a $(\bigwedge)$ with $A^{(\sigma)} \in \Pi^{(\cdot)}$.
Case 4.3. Third the case when $A \in \Gamma_{\mathtt{u}}$ is introduced by a $(\bigwedge)$ is seen from IH.

The lemma follows from IH when the last inference is a $(\Sigma(\Omega)\text{-rfl})$. □

Definition 5.3 For a finite family $\mathtt{Q} = ((\mathtt{Q})_0, \gamma^{\mathtt{Q}}_{\cdot})$ for γ_0 with thresholds, let $\kappa_i \in L^{\mathtt{Q}}_{\rho_i}(\Theta, \emptyset)$ with a $(\mathbb{T}_i, \rho_i) \in \mathtt{Q}$ for each i.

$\mathtt{Q}^{[\kappa/\rho]} = ((\mathtt{Q}^{[\kappa/\rho]})_0, \gamma^{\mathtt{Q}}_{\cdot})$ denotes a finite family for γ_0 with thresholds defined as follows. $dom(\mathtt{Q}^{[\kappa/\rho]}) = dom(\mathtt{Q})$, and $\mathtt{Q}^{[\kappa/\rho]}(\mathbb{T}) = \{\kappa_i : \mathbb{T}_i = \mathbb{T}\} \cup \{\mu \in \mathtt{Q}(\mathbb{T}) : \mu \notin \{\rho_i : \mathbb{T}_i = \mathbb{T}\}\}$.

Lemma 5.4 (Recapping) *Let* $\mathtt{Q}$ *be a finite family for* γ_0 *with thresholds, b and d ordinals, and* $\mathbb{T} \leq b$ *a stable ordinal such that* $b \in \mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)]$. *Let* $\Xi = \bigcup\{\Xi_j^{(\tau_j)}\}_j$ *be a set of formulas,* $\Gamma = \bigcup\{\Gamma_i^{(\rho_i)}\}_i$ *a set of formulas such that* $\mathrm{rk}(\bigvee \Gamma_i) < b < s(\rho_i)$ *for each i, and* $\Pi = \bigcup\{\Pi_k^{(\lambda_k)}\}_k$ *a set of formulas such that* $\mathrm{rk}(\Pi_k) < d$ *for each k.*

Suppose $\{\rho_i\}_i \cup \{\lambda_k\}_k \subset \bigcup \partial\mathtt{Q}$, $\max\{a, b, d\} < \Lambda$, $d > b$ *and*

$$(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_{d,d,\mathbb{T}^\dagger,\beta,\gamma_0} \Xi, \Pi, \Gamma \tag{33}$$

For each i, let $\kappa_i \in H^{\mathtt{Q}}_{\rho_i}(h^b(g_i; 2b + \omega a), \Theta, \emptyset) \subset L^{\mathtt{Q}}_{\rho_i}(\Theta, \emptyset)$ *with* $g_i = m(\rho_i)$, *and* $\sigma_k \in L^{\mathtt{Q}}_{\lambda_k}(\Theta, \emptyset)$ *for each k. Let* $\Gamma_1 = \bigcup\{\Gamma_i^{(\kappa_i)}\}_i$ *and* $\Pi_1 = \bigcup\{\Pi_k^{(\sigma_k)}\}_k$. $\mathtt{Q}_1$ *denotes a finite family obtained from* $\mathtt{Q}$ *by replacing* ρ_i *by* κ_i, *and* λ_k *by* σ_k, *cf. Definition 5.3. Then*

$$(\mathcal{H}_\gamma, \Theta, \mathtt{Q}_1) \vdash^{2b+\omega a}_{d,b,\mathbb{T}^\dagger,\beta,\gamma_0} \Xi, \Pi_1, \Gamma_1 \tag{34}$$

holds.

Proof. By induction on a. The third, fourth and fifth subscripts $\mathbb{T}^\dagger$, β and γ_0 are fixed, and omitted in the proof. We write $\vdash^a_{c,d}$ for $\vdash^a_{c,d,\mathbb{T}^\dagger,\beta,\gamma_0}$. A special finite function $h^b(g; a)$ is defined from ordinals a, b and a function g in Definition 4.19. Note that $[\kappa_i]J \subset [\rho_i]J$ holds by $\kappa_i < \rho_i$.

Let $\kappa = \kappa_i \in L^{\mathtt{Q}}_\rho(\Theta, \emptyset)$ with $g = m(\rho)$, $\rho = \rho_i \in \partial\mathtt{Q}(\mathbb{S})$ and $\mathbb{T}^\dagger > \mathbb{S} \in dom(\mathtt{Q})$. By Definitions 4.22 and 4.38 we obtain $\Theta \cup \{\mathtt{p}_0(\rho)\} \cup SC(m(\rho)) \cup \mathtt{Q}^\circ(\mathbb{S}) \subset M_\kappa$, $\kappa \in \mathcal{H}_{\gamma^{\mathtt{Q}}_{\mathbb{S}}+\mathbb{I}_N}[\Theta(\mathtt{Q}^\circ \upharpoonright \mathbb{S})]$ and $\gamma^{\mathtt{Q}}_{\mathbb{S}} \leq \mathtt{p}_0(\kappa)$. Then $\kappa \in \partial\mathtt{Q}_1(\mathbb{S})$ and $\Theta \subset M_{\partial\mathtt{Q}_1}$. On the other hand we have $\{a, b, d\} \subset \mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)]$ by the assumption, where $\mathtt{Q}^\circ = \mathtt{Q}_1^\circ$. Moreover we have $SC(m(\kappa)) \cup \{\mathtt{p}_0(\kappa)\} \subset M_\kappa$ by Proposition 3.38. Hence (27) and (28) are enjoyed in $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}_1) \vdash^{2b+\omega a}_{d,b,\mathbb{T}^\dagger,\beta,\gamma_0} \Xi, \Pi_1, \Gamma_1$.

We have $\mathtt{Q}(\mathbb{S}) \cup \{\kappa\} \subset \mathcal{H}_{\gamma^{\mathtt{Q}}_{\mathbb{S}}+\mathbb{I}_N}[\Theta(\mathtt{Q}^\circ \upharpoonright \mathbb{S})]$ by (29) and $\kappa \in L^{\mathtt{Q}}_\rho(\Theta, \emptyset)$. This together with $\mathtt{Q}^\circ = \mathtt{Q}_1^\circ$ yields $\mathtt{Q}_1(\mathbb{S}) \subset \mathcal{H}_{\gamma^{\mathtt{Q}}_{\mathbb{S}}+\mathbb{I}_N}[\Theta(\mathtt{Q}_1^\circ \upharpoonright \mathbb{S})]$. Hence (29) is enjoyed for $\mathtt{Q}_1$.

By Lemma 4.37, (29) and $\Theta \subset M_\kappa$ we obtain $\Theta(\mathtt{Q}^\circ) \subset M_\kappa$ for $\kappa \in \partial\mathtt{Q}_1(\mathbb{S})$. From $\{a, b\} \subset \mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)]$ and $\Theta \cup SC(m(\rho)) \subset M_\kappa$ we see $SC(h^b(g; 2b+\omega a)) \subset M_\kappa$ by Lemma 3.43.1 and $\gamma \leq \gamma^{\mathtt{Q}}_{\mathbb{S}} \leq \mathtt{p}_0(\kappa)$. Also $\mathtt{p}_0(\rho) \in M_\kappa$ by $\kappa \in L^{\mathtt{Q}}_\rho(\Theta, \emptyset)$, cf. (12).

Case 1. First consider the case when the last inference is an $(i{-}\mathrm{rfl}_{\mathbb{S}}(\rho, f, \Theta_1))$ for an $\mathbb{S} \leq \mathbb{T}$: We have $\{\mathbb{S}, \mathbb{T}\} \subset \mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)]$ by (30) and (28). We have $\Theta \subset M_\rho$ by (31). Let $\Gamma_\rho = \Gamma_i^{(\rho_i)}$ if $\rho = \rho_i$, and $\Gamma_\rho = \emptyset$ else.

Let $g = m(\rho)$ and $s \in \mathrm{supp}(g)$. D is a finite conjunction with $D \simeq \bigwedge(D_n)_{n<m}$ and $\mathrm{rk}(D) < \min\{s, d\}$ by (r1) with $s \leq s(\rho)$, and $a_0 < a$ is an ordinal such that for $\mathtt{R} = \mathtt{Q} \cup \{(\mathbb{S}, \rho)\}$ and each $n < m$

$$(\mathcal{H}_\gamma, \Theta, \mathtt{R}) \vdash^{a_0}_{d,d} \Xi, \Pi, \Gamma, D_n^{(\rho)} \tag{35}$$

where $\rho \in \partial\mathtt{R}(\mathbb{S})$.

On the other side for each $\sigma\in H^{\mathtt{R}}_{\rho}(f,\Theta,\Theta_1)$ we have

$$(\mathcal{H}_\gamma,\Theta,\mathtt{R}^\sigma)\vdash^{a_0}_{d,d}\neg D^{(\sigma)},\Xi,\Pi,\Gamma$$

f is a special finite function such that $f_s=g_s$, $f^s<^s g'(s)$ and $SC(f)\subset\mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)]$. We obtain by IH

$$(\mathcal{H}_\gamma,\Theta,\mathtt{R}_1^\sigma)\vdash^{2b+\omega a_0}_{d,b}\neg D^{(\sigma)},\Xi,\Pi_1,\Gamma_1 \tag{36}$$

Let $c=\mathrm{rk}(D)<d$. Then $c\in\mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)]$ by (28).

Case 1.1. $c<b$: Then $\mathrm{rk}(D_n)+1<\mathrm{rk}(D)+1\leq c+1\leq b$ and $\mathrm{rk}(\bigvee(\Gamma_\rho\cup\{D_n\}))<b$. If $b\geq s(\rho)$, then let $\kappa=\rho$. If $b<s(\rho)$, then let $\Theta^+=\Theta_1\cup SC(m(\rho))\cup\{\mathtt{p}_0(\rho)\}$ and $\kappa\in H^{\mathtt{R}}_{\rho}(h^b(g;2b+\omega a),\Theta,\emptyset)$ for $g=m(\rho)$.

IH with (35) yields for $n<m$

$$(\mathcal{H}_\gamma,\Theta,\mathtt{R}_1)\vdash^{2b+\omega a_0}_{d,b}\Xi,\Pi_1,\Gamma_1,D_n^{(\kappa)} \tag{37}$$

Case 1.1.1. $b\geq s(\rho)$: Then $\rho\neq\rho_i$ for every i, and $\Gamma_\rho=\emptyset$. By (36) and (37) an $(i\text{–}\mathrm{rfl}_{\mathbb{S}}(\kappa,f,\Theta_1))$ yields (34) with $\kappa=\rho$ and $\mathrm{rk}(D)<\min\{s,b\}$.

Case 1.1.2. $b<s(\rho)$: We claim for the special finite function $h=h^b(g;2b+\omega a)\leq m(\kappa)$ and $s_1=\min\{b,s\}$ that if $b<s(\rho)$

$$f_{s_1}=h_{s_1}\ \&\ f^{s_1}<^{s_1}h'(s_1) \tag{38}$$

If $s_1=s\leq b$, then $h_s=g_s=f_s$ and $g'(s)=g(s)\leq h'(s)$. Proposition 3.6 yields the claim. If $s_1=b<s$, then Proposition 4.20.1 yields the claim.

Let $\sigma\in H^{\mathtt{R}_1}_{\kappa}(f,\Theta,\Theta^+)$. Then $\Theta\cup\Theta^+=\Theta\cup\Theta_1\cup SC(m(\rho))\cup\{\mathtt{p}_0(\rho)\}\subset M_\sigma$. Therefore $\sigma\in H^{\mathtt{R}}_{\rho}(f,\Theta,\Theta_1)$.

By (38), (37) and (36), an $(i\text{–}\mathrm{rfl}_{\mathbb{S}}(\kappa,f,\Theta^+))$ yields (34), where $\mathrm{rk}(D)<s_1\leq b$, $c<b$ and $s_1\in\mathrm{supp}(m(\kappa))$.

Case 1.2. $b\leq c$: Let $\sigma\in L:=H^{\mathtt{R}_1}_{\kappa}(h,\Theta,\Theta^+)$ for $\Theta^+=\Theta_1\cup SC(m(\rho))\cup\{\mathtt{p}_0(\rho)\}\subset M_\sigma$ and $h=(h^c(g;2b+\omega a_0))*f^{c+1}$. We obtain $L\subset L^{\mathtt{R}}_{\rho}(\Theta,\emptyset)\cap H^{\mathtt{R}}_{\rho}(f,\Theta,\Theta_1)$ and $SC(h)\subset\mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)]$.

IH with (35) for $\sigma\in L\subset L^{\mathtt{R}}_{\rho}(\Theta,\emptyset)$ and $\mathrm{rk}(\{D_n^{(\rho)}\}\cup\Gamma_\rho)<c<d$ yields $(\mathcal{H}_\gamma,\Theta,\mathtt{R}_2)\vdash^{2b+\omega a_0}_{d,b}\Xi,\Pi_1,\Gamma^{(\sigma)}_{\rho},D^{(\sigma)}_n,(\Gamma\setminus\Gamma_\rho)_1$ for each $n<m$, where $\Xi,\Pi,\Gamma,D_n^{(\rho)}=\Xi,\Pi\cup\{D_n^{(\rho)}\}\cup\Gamma_\rho,(\Gamma\setminus\Gamma_\rho)$, and each $A^{(\rho)}\in\Gamma_\rho$ is replaced by $A^{(\sigma)}$ in $\mathtt{R}_2$, while $B^{(\rho_i)}\in(\Gamma\setminus\Gamma_\rho)$ by $B^{(\kappa_i)}$ in $(\Gamma\setminus\Gamma_\rho)_1$. A $(\bigwedge)$ with Lemma 4.41 yields

$$(\mathcal{H}_\gamma,\Theta,(\mathtt{R}_1)^\sigma)\vdash^{2b+\omega a_0+1}_{d,b}\Xi,\Pi_1,\Gamma^{(\sigma)}_{\rho},D^{(\sigma)},(\Gamma\setminus\Gamma_\rho)_1 \tag{39}$$

where $(\mathtt{R}_1)^\sigma=\mathtt{R}_1\cup\{(\mathbb{S},\sigma)\}=\mathtt{R}_2\cup\{(\mathbb{S},\kappa)\}$.

On the other side, IH with $\sigma\in L\subset H^{\mathtt{R}}_{\rho}(f,\Theta,\Theta_1)$ yields (36).

A (cut) with $\mathrm{rk}(D)<d$, (39) and (36) yields

$$(\mathcal{H}_\gamma,\Theta,(\mathtt{R}_1)^\sigma)\vdash^{a_1}_{d,b}\Xi,\Pi_1,\Gamma_1,\Gamma^{(\sigma)}_{\rho}$$

for $2b\leq a_1=2b+\omega a_0+2<2b+\omega a$. Several $(\bigvee)$'s yield for a $p<\omega$

$$\forall\sigma\in L\left[(\mathcal{H}_\gamma,\Theta,(\mathtt{R}_1)^\sigma)\vdash^{a_1+p}_{d,b}\Xi,\Pi_1,\Gamma_1,\bigvee\Gamma^{(\sigma)}_{\rho}\right] \tag{40}$$

where $\bigvee\Gamma_\rho \equiv (A_0 \vee \cdots \vee A_{n-1})$ with $n = 0$ when $\Gamma_\rho = \emptyset$.

On the other, Tautology 4.40 yields $(\mathcal{H}_\gamma, \Theta, \mathtt{R}_1) \vdash^{2b}_{0,b} \Gamma_1, \neg\theta^{(\kappa)}$ for each $\theta^{(\rho)} \in \Gamma_\rho$. We obtain

$$(\mathcal{H}_\gamma, \Theta, \mathtt{R}_1) \vdash^{2b+p}_{0,b} \Gamma_1, \neg \bigvee \Gamma^{(\kappa)}_\rho \tag{41}$$

Let $k = h^b(g; 2b + \omega a)$. Then $h_b = g_b = k_b$ and $h <^b k'(b)$ for $h = (h^c(g; 2b + \omega a_0)) * f^{c+1}$ by Proposition 4.20.2.

By (41), (40) with $\max\{2b, a_1\} + p < 2b + \omega a$, $\mathrm{rk}(\bigvee\Gamma_\rho) < b$, (34) follows from an $(i\text{−rfl}_{\mathbb{S}}(\kappa, h, \Theta^+))$ with the resolvent class $L = H^{\mathtt{R}_1}_\kappa(h, \Theta, \Theta^+)$.

Case 2. Second consider the case when the last inference introduces a formula $B^{(\rho)} \in \Gamma$: For example let $B \simeq \bigwedge(B_\iota)_{\iota\in J}$. For each $\iota \in [\kappa]J \subset [\rho]J$, we obtain $\mathrm{rk}(\bigvee(\Gamma \cup \{B_\iota\})) = \mathrm{rk}(\bigvee\Gamma)$. IH followed by a $(\bigwedge)$ yields (34).

Case 3. Third consider the case when the last inference is a (cut) with a cut formula $C^{(\rho)}$: We have $\mathrm{rk}(C) < d$, and IH followed by a (cut) with the cut formula $C^{(\kappa)}$ yields (34).

Other cases are seen from IH. □

5.2 Eliminations of inferences (rfl)

In this subsection, inferences $(i\text{−rfl}_{\mathbb{S}}(\rho, f, \Theta_1))$ are removed from operator controlled derivations of sequents of formulas in $\Sigma(\Omega) \cup \Pi(\Omega)$.

Definition 5.5 We define the *S-rank* $\mathrm{srk}(A^{(\rho)})$ of a capped formula $A^{(\rho)}$ as follows. Let $\mathrm{srk}(\rho) = \mathbb{S} \in SSt$ for $\rho \in \Psi_{\mathbb{S}}$, and $\mathrm{srk}(\mathtt{u}) = 0$. $\mathrm{srk}(A^{(\rho)}) = \mathrm{srk}(\rho)$. $\mathrm{srk}(\Gamma) = \max\{\mathrm{srk}(A^{(\rho)}) : A^{(\rho)} \in \Gamma\}$.

Proposition 5.6 *Let* $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a}_{\mathbb{S},\mathbb{S},\mathbb{S}^\dagger,\beta,\gamma_0} \Xi, \Gamma^{(\cdot)}$ *with a finite family* $\mathtt{Q}$ *for* γ_0 *with thresholds, where* $\Gamma^{(\cdot)} = \bigcup\{\Gamma^{(\sigma)}_\sigma : (\mathbb{S}, \sigma) \in \mathtt{Q}\}$ *for* $\Gamma = \bigcup\{\Gamma_\sigma : (\mathbb{S}, \sigma) \in \mathtt{Q}\}$. *Assume that* $\max\{\mathrm{srk}(\Xi), \mathrm{rk}(\Xi \cup \Gamma^{(\cdot)})\} < \mathbb{S}$. *Let* $\gamma_1 = \gamma^{\mathtt{Q}}_{\mathbb{S}} + \mathbb{I}_N$ *if* $\mathbb{S} \in dom(\mathtt{Q})$, *and* $\gamma_1 = \gamma$ *else. Then* $(\mathcal{H}_{\gamma_1}, \Theta, \mathtt{R}) \vdash^{2a}_{\mathbb{S},\mathbb{S},\mathbb{S},\beta,\gamma_0} \Xi, \Gamma^{(\mathtt{u})}$ *holds for* $\mathtt{R} = \mathtt{Q} \restriction \mathbb{S}$ *and* $\Gamma^{(\mathtt{u})} = \{C^{(\mathtt{u})} : C \in \Gamma\}$.

Proof. By induction on a. The fourth and fifth subscripts β, γ_0 are omitted in the proof. If $\mathbb{S} \in dom(\mathtt{Q})$, then we have $\mathtt{Q}^\circ(\mathbb{S}) \subset \mathcal{H}_{\gamma_1}[\Theta(\mathtt{R}^\circ)]$ by (29), where $\mathtt{R}^\circ = \mathtt{Q}^\circ \restriction \mathbb{S}$. Hence (28) is enjoyed in $(\mathcal{H}_{\gamma_1}, \Theta, \mathtt{R}) \vdash^{2a}_{\mathbb{S},\mathbb{S},\mathbb{S},\beta,\gamma_0} \Xi, \Gamma^{(\mathtt{u})}$.

Case 1. First consider the case when the last inference is a $(\bigwedge)$ with its major formula $C^{(\sigma)} \in \Xi \cup \Gamma^{(\cdot)}$ with $C \simeq \bigwedge(C_\iota)_{\iota\in J}$: We have $(\mathcal{H}_\gamma, \Theta_\iota, \mathtt{Q}) \vdash^{a(\iota)}_{\mathbb{S},\mathbb{S},\mathbb{S}^\dagger}$ $\Xi, \Gamma^{(\cdot)}, C^{(\sigma)}_\iota$ for each $\iota \in [\partial\mathtt{Q}]J \cap [\sigma]J$. IH yields $(\mathcal{H}_{\gamma_1}, \Theta_\iota, \mathtt{R}) \vdash^{2a(\iota)}_{\mathbb{S},\mathbb{S},\mathbb{S}} \Xi, \Gamma^{(\mathtt{u})}, C^{(\mathtt{u})}_\iota$.

Let $\sigma_0 = \sigma$ if $\mathrm{srk}(\sigma) < \mathbb{S}$, and $\sigma_0 = \mathtt{u}$ else. We claim that $\iota \in [\partial\mathtt{Q}]J \cap [\sigma]J$ iff $\iota \in [\partial\mathtt{R}]J \cap [\sigma_0]J$ for each $\iota \in J$. We may assume that $\mathsf{k}(\iota) \subset \mathsf{k}(C_\iota)$. By the assumption and Proposition 4.5.6 we have $\mathrm{rk}(C_\iota) < \mathrm{rk}(C) < \mathbb{S}$ for each $\iota \in J$.

Let $\rho \in \partial\mathtt{Q}(\mathbb{S})$ and $\iota \in [\partial\mathtt{R}]J \cap [\sigma_0]J$. First let $C^{(\sigma)} \in \Gamma^{(\cdot)}$. We show $\iota \in [\sigma]J \cap [\rho]J$. We obtain $\mathsf{k}(C) \subset M_\sigma \cap \mathbb{S} = \sigma \leq \rho$, and hence $\mathsf{k}(\iota) \subset \sigma \subset M_\sigma \subset M_\rho$. Next let $C^{(\sigma)} \in \Xi$. We show $\iota \in [\rho]J$. We obtain $\mathsf{k}(C) \subset M_\rho \cap \mathbb{S} = \rho$, and hence $\mathsf{k}(\iota) \subset \rho \subset M_\rho$. The claim is shown.

A $(\bigwedge)$ yields $(\mathcal{H}_{\gamma_1}, \Theta, \mathtt{R}) \vdash^{2a}_{\mathbb{S},\mathbb{S},\mathbb{S}} \Xi, \Gamma^{(\mathtt{u})}$ with $C^{(\mathtt{u})} \in \Gamma^{(\mathtt{u})}$..

Case 2. Second consider the case when the last inference is an $(i\text{−rfl}_{\mathbb{S}}(\rho, f, \Theta_1))$: We have a finite conjunction $D \equiv \bigwedge(D_n)_{n<m}$ and an ordinal $a_0 < a$ such that $\text{rk}(D) < \text{srk}(\rho) = \text{srk}(\sigma) = \mathbb{S}$ by (r1), and

$$\frac{\{(\mathcal{H}_\gamma, \Theta, \mathtt{Q}^\rho) \vdash^{a_0}_{\mathbb{S},\mathbb{S},\mathbb{S}^\dagger} \Xi, \Gamma^{(\cdot)}, D_n^{(\rho)}\}_{n<m} \quad \{(\mathcal{H}_\gamma, \Theta, \mathtt{Q}^{\rho\sigma}) \vdash^{a_0}_{\mathbb{S},\mathbb{S},\mathbb{S}^\dagger} \Xi, \Gamma^{(\cdot)}, \neg D^{(\sigma)}\}_\sigma}{(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a}_{\mathbb{S},\mathbb{S},\mathbb{S}^\dagger} \Xi, \Gamma^{(\cdot)}}$$

We have $X = \Theta \cup \Theta_1 \cup \{\mathtt{p}_0(\rho)\} \cup SC(m(\rho)) \cup \mathtt{Q}^\circ(\mathbb{S}) \subset M_\rho$ for $\rho \in \partial\mathtt{Q}^\rho(\mathbb{S})$. Pick a $\sigma \in H^{\mathtt{Q}^\rho}_\rho(f, \Theta, \Theta_1)$. For example $\sigma = \psi^f_\rho(\alpha + \eta)$ for $\rho = \psi^g_\kappa(\alpha)$ and $\eta = \max(\{1\} \cup E_{\mathbb{S}}(X))$. IH yields

$$\dfrac{\dfrac{\{(\mathcal{H}_{\gamma_1}, \Theta, \mathtt{R}) \vdash^{2a_0}_{\mathbb{S},\mathbb{S},\mathbb{S}} \Xi, \Gamma^{(\mathtt{u})}, D_n^{(\mathtt{u})}\}_{n<m}}{(\mathcal{H}_{\gamma_1}, \Theta, \mathtt{R}) \vdash^{2a_0+1}_{\mathbb{S},\mathbb{S},\mathbb{S}} \Xi, \Gamma^{(\mathtt{u})}, D^{(\mathtt{u})}}\,(\bigwedge) \quad (\mathcal{H}_{\gamma_1}, \Theta, \mathtt{R}) \vdash^{2a_0}_{\mathbb{S},\mathbb{S},\mathbb{S}} \Xi, \Gamma^{(\mathtt{u})}, \neg D^{(\mathtt{u})}}{(\mathcal{H}_{\gamma_1}, \Theta, \mathtt{R}) \vdash^{2a}_{\mathbb{S},\mathbb{S},\mathbb{S}} \Xi, \Gamma^{(\mathtt{u})}}\,(cut)$$

where $\mathtt{Q}^\rho \restriction \mathbb{S} = \mathtt{Q}^{\rho\sigma} \restriction \mathbb{S} = \mathtt{Q} \restriction \mathbb{S} = \mathtt{R}$.

Case 3. Third the last inference is a (cut) with a cut formula $C^{(\sigma)}$ with $\text{srk}(\sigma) = \mathbb{S}$: Then $\text{rk}(C) < \mathbb{S} = \text{srk}(\sigma)$ for the cut formula $C^{(\sigma)}$. IH followed by a (cut) with the cut formula $C^{(\mathtt{u})}$ yields the proposition.

Other case are seen from IH. □

Lemma 5.7 (Elimination of one stable ordinal)
Suppose $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_{\mathbb{S}^\dagger,\mathbb{S}^\dagger,\mathbb{S}^\dagger,\beta,\gamma_0} \Xi$ *with a finite family* $\mathtt{Q} = ((\mathtt{Q})_0, \gamma^{\mathtt{Q}}_{\cdot})$ *for* γ_0, *where* $\mathbb{S} \in St$, *and* $\max\{\text{rk}(\Xi), \text{srk}(\Xi)\} < \mathbb{S}$.

Let $\gamma_1 = \gamma^{\mathtt{Q}}_{\mathbb{S}} + \mathbb{I}_N$ *if* $\mathbb{S} \in dom(\mathtt{Q})$, *and* $\gamma_1 = \gamma$ *else.*

Then $(\mathcal{H}_{\gamma_1}, \Theta, \mathtt{R}) \vdash^{\tilde{a}}_{\mathbb{S},\mathbb{S},\mathbb{S},\beta,\gamma_0} \Xi$ *holds for* $\tilde{a} = \varphi_{\mathbb{S}^\dagger}(\mathbb{S} + \omega a)$ *and* $\mathtt{R} = \mathtt{Q} \restriction \mathbb{S}$.

Proof. We have $\mathcal{B}(\{\mathbb{S}^\dagger, a\}) \subset \mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)]$ by (28) and Propositions 4.15.6 and 4.15.9 with $\mathcal{B}(\Theta(\mathtt{Q}^\circ)) \subset \Theta(\mathtt{Q}^\circ)$. We see $E(\mathbb{S}) \subset \{\mathbb{S}\} \cup E(\mathbb{S}^\dagger)$ and $\mathcal{B}_0(\mathbb{S}) \subset \{\mathbb{S}\} \cup \mathcal{B}_0(\mathbb{S}^\dagger)$ with $\mathbb{S} \in \text{trail}(\mathbb{S}^\dagger)$. Hence $\mathcal{B}(\mathbb{S}) \subset \mathcal{B}(\mathbb{S}^\dagger)$, and $\mathcal{B}(\{\mathbb{S}, \tilde{a}\}) \subset \mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)]$. On the other hand we have $\mathtt{Q}^\circ(\mathbb{S}) \subset \mathcal{H}_{\gamma_1}[\Theta(\mathtt{R}^\circ)]$ by (29) when $\mathbb{S} \in dom(\mathtt{Q})$, where $\mathtt{R}^\circ = \mathtt{Q}^\circ \restriction \mathbb{S}$. Therefore $\{\mathbb{S}, \tilde{a}\} \subset \mathcal{H}_{\gamma_1}[\Theta(\mathtt{R}^\circ)]$.

$(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{\mathbb{S}+\omega a}_{\mathbb{S}^\dagger,\mathbb{S},\mathbb{S}^\dagger,\beta,\gamma_0} \Xi$ follows from Recapping 5.4 for $\mathbb{S} = 2\mathbb{S}$. Cut-elimination 4.44 yields $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{\tilde{a}}_{\mathbb{S},\mathbb{S},\mathbb{S}^\dagger,\beta,\gamma_0} \Xi$. We obtain $(\mathcal{H}_{\gamma_1}, \Theta, \mathtt{R}) \vdash^{\tilde{a}}_{\mathbb{S},\mathbb{S},\mathbb{S},\beta,\gamma_0} \Xi$ by Proposition 5.6 with $2\tilde{a} = \tilde{a}$. □

Definition 5.8 Let $\mathtt{Q}$ be a finite family for γ_0 with thresholds $\gamma^{\mathtt{Q}}_{\mathbb{S}}$, and γ an ordinal. Let

$$\mathtt{s}(\gamma, \mathtt{Q}) := \min\{\mathbb{S} \in SSt : \gamma \geq \gamma^{\mathtt{Q}}_{\mathbb{S}} + \mathbb{I}_N\}$$

if there exists an $\mathbb{S} \in SSt$ such that $\gamma \geq \gamma^{\mathtt{Q}}_{\mathbb{S}} + \mathbb{I}_N$. Otherwise $\mathtt{s}(\gamma, \mathtt{Q}) := \beta$ for the fixed ordinal β.

We say that a non-zero ordinal γ is a *multiple* of $\mathbb{I}_N$ if $\gamma = \mathbb{I}_N \cdot \alpha$ for an $\alpha \neq 0$. For a multiple γ of $\mathbb{I}_N$ we obtain for $\mathtt{s} = \mathtt{s}(\gamma, \mathtt{Q})$

$$\forall \mathbb{S} \in dom(\mathtt{Q})(\mathbb{S} < \mathtt{s} \Rightarrow \gamma \leq \gamma^{\mathtt{Q}}_{\mathbb{S}}) \,\&\, \forall \mathbb{T} \in dom(\mathtt{Q})(\mathtt{s} \leq \mathbb{T} \Rightarrow \gamma^{\mathtt{Q}}_{\mathbb{T}} + \mathbb{I}_N \leq \gamma) \tag{42}$$

Definition 5.9 Let $\mathtt{Q} = ((\mathtt{Q})_0, \gamma^{\mathtt{Q}}_{\cdot})$ be a finite family for γ_0 with thresholds function $\gamma^{\mathtt{Q}}_{\cdot}$, $\mathbb{W}$ a successor stable ordinal, and γ an ordinal. Let $\mathbb{W} < e$ and $\mathtt{s} = \mathtt{s}(\gamma, \mathtt{Q})$. $\delta^{\mathtt{Q}}_{\mathbb{W}}(\gamma)$ denotes an ordinal defined as follows. If $[\mathbb{W}, \mathtt{s}) \cap dom(\mathtt{Q}) = \emptyset$, then $\delta^{\mathtt{Q}}_{\mathbb{W}}(\gamma) = \gamma$. Otherwise $\delta^{\mathtt{Q}}_{\mathbb{W}}(\gamma) = \gamma^{\mathtt{Q}}_{\mathbb{U}} = \gamma^{\mathtt{Q}}_{\mathbb{W}}$ for the least $\mathbb{U} \in [\mathbb{W}, \mathtt{s}) \cap dom(\mathtt{Q})$.

Proposition 5.10 *Let $\mathtt{Q} = ((\mathtt{Q})_0, \gamma^{\mathtt{Q}}_{\cdot})$ be a finite family for γ_0 with thresholds function $\gamma^{\mathtt{Q}}_{\cdot}$, and $\mathbb{W} < \mathbb{S} < e$ successor stable ordinals. Then $\gamma \leq \delta^{\mathtt{Q}}_{\mathbb{S}}(\gamma) \leq \delta^{\mathtt{Q}}_{\mathbb{W}}(\gamma)$ for a multiple γ of $\mathbb{I}_N$.*

Proof. Let $\mathtt{s} = \mathtt{s}(\gamma, \mathtt{Q})$. If $[\mathbb{W}, \mathtt{s}) \cap dom(\mathtt{Q}) = \emptyset$, then $\delta^{\mathtt{Q}}_{\mathbb{W}}(\gamma) = \delta^{\mathtt{Q}}_{\mathbb{S}}(\gamma) = \gamma$. Otherwise $\delta^{\mathtt{Q}}_{\mathbb{W}}(\gamma) = \gamma^{\mathtt{Q}}_{\mathbb{U}} = \gamma^{\mathtt{Q}}_{\mathbb{W}}$ for the least $\mathbb{U} \in [\mathbb{W}, \mathtt{s}) \cap dom(\mathtt{Q})$. By (42) we obtain $\gamma^{\mathtt{Q}}_{\mathbb{U}} \geq \gamma$. If $\mathbb{S} \leq \mathbb{U}$, then $\delta^{\mathtt{Q}}_{\mathbb{S}}(\gamma) = \gamma^{\mathtt{Q}}_{\mathbb{U}}$. Assume $\mathbb{U} < \mathbb{S}$. If $[\mathbb{S}, \mathtt{s}) \cap dom(\mathtt{Q}) = \emptyset$, then $\delta^{\mathtt{Q}}_{\mathbb{S}}(\gamma) = \gamma \leq \gamma^{\mathtt{Q}}_{\mathbb{U}}$. Otherwise let $\delta^{\mathtt{Q}}_{\mathbb{S}}(\gamma) = \gamma^{\mathtt{Q}}_{\mathbb{T}}$ for the least $\mathbb{T} \in [\mathbb{S}, \mathtt{s}) \cap dom(\mathtt{Q})$. Then $\mathbb{U} < \mathbb{T}$, and $\gamma \leq \gamma^{\mathtt{Q}}_{\mathbb{T}} < \gamma^{\mathtt{Q}}_{\mathbb{U}}$ by Definition 4.36.4a. □

Lemma 5.11 (Elimination of stable ordinals)
Let $\mathtt{Q} = ((\mathtt{Q})_0, \gamma^{\mathtt{Q}}_{\cdot})$ be a finite family for γ_0, and $f(e,a) = \varphi_{e+1}(a)$. Suppose $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_{e,e,e,\beta,\gamma_0} \Xi$ for a multiple γ of $\mathbb{I}_N$, and $\max\{\mathrm{rk}(\Xi), \mathrm{srk}(\Xi)\} < \mathbb{W} < e$, where e is a stable ordinal, $a, e < \Lambda < \mathbb{I}_N$, $\mathbb{W}$ is a successor stable ordinal such that $\mathbb{W} \in \mathcal{H}_{\delta^{\mathtt{Q}}_{\mathbb{W}}(\gamma)+\mathbb{I}_N}[\Theta(\mathtt{Q}^\circ \restriction \mathbb{W})]$.

Assume that $\mathtt{Q}$ has gaps $f(e,a)+1$. Then $(\mathcal{H}_{\gamma_{\mathbb{W}}}, \Theta, \mathtt{Q}_{\mathbb{W}}) \vdash^{f(e,a)}_{\mathbb{W},\mathbb{W},\mathbb{W},\beta,\gamma_0} \Xi$ holds for $\gamma_{\mathbb{W}} = \delta^{\mathtt{Q}}_{\mathbb{W}}(\gamma) + \mathbb{I}_N \cdot (f(e,a)) < \gamma_0 + (\mathbb{I}_N)^2$ and $\mathtt{Q}_{\mathbb{W}} = \mathtt{Q} \restriction \mathbb{W}$.

Proof. By main induction on e with subsidiary induction on a. In the proof let us omit the fourth and fifth subscripts β, γ_0.

Let $\mathbb{W} \leq \mathbb{S} \in dom(\mathtt{Q})$. We have $\mathtt{Q}(\mathbb{S}) \subset \mathcal{H}_{\gamma^{\mathtt{Q}}_{\mathbb{S}}+\mathbb{I}_N}[\Theta(\mathtt{Q}^\circ \restriction \mathbb{S})]$ by (29). We see $\gamma^{\mathtt{Q}}_{\mathbb{S}} \leq \delta^{\mathtt{Q}}_{\mathbb{W}}(\gamma)$ from Definition 4.36.4a and (42). Hence $\mathtt{Q}(\mathbb{S}) \subset \mathcal{H}_{\delta^{\mathtt{Q}}_{\mathbb{W}}(\gamma)+\mathbb{I}_N}[\Theta(\mathtt{Q}^\circ_{\mathbb{W}})]$, and $\mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)] \subset \mathcal{H}_{\delta^{\mathtt{Q}}_{\mathbb{W}}(\gamma)+\mathbb{I}_N}[\Theta(\mathtt{Q}^\circ_{\mathbb{W}})]$, where $\mathtt{Q}^\circ_{\mathbb{W}} = \mathtt{Q}^\circ \restriction \mathbb{W}$.

By the assumption and (28), $\{\mathbb{W}, f(e,a), \gamma_{\mathbb{W}}\} \subset \mathcal{H}_{\delta^{\mathtt{Q}}_{\mathbb{W}}(\gamma)+\mathbb{I}_N}[\Theta(\mathtt{Q}^\circ_{\mathbb{W}})]$ follows, and (28) is enjoyed in $(\mathcal{H}_{\gamma_{\mathbb{W}}}, \Theta, \mathtt{Q}_{\mathbb{W}}) \vdash^{f(e,a)}_{\mathbb{W},\mathbb{W},\mathbb{W},\beta,\gamma_0} \Xi$.

We see $\gamma_{\mathbb{W}} \leq \gamma^{\mathtt{Q}}_{\mathbb{S}}$ for every $\mathbb{S} \in dom(\mathtt{Q}) \cap \mathbb{W}$ from the assumption that $\mathtt{Q}$ has gaps $f(e,a)+1$ as follows. If $\delta^{\mathtt{Q}}_{\mathbb{W}}(\gamma) = \gamma_0 = \gamma$, then $\gamma_{\mathbb{W}} = \gamma_0 + \mathbb{I}_N \cdot (f(e,a)) < \gamma_0 + \mathbb{I}_N \cdot (f(e,a)+1) \leq \gamma^{\mathtt{Q}}_{\mathbb{S}}$. Otherwise let $\delta^{\mathtt{Q}}_{\mathbb{W}}(\gamma) = \gamma^{\mathtt{Q}}_{\mathbb{U}}$ for $\mathbb{S} < \mathbb{W} \leq \mathbb{U} \in dom(\mathtt{Q})$. Then $\gamma_{\mathbb{W}} = \gamma^{\mathtt{Q}}_{\mathbb{U}} + \mathbb{I}_N \cdot (f(e,a)) < \gamma^{\mathtt{Q}}_{\mathbb{U}} + \mathbb{I}_N(f(e,a)+1) \leq \gamma^{\mathtt{Q}}_{\mathbb{S}}$.
Case 1. Consider the case when the last inference is an $(i\text{–rfl}_{\mathbb{S}}(\rho, f, \Theta_1))$ for a successor i-stable ordinal $\mathbb{S} < e$ such that $\mathbb{S} \in \mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)]$ by (30).

Let $\mathtt{R} = \mathtt{Q} \cup \{(\mathbb{S}, \rho)\}$. $a_0 < a$ is an ordinal, and $D \equiv \bigwedge(D_n)_{n<m}$ is a finite conjunction such that $(\mathcal{H}_\gamma, \Theta, \mathtt{R}) \vdash^{a_0}_{e,e,e} \Xi, D^{(\rho)}_n$ for each $n < m$, and $(\mathcal{H}_\gamma, \Theta, \mathtt{R}^\sigma) \vdash^{a_0}_{e,e,e} \Xi, \neg D^{(\sigma)}$ for every $\sigma \in L = H^{\mathtt{R}}_\rho(f, \Theta, \Theta_1)$ and $\mathrm{rk}(D) < \min\{s, e\}$. Since $(f(e,a_0)+1) \cdot 2 < f(e,a)$, we may assume that the finite family $\mathtt{R}$ for γ_0 has gaps $f(e,a_0)+1$. We have $\mathrm{srk}(D^{(\rho)}) = \mathrm{srk}(D^{(\sigma)}) = \mathbb{S} < \mathbb{S}^\dagger \leq e \in St$.

Let $\mathbb{U}^\dagger = \max\{\mathbb{W}, \mathrm{rk}(D)^\dagger, \mathbb{S}^\dagger\}$. We obtain $\mathbb{U}^\dagger \leq e$. We claim that $\mathbb{U}^\dagger \in \mathcal{H}_{\delta^{\mathtt{Q}}_{\mathbb{U}^\dagger}(\gamma)+\mathbb{I}_N}[\Theta(\mathtt{Q}^\circ_{\mathbb{U}^\dagger})]$, where $\delta^{\mathtt{Q}}_{\mathbb{U}^\dagger}(\gamma) = \delta^{\mathtt{R}}_{\mathbb{U}^\dagger}(\gamma) = \delta^{\mathtt{R}^\sigma}_{\mathbb{U}^\dagger}(\gamma)$ by $\mathbb{U}^\dagger > \mathbb{S}$. We may assume that $\mathbb{U}^\dagger \neq \mathbb{W}$ by the assumption. First let $\mathbb{U}^\dagger = \mathbb{S}^\dagger$. We see $E(\mathbb{S}^\dagger) \subset$

$\{\mathbb{S}^\dagger\} \cup E(\mathbb{S})$ with $E_{\mathbb{S}}(\mathbb{S}^\dagger) = \emptyset$. Moreover $\mathcal{B}_0(\mathbb{S}^\dagger) \subset \{\mathbb{S}^\dagger\} \cup \mathcal{B}_0(\mathbb{S})$ since $\mathrm{trail}(\mathbb{S}^\dagger) \subset \mathrm{trail}(\mathbb{S}) \cup \{\mathbb{S}^\dagger\}$. Hence $\mathcal{B}(\mathbb{S}^\dagger) \subset \{\mathbb{S}^\dagger\} \cup \mathcal{B}(\mathbb{S})$. Therefore $\mathbb{S}^\dagger \in \mathcal{H}_{\delta^{\mathtt{Q}}_{\mathbb{S}^\dagger}(\gamma)+\mathbb{I}_N}[\Theta(\mathtt{Q}^\circ_{\mathbb{S}^\dagger})]$ by $\mathbb{S} \in \mathcal{B}(\mathbb{S}) \subset \mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)]$ and $\gamma \leq \delta^{\mathtt{Q}}_{\mathbb{S}^\dagger}(\gamma)$. Next let $\mathbb{U}^\dagger = \mathrm{rk}(D)^\dagger > \max\{\mathbb{W}, \mathbb{S}^\dagger\}$. Then $\delta^{\mathtt{Q}}_{\mathbb{U}^\dagger}(\gamma) = \gamma^{\mathtt{Q}}_{\mathbb{V}}$ for $\mathbb{V} = \min\{\mathbb{V} \in dom(\mathtt{Q}) : \mathrm{rk}(D) < \mathbb{V} < \mathtt{s}(\gamma, \mathtt{Q})\}$ if such a $\mathbb{V}$ exists. Otherwise $\delta^{\mathtt{Q}}_{\mathbb{U}^\dagger}(\gamma) = \gamma$.

By (28) we obtain $\mathsf{k}(D) \subset \mathcal{H}_\gamma[\Theta(\mathtt{Q}^\circ)]$, and $\mathsf{k}(D) \subset \mathcal{H}_{\delta^{\mathtt{Q}}_{\mathbb{U}^\dagger}(\gamma)+\mathbb{I}_N}[\Theta(\mathtt{Q}^\circ_{\mathbb{U}^\dagger})]$. Hence $\mathrm{rk}(D)^\dagger \in \mathcal{H}_{\delta^{\mathtt{Q}}_{\mathbb{U}^\dagger}(\gamma)+\mathbb{I}_N}[\Theta(\mathtt{Q}^\circ_{\mathbb{U}^\dagger})]$ follows from Proposition 4.5.3. Thus the claim is shown.

Let $a_1 = f(e, a_0)$ and $\gamma_{\mathbb{U}^\dagger} = \delta^{\mathtt{Q}}_{\mathbb{U}^\dagger}(\gamma) + \mathbb{I}_N \cdot (f(e, a_0))$. For each $n < m$, SIH yields $(\mathcal{H}_{\gamma_{\mathbb{U}^\dagger}}, \Theta, \mathtt{Q}_{\mathbb{U}^\dagger}) \vdash^{a_1}_{\mathbb{U}^\dagger,\mathbb{U}^\dagger,\mathbb{U}^\dagger} \Xi, D^{(\rho)}_n$, and $(\mathcal{H}_{\gamma_{\mathbb{U}^\dagger}}, \Theta, \mathtt{Q}_{\mathbb{U}^\dagger}) \vdash^{a_1}_{\mathbb{U}^\dagger,\mathbb{U}^\dagger,\mathbb{U}^\dagger} \Xi, \neg D^{(\sigma)}$ for each $\sigma \in L$. We obtain by an $(i\text{−}\mathrm{rfl}_{\mathbb{S}}(\rho, f, \Theta_1))$, $(\mathcal{H}_{\gamma_{\mathbb{U}^\dagger}}, \Theta, \mathtt{Q}_{\mathbb{U}^\dagger}) \vdash^{a_1+1}_{\mathbb{U}^\dagger,\mathbb{U}^\dagger,\mathbb{U}^\dagger} \Xi$. If $\mathbb{U}^\dagger = \mathbb{W}$, then $\mathbb{S} < \mathbb{W}$. We are done. Assume $\mathbb{W} < \mathbb{U}^\dagger$. Then $\mathbb{W} \leq \mathbb{U} \in St$.

Let $a_2 = \varphi_{\mathbb{U}^\dagger}(\mathbb{U} + \omega(a_1 + 1)) < f(e, a)$. Lemma 5.7 yields $(\mathcal{H}_{\gamma_1}, \Theta, \mathtt{Q}_{\mathbb{U}}) \vdash^{a_2}_{\mathbb{U},\mathbb{U},\mathbb{U}} \Xi$, where $\gamma_1 = \gamma^{\mathtt{Q}}_{\mathbb{U}} + \mathbb{I}_N$ if $\mathbb{U} \in dom(\mathtt{Q})$, and $\gamma_1 = \gamma_{\mathbb{U}^\dagger}$ else. In each case γ_1 is a multiple of $\mathbb{I}_N$.

Claim 5.12 $\gamma_2 = \delta^{\mathtt{Q}_{\mathbb{U}}}_{\mathbb{W}}(\gamma_1) + \mathbb{I}_N \cdot f(\mathbb{U}, a_2) \leq \delta^{\mathtt{Q}}_{\mathbb{W}}(\gamma) + \mathbb{I}_N \cdot f(e, a) = \gamma_{\mathbb{W}}$.

Proof of Claim 5.12. Let $\mathtt{s} = \mathtt{s}(\gamma, \mathtt{Q})$, $\delta = \delta^{\mathtt{Q}}_{\mathbb{W}}(\gamma)$, $\mathtt{s}_1 = \mathtt{s}(\gamma_1, \mathtt{Q}_{\mathbb{U}})$ and $\delta_1 = \delta^{\mathtt{Q}_{\mathbb{U}}}_{\mathbb{W}}(\gamma_1)$.
Case 1. $\mathbb{U} \in dom(\mathtt{Q})$: Then $\gamma_1 = \gamma^{\mathtt{Q}}_{\mathbb{U}} + \mathbb{I}_N$ and $\mathtt{s}_1 \leq \mathbb{U}$.
Case 1.1. $\mathtt{s} \leq \mathbb{U}$: Then $\gamma_1 \leq \gamma$. First let $[\mathbb{W}, \mathtt{s}) \cap dom(\mathtt{Q}) = \emptyset$. In this case we show that $\delta_1 \leq \gamma = \delta$, which yields the claim by $f(\mathbb{U}, a_2) < f(e, a)$. If $[\mathbb{W}, \mathtt{s}_1) \cap dom(\mathtt{Q}_{\mathbb{U}}) = \emptyset$, then $\delta_1 = \gamma_1 \leq \gamma$. Otherwise let $\mathbb{V} \in [\mathbb{W}, \mathtt{s}_1) \cap dom(\mathtt{Q}_{\mathbb{U}})$ be the least one. Then $\mathtt{s} \leq \mathbb{V}$, and $\delta_1 = \gamma^{\mathtt{Q}}_{\mathbb{V}} < \gamma^{\mathtt{Q}}_{\mathbb{V}} + \mathbb{I}_N \leq \gamma$.

Second let $\delta = \gamma^{\mathtt{Q}}_{\mathbb{V}}$ for the least $\mathbb{V} \in [\mathbb{W}, \mathtt{s}) \cap dom(\mathtt{Q})$. From $\gamma_1 \leq \gamma$ we see $\mathtt{s}_1 \geq \mathtt{s}$. Hence $\mathbb{V} \in [\mathbb{W}, \mathtt{s}_1) \cap dom(\mathtt{Q}_{\mathbb{U}})$ and $\delta_1 = \gamma^{\mathtt{Q}}_{\mathbb{V}}$. The claim follows from $f(\mathbb{U}, a_2) < f(e, a)$.
Case 1.2. $\mathbb{U} < \mathtt{s}$: Then $\mathbb{U} \in [\mathbb{W}, \mathtt{s}) \cap dom(\mathtt{Q})$ and $\delta = \gamma^{\mathtt{Q}}_{\mathbb{V}}$ with $\mathbb{V} \leq \mathbb{U}$. If $\mathbb{V} < \mathtt{s}_1$, then $\delta_1 = \gamma^{\mathtt{Q}}_{\mathbb{V}} = \delta$. The claim follows from $f(\mathbb{U}, a_2) < f(e, a)$. Let $\mathtt{s}_1 \leq \mathbb{V} \leq \mathbb{U}$. Then $\delta_1 = \gamma_1 = \gamma^{\mathtt{Q}}_{\mathbb{U}} + \mathbb{I}_N$ and $\gamma^{\mathtt{Q}}_{\mathbb{U}} \leq \gamma^{\mathtt{Q}}_{\mathbb{V}}$. $1 + f(\mathbb{U}, a_2) < f(e, a)$ yields the claim.
Case 2. $\mathbb{U} \notin dom(\mathtt{Q})$: Then $\gamma_1 = \gamma_{\mathbb{U}^\dagger} = \delta^{\mathtt{Q}}_{\mathbb{U}^\dagger}(\gamma) + \mathbb{I}_N \cdot f(e, a_0)$.
Case 2.1. $[\mathbb{W}, \mathtt{s}) \cap dom(\mathtt{Q}) = \emptyset$: Then $\delta = \gamma = \delta^{\mathtt{Q}}_{\mathbb{U}^\dagger}(\gamma)$ by $\mathbb{W} \leq \mathbb{U}^\dagger$, and $\gamma_1 = \gamma + \mathbb{I}_N \cdot f(e, a_0)$. We have either $\delta_1 = \gamma_1$ or $\delta_1 = \gamma^{\mathtt{Q}}_{\mathbb{V}}$ for a $\mathtt{s} \leq \mathbb{V} \in dom(\mathtt{Q})$. In each case we obtain $\delta_1 \leq \gamma + \mathbb{I}_N \cdot f(e, a_0)$. The claim follows from $f(e, a_0) + f(\mathbb{U}, a_2) < f(e, a)$.
Case 2.2. Otherwise: Let $\delta = \gamma^{\mathtt{Q}}_{\mathbb{V}}$ for the least $\mathbb{V} \in [\mathbb{W}, \mathtt{s}) \cap dom(\mathtt{Q})$. If $\mathbb{V} < \mathtt{s}_1$, then $\delta_1 = \gamma^{\mathtt{Q}}_{\mathbb{V}} = \delta$. The claim follows from $f(e, a_0) < f(e, a)$. Let $\mathtt{s}_1 \leq \mathbb{V} < \mathtt{s}$. Then $\delta_1 = \gamma_1$. We show $\delta^{\mathtt{Q}}_{\mathbb{U}^\dagger}(\gamma) \leq \delta + \mathbb{I}_N$, which yields the claim by $f(e, a_0) + f(\mathbb{U}, a_2) < f(e, a) = 1 + f(e, a)$. If $\mathbb{U}^\dagger \leq \mathbb{V}$, then $\delta^{\mathtt{Q}}_{\mathbb{U}^\dagger}(\gamma) = \gamma^{\mathtt{Q}}_{\mathbb{V}} = \delta$. If $\delta^{\mathtt{Q}}_{\mathbb{U}^\dagger}(\gamma) = \gamma^{\mathtt{Q}}_{\mathbb{X}}$ for an $\mathbb{V} < \mathbb{U}^\dagger \leq \mathbb{X} < \mathtt{s}$, then $\gamma^{\mathtt{Q}}_{\mathbb{X}} < \gamma^{\mathtt{Q}}_{\mathbb{V}}$. Otherwise $\delta^{\mathtt{Q}}_{\mathbb{U}^\dagger}(\gamma) = \gamma < \gamma^{\mathtt{Q}}_{\mathbb{V}} + \mathbb{I}_N = \delta + \mathbb{I}_N$ by $\mathbb{V} < \mathtt{s}$. □

We have $a_3 = f(\mathbb{U}, a_2) = \varphi_{\mathbb{U}+1}(a_2) < f(e,a)$, and hence $\mathtt{Q}$ has gaps $a_3 + 1 < f(e,a)$. By MIH with $\mathbb{U} < e$ we obtain $(\mathcal{H}_{\gamma_2}, \Theta, \mathtt{Q}_{\mathbb{W}}) \vdash^{a_3}_{\mathbb{W},\mathbb{W},\mathbb{W}} \Xi$ for $\gamma_2 = \delta^{\mathtt{Q}_{\mathbb{U}}}_{\mathbb{W}}(\gamma_1) + \mathbb{I}_N \cdot f(\mathbb{U}, a_2)$. On the other hand we have $\gamma_2 \leq \gamma_{\mathbb{W}}$ by Claim 5.12 and $a_3 < f(e,a)$. Therefore $(\mathcal{H}_{\gamma_{\mathbb{W}}}, \Theta, \mathtt{Q}_{\mathbb{W}}) \vdash^{f(e,a)}_{\mathbb{W},\mathbb{W},\mathbb{W}} \Xi$.

Case 2. Next consider the case when the last inference is a (cut) of a cut formula $C^{(\sigma)}$ wth $\max\{\mathrm{rk}(C), \mathrm{srk}(\sigma)\} < e$. We have an ordinal $a_0 < a$ such that $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a_0}_{e,e,e} \neg C^{(\sigma)}, \Xi$ and $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a_0}_{e,e,e} C^{(\sigma)}, \Xi$.

We may assume that $\mathrm{rk}(C) \geq \mathrm{srk}(\sigma)$ by Proposition 5.6. Let $\mathbb{U}^\dagger = \max\{\mathbb{W}, \mathrm{rk}(C)^\dagger\}$. We obtain $\mathbb{U}^\dagger \leq e$. We see $\mathbb{U}^\dagger \in \mathcal{H}_{\delta^{\mathtt{Q}}_{\mathbb{U}^\dagger}(\gamma)+\mathbb{I}_N}[\Theta(\mathtt{Q}^\circ_{\mathbb{U}^\dagger})]$ as in **Case 1**. Let $\gamma_{\mathbb{U}^\dagger} = \delta^{\mathtt{Q}}_{\mathbb{U}^\dagger}(\gamma) + \mathbb{I}_N \cdot a_1$ for $a_1 = f(e, a_0) = \varphi_{e+1}(a_0)$. SIH yields $(\mathcal{H}_{\gamma_{\mathbb{U}^\dagger}}, \Theta, \mathtt{Q}_{\mathbb{U}^\dagger}) \vdash^{a_1}_{\mathbb{U}^\dagger,\mathbb{U}^\dagger,\mathbb{U}^\dagger} \neg C^{(\sigma)}, \Xi$ and $(\mathcal{H}_{\gamma_{\mathbb{U}^\dagger}}, \Theta, \mathtt{Q}_{\mathbb{U}^\dagger}) \vdash^{a_1}_{\mathbb{U}^\dagger,\mathbb{U}^\dagger,\mathbb{U}^\dagger} C^{(\sigma)}, \Xi$. A (cut) yields $(\mathcal{H}_{\gamma_{\mathbb{U}^\dagger}}, \Theta, \mathtt{Q}_{\mathbb{U}^\dagger}) \vdash^{a_1+1}_{\mathbb{U}^\dagger,\mathbb{U}^\dagger,\mathbb{U}^\dagger} \Xi$. If $\mathbb{U}^\dagger = \mathbb{W}$, then we are done. Assume $\mathbb{W} < \mathbb{U}^\dagger$. Then $\mathbb{W} \leq \mathbb{U} \in St$. Let $a_2 = \varphi_{\mathbb{U}^\dagger}(\mathbb{U} + \omega(a_1+1)) < f(e,a)$. Lemma 5.7 yields $(\mathcal{H}_{\gamma_1}, \Theta, \mathtt{Q}_{\mathbb{U}}) \vdash^{a_2}_{\mathbb{U},\mathbb{U},\mathbb{U}} \Xi$, where $\gamma_1 = \gamma^{\mathtt{Q}}_{\mathbb{U}} + \mathbb{I}_N$ if $\mathbb{U} \in dom(\mathtt{Q})$, and $\gamma_1 = \gamma_{\mathbb{U}^\dagger}$ else. In each case γ_1 is a multiple of $\mathbb{I}_N$. We have $a_3 = f(\mathbb{U}, a_2) = \varphi_{\mathbb{U}+1}(a_2) < f(e,a)$, and hence $\mathtt{Q}$ has gaps $a_3 + 1 < f(e,a)$.

By MIH with $\mathbb{U} < e$ we obtain $(\mathcal{H}_{\gamma_2}, \Theta, \mathtt{Q}) \vdash^{a_3}_{\mathbb{W},\mathbb{W},\mathbb{W}} \Xi$ for $\gamma_2 = \delta^{\mathtt{Q}_{\mathbb{U}}}_{\mathbb{W}}(\gamma_1) + \mathbb{I}_N \cdot f(\mathbb{U}, a_2) \leq \gamma_{\mathbb{W}}$ by Claim 5.12.

Other cases $(\bigvee), (\bigwedge)$ and $(\Sigma-\mathrm{rfl})$ on Ω are seen from SIH. □

Let us prove Theorem 1.1. Let $S_{\mathbb{I}_N} \vdash \theta^{L_\Omega}$ for a Σ-sentence θ. By Embedding 4.27 pick an $m > 0$ so that $(\mathcal{H}_{\mathbb{I}_N}, \emptyset; \emptyset) \vdash^{*\mathbb{I}_N \cdot 2+m}_{\mathbb{I}_N+m} \theta^{L_\Omega}; \emptyset$. Cut-elimination 4.32 yields $(\mathcal{H}_{\mathbb{I}_N}, \emptyset; \emptyset) \vdash^{*a}_{\mathbb{I}_N} \theta^{L_\Omega}; \emptyset$ for $a = \omega_m(\mathbb{I}_N \cdot 2 + m) < \omega_{m+1}(\mathbb{I}_N + 1)$. Then Collapsing 4.34 yields $(\mathcal{H}_{\hat{a}+1}, \emptyset; \emptyset) \vdash^{*\beta}_{\beta} \theta^{L_\Omega}; \emptyset$ for $\beta = \psi_{\mathbb{I}_N}(\hat{a}) \in LSt_N$ with $\hat{a} = \mathbb{I}_N + \omega^a = \omega_{m+1}(\mathbb{I}_N \cdot 2 + m) > \beta$. Now let $\gamma_0 = \hat{a} + \mathbb{I}_N$. Capping 5.1 then yields $(\mathcal{H}_{\gamma_0}, \emptyset, \emptyset) \vdash^{\beta}_{\beta,\beta,\beta,\beta,\gamma_0} \theta^{L_\Omega}$ where $2\beta = \beta$, $\theta^{L_\Omega} \equiv (\theta^{L_\Omega})^{(\mathtt{u})}$, and $\emptyset$ is a finite family for γ_0 with thresholds and gaps $\varphi_{\beta+1}(\beta) + 1$. For the empty family $\emptyset$ this means that each finite family $\mathtt{Q}$ with thresholds $\gamma^{\mathtt{Q}}_{\mathbb{S}}$ have gaps $\varphi_{\beta+1}(\beta) + 1$ in a sequent $(\mathcal{H}_{\gamma_0}, \Theta, \mathtt{Q}) \vdash^{a}_{\beta,\beta,\beta,\beta,\gamma_0} \Gamma$ occurring in the derivation of $(\mathcal{H}_{\gamma_0}, \emptyset, \emptyset) \vdash^{\beta}_{\beta,\beta,\beta,\beta,\gamma_0} \theta^{L_\Omega}$.

Let $\beta < \Lambda = \Gamma(\beta) < \mathbb{I}_N$ be the next strongly critical number as the base of the $\tilde{\theta}$-function. In what follows each finite function is an $f : \Lambda \to \Gamma(\Lambda)$. Let $\alpha = \varphi_{\beta+1}(\beta)$ and $\mathbb{S}_0 = \Omega^\dagger$ be the least stable ordinal with $\mathcal{B}(\mathbb{S}_0) = \{\mathbb{S}_0\} \subset \mathcal{H}_0[\emptyset]$. By Lemma 5.11 for the multiple γ_0 of $\mathbb{I}_N$ we obtain $(\mathcal{H}_{\gamma_{\mathbb{S}_0}}, \emptyset, \emptyset) \vdash^{\alpha}_{\mathbb{S}_0,\mathbb{S}_0,\mathbb{S}_0,\beta,\gamma_0} \theta^{L_\Omega}$ for $\gamma_{\mathbb{S}_0} = \delta^{\emptyset}_{\mathbb{S}_0} + \mathbb{I}_N \cdot f(\beta,\beta) = \gamma_0 + \mathbb{I}_N \cdot \alpha < \gamma_0 + (\mathbb{I}_N)^2$. Cut-elimination 4.44 yields $(\mathcal{H}_{\gamma_{\mathbb{S}_0}}, \emptyset, \emptyset) \vdash^{\alpha_1}_{\Omega,\mathbb{S}_0,\mathbb{S}_0,\beta,\gamma_0} \theta^{L_\Omega}$ for $\alpha_1 = \varphi_{\mathbb{S}_0}(\alpha)$.

In a witnessed derivation of this fact, there occurs no inference $(i-\mathrm{rfl}_{\mathbb{S}}(\rho, f, \Theta_1))$ since there is no successor stable ordinal $\mathbb{S} < \mathbb{S}_0$, cf. Definition 4.39. Hence $(\mathcal{H}_{\gamma_{\mathbb{S}_0}}, \emptyset; \emptyset) \vdash^{*\alpha_1}_{\Omega,\gamma_1} \theta^{L_\Omega}; \emptyset$ for $\gamma_1 = \gamma_{\mathbb{S}_0} + \alpha_1 + 1$. $(\mathcal{H}_\gamma, \emptyset; \emptyset) \vdash^{*\delta}_{\delta,\gamma_1} \theta^{L_\delta}; \emptyset$ follows from Collapsing 4.35, where $\delta = \psi_\Omega(\gamma_{\mathbb{S}_0} + \alpha_1)$ with the epsilon number α_1. Cut-elimination 4.32 yields $(\mathcal{H}_\gamma, \emptyset; \emptyset) \vdash^{*\varphi_\delta(\delta)}_{0,\gamma_1} \theta^{L_\delta}; \emptyset$. We see that θ^{L_δ} is true by induction up to $\varphi_\delta(\delta)$, where $\delta < \psi_\Omega(\omega_{m+2}(\mathbb{I}_N + 1)) < \psi_\Omega(\varepsilon_{\mathbb{I}_N+1})$.

6 Some ordinals in well-foundedness proof

In this section we introduce some ordinals needed in our well-foundedness proof.

In [4] the following Lemmas 6.2 and 6.3 are shown. Lemma 6.2 is used in showing the finiteness of the sequence $\rho_0 \succ \rho_1 \succ \rho_2 \succ \cdots$, cf. Definition 3.28 and Lemma 6.15. Lemma 6.3 is needed in showing Corollary 7.38.

Definition 6.1 Let $\Lambda \leq \mathbb{I}_N$ be a strongly critical number.

1. For $\xi < \varphi_\Lambda(0)$, $a_\Lambda(\xi)$ denotes an ordinal defined recursively by $a_\Lambda(0) = 0$, and $a_\Lambda(\xi) = \sum_{i \leq m} \tilde{\theta}_{b_i}(\omega \cdot a_\Lambda(\xi_i); \Lambda) \cdot a_i$ when $\xi =_{NF} \sum_{i \leq m} \tilde{\theta}_{b_i}(\xi_i; \Lambda) \cdot a_i$ in (6).

2. For irreducible functions $f : \Lambda \to \varphi_\Lambda(0)$ with base Λ let us associate ordinals $o_\Lambda(f) < \varphi_\Lambda(0)$ as follows. $o_\Lambda(\emptyset) = 0$ for the empty function $f = \emptyset$. Let $\{0\} \cup \mathrm{supp}(f) = \{0 = c_0 < c_1 < \cdots < c_n\}$, $f(c_i) = \xi_i < \varphi_\Lambda(0)$ for $i > 0$, and $\xi_0 = 0$. Define ordinals $\zeta_i = o_\Lambda(f; c_i)$ by $\zeta_n = \omega \cdot a_\Lambda(\xi_n)$, and $\zeta_i = \omega \cdot a_\Lambda(\xi_i) + \tilde{\theta}_{c_{i+1}-c_i}(\zeta_{i+1} + 1; \Lambda)$. Finally let $o_\Lambda(f) = \zeta_0 = o_\Lambda(f; c_0)$.

3. For $d \notin \{0\} \cup \mathrm{supp}(f)$, let $o_\Lambda(f; d) = 0$ if $f^d = \emptyset$. Otherwise $o_\Lambda(f; d) = \tilde{\theta}_{c-d}(o_\Lambda(f; c) + 1; \Lambda)$ for $c = \min(\mathrm{supp}(f^d))$.

Lemma 6.2 *Let $f : \Lambda \to \varphi_\Lambda(0)$ be an irreducible finite function with base Λ defined from an irreducible function $g : \Lambda \to \varphi_\Lambda(0)$ and ordinals c, d as follows. $f_c = g_c$, $c < d \in \mathrm{supp}(g)$ with $(c, d) \cap \mathrm{supp}(g) = (c, d) \cap \mathrm{supp}(f) = \emptyset$, $f(c) < g(c) + \tilde{\theta}_{d-c}(g(d); \Lambda) \cdot \omega$, and $f <^d_\Lambda g(d)$, cf. Definition 3.31.6. Then $o_\Lambda(f) < o_\Lambda(g)$ holds.*

Lemma 6.3 *For irreducible finite functions $f, g : \Lambda \to \varphi_\Lambda(0)$ with base Λ, assume $f <^0_{lx} g$. Then $o_\Lambda(f) < o_\Lambda(g)$ holds.*

6.1 A preview of well-foundedness proof

To prove the well-foundedness of a computable notation system, we utilize the distinguished class introduced by W. Buchholz[7]. Also cf. [11] for a well-foundedness in terms of a maximal distinguished class.

Let OT be a computable notation system of ordinals with an ordinal term Ω_1. Ω_1 denotes the least recursively regular ordinal ω_1^{CK}. Assume that we are working in a theory in which the well-founded part $W(OT)$ of OT exists as a set. A parameter-free Π^{1-}_1+CA suffices to show the existence. Then the well-foundedness of such a notation system OT is provable. When the next recursively regular ordinal Ω_2 is in OT, we further assume that a well-founded part $W(\mathcal{C}^{\Omega_1}(W_0))$ of a set $\mathcal{C}^{\Omega_1}(W_0)$ exists, where $W_0 = W(OT) \cap \Omega_1$, and $\alpha \in \mathcal{C}^{\Omega_1}(W_0)$ iff each component$<\Omega_1$ of α is in W_0. Likewise when OT contains α-many terms denoting increasing sequence of recursively regular ordinals, we need to iterate the process of defining the well-founded parts α-times.

Let us consider a notation system OT for recursively inaccessible universes. There are α-many ordinal terms denoting recursively regular ordinals in OT

with the order type α of OT. The whole process then should be internalized. We need to specify a feature of sets arising in the process. Then *distinguished sets* emerge. $D[P]$ denotes the fact that P is a distinguished class and defined by

$$D[P] :\Leftrightarrow \forall\alpha\left(\alpha \leq P \to W(C^{\alpha}(P)) \cap \alpha^{+} = P \cap \alpha^{+}\right)$$

where $\alpha \leq P \Leftrightarrow \exists\beta \in P(\alpha \leq \beta)$ and α^{+} denotes the next recursively regular ordinal above α if such an ordinal exists.

$W_0 = W(OT) \cap \Omega_1$ is the smallest distinguished set, and $W_1 = W(\mathcal{C}^{\Omega_1}(W_0)) \cap \Omega_2$ is the next one. Given two distinguished sets, it turns out that one is an initial segment of the other, and the union $\mathcal{W}_0 = \bigcup\{P \subset OT : D[P]\}$ of all distinguished sets is distinguished, the *maximal distinguished class*. The maximal distinguished class $\mathcal{W}_0$ is Σ^{1-}_2-definable, and a proper class without assuming Σ^{1-}_2-CA.

Assuming the maximal distinguished class $\mathcal{W}_0$ exists as a set, the well-foundedness of OT for a single stable ordinal is provable in [4]. Consider now a notation system OT for several stable ordinals $\mathbb{S}_0, \mathbb{S}_1, \ldots$. We then need several maximal distinguished sets $\mathcal{W}_0, \mathcal{W}_1, \ldots$ to prove the well-foundedness. $\mathcal{W}_0$ is the maximal distinguished set in an absolute sense as for the well-founded part $W_0 = W(OT) \cap \Omega_1$.

A moment reflection on the emergence of distinguished sets shows that $\mathcal{W}_1$ could be a *maximal distinguished set relative to* $\mathcal{W}_0$ and $\mathbb{S}_0$. Specifically cf. (46), a set P is said to be a 0-*distinguished set* for γ and X, denoted by $D^{\gamma}[P; X]$, iff P is well-founded and

$$P \cap \gamma^{-\dagger} = X \cap \gamma^{-\dagger} \,\&\, \forall\alpha \geq \gamma^{-\dagger}\left(\alpha \leq P \to W(C^{\alpha}(P)) \cap \alpha^{+} = P \cap \alpha^{+}\right)$$

where $\gamma^{-\dagger} = \max\{\mathbb{S} \in St \cup \{0\} : \mathbb{S} \leq \gamma\}$. Then let, cf. (47)

$$W^{\gamma}_1(X) := \bigcup\{P \subset OT : D^{\gamma}[P; X]\}.$$

Observe that $W^{\gamma}_1(X)$ is a Σ^1_2-definable class, and hence a set assuming Σ^1_2-CA. We see in Lemma 7.8.2 that $W^{\gamma}_1(X)$ is the maximal 0-distinguished class for γ and X provided that $X \cap \gamma^{-\dagger}$ is well-founded.

Assume that there are α-many stable ordinals with the order type α of a notation system OT of ordinals. Then we have to introduce distinguished sets in the next level. In the higher level the recursive regularity is replaced by the stability, and the Π^1_1-sets $W(\mathcal{C}^{\alpha}(P))$ by Σ^1_2-sets $\mathcal{W}^{\gamma}_1(X)$.

A set X is a 1-*distinguished set*, denoted by $D_1[X]$ iff X is well-founded and

$$\forall\gamma\left(\gamma \leq X \to \mathcal{W}^{\gamma}_1(X) \cap \gamma^{\dagger} = X \cap \gamma^{\dagger}\right).$$

where $\alpha^{\dagger} = \min\{\mathbb{S} \in St : \alpha < \mathbb{S}\}$ if such a stable ordinal $\mathbb{S}$ exists. We see that $\mathcal{W}_0 = W^0_1(\emptyset)$ is the smallest 1-distinguished set, and $\mathcal{W}_1 = W^{\mathbb{S}_0}_1(\mathcal{W}_0)$ is the next 1-distinguished set, and so forth. In Lemma 7.10 it is shown that if $D_1[X]$ and $\gamma \in X$, then X is a 0-distinguished set for γ and X, i.e., $D^{\gamma}[X; X]$, and $\gamma \in W(\mathcal{C}^{\gamma}(X)) \cap \gamma^{+} = X \cap \gamma^{+}$, where $\gamma \in W^{\gamma}_1(X) \cap \gamma^{\dagger} = X \cap \gamma^{\dagger}$. This crucial lemma allows us to prove facts by going down to the lowest level, i.e., to the well-foundedness.

$\mathcal{W} := \bigcup\{X \subset OT : D_1[X]\}$ is then the 1-maximal distinguished class, which is a Σ^{1-}_3-definable class. Although $\mathcal{W}$ is a proper class in a set theory with Π_1-Collection or equivalently in $\Sigma^1_3{-}\mathrm{DC} + \mathrm{BI}$, the theories proves that if $\mathbb{S} \in \mathcal{W}$ for $\mathbb{S} \in St \cup \{0\}$, then $\mathbb{S}^\dagger \in \mathcal{W}$, cf. Lemma 7.20. In showing that a limit of stable ordinals is in $\mathcal{W}$, we invoke $\Sigma^1_3{-}\mathrm{DC}$ in Lemma 7.22: if $\alpha \in \mathcal{G}^{\mathcal{W}}$, then there exists a 1-distinguished *set* Z such that Z is closed under $\mathbb{S} \mapsto \mathbb{S}^\dagger$ and $\alpha \in \mathcal{G}^Z$ for a Π^1_0-set $\mathcal{G}^Z$ in Definition 7.14 of subsection 7.2.

By iterating this 'jump' operators, we arrive at a Σ_{N+1}-formula $D_N[X]$ denoting the fact that X is an N-distinguished set for positive integers N, cf. Definition 7.4. The maximal N-distinguished class $\bigcup\{X \subset OT : D_N[X]\}$ is Σ^{1-}_{N+2}-definable proper class in Π_N-Collection or in $\Sigma^1_{N+2}{-}\mathrm{DC} + \mathrm{BI}$.

Up to this, everything seems to go well. But as long as we have an infinite increasing sequence $\{\mathbb{S}_n\}_n = \{\mathbb{S}_0 < \mathbb{S}_1 < \cdots\}$ of successor stable ordinals, a technical difficulty is hidden as follows. Above a successor stable ordinal $\mathbb{S}_0$, there are increasing sequence $\mathbb{S}_1 = \mathbb{S}_0^\dagger < \mathbb{S}_2 = \mathbb{S}_1^\dagger < \cdots$ of successor stable ordinals. Let $\rho_n \prec \mathbb{S}_n$. Let us define ordinals $\kappa_{n,i}$ and $\sigma_{n,i}$ for $i \leq n$ recursively by $\kappa_{n,n} = \mathbb{S}_n$, $\kappa_{n,i} = \kappa_{n,i+1}[\rho_i/\mathbb{S}_i]$, $\sigma_{n,n} = \rho_n$ and $\sigma_i = \sigma_{n,i+1}[\rho_i/\mathbb{S}_i]$. Let $\kappa_n = \kappa_{n,0}$ and $\sigma_n = \sigma_{n,0}$. Then we see that $\sigma_0 < \sigma_1 < \sigma_2 < \cdots < \kappa_2 < \kappa_1 < \kappa_0$. This might yield an infinite decreasing chain $\{\kappa_n\}_n$ of collapsed ordinals.

For simplicity let $\rho_i = \psi^{f_i}_{\mathbb{S}_i}(\alpha_i)$. Then $M_{\rho_i} = \mathcal{H}_{\alpha_i}(\rho_i)$. In order to collapse $\kappa_{n,i+1}$ by ρ_i, $\rho_j \in M_{\rho_i}$ has to be enjoyed for $j > i$. Since $\rho_j > \rho_i$, this means that $\alpha_j < \alpha_i$. Namely there must exist an infinite decreasing chain $\alpha_0 > \alpha_1 > \alpha_2 > \cdots$ in advance to have another chain $\kappa_0 > \kappa_1 > \kappa_2 > \cdots$. Here α_i is the ordinal $\mathtt{p}_0(\rho_i)$ in Definition 3.30.2. Let $\eta \in L(\mathbb{S})$ be an ordinal in the layer $L(\mathbb{S})$ of a successor stable ordinal $\mathbb{S}$, cf. Definition 3.34. A pair $(\mathtt{g}_1(\eta), \mathtt{g}_2(\eta))$ of ordinals is associated with such an ordinal η in Definitions 6.7 and 6.14, and we show in Lemma 6.15 that $(\mathtt{g}_1(\gamma), \mathtt{g}_2(\gamma)) <_{lx} (\mathtt{g}_1(\eta), \mathtt{g}_2(\eta))$ when $\gamma \in R(\eta)$ for the set $R(\eta)$ in Definition 6.12. It turns out that this suffices to prove the well-foundedness in Lemma 7.32.

6.2 Props

In this subsection an ordinal $\mathtt{p}_{\mathbb{S}}(\alpha)$ and a pair $\mathtt{g}(\alpha) = (\mathtt{g}_1(\alpha), \mathtt{g}_2(\alpha))$ are introduced for ordinal terms α. These are needed to show that there is no infinite sequence $\{\rho_n, \kappa_n\}_n$ such that $\rho_0 \prec \mathbb{S}_0$, $\kappa_n \in \{\mathbb{I}_N[\rho_n]\} \cup \{\rho_n^{\dagger\vec{i}_n}, \mathbb{S}_n^{\dagger\vec{i}_n}[\rho_n/\mathbb{S}_n]\}$ and either $\rho_{n+1} \prec \mathbb{S}_n^{\dagger\vec{i}_n}[\rho_n/\mathbb{S}_n] = \kappa_n$ or $\rho_{n+1} \prec \tau^{\dagger\vec{i}_n}$ for $\tau \prec \mathbb{I}_N[\rho_n] = \kappa_n$, cf. Proposition 6.10, Lemmas 6.15 and 7.32.

Recall that $\alpha \in SSt^M$ iff either α is a successor stable ordinal in SSt or $\alpha = \beta[\rho/\mathbb{S}]$ for a $\beta \in SSt^M$ and a successor stable ordinal $\mathbb{S}$, cf. Definition 3.31.8.

Definition 6.4 For $\rho \in \Psi_{\mathbb{S}}$ with $\mathbb{S} \in SSt^M$, let $N(\rho) = \{\mathbb{I}_N[\rho]\} \cup \{\rho^{\dagger\vec{i}}, \mathbb{S}^{\dagger\vec{i}}[\rho/\mathbb{S}] : \vec{i} \neq \emptyset\} \cap OT(\mathbb{I}_N)$ if $\mathbb{S} \notin SSt$. Otherwise $N(\rho) = \{\mathbb{I}_N[\rho]\} \cup \{\mathbb{S}^{\dagger\vec{i}}[\rho/\mathbb{S}] : \vec{i} \neq$

$\emptyset\} \cap OT(\mathbb{I}_N)$.

Note that $\rho^{\dagger\vec{i}} \in SSt$ when $\mathbb{S} \in SSt$, and $N(\rho) \cap \Psi = \emptyset$. Recall that, cf. Definition 3.34, $L(\mathbb{S})$ denotes the layer of $\mathbb{S}$, and $\alpha \in L(\mathbb{S})$ iff $\alpha \prec^R \mathbb{S}$ iff there are ordinals $\{\rho_i, \kappa_i\}_i$ such that $\kappa_0 = \mathbb{S}$, $\rho_i \prec \kappa_i$, $\kappa_{i+1} \in N(\rho_i)$, and $\alpha \in \{\rho_0\} \cup \{\rho_i, \kappa_i\}_{i>0}$.

Definition 6.5 Let $\mathbb{S} \in SSt_i$ and $\mathbb{T} \in St \cup \{\Omega\}$ be the least such that $\mathbb{S} = \mathbb{T}^{\dagger i}$. For $a \in OT(\mathbb{I}_N)$, the *prop* $\mathtt{p}_{\mathbb{S}}(a)$ of a denotes an ordinal term defined recursively as follows.

1. $\mathtt{p}_{\mathbb{S}}(\mathbb{I}_N) = \mathtt{p}_{\mathbb{S}}(a) = 0$ if $a \leq \mathbb{T}$

   In what follows assume $\mathbb{I}_N \neq a > \mathbb{T}$.

2. $\mathtt{p}_{\mathbb{S}}(a) = \max_{i \leq m} \mathtt{p}_{\mathbb{S}}(a_i)$ if $a = a_0 + \cdots + a_m$.

   $\mathtt{p}_{\mathbb{S}}(a) = \max\{\mathtt{p}_{\mathbb{S}}(b), \mathtt{p}_{\mathbb{S}}(c)\}$ if $a = \varphi bc$.

3. $\mathtt{p}_{\mathbb{S}}(a) = \mathtt{p}_{\mathbb{S}}(\kappa)$ if $a \in N(\kappa)$ for a $\kappa \in L(\mathbb{U}) \cap \Psi$ with a $\mathbb{U} > \mathbb{T}$.

4. $\mathtt{p}_{\mathbb{S}}(\mathbb{U}^{\dagger k}) = \mathtt{p}_{\mathbb{S}}(\mathbb{U})$ for $\mathbb{T} \leq \mathbb{U} \in St$.

5. $\mathtt{p}_{\mathbb{S}}(\psi_{\mathbb{I}_N}(a)) = \mathtt{p}_{\mathbb{S}}(a)$.

6. For $\mathtt{p}_{\mathbb{S}}(SC(f)) = \max\{\mathtt{p}_{\mathbb{S}}(b) : b \in SC(f)\}$, let

$$\mathtt{p}_{\mathbb{S}}(\psi^f_\kappa(a)) = \begin{cases} \max\{\mathtt{p}_{\mathbb{S}}(\kappa), \mathtt{p}_{\mathbb{S}}(a), \mathtt{p}_{\mathbb{S}}(SC(f))\} & \text{if } \kappa > \mathbb{S} \\ \max\{a, \mathtt{p}_{\mathbb{S}}(a)\} & \text{if } \kappa = \mathbb{S} \\ \mathtt{p}_{\mathbb{S}}(\kappa) & \text{if } \kappa < \mathbb{S} \end{cases}$$

Proposition 6.6 *Let $\mathbb{S} \in SSt$ and $\alpha = \psi^f_{\mathbb{S}}(a)$, $\beta = \psi^g_{\mathbb{S}}(b)$ with $\{\alpha, \beta\} \subset OT(\mathbb{I}_N)$,*

1. *Let $c \in \mathcal{H}_b(\beta)$ with $\mathtt{p}_{\mathbb{S}}(c) \neq 0$. Then there exists a subterm $\gamma \in \mathcal{H}_b(\beta)$ of c such that $\gamma \prec \mathbb{S}$ and $\mathtt{p}_{\mathbb{S}}(\gamma) = \mathtt{p}_{\mathbb{S}}(c)$.*

2. *$\mathtt{p}_{\mathbb{S}}(SC(f)) \leq \mathtt{p}_{\mathbb{S}}(\alpha) = \max\{a, \mathtt{p}_{\mathbb{S}}(a)\}$ holds.*

3. *$\mathtt{p}_{\mathbb{S}}(\beta) \leq \mathtt{p}_{\mathbb{S}}(\alpha)$ if $\beta < \alpha$.*

4. *Let $\delta < \alpha < \beta$ with $\delta \prec \beta$. Then $\mathtt{p}_{\mathbb{S}}(\beta) \leq \mathtt{p}_{\mathbb{S}}(\alpha)$.*

5. *Let $\{\gamma, \delta\} \subset OT(\mathbb{I}_N)$. Then $\mathtt{p}_{\mathbb{S}}(\gamma) \leq \mathtt{p}_{\mathbb{S}}(\delta)$ if $\gamma < \delta$.*

Proof. 6.6.1. By induction on ℓc.
6.6.2. By (11) in Definition 3.31.5 we obtain $SC(f) \subset \mathcal{H}_a(SC(a))$.

By induction on ℓb, we see $b \in \mathcal{H}_a(SC(a)) \Rightarrow \mathtt{p}_{\mathbb{S}}(b) \leq \max\{a, \mathtt{p}_{\mathbb{S}}(a)\}$.

We show Propositions 6.6.3 and 6.6.4 simultaneously by induction on $\ell\beta + \ell\alpha$.
6.6.3. If $a = b$, then $\mathtt{p}_{\mathbb{S}}(\beta) = \mathtt{p}_{\mathbb{S}}(\alpha)$. Let $b < a$. We can assume $a < c = \mathtt{p}_{\mathbb{S}}(b)$.

By Proposition 6.6.1 pick a shortest subterm $\gamma \in \mathcal{H}_b(\beta) \cap \mathbb{S} \subset \beta$ of b such that $\gamma \prec \mathbb{S}$ and $\mathtt{p}_{\mathbb{S}}(\gamma) = \mathtt{p}_{\mathbb{S}}(b) = c$ for $b \in \mathcal{H}_b(\beta)$. Then $\gamma \preceq \delta = \psi^h_{\mathbb{S}}(c)$ for some h and $\gamma < \beta$. If $\delta \leq \alpha$, then IH yields $c = \mathtt{p}_{\mathbb{S}}(\delta) \leq \mathtt{p}_{\mathbb{S}}(\alpha)$. Assume $\gamma < \alpha < \delta$ with $\gamma \prec \delta$. IH for Proposition 6.6.4 then yields $c \leq \mathtt{p}_{\mathbb{S}}(\alpha)$.

Next let $a < b$. Pick a subterm η of a term in $\{a\} \cup SC(f)$ such that $\beta \leq \eta \in \mathcal{H}_a(\alpha)$ and $\eta \prec \mathbb{S}$. Let $\eta \preceq \psi^h_{\mathbb{S}}(d) = \sigma$ for some h and d. Then we obtain $\beta \leq \sigma$, and IH yields $\mathtt{p}_{\mathbb{S}}(\beta) \leq \mathtt{p}_{\mathbb{S}}(\sigma) = \mathtt{p}_{\mathbb{S}}(\eta)$. On the other hand we have $\mathtt{p}_{\mathbb{S}}(\eta) \leq \max\{a, \mathtt{p}_{\mathbb{S}}(a)\}$ by Proposition 6.6.2. Hence $\mathtt{p}_{\mathbb{S}}(\beta) \leq \mathtt{p}_{\mathbb{S}}(\alpha)$.
6.6.4. Pick a subterm η of a term in $\{a\} \cup SC(f)$ such that $\delta \leq \eta \in \mathcal{H}_a(\alpha)$, $\eta \prec \mathbb{S}$ and $\mathtt{p}_{\mathbb{S}}(\eta) \leq \max\{a, \mathtt{p}_{\mathbb{S}}(a)\}$ by Proposition 6.6.2. Let $\eta \preceq \psi^h_{\mathbb{S}}(d) = \sigma$ for some h and d. Then we obtain $\delta \leq \sigma$. If $\beta \leq \sigma$, then IH for Proposition 6.6.3 yields $\mathtt{p}_{\mathbb{S}}(\beta) \leq \mathtt{p}_{\mathbb{S}}(\sigma) = \mathtt{p}_{\mathbb{S}}(\eta) \leq \mathtt{p}_{\mathbb{S}}(\alpha)$. Otherwise we obtain $\delta \leq \sigma < \beta$ with $\delta \prec \beta$. IH yields $\mathtt{p}_{\mathbb{S}}(\beta) \leq \mathtt{p}_{\mathbb{S}}(\sigma) \leq \mathtt{p}_{\mathbb{S}}(\alpha)$.

6.6.5. This is seen by induction on $\ell\gamma + \ell\delta$ using Definition 3.35, and Propositions 6.6.3 and 6.6.4. □

The set Cr of strongly critical numbers in $OT(\mathbb{I}_N)$ is divided to $Cr = LSt_N \cup SSt \cup \bigcup\{L(\mathbb{S}) : \mathbb{S} \in SSt\} \cup (Cr \cap (\Omega+1))$, where $LSt_N = \{\psi_{\mathbb{I}_N}(a) : a \in OT(\mathbb{I}_N)\}$, cf. Definition 3.34.

Definition 6.7 Let $\mathbb{S} \in SSt$ and $\alpha \in L(\mathbb{S})$. Let us define ordinals $\mathtt{g}_0(\alpha), \mathtt{g}^*_0(\alpha)$ and $\mathtt{g}_2(\alpha)$ as follows.

1. $\mathtt{g}_0(\alpha) = \mathtt{g}_2(\alpha) = 0$ for $\alpha \notin \Psi$.
2. If $\rho \prec \mathbb{S}$, then let $\mathtt{g}_0(\rho) = \mathtt{g}^*_0(\rho) = \mathtt{g}_0(\psi_{\mathbb{I}_N[\rho]}(b)) = \mathtt{g}^*_0(\psi_{\mathbb{I}_N[\rho]}(b)) = \mathtt{p}_{\mathbb{S}}(\rho)$ for every b. Also $\mathtt{g}_2(\rho) = o_{\mathbb{I}_N}(m(\rho)) + 1$ for $m(\rho) : \mathbb{I}_N \to \varphi_{\mathbb{I}_N}(0)$ with base $\mathbb{I}_N$, and $\mathtt{g}_2(\psi_{\mathbb{I}_N[\rho]}(b)) = 0$.
3. Let $\rho \prec \mathbb{S}$ and $\alpha \prec^R \tau \in N(\rho)$, where $\alpha \neq \psi_{\mathbb{I}_N[\rho]}(b)$ for any b if $\tau = \mathbb{I}_N[\rho]$. Let $\mathtt{g}^*_0(\alpha) = \mathtt{g}_0(\rho) = \mathtt{p}_{\mathbb{S}}(\rho)$. Let $\beta \in M_\rho$ be such that $\alpha = \beta[\rho/\mathbb{S}]$. If $\alpha \in \Psi$, let $\mathtt{g}_i(\alpha) = \mathtt{g}_i(\beta)$ for $i = 0, 2$.

Proposition 6.8 *Let $b = \mathtt{p}_0(\alpha)$ for $\alpha \in L(\mathbb{S}) \cap \Psi$ with $\mathbb{S} \in SSt$. Then $SC(\mathtt{g}_2(\alpha)) \subset \psi_{\mathbb{I}_N}(b)$. Moreover $\mathtt{p}_0(\alpha) \leq \mathtt{g}^*_0(\alpha)$.*

Proof. By induction on $\ell\alpha$. Cf. Definition 3.30.2 for $\mathtt{p}_0(\alpha)$.
Case 1. First let $\alpha \preceq \psi^g_{\mathbb{S}}(b)$ with an $\mathbb{S} \in SSt$ and $f = m(\alpha)$. By Proposition 3.32.2 let $\mathbb{T} \in LSt \cup \{\Omega\}$ be such that $\mathbb{S} = \mathbb{T}^{\dagger\vec{i}}$ for a sequence $\vec{i}$. We obtain $SC(\mathtt{g}_2(\alpha)) \subset SC(f)$ for $\mathtt{g}_2(\alpha) = o_{\mathbb{I}_N}(f) + 1$. By (12) in Definition 3.31 we obtain $SC(f) \subset M_\alpha \cap \mathbb{I}_N = \mathcal{H}_b(\alpha) \cap \mathbb{I}_N$. On the other hand we have $\mathtt{p}_0(\alpha) = b \leq \mathtt{p}_{\mathbb{S}}(\alpha) = \mathtt{g}^*_0(\alpha)$.

We claim that $\alpha < \psi_{\mathbb{I}_N}(b)$. $SC(\mathtt{g}_2(\alpha)) \subset \mathcal{H}_b(\psi_{\mathbb{I}_N}(b)) \cap \mathbb{I}_N \subset \psi_{\mathbb{I}_N}(b)$ follows from the claim. For the claim it suffices to show $\mathbb{S} < \psi_{\mathbb{I}_N}(b)$. Let $\{(\mathbb{T}_m, \mathbb{S}_m, \vec{i}_m)\}_{m \leq n}$ be the sequence such that $\mathbb{T}_0 \in LSt_N \cup \{\Omega\}$, $\mathbb{S}_m = \mathbb{T}_m^{\dagger\vec{i}_m}$ and $\mathbb{T}_{m+1} \prec \mathbb{S}_m$ $(m < n)$, and $\mathbb{S} = \mathbb{S}_n$, cf. the trail to $\mathbb{S}$ in Proposition 4.12. If

$\mathbb{T}_0 = \Omega$, then $\mathbb{S} \leq \mathbb{S}_0 < \psi_{\mathbb{I}_N}(b) \in LSt_N$. Let $\mathbb{T}_0 = \psi_{\mathbb{I}_N}(c)$. Proposition 3.32.3 yields $c < b$, and $\mathbb{T}_0 = \psi_{\mathbb{I}_N}(c) < \psi_{\mathbb{I}_N}(b) \in LSt_N$. Hence $\mathbb{S} \leq \mathbb{S}_0 < \psi_{\mathbb{I}_N}(b) \in LSt_N$.
Case 2. Next let $LSt_i \ni \rho \prec \mathbb{S} \in SSt$, $\alpha \prec^R \tau \in N(\rho)$ and $\alpha = \beta[\rho/\mathbb{S}]$ for a $\beta \in M_\rho$. Then $b = \mathtt{p}_0(\alpha) = \mathtt{p}_0(\beta)$, and IH yields $SC(\mathtt{g}_2(\alpha)) = SC(\mathtt{g}_2(\beta)) \subset \psi_{\mathbb{I}_N}(b)$ for $\mathtt{g}_2(\alpha) = \mathtt{g}_2(\beta)$, and $\mathtt{p}_0(\beta) \leq \mathtt{g}_0^*(\beta)$.

On the other hand we have $\mathtt{g}_0^*(\alpha) = \mathtt{g}_0(\rho) = \mathtt{p}_{\mathbb{S}}(\rho) \geq \mathtt{p}_0(\rho) = c$ with $M_\rho = \mathcal{H}_c(\rho)$. Thus it suffices to show $\mathtt{g}_0^*(\delta) \leq \mathtt{p}_0(\rho)$ for $\rho < \delta \in \mathcal{H}_c(\rho)$ by induction on $\ell\delta$. If $\delta \preceq \psi_{\mathbb{T}}^f(d)$ with a $\mathbb{S} < \mathbb{T} \in SSt$, then $\mathtt{g}_0^*(\delta) = \mathtt{g}_0(\delta) = \mathtt{p}_{\mathbb{T}}(\delta) = \max\{d, \mathtt{p}_{\mathbb{T}}(d)\}$. We obtain $d < c$ and $d \in \mathcal{H}_c(\rho)$. IH yields $\mathtt{p}_{\mathbb{T}}(d) < c$.

Next let $\delta = \gamma[\tau/\mathbb{T}]$ with a $\gamma \in M_\tau$. Then $\mathtt{g}_0^*(\delta) = \mathtt{g}_0^*(\tau)$ and $\tau \in M_\rho$. IH yields $\mathtt{g}_0^*(\tau) \leq \mathtt{p}_0(\rho)$. □

Proposition 6.9 *Let $\tau \in L(\mathbb{S}) \cup \{\mathbb{S}\}$ and $\mathbb{S} \in SSt$.*
For $\rho, \eta \prec \tau$, if $\rho < \eta$, then $\mathtt{g}_0(\rho) \leq \mathtt{g}_0(\eta)$.

Proof. By induction on $\ell\rho$.
Case 1. $\tau = \mathbb{S}$: Let $\eta \preceq \alpha = \psi_{\mathbb{S}}^f(a)$ and $\rho \preceq \beta = \psi_{\mathbb{S}}^g(b)$. Then $\mathtt{g}_0(\eta) = \mathtt{p}_{\mathbb{S}}(\eta) = \mathtt{p}_{\mathbb{S}}(\alpha)$ and $\mathtt{g}_0(\rho) = \mathtt{p}_{\mathbb{S}}(\rho) = \mathtt{p}_{\mathbb{S}}(\beta)$. If $\beta \leq \alpha$, then Proposition 6.6.3 yields $\mathtt{p}_{\mathbb{S}}(\beta) \leq \mathtt{p}_{\mathbb{S}}(\alpha)$. Suppose $\rho < \alpha < \beta$ with $\rho \prec \beta$. We obtain $\mathtt{p}_{\mathbb{S}}(\beta) \leq \mathtt{p}_{\mathbb{S}}(\alpha)$ by Proposition 6.6.4.
Case 2. $\tau \neq \mathbb{S}$: Let $\kappa \prec \mathbb{S}$ be such that either $\tau \preceq^R \mathbb{S}^{\dagger\vec{i}}[\kappa/\mathbb{S}]$ or $\tau \prec^R \mathbb{I}_N[\kappa]$. Then $\mathtt{g}_0(\rho) = \mathtt{g}_0(\rho_1)$ and $\mathtt{g}_0(\eta) = \mathtt{g}_0(\eta_1)$ for $\rho_1 = \rho[\kappa/\mathbb{S}]^{-1}$ and $\eta_1 = \eta[\kappa/\mathbb{S}]^{-1}$, cf. Definition 3.44 for uncollapsing. We obtain $\rho_1 < \eta_1$, $\rho_1 \prec \tau_1$ and $\eta_1 \prec \tau_1$ for $\tau_1 = \tau[\kappa/\mathbb{S}]^{-1}$. IH with $\ell\rho_1 < \ell\rho$ yields $\mathtt{g}_0(\rho_1) \leq \mathtt{g}_0(\eta_1)$. □

Proposition 6.10 *Let $\mathbb{S} \in SSt$, $\rho \prec \tau \in (L(\mathbb{S}) \cup \{\mathbb{S}\}) \cap SSt^M$, and $\alpha \prec \sigma \in SSt^M$, where $\sigma \preceq^R \kappa \in N(\rho)$. Then $\mathtt{g}_0(\alpha) < \mathtt{g}_0(\rho)$.*

Proof. We may assume that either $\sigma = \kappa = \mathbb{S}^{\dagger\vec{i}}[\rho/\mathbb{S}]$ or $\kappa = \mathbb{I}_N[\rho] \,\&\, \sigma = (\psi_{\mathbb{I}_N[\rho]}(\gamma))^{\dagger\vec{i}}$ for a γ and an $\vec{i}$. By induction on $\ell\alpha$ we show $\mathtt{g}_0(\alpha) < \mathtt{g}_0(\rho)$.
Case 1. $\rho \prec \mathbb{S}$: Let $\rho \preceq \beta = \psi_{\mathbb{S}}^g(b)$. Then $\mathtt{g}_0(\rho) = \mathtt{p}_{\mathbb{S}}(\psi_{\mathbb{S}}^g(b))$. From $b \in \mathcal{H}_b(\psi_{\mathbb{S}}^g(b))$ we see $\mathtt{p}_{\mathbb{T}}(b) < \mathtt{p}_{\mathbb{T}}(\psi_{\mathbb{T}}(b)) = b \leq \mathtt{p}_{\mathbb{S}}(\psi_{\mathbb{S}}^g(b)) = \mathtt{g}_0(\rho)$ for any $\mathbb{S} < \mathbb{T} \in SSt$.
Case 1.1. $\sigma = \mathbb{S}^{\dagger\vec{i}}[\rho/\mathbb{S}]$: Let $\alpha \preceq \psi_\sigma^{h_1}(c_1) = \left(\psi_{\mathbb{S}^{\dagger\vec{i}}}^h(c)\right)[\rho/\mathbb{S}]$, where $h_1 = h[\rho/\mathbb{S}] \neq \emptyset$, $c_1 = c[\rho/\mathbb{S}]$ and $\sigma = \mathbb{S}^{\dagger\vec{i}}[\rho/\mathbb{S}] = (\mathbb{S}^{\dagger\vec{i}})[\rho/\mathbb{S}]$. Then $\mathtt{g}_0(\alpha) = \mathtt{p}_{\mathbb{S}^{\dagger\vec{i}}}(\psi_{\mathbb{S}^{\dagger\vec{i}}}^h(c))$. We have $\rho < \psi_{\mathbb{S}^{\dagger\vec{i}}}^h(c) \in M_\rho = \mathcal{H}_b(\rho)$, and hence $c < b$. We obtain $\mathtt{p}_{\mathbb{S}^{\dagger\vec{i}}}(c) \leq \mathtt{p}_{\mathbb{S}^{\dagger\vec{i}}}(b)$ by Proposition 6.6.5.
Case 1.2. $\sigma = (\psi_{\mathbb{I}_N[\rho]}(\gamma_1))^{\dagger\vec{i}}$ for a γ_1: Let $\alpha \preceq \psi_\sigma^{h_1}(c_1) = \left(\psi_{\mathbb{T}^{\dagger\vec{i}}}^h(c)\right)[\rho/\mathbb{S}]$, where $h_1 = h[\rho/\mathbb{S}] \neq \emptyset$, $c_1 = c[\rho/\mathbb{S}]$ and $\sigma = \mathbb{T}[\rho/\mathbb{S}]$ with $\mathbb{T} = \psi_{\mathbb{I}_N}(\gamma)$ and $\gamma_1 = \gamma[\rho/\mathbb{S}]$. Then $\mathtt{g}_0(\alpha) = \mathtt{p}_{\mathbb{T}^{\dagger\vec{i}}}(\psi_{\mathbb{T}^{\dagger\vec{i}}}^h(c))$. We have $\psi_{\mathbb{T}^{\dagger\vec{i}}}^h(c) \in M_\rho$. As in **Case 1.1** we see $c < b$ and $\mathtt{p}_{\mathbb{T}^{\dagger\vec{i}}}(c) \leq \mathtt{p}_{\mathbb{T}^{\dagger\vec{i}}}(b)$ from $\mathbb{S} < \mathbb{T}^{\dagger\vec{i}}$, i.e., from $\mathbb{S} < \mathbb{T} = \psi_{\mathbb{I}_N}(\gamma) \in LSt_N$.
Case 2. $\rho \prec \tau \neq \mathbb{S}$: Let $\lambda \prec \mathbb{S}$ be such that either $\alpha \prec^R \mathbb{S}^{\dagger\vec{j}}[\lambda/\mathbb{S}]$ or $\alpha \prec^R \mathbb{I}_N[\lambda]$. Then $\mathtt{g}_0(\alpha) = \mathtt{g}_0(\alpha_1)$ with $\alpha = \alpha_1[\lambda/\mathbb{S}]$ and $\mathtt{g}_0(\rho) = \mathtt{g}_0(\rho_1)$

with $\rho = \rho_1[\lambda/\mathbb{S}]$. We have $\rho_1 \prec \tau[\lambda/\mathbb{S}]^{-1}$ and $\alpha_1 \prec \sigma[\lambda/\mathbb{S}]^{-1}$ with $\sigma = \mathbb{S}^{\dagger\vec{i}}[\rho/\mathbb{S}]$ or $\sigma = (\psi_{\mathbb{I}_N[\rho]}(\gamma))^{\dagger\vec{i}}$ for a γ. If $\tau[\lambda/\mathbb{S}]^{-1} \in SSt$, then we obtain $\mathsf{g}_0(\alpha_1) < \mathsf{g}_0(\rho_1)$ by **Case 1**. Otherwise IH with $\ell\alpha_1 < \ell\alpha$ yields the proposition. □

Proposition 6.11 *Let $\{\alpha, \beta\} \subset L(\mathbb{S})$ with an $\mathbb{S} \in SSt$. If $\alpha < \beta$, then $\mathsf{g}_0^*(\alpha) \leq \mathsf{g}_0^*(\beta)$.*

Proof. Let $\rho, \eta \prec \mathbb{S}$ be such that either $\alpha = \rho$ or $\alpha \preceq^R \kappa \in N(\rho)$, and either $\beta = \eta$ or $\beta \preceq^R \sigma \in N(\eta)$. Then $\rho \leq \eta$ by $\alpha < \beta$. Proposition 6.9 yields $\mathsf{g}_0^*(\alpha) = \mathsf{g}_0(\rho) \leq \mathsf{g}_0(\eta) = \mathsf{g}_0^*(\beta)$. □

Definition 6.12 A set $R(\eta) \subset \Psi$ is defined.

1. Let $\eta \prec \mathbb{I}_N$. $\gamma \in R(\eta)$ holds iff there exists an $SSt \ni \mathbb{S} < \eta$ such that $\gamma \in L(\mathbb{S}) \cap \Psi$.
2. Let $\eta \in L(\mathbb{S})$ with an $\mathbb{S} \in SSt$. $\gamma \in R(\eta) \cap L(\mathbb{S})$ holds iff $\gamma \in \Psi$, $\gamma < \eta$ and one of the following holds:
   (a) $\gamma \prec \eta$.
   (b) There exist $\tau \in L(\mathbb{S})$ and $\vec{j}, \vec{i}$ such that $\eta \preceq \tau^{\dagger\vec{j}}$ and one of the following holds:
      i. $\gamma \prec^R \tau^{\dagger\vec{i}}$ and $\vec{i} <_{lx} \vec{j}$.
      ii. $\gamma \prec^R \tau^{\dagger\vec{i}}$, $\eta = \tau^{\dagger\vec{j}}$ and $\vec{i} = \vec{j}$.
      iii. $\gamma \prec^R \mathbb{I}_N[\tau]$.
      iv. $\gamma \prec^R \mathbb{S}^{\dagger\vec{i}}[\tau/\mathbb{S}]$.
   (c) There exist $\tau \in L(\mathbb{S})$ and $\vec{i}$ such that $\eta \preceq \mathbb{I}_N[\tau]$, and and one of the following holds:
      i. $\gamma \prec^R \mathbb{I}_N[\tau]$ and $\eta = \mathbb{I}_N[\tau]$.
      ii. $\gamma \prec^R \mathbb{S}^{\dagger\vec{i}}[\tau/\mathbb{S}]$.
   (d) There exist $\tau \in L(\mathbb{S})$ and $\vec{j}, \vec{i}$ such that $\eta \preceq \mathbb{S}^{\dagger\vec{j}}[\tau/\mathbb{S}]$, and and one of the following holds:
      i. $\gamma \prec^R \mathbb{S}^{\dagger\vec{i}}[\tau/\mathbb{S}]$ and $\vec{i} <_{lx} \vec{j}$.
      ii. $\gamma \prec^R \mathbb{S}^{\dagger\vec{i}}[\tau/\mathbb{S}]$, $\eta = \mathbb{S}^{\dagger\vec{j}}[\tau/\mathbb{S}]$ and $\vec{i} = \vec{j}$.
   (e) There exist $\tau \in L(\mathbb{S})$, ρ and $\vec{i}$ such that $\eta, \rho \prec \mathbb{I}_N[\tau]$, $\rho < \eta$ and $\gamma \prec^R \rho^{\dagger\vec{i}}$.
   (f) There exist $\tau \in (L(\mathbb{S}) \cup \{\mathbb{S}\}) \cap SSt^M$, ρ and κ such that $\eta, \rho \prec \tau$, $\rho < \eta$, $\gamma \prec^R \kappa \in N(\rho)$.

Proposition 6.13 *Let $\eta, \gamma \in L(\mathbb{S})$ for an $\mathbb{S} = \mathbb{T}^{\dagger k} \in SSt$ with $\mathbb{T} \in \{\Omega\} \cup (LSt \cap \Psi)$. Assume $\eta > \gamma \notin R(\eta)$, and let τ be maximal such that $\gamma \prec \tau \leq \eta$. Then $\eta > \tau \in \Psi$.*

Proof. This is seen by an inspection to Definition 3.35. □

Definition 6.14 Let $\mathbb{S} \in SSt$ and $\alpha \in L(\mathbb{S})$.

Let $\vec{i} = (i_0 \geq i_1 \geq \cdots \geq i_m)$ be a weakly descending chain of positive integers with $i_0 \leq N$. Then let $o(\vec{i}) := \omega^{i_0-1} + \omega^{i_1-1} + \cdots + \omega^{i_m-1} < \omega^N$.

Let us define ordinals $\mathsf{g}_1'(\alpha)$ and $\mathsf{g}_1(\alpha)$ as follows. Let $\lambda = \omega^{N+1}$.

1. Let $\rho \prec \mathbb{S}$. Then $\mathsf{g}_1'(\rho) = \lambda^{\mathsf{g}_0(\rho)}$ and $\mathsf{g}_1(\rho) = \lambda^{\mathsf{g}_0(\rho)+1}$.
2. Let $\rho \in L(\mathbb{S})$ be such that $\rho \prec \mathbb{T} \in SSt^M \cap (L(\mathbb{S}) \cup \{\mathbb{S}\})$, $\alpha \prec \kappa \in N(\rho) \cup \{(\psi_{\mathbb{I}_N[\rho]}(a))^{\dagger\vec{i}} : \vec{i} \neq \emptyset\}$, where $\alpha \neq \psi_{\mathbb{I}_N[\rho]}(b)$ for any b if $\kappa = \mathbb{I}_N[\rho]$. Let $\mathsf{g}_1(\alpha) = \mathsf{g}_1'(\alpha) + \lambda^{\mathsf{g}_0(\rho)}$.
   (a) $\mathsf{g}_1'(\mathbb{T}^{\dagger\vec{i}}[\rho/\mathbb{T}]) = \mathsf{g}_1'(\rho) + \lambda^{\mathsf{g}_0(\rho)} \cdot (o(\vec{i}) + 1)$.
   (b) $\alpha \prec \mathbb{T}^{\dagger\vec{i}}[\rho/\mathbb{T}]$: $\mathsf{g}_1'(\alpha) = \mathsf{g}_1'(\rho) + \lambda^{\mathsf{g}_0(\rho)} \cdot o(\vec{i})$.
   (c) $\mathsf{g}_1'(\mathbb{I}_N[\rho]) = \mathsf{g}_1'(\rho) + \lambda^{\mathsf{g}_0(\rho)} \cdot (\omega^N + 1)$.
   (d) $\alpha \prec \mathbb{I}_N[\rho]$: $\mathsf{g}_1'(\alpha) = \mathsf{g}_1'(\rho) + \lambda^{\mathsf{g}_0(\rho)} \cdot \omega^N$.
   (e) $\mathsf{g}_1'((\psi_{\mathbb{I}_N[\rho]}(a))^{\dagger\vec{i}}) = \mathsf{g}_1'(\rho) + \lambda^{\mathsf{g}_0(\rho)} \cdot (\omega^N + o(\vec{i}) + 1)$.
   (f) $\alpha \prec (\psi_{\mathbb{I}_N[\rho]}(a))^{\dagger\vec{i}}$: $\mathsf{g}_1'(\alpha) = \mathsf{g}_1'(\rho) + \lambda^{\mathsf{g}_0(\rho)} \cdot (\omega^N + o(\vec{i}))$.
   (g) $\mathsf{g}_1'(\rho^{\dagger\vec{i}}) = \mathsf{g}_1'(\rho) + \lambda^{\mathsf{g}_0(\rho)} \cdot (\omega^N + \omega^N + o(\vec{i}) + 1)$.
   (h) $\alpha \prec \rho^{\dagger\vec{i}}$: $\mathsf{g}_1'(\alpha) = \mathsf{g}_1'(\rho) + \lambda^{\mathsf{g}_0(\rho)} \cdot (\omega^N + \omega^N + o(\vec{i}))$.

Let $\mathsf{g}(\alpha) = (\mathsf{g}_1(\alpha), \mathsf{g}_2(\alpha))$.

Lemma 6.15 *Let $\eta \in L(\mathbb{S})$ with $\mathbb{S} \in SSt$. Then $\mathsf{g}_0^*(\gamma) \leq \mathsf{g}_0^*(\eta)$, $\mathsf{g}(\gamma) <_{lx} \mathsf{g}(\eta)$ and $SC(\mathsf{g}_2(\gamma)) \subset \psi_{\mathbb{I}_N}(b)$ for $\gamma \in R(\eta)$ and $b = \mathsf{g}_0^*(\eta)$.*

Proof.
Case 1. $\gamma \prec \eta$: We have $\mathsf{g}_0^*(\gamma) = \mathsf{g}_0^*(\eta)$. If $\eta \in \Psi$, then $\mathsf{g}_1(\eta) = \mathsf{g}_1(\gamma)$ and $\mathsf{g}_2(\gamma) < \mathsf{g}_2(\eta)$ by Lemma 6.2. Otherwise $\mathsf{g}_1(\gamma) < \mathsf{g}_1(\eta)$. In what follows assume $\gamma \not\prec \eta$. We claim that $\mathsf{g}_1(\gamma) < \mathsf{g}_1(\eta)$.
Case 2. $\eta \preceq \tau_1$, $\gamma \prec^R \tau_2$ with $\{\tau_2 \leq \tau_1\} \subset N(\tau)$ for a $\tau \in L(\mathbb{S})$, cf. Definitions 6.12.2b, 6.12.2(b)iii, 6.12.2(b)iv, 6.12.2c, 6.12.2d: We have $\mathsf{g}_1(\eta) = \mathsf{g}_1'(\tau) + \lambda^{\mathsf{g}_0(\tau)} \cdot (\alpha+1)$ for an $\alpha < \omega^{N+1}$. If $\gamma \prec \tau_2$, then $\mathsf{g}_1(\gamma) = \mathsf{g}_1'(\tau) + \lambda^{\mathsf{g}_0(\tau)} \cdot (\beta+1)$ with $\beta < \alpha$. Otherwise let $\sigma \prec \tau_2$ be such that $\gamma \prec \sigma_1 \in SSt^M$, $\sigma_1 \preceq^R \kappa_1 \in N(\sigma)$. We obtain $\mathsf{g}_0(\sigma) < \mathsf{g}_0(\tau)$ by Proposition 6.10, and $\mathsf{g}_1(\gamma) = \mathsf{g}_1'(\tau) + \lambda^{\mathsf{g}_0(\tau)} \cdot \beta + \delta$ with $\delta < \lambda^{\mathsf{g}_0(\sigma)+1} \leq \lambda^{\mathsf{g}_0(\tau)}$.
Case 3. $\rho, \eta \prec \tau_1 \in N(\tau)$, $\rho < \eta$ and $\gamma \prec^R \kappa \in N(\rho)$, cf. Definitions 6.12.2e and 6.12.2f: We have $\mathsf{g}_1'(\rho) = \mathsf{g}_1'(\tau) + \lambda^{\mathsf{g}_0(\tau)} \cdot \alpha$ for an $\alpha < \omega^{N+1}$, $\mathsf{g}_1(\eta) = \mathsf{g}_1'(\rho) + \lambda^{\mathsf{g}_0(\tau)}$, and $\mathsf{g}_1(\gamma) = \mathsf{g}_1'(\rho) + \delta$ for $\delta < \lambda^{\mathsf{g}_0(\tau)}$ by Proposition 6.10.
Case 4. $\rho, \eta \prec \mathbb{S}$, $\rho < \eta$ and $\gamma \prec^R \kappa \in N(\rho)$, cf. Definition 6.12.2f: We have $\mathsf{g}_1(\eta) = \lambda^{\mathsf{g}_0(\eta)+1}$, and $\mathsf{g}_1'(\rho) = \lambda^{\mathsf{g}_0(\rho)}$, where $\mathsf{g}_0(\rho) \leq \mathsf{g}_0(\eta)$ by Proposition 6.9. On the other hand we have $\mathsf{g}_1(\gamma) = \mathsf{g}_1'(\rho) + \delta$ with $\delta < \lambda^{\mathsf{g}_0(\rho)+1}$.

Thus $\mathtt{g}(\gamma) <_{lx} \mathtt{g}(\eta)$ is shown. In each case $c = \mathtt{p}_0(\gamma) \leq \mathtt{g}^*_0(\gamma) \leq \mathtt{g}^*_0(\eta) = b$ holds by Proposition 6.8. We obtain $\psi_{\mathbb{I}_N}(c) \leq \psi_{\mathbb{I}_N}(b)$ by $c \in \mathcal{H}_c(\psi_{\mathbb{I}_N}(c))$ and $b \in \mathcal{H}_b(\psi_{\mathbb{I}_N}(b))$. On the other hand we have $SC(\mathtt{g}_2(\gamma)) \subset \psi_{\mathbb{I}_N}(c)$ by Proposition 6.8. Hence $SC(\mathtt{g}_2(\gamma)) \subset \psi_{\mathbb{I}_N}(b)$. □

Proposition 6.16 *Let* $\{\alpha_1, \beta\} \subset L(\mathbb{S})$ *for an* $\mathbb{S} \in SSt$*,* $\alpha_1 = \psi^f_\kappa(a) \leq \psi^h_\sigma(c) = \beta$ *and* $\beta \in \mathcal{H}_a(\alpha_1)$*. Then* $c < a$ *and* $\mathtt{g}^*_0(\beta) \leq \mathtt{g}^*_0(\alpha_1)$*.*

Proof. By induction on $\ell\beta$. We have $c \in K_{\alpha_1}(\beta) < a$, and $\{\sigma, c\} \subset \mathcal{H}_a(\alpha_1)$. We show $\mathtt{g}^*_0(\beta) \leq \mathtt{g}^*_0(\alpha_1)$. First let $\beta \prec \mathbb{S}$. We show $\mathtt{g}^*_0(\beta) = \mathtt{p}_{\mathbb{S}}(\beta) \leq \mathtt{g}^*_0(\alpha_1)$. We can assume $\sigma = \mathbb{S}$ by IH. Let $\gamma = \psi^g_{\mathbb{S}}(b)$ be a proper subterm of β. If $\gamma \in K_{\alpha_1}(\beta)$, then $b < a$. If $\gamma < \alpha_1$, then $\mathtt{g}^*_0(\gamma) \leq \mathtt{g}^*_0(\alpha_1)$ by Proposition 6.11.

Second let $\rho \prec \mathbb{S}$ and $\beta \prec^R \kappa \in N(\rho)$. Then $\mathtt{g}^*_0(\beta) = \mathtt{p}_{\mathbb{S}}(\rho)$. If $\alpha_1 \leq \rho$, then $\rho \in \mathcal{H}_a(\alpha_1)$ and $\mathtt{p}_{\mathbb{S}}(\rho) \leq \mathtt{g}^*_0(\alpha_1)$ by the first case. Let $\rho < \alpha_1 < \beta$. Then we obtain $\mathtt{g}^*_0(\alpha) = \mathtt{p}_{\mathbb{S}}(\rho) = \mathtt{g}^*_0(\beta)$. □

6.3 Coefficients

In this subsection we introduce coefficient sets $\mathcal{E}(\alpha), G_\delta(\alpha), F_X(\alpha), k_X(\alpha)$ of $\alpha \in OT(\mathbb{I}_N)$ for $X \subset OT(\mathbb{I}_N)$, each of which is a finite set of subterms of α. These are utilized in our well-foundedness proof. Roughly $\mathcal{E}(\alpha)$ is the set of subterms of the form $\psi^f_\pi(a)$, and $F_X(\alpha)$ $[k_X(\alpha)]$ the set of subterms in X [subterms not in X], resp.

Let us write for $\alpha < \mathbb{I}_N$, $\alpha^{\dagger 0} = \min\{\sigma \in Reg : \sigma > \alpha\}$ for the next regular ordinal α^+ above α. Let $\alpha^{\dagger i} := \infty$ if $\alpha \geq \mathbb{I}_N$. For $0 \leq i \leq N$, let $\alpha^{-i} := \max\{\sigma \in St_i \cup \{0\} : \sigma \leq \alpha\}$ when $\alpha < \mathbb{I}_N$, and $\alpha^{-i} := \mathbb{I}_N$ if $\alpha \geq \mathbb{I}_N$.

Although α^{-1} looks alike the Mostowski uncollapsing $\alpha[\rho/\mathbb{S}]^{-1}$ in Definition 3.44, no confusion likely occurs.

Since $St_{i+1} \subset St_i$, we obtain $\alpha^{\dagger i} \leq \alpha^{\dagger(i+1)}$ and $\beta^{\dagger 0} < \sigma$ if $\beta < \sigma \in St \cap \mathbb{I}_N$ since each $\sigma \in St$ is a limit of regular ordinals.

Note that $R(\eta) \subset L(\mathbb{S})$ if $\eta \in L(\mathbb{S})$, and $\gamma^{-N} = \eta^{-N}$ for every $\gamma, \eta \in L(\mathbb{S})$.

Definition 6.17 For terms $\alpha, \delta \in OT(\mathbb{I}_N)$ and $X \subset OT(\mathbb{I}_N)$, finite sets $\mathcal{E}(\alpha)$, $G_\delta(\alpha)$, $F_X(\alpha)$, $k_X(\alpha)$ of terms are defined recursively as follows.

1. $\mathcal{E}(\alpha) = \emptyset$ for $\alpha \in \{0, \Omega, \mathbb{I}_N\}$. $\mathcal{E}(\alpha_m + \cdots + \alpha_0) = \bigcup_{i \leq m} \mathcal{E}(\alpha_i)$. $\mathcal{E}(\varphi\beta\gamma) = \mathcal{E}(\beta) \cup \mathcal{E}(\gamma)$. $\mathcal{E}(\mathbb{I}_N[\rho]) = \mathcal{E}(\rho^{\dagger\vec{i}}) = \mathcal{E}(\mathbb{S}^{\dagger\vec{i}}[\rho/\mathbb{S}]) = \mathcal{E}(\rho)$. $\mathcal{E}(\psi^f_\pi(a)) = \{\psi^f_\pi(a)\}$. $\mathcal{E}(\psi_{\mathbb{I}_N}(a)) = \{\psi_{\mathbb{I}_N}(a)\}$.
2. $\mathcal{A}(\alpha) = \bigcup\{\mathcal{A}(\beta) : \beta \in \mathcal{E}(\alpha)\}$ for $\mathcal{A} \in \{G_\delta, F_X, k_X\}$.
3. $G_\delta(\psi_{\mathbb{I}_N}(a)) = G_\delta(a)$. $F_X(\psi_{\mathbb{I}_N}(a)) = F_X(a)$ if $\psi_{\mathbb{I}_N}(a) \notin X$, and $F_X(\psi_{\mathbb{I}_N}(a)) = \{\psi_{\mathbb{I}_N}(a)\}$ if $\psi_{\mathbb{I}_N}(a) \in X$. $k_X(\psi_{\mathbb{I}_N}(a)) = \{\psi_{\mathbb{I}_N}(a)\} \cup k_X(a)$ if $\psi_{\mathbb{I}_N}(a) \notin X$, and $k_X(\psi_{\mathbb{I}_N}(a)) = \emptyset$ if $\psi_{\mathbb{I}_N}(a) \in X$.

$$G_\delta(\psi^f_\pi(a)) = \begin{cases} G_\delta(\{\pi, a\} \cup SC(f)) & \delta < \pi \\ \{\psi^f_\pi(a)\} & \pi \leq \delta \end{cases}$$

$$F_X(\psi_\pi^f(a)) = \left\{ \begin{array}{ll} F_X(\{\pi, a\} \cup SC(f)) & \psi_\pi^f(a) \not\in X \\ \{\psi_\pi^f(a)\} & \psi_\pi^f(a) \in X \end{array} \right.$$

$$k_X(\psi_\pi^f(a)) = \left\{ \begin{array}{ll} \{\psi_\pi^f(a)\} \cup k_X(\{\pi, a\} \cup SC(f)) & \psi_\pi^f(a) \not\in X \\ \emptyset & \psi_\pi^f(a) \in X \end{array} \right.$$

4. For $\alpha \in N(\rho)$

$$G_\delta(\alpha) = \left\{ \begin{array}{ll} \{\alpha\} & \alpha < \delta \\ G_\delta(\rho) & \delta \leq \alpha \end{array} \right.$$

$F_X(\alpha) = F_X(\rho)$ and $k_X(\alpha) = k_X(\rho)$.

For $\mathcal{A} \in \{K_\delta, G_\delta, F_X, k_X\}$ and sets $Y \subset OT(\mathbb{I}_N)$, $\mathcal{A}(Y) := \bigcup\{\mathcal{A}(\alpha) : \alpha \in Y\}$.

Definition 6.18 $S(\eta)$ denotes the set of immediate subterms of η. For example $S(\varphi\beta\gamma) = \{\beta, \gamma\}$. $S(\eta) := \emptyset$ when $\eta \in \{0, \Omega, \mathbb{I}_N\}$, $S(\alpha) = \{\rho\}$ for $\alpha \in N(\rho)$, $S(\eta) = \{\eta\}$ when $\eta \in \Psi$.

Proposition 6.19 *For $\{\alpha, \delta, a, b, \rho\} \subset OT(\mathbb{I}_N)$,*

1. *$G_\delta(\alpha) \leq \alpha$.*
2. *$\alpha \in \mathcal{H}_a(b) \Rightarrow G_\delta(\alpha) \subset \mathcal{H}_a(b)$.*

Proof. These are shown simultaneously by induction on $\ell\alpha$. It is easy to see that

$$G_\delta(\alpha) \ni \beta \Rightarrow \beta < \delta \,\&\, \ell\beta \leq \ell\alpha \tag{43}$$

6.19.1. Consider the case $\alpha = \psi_\pi^f(a)$ with $\delta < \pi$. Then $G_\delta(\alpha) = G_\delta(SC(f) \cup \{\pi, a\})$. On the other hand we have $SC(f) \cup \{\pi, a\} \subset \mathcal{H}_a(\alpha)$. Proposition 6.19.2 with (43) yields $G_\delta(SC(f) \cup \{\pi, a\}) \subset \mathcal{H}_a(\alpha) \cap \pi \subset \alpha$. Hence $G_\delta(\alpha) < \alpha$.

Next let $\alpha \in N(\rho)$ with $\delta \leq \alpha$. Then $G_\delta(\alpha) = G_\delta(\rho)$. By IH we have $G_\delta(\rho) \leq \rho < \alpha$. Hence $G_\delta(\alpha) < \alpha$.

6.19.2. Since $G_\delta(\alpha) \leq \alpha$ by Proposition 6.19.1, we can assume $\alpha \geq b$.

Consider the case $\alpha = \psi_\pi^f(a)$ with $\delta < \pi$. Then $SC(f) \cup \{\pi, a\} \subset \mathcal{H}_a(b)$ and $G_\delta(\alpha) = G_\delta(SC(f) \cup \{\pi, a\})$. IH yields the lemma.

Next let $\alpha \in N(\rho)$ with $\delta \leq \alpha$. Then $G_\delta(\alpha) = G_\delta(\rho)$ and $\rho < \alpha$. $b \leq \alpha \in \mathcal{H}_a(b)$ yields $\rho \in \mathcal{H}_a(b)$. IH yields the lemma. □

Proposition 6.20 *If $\beta \not\in \mathcal{H}_a(Y)$ and $K_X(\beta) < a$, then there exists a $\gamma \in F_X(\beta)$ such that $\mathcal{H}_a(Y) \not\ni \gamma \in X$.*

Proof. By induction on $\ell\beta$. Assume $\beta \not\in \mathcal{H}_a(\alpha)$ and $K_X(\beta) < a$. By IH we can assume that $\beta = \psi_\kappa^f(b)$. If $\beta \in X$, then $\beta \in F_X(\beta)$, and $\gamma = \beta$ is a desired one. Assume $\beta \not\in X$. Then we obtain $K_X(\beta) = \{b\} \cup K_X(\{b, \kappa\} \cup SC(f)) < a$. In particular $b < a$, and hence $\{b, \kappa\} \cup SC(f) \not\subset \mathcal{H}_a(Y)$. By IH there exists a $\gamma \in F_X(\{b, \kappa\} \cup SC(f)) = F_X(\beta)$ such that $\mathcal{H}_a(Y) \not\ni \gamma \in X$. □

7 Well-foundedness proof with the maximal distinguished sets

In this section working in the second order arithmetic Σ^1_{N+2}-DC + BI, we show the well-foundedness of the notation system $OT(\mathbb{I}_N)$ up to *each* $\alpha < \Omega$. The proof is based on distinguished classes, which was first introduced by Buchholz[7]. Each ordinal term $\alpha \in OT(\mathbb{I}_N)$ is identified with its code $\lceil \alpha \rceil \in \mathbb{N}$, cf. Lemma 3.36.

7.1 Distinguished sets

In this subsection we establish elementary facts on distinguished classes.

$X, Y, Z, \ldots$ range over *subsets* of $OT(\mathbb{I}_N)$, while $\mathcal{X}, \mathcal{Y}, \ldots$ range over *classes*, which are definable by second-order formulas in the language of arithmetic. Following [10], we define sets $C^\alpha(X) \subset OT(\mathbb{I}_N)$ for $\alpha \in OT(\mathbb{I}_N)$ and $X \subset OT(\mathbb{I}_N)$ as follows.

Definition 7.1 For $\alpha, \beta \in OT(\mathbb{I}_N)$ and $X \subset OT(\mathbb{I}_N)$, let us define a set $C^\alpha(X)$ recursively as follows.

1. $\{0, \Omega, \mathbb{I}_N\} \cup (X \cap \alpha) \subset C^\alpha(X)$.
2. Let $(\alpha_1 + \cdots + \alpha_n) \in OT(\mathbb{I}_N)$ with $\{\alpha_1, \ldots, \alpha_n\} \subset C^\alpha(X)$. Then $(\alpha_1 + \cdots + \alpha_n) \in C^\alpha(X)$.
3. Let $\varphi\beta\gamma \in OT(\mathbb{I}_N)$ with $\{\beta, \gamma\} \subset C^\alpha(X)$. Then $\varphi\beta\gamma \in C^\alpha(X)$.
4. Let $\psi_{\mathbb{I}_N}(\beta) \in OT(\mathbb{I}_N)$ with $\beta \in C^\alpha(X)$. Then $\psi_{\mathbb{I}_N}(\beta) \in C^\alpha(X)$ if $\mathbb{I}_N > \alpha$.
5. Let $\psi^f_\sigma(\beta) \in OT(\mathbb{I}_N)$ with $\{\sigma, \beta\} \cup SC(f) \subset C^\alpha(X)$. Then $\psi^f_\sigma(\beta) \in C^\alpha(X)$ if $\sigma > \alpha$.
6. Let $\beta \in N(\rho)$ with $\rho \in C^\alpha(X)$. Then $\beta \in C^\alpha(X)$ if $\beta \geq \alpha$.

Proposition 7.2 *Assume* $\forall \gamma \geq \alpha[\gamma \in P \Rightarrow \gamma \in C^\gamma(P)]$ *for a set* $P \subset OT(\mathbb{I}_N)$.

1. $\alpha \leq \beta \Rightarrow C^\beta(P) \subset C^\alpha(P)$.
2. $\alpha \leq \beta < \alpha^{\dagger 0} \Rightarrow C^\beta(P) = C^\alpha(P)$.

Proof. 7.2.1. We see by induction on $\ell\gamma\,(\gamma \in OT(\mathbb{I}_N))$ that

$$\forall \beta \geq \alpha[\gamma \in C^\beta(P) \Rightarrow \gamma \in C^\alpha(P) \cup (P \cap \beta)] \tag{44}$$

For example, if $\psi^f_\pi(\delta) \in C^\beta(P)$ with $\pi > \beta \geq \alpha$ and $\{\pi, \delta\} \cup SC(f) \subset C^\alpha(P) \cup (P \cap \beta)$, then $\pi \in C^\alpha(P)$, and for any $\gamma \in \{\delta\} \cup SC(f)$, either $\gamma \in C^\alpha(P)$ or $\gamma \in P \cap \beta$. If $\gamma < \alpha$, then $\gamma \in P \cap \alpha \subset C^\alpha(P)$. If $\alpha \leq \gamma \in P \cap \beta$, then $\gamma \in C^\gamma(P)$ by the assumption, and by IH we have $\gamma \in C^\alpha(P) \cup (P \cap \gamma)$, i.e., $\gamma \in C^\alpha(P)$. Therefore $\{\pi, \delta\} \cup SC(f) \subset C^\alpha(P)$, and $\psi^f_\pi(\delta) \in C^\alpha(P)$.

Using (44) we see from the assumption that $\forall\beta \geq \alpha[\gamma \in C^{\beta}(P) \Rightarrow \gamma \in C^{\alpha}(P)]$.

7.2.2. Assume $\alpha \leq \beta < \alpha^{\dagger 0}$. Then by Proposition 7.2.1 we have $C^{\beta}(P) \subset C^{\alpha}(P)$. $\gamma \in C^{\alpha}(P) \Rightarrow \gamma \in C^{\beta}(P)$ is seen by induction on $\ell\gamma$ using the facts $\beta^{-0} = \alpha^{-0}$ and $\beta^{\dagger 0} = \alpha^{\dagger 0}$. □

Definition 7.3 1. $\mathrm{Prg}[X, Y] :\Leftrightarrow \forall\alpha \in X(X \cap \alpha \subset Y \to \alpha \in Y)$.

2. For a definable class $\mathcal{X}$, $\mathrm{TI}[\mathcal{X}]$ denotes the schema:
   $\mathrm{TI}[\mathcal{X}] :\Leftrightarrow \mathrm{Prg}[\mathcal{X}, \mathcal{Y}] \to \mathcal{X} \subset \mathcal{Y}$ holds for any definable classes $\mathcal{Y}$.

3. For $X \subset OT(\mathbb{I}_N)$, $W(X)$ denotes the *well-founded part* of X.

4. $Wo[X] :\Leftrightarrow X \subset W(X)$.

Note that for $\alpha \in OT(\mathbb{I}_N)$, $W(X) \cap \alpha = W(X \cap \alpha)$.

Definition 7.4 For $P, X \subset OT(\mathbb{I}_N) \cap \mathbb{I}_N$ and $\alpha, \gamma \in OT(\mathbb{I}_N)$ with $\gamma < \mathbb{I}_N$, define $W_i^{\alpha}(P)\,(0 \leq i \leq N)$ and $D_i^{\gamma}[P; X]\,(0 \leq i \leq N)$ recursively on $i \leq N$ as follows.

$$
\begin{aligned}
W_0^{\alpha}(P) &:= W(C^{\alpha}(P)) && (45)\\
D_i^{\gamma}[P; X] &:\Leftrightarrow Wo[P]\,\&\, P \cap \gamma^{-(i+1)} = X \cap \gamma^{-(i+1)}\,\& && (46)\\
&\qquad \forall\alpha < \mathbb{I}_N \left(\gamma^{-(i+1)} \leq \alpha \leq P \to W_i^{\alpha}(P) \cap \alpha^{\dagger i} = P \cap \alpha^{\dagger i}\right)\\
W_{i+1}^{\gamma}(X) &:= \bigcup\{P \subset OT(\mathbb{I}_N) \cap \mathbb{I}_N : D_i^{\gamma}[P; X]\}\,(i < N) && (47)
\end{aligned}
$$

where $\gamma^{-(N+1)} := 0$. Obviously $D_N^{\gamma}[X; Y] \Leftrightarrow D_N^{\delta}[X; Z]$ for every γ, δ, Y, Z. From $\mathcal{W}_N^{\gamma}(X)$ define

$$
\begin{aligned}
D_N[X] &:\Leftrightarrow D_N[X; X]\\
&\Leftrightarrow Wo[X]\,\&\,\forall\gamma\left(\gamma \leq X \to \mathcal{W}_N^{\gamma}(X) \cap \gamma^{\dagger N} = X \cap \gamma^{\dagger N}\right)\\
\mathcal{W}_{N+1} &:= \bigcup\{X \subset OT(\mathbb{I}_N) \cap \mathbb{I}_N : D_N[X]\}
\end{aligned}
$$

A set P is said to be an *i-distinguished set* for γ and X if $D_i^{\gamma}[P; X]$, and a set X is an *N-distinguished set* if $D_N[X]$.

Observe that in $S_{\mathbb{I}_N}$, $W_0^{\alpha}(P)$ as well as $D_0^{\gamma}[P; X]$ are Δ_1. Assuming that $D_i^{\gamma}[P; X]$ is Δ_{i+1}, $\mathcal{W}_{i+1}^{\gamma}(X)$ is Σ_{i+1}, and $D_{i+1}^{\gamma}[P; X]$ is Δ_{i+2}. Hence $D_N[X]$ is Δ_{N+1}, and $\mathcal{W} = \mathcal{W}_{N+1}$ is a Σ_{N+1}^{-}-class. In $S_{\mathbb{I}_N}$, each $\mathcal{W}_i^{\gamma}(X)$ is a set, i.e., $\forall\gamma \in OT(\mathbb{I}_N) \cap \mathbb{I}_N \forall X \subset OT(\mathbb{I}_N) \exists Y[Y = \mathcal{W}_i^{\gamma}(X)]$ for $0 \leq i \leq N$, and $\mathcal{W}_{N+1}$ is a proper class.

Proposition 7.5 *Let* $D_0^{\gamma}[P; X]$ *and* $\gamma^{-1} \leq \alpha \in P$. *Then* $\forall\beta \geq \gamma^{-1}[\alpha \in C^{\beta}(P)]$.

Proof. Let $D_0^\gamma[P;X]$ and $\gamma^{-1}\leq\alpha\in P$. We obtain $\alpha\in P\cap\alpha^{\dagger 0}=W(C^\alpha(P))\cap\alpha^{\dagger 0}\subset C^\alpha(P)$ by (45) and (46). Hence $\forall\delta\geq\gamma^{-1}(\delta\in P\Rightarrow\delta\in C^\delta(P))$, and $\alpha\in C^\beta(P)$ for any $\gamma^{-1}\leq\beta\leq\alpha$ by Proposition 7.2.1. Moreover for $\beta>\alpha$ we have $\alpha\in P\cap\beta\subset C^\beta(P)$. □

Proposition 7.6 *If $P\cap\alpha=Q\cap\alpha$, then $W_i^\alpha(P)=W_i^\alpha(Q)$.*

Proof. For $i>0$, this follows from (46) and $\alpha^{-i}\leq\alpha$. For $i=0$, we obtain $C^\alpha(P)=C^\alpha(Q)$ by $P\cap\alpha=Q\cap\alpha$. Hence $W_0^\alpha(P)=W(C^\alpha(P))=W(C^\alpha(Q))=W_0^\alpha(Q)$ by (45). □

Lemma 7.7 $\alpha\leq P\,\&\,\alpha\leq Q\Rightarrow P\cap\alpha^{\dagger i}=Q\cap\alpha^{\dagger i}$ *if $D_i^\gamma[P;X]$ and $D_i^\gamma[Q;X]$.*

Proof. Suppose $\alpha\leq P$, $\alpha\leq Q$, $D_i^\gamma[P;X]$ and $D_i^\gamma[Q;X]$. We have $P\cap\gamma^{-(i+1)}=X\cap\gamma^{-(i+1)}=Q\cap\gamma^{-(i+1)}$. We may assume that $\gamma^{-(i+1)}\leq\alpha$ since $\alpha^{\dagger i}\leq\gamma^{-(i+1)}$ when $\alpha<\gamma^{-(i+1)}$.

By (46) we obtain $\mathcal{W}_i^\alpha(P)\cap\alpha^{\dagger i}=P\cap\alpha^{\dagger i}$ and $\mathcal{W}_i^\alpha(Q)\cap\alpha^{\dagger i}=Q\cap\alpha^{\dagger i}$. We obtain $Wo[P\cup Q]$ by $Wo[P]$ and $Wo[Q]$. We show $\beta\in P\cap Q$ by induction on $\beta\in(P\cup Q)\cap\alpha^{\dagger i}$. Let $\beta\in(P\cup Q)\cap\alpha^{\dagger i}$ and $P\cap\beta=Q\cap\beta$. If $\beta<\gamma^{-(i+1)}$, then $\beta\in P\cap Q$ by $P\cap\gamma^{-(i+1)}=Q\cap\gamma^{-(i+1)}$. Let $\gamma^{-(i+1)}\leq\beta$.

If $\alpha\leq\beta$, then $P\cap\alpha=Q\cap\alpha$, and $\mathcal{W}_i^\alpha(P)\cap\alpha^{\dagger i}=\mathcal{W}_i^\alpha(Q)\cap\alpha^{\dagger i}$ by Proposition 7.6. Hence $\beta\in P\cap Q$.

Let $\gamma^{-(i+1)}\leq\beta<\alpha$. We obtain $\mathcal{W}_i^\beta(P)\cap\beta^{\dagger i}=\mathcal{W}_i^\beta(Q)\cap\beta^{\dagger i}$ by $P\cap\beta=Q\cap\beta$ and Proposition 7.6. By (46), $\beta\leq P$ and $\beta\leq Q$, we obtain $P\cap\beta^{\dagger i}=\mathcal{W}_i^\beta(P)\cap\beta^{\dagger i}=\mathcal{W}_i^\beta(Q)\cap\beta^{\dagger i}=Q\cap\beta^{\dagger i}$. Hence $\beta\in P\cap Q$. □

Lemma 7.8 (Σ^1_{N+1}-CA)
For each $i\leq N$, $\forall\gamma<\mathbb{I}_N\forall X\exists Y(Y=W_i^\gamma(X))$. Let $\gamma<\mathbb{I}_N$.

1. *For $i\leq N$, $W_i^\gamma(X)$ is a well order: $Wo[W_i^\gamma(X)]$.*
2. *For $i<N$, $W_{i+1}^\gamma(X)$ is the maximal i-distinguished* set *for γ and X if $X\cap\gamma^{-(i+1)}$ is a well order: $Wo[X\cap\gamma^{-(i+1)}]\Rightarrow D_i^\gamma[W_{i+1}^\gamma(X);X]$. In particular $W_{i+1}^\gamma(X)\cap\gamma^{-(i+1)}=X\cap\gamma^{-(i+1)}$ holds.*

Proof. 7.8.1. Clearly $W_0^\gamma(X)=W(C^\gamma(X))$ is a well order. We show $Wo[W_{i+1}^\gamma(X)]$. Let $\{\beta<\alpha\}\subset W_{i+1}^\gamma(X)$. Pick a P and a Q such that $D_i^\gamma[P;X]$, $\alpha\in P$, $D_i^\gamma[Q;X]$ and $\beta\in Q$ by (47). Lemma 7.7 yields $\beta\in Q\cap\beta^{\dagger i}\subset P$. We obtain $Wo[W_{i+1}^\gamma(X)\cap\alpha]$ by $Wo[P]$.
7.8.2. Assuming that $X\cap\gamma^{-(i+1)}$ is a well order, we see that $X\cap\gamma^{-(i+1)}$ is the minimal i-distinguished set for γ and X: $D_i^\gamma[X\cap\gamma^{-(i+1)};X]$. We obtain $W_{i+1}^\gamma(X)\cap\gamma^{-(i+1)}=X\cap\gamma^{-(i+1)}$. Lemma 7.8.1 yields $Wo[W_{i+1}^\gamma(X)]$.

Let $\gamma^{-(i+1)}\leq\alpha\leq W_{i+1}^\gamma(X)$. We show $W_i^\alpha(W_{i+1}^\gamma(X))\cap\alpha^{\dagger i}=W_{i+1}^\gamma(X)\cap\alpha^{\dagger i}$. Pick a P such that $D_i^\gamma[P;X]$ and $\alpha\leq P$. We obtain $W_i^\alpha(P)\cap\alpha^{\dagger i}=P\cap\alpha^{\dagger i}\subset W_{i+1}^\gamma(X)\cap\alpha^{\dagger i}$ by (46). Let $D_i^\gamma[Q;X]$ and $\beta\in Q\cap\alpha^{\dagger i}$. Lemma 7.7 yields $\beta\in Q\cap\beta^{\dagger i}=P\cap\beta^{\dagger i}$ for $\beta^{\dagger i}\leq\alpha^{\dagger i}$.

Therefore we obtain $W_i^\alpha(P) \cap \alpha^{\dagger i} = P \cap \alpha^{\dagger i} = W_{i+1}^\gamma(X) \cap \alpha^{\dagger i}$, a fortiori $P \cap \alpha = \mathcal{W}_{i+1}^\gamma(X) \cap \alpha$. Hence $W_{i+1}^\gamma(X) \cap \alpha^{\dagger i} = P \cap \alpha^{\dagger i} = W_i^\alpha(P) \cap \alpha^{\dagger i} = W_i^\alpha(W_{i+1}^\gamma(X)) \cap \alpha^{\dagger i}$ by Proposition 7.6. □

Lemma 7.9 *1. Let X and Y be N-distinguished sets, and $\gamma < \mathbb{I}_N$. Then $\gamma \leq X \,\&\, \gamma \leq Y \Rightarrow X \cap \gamma^{\dagger N} = Y \cap \gamma^{\dagger N}$.*

2. $\mathcal{W}_{N+1}$ is the N-maximal distinguished class, *i.e., $D_N[\mathcal{W}_{N+1}]$.*

3. For a family $\{Y_j\}_{j \in J}$ of N-distinguished sets, the union $Y = \bigcup_{j \in J} Y_j$ is also an N-distinguished set.

Proof. 7.9.1 is seen as in Lemma 7.7. 7.9.2 and 7.9.3 follow from Lemma 7.9.1 as in Lemma 7.8. □

Lemma 7.10 *Let $D_N[X]$ and $\gamma \in X \subset \mathbb{I}_N$. Then for each $0 \leq i \leq N$, $\gamma \in W_i^\gamma(X) \cap \gamma^{\dagger i} = X \cap \gamma^{\dagger i}$ and $D_i^\gamma[X; X]$ holds. In particular $\gamma \in C^\gamma(X)$.*

Proof. By induction on $N - i$. We obtain $\gamma \in W_N^\gamma(X) \cap \gamma^{\dagger N} = X \cap \gamma^{\dagger N}$ by $D_N[X]$ and $\gamma \in X$. Lemma 7.8 with $Wo[X]$ yields $D_{N-1}^\gamma[W_N^\gamma(X); X]$, and $D_{N-1}^\gamma[X; X]$ follows.

Assuming $D_{i+1}^\gamma[X; X]$, we obtain $W_{i+1}^\gamma(X) \cap \gamma^{\dagger(i+1)} = X \cap \gamma^{\dagger(i+1)}$ by $\gamma^{-(i+1)} \leq \gamma \in X$, and $D_i^\gamma[W_{i+1}^\gamma(X); X]$ by Lemma 7.8. Hence $D_i^\gamma[X; X]$ and $\gamma \in W_i^\gamma(X) \cap \gamma^{\dagger i} = X \cap \gamma^{\dagger i}$. □

Proposition 7.11 *Let $D_N[X]$, $\alpha \leq \gamma \in X$ and $\alpha \in C^\gamma(X)$. Then $\alpha \in X$.*

Proof. Lemma 7.10 yields $\gamma \in W(C^\gamma(X)) \cap \gamma^{\dagger 0} = W_0^\gamma(X) \cap \gamma^{\dagger 0} = X \cap \gamma^{\dagger 0}$. $\gamma \geq \alpha \in C^\gamma(X)$ yields $\alpha \in W_0^\gamma(X) \cap \gamma^{\dagger 0} = X \cap \gamma^{\dagger 0}$. □

Proposition 7.12 *Let $D_N[X]$ and $\alpha, \beta < \mathbb{I}_N$.*

1. *Let $\{\alpha, \beta\} \subset X$ with $\alpha + \beta = \alpha \# \beta$ and $\alpha > 0$. Then $\gamma = \alpha + \beta \in X$.*

2. *If $\{\alpha, \beta\} \subset X$, then $\varphi\alpha\beta \in X$.*

Proof. Proposition 7.12.2 is seen by main induction on $\alpha \in X$ with subsidiary induction on $\beta \in X$ using Proposition 7.12.1. We show Proposition 7.12.1. By Lemma 7.10 we obtain $\alpha \in X \cap \alpha^{\dagger 0} = W_0^\alpha(X) \cap \alpha^{\dagger 0}$. We see that $\alpha + \beta \in W_0^\alpha(X) = W(C^\alpha(X))$ by induction on $\beta \in X \cap (\alpha + 1) \subset C^\alpha(X)$. □

Lemma 7.13 *1. $C^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap \mathbb{I}_N = \mathcal{W}_{N+1} \cap \mathbb{I}_N = W(C^{\mathbb{I}_N}(\mathcal{W}_{N+1})) \cap \mathbb{I}_N$.*

2. (BI) *For* each *$n < \omega$,* $\mathrm{TI}[C^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap \omega_n(\mathbb{I}_N + 1)]$, *i.e., for each class $\mathcal{X}$,* $\mathrm{Prg}[C^{\mathbb{I}_N}(\mathcal{W}_{N+1}), \mathcal{X}] \to C^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap \omega_n(\mathbb{I}_N + 1) \subset \mathcal{X}$.

3. For each *$n < \omega$, $C^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap \omega_n(\mathbb{I}_N + 1) \subset W(C^{\mathbb{I}_N}(\mathcal{W}_{N+1}))$. In particular $\{\mathbb{I}_N, \omega_n(\mathbb{I}_N + 1)\} \subset W(C^{\mathbb{I}_N}(\mathcal{W}_{N+1}))$.*

Proof. 7.13.1. $\alpha \in C^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap \mathbb{I}_N \Rightarrow \alpha \in \mathcal{W}_{N+1}$ is seen by induction on $\ell\alpha$ using Proposition 7.12 and Lemma 7.9.2. Since $\mathcal{W}_{N+1}$ is well-founded, we obtain $C^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap \mathbb{I}_N = W(C^{\mathbb{I}_N}(\mathcal{W}_{N+1})) \cap \mathbb{I}_N$.
7.13.2. We show $\mathrm{TI}[C^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap \omega_n(\mathbb{I}_N + 1)]$ by metainduction on $n < \omega$. Let $D_N[Y]$. We obtain $Wo[Y]$, and $\mathrm{TI}[Y]$ follows from (BI). We have $C^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap \mathbb{I}_N = \mathcal{W}_{N+1} \cap \mathbb{I}_N$, and $\mathcal{W}_{N+1} \cap \gamma^{\dagger N} = Y \cap \gamma^{\dagger N}$ for $\gamma \in Y \cap \mathbb{I}_N$ by Lemma 7.9.1. We obtain $\mathrm{TI}[\mathcal{W}_{N+1} \cap \mathbb{I}_N]$, from which $\mathrm{TI}[C^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap (\mathbb{I}_N + 1)]$ follows.

Assuming $\mathrm{TI}[C^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap \omega_n(\mathbb{I}_N + 1)]$, $\mathrm{TI}[C^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap \omega_{n+1}(\mathbb{I}_N + 1)]$ is seen from the fact that $\mathrm{Prg}[C^{\mathbb{I}_N}(\mathcal{W}_{N+1}), A] \to \mathrm{Prg}[C^{\mathbb{I}_N}(\mathcal{W}_{N+1}), \mathtt{j}[A]]$, where for a given formula A, $\mathtt{j}[A](\alpha)$ denotes the formula
$\forall \beta \in C^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \left[\forall \gamma \in C^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap \beta\, A(\gamma) \to \forall \gamma \in C^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap (\beta + \omega^{\alpha}) A(\gamma)\right]$.
□

7.2 Sets $\mathcal{G}^X$

In this subsection we establish a key fact, Lemma 7.25 on distinguished sets.

Definition 7.14 $\mathcal{G}^X := \{\alpha \in OT(\mathbb{I}_N) \cap \mathbb{I}_N : \alpha \in C^{\alpha}(X) \,\&\, C^{\alpha}(X) \cap \alpha \subset X\}$.

Proposition 7.15 *Let $D_N[X]$ and $\alpha \in X$. Then $\alpha \in \mathcal{G}^X$.*

Proof. By Lemma 7.10 we obtain $\alpha \in W_0^{\alpha}(X) = W(C^{\alpha}(X))$. Hence $\alpha \in C^{\alpha}(X)$. On the other side Proposition 7.11 yields $C^{\alpha}(X) \cap \alpha \subset X$. □

Lemma 7.16 (Σ^1_{N+1}-CA)
Suppose $D_N[Y]$ and $\alpha \in \mathcal{G}^Y$. Let $P_N = W_N^{\alpha}(Y) \cap \alpha^{\dagger N}$. Assume that the following condition (48) is fulfilled. Then $\alpha \in P_N$ and $D_N[P_N]$. In particular $\alpha \in \mathcal{W}_{N+1}$ holds.

Moreover if there exists a set Z and an ordinal γ such that $Y = W_N^{\gamma}(Z)$ and $\alpha^{-N} = \gamma^{-N}$, then $\alpha \in Y$ holds.

$$\forall \beta \geq \alpha^{-1} \left(Y \cap \alpha^{\dagger 1} < \beta \,\&\, \beta^{\dagger 0} < \alpha^{\dagger 0} \to W_0^{\beta}(Y) \cap \beta^{\dagger 0} \subset Y\right) \tag{48}$$

Proof. If $Y = W_N^{\gamma}(Z)$ with $\alpha^{-N} = \gamma^{-N}$, then $Y \cap \alpha^{-N} = Z \cap \alpha^{-N}$ and $W_N^{\gamma}(Z) = W_N^{\alpha}(Y)$. Hence if $\alpha \in W_N^{\alpha}(Y)$, then $\alpha \in Y$.

Lemma 7.8.2 yields

$$\forall i < N \left[W_{i+1}^{\beta}(Y) \cap \beta^{\dagger(i+1)} = Y \cap \beta^{\dagger(i+1)}\right] \tag{49}$$

Let $P_i = W_i^{\alpha}(Y) \cap \alpha^{\dagger i}$ for $0 \leq i \leq N$. By $C^{\alpha}(Y) \cap \alpha \subset Y$ and $Wo[Y]$ we obtain for $P_0 = W(C^{\alpha}(Y)) \cap \alpha^{\dagger 0}$

$$P_0 \cap \alpha = Y \cap \alpha = C^{\alpha}(Y) \cap \alpha \tag{50}$$

Hence $\alpha \in P_0$. On the other hand we have $D_{i-1}^{\alpha}[W_i^{\alpha}(Y); Y]$ for $i > 0$. This together with (50) yields for $0 \leq i \leq N$

$$P_i \cap \alpha^{-i} = Y \cap \alpha^{-i} \tag{51}$$

Claim 7.17 $\alpha^{\dagger 0}=\gamma^{\dagger 0}\,\&\,\gamma\in P_0\Rightarrow\gamma\in C^{\gamma}(P_0)$.

Proof of Claim 7.17. Let $\alpha^{\dagger 0}=\gamma^{\dagger 0}$ and $\gamma\in P_0=W(C^{\alpha}(Y))\cap\alpha^{\dagger 0}$. We obtain $\gamma\in C^{\alpha}(Y)=C^{\gamma}(Y)$ by Propositions 7.15 and 7.2. Hence $Y\cap\gamma\subset C^{\gamma}(Y)\cap\gamma=C^{\alpha}(Y)\cap\gamma$. $\gamma\in W(C^{\alpha}(Y))$ yields $Y\cap\gamma\subset P_0$. Therefore we obtain $\gamma\in C^{\gamma}(Y)\subset C^{\gamma}(P_0)$. □ of Claim 7.17.

Claim 7.18 $D_i^{\alpha}[P_i;Y]$ *and* $\alpha\in P_{i+1}$ *for each* $0\leq i<N$.

Proof of Claim 7.18. Obviously $Wo[P_i]$. (51) yields $P_i\cap\alpha^{-(i+1)}=Y\cap\alpha^{-(i+1)}$. Let $\alpha^{-(i+1)}\leq\beta\leq P_i$. We show $W_i^{\beta}(P_i)\cap\beta^{\dagger i}=P_i\cap\beta^{\dagger i}$.
Case 1. $\beta^{\dagger i}=\alpha^{\dagger i}$: First let $i=0$. We obtain $C^{\beta}(P_0)=C^{\alpha}(P_0)$ by Proposition 7.2 and Claim 7.17. Hence the assertion follows from (50).

Next let $i>0$. (51) with $\beta^{-i}=\alpha^{-i}$ yields $W_i^{\beta}(P_i)=W_i^{\alpha}(P_i)=W_i^{\alpha}(Y)$.
Case 2. $\beta^{\dagger i}<\alpha^{\dagger i}$: For $i>0$, (49) yields $W_i^{\beta}(Y)\cap\beta^{\dagger i}=Y\cap\beta^{\dagger i}$. We obtain $W_i^{\beta}(P_i)\cap\beta^{\dagger i}=W_i^{\beta}(Y)\cap\beta^{\dagger i}=Y\cap\beta^{\dagger i}=P_i\cap\beta^{\dagger i}$ by (51).

Let $i=0$. We have $\beta^{\dagger 0}\leq\alpha^{-0}$. First let $Y\cap\alpha^{\dagger 1}<\beta$. Then the assumption (48) with $\alpha^{-1}\leq\beta$ yields $W_0^{\beta}(Y)\cap\beta^{\dagger 0}\subset Y$. We obtain $W_0^{\beta}(P_0)\cap\beta^{\dagger 0}=W_0^{\beta}(Y)\cap\beta^{\dagger 0}\subset Y\cap\beta^{\dagger 0}=P_0\cap\beta^{\dagger 0}$ by (50). It remains to show $Y\cap\beta^{\dagger 0}\subset W_0^{\beta}(Y)$. Let $\gamma\in Y\cap\beta^{\dagger 0}$. We obtain $\gamma\in W_0^{\gamma}(Y)$ by Lemma 7.10. On the other hand we have $\mathcal{C}^{\beta}(Y)\subset\mathcal{C}^{\gamma}(Y)$ by Propositions 7.15 and 7.2. Moreover (50) with Propositions 7.15 and 7.2 yields $\gamma\in\mathcal{C}^{\alpha}(Y)\subset\mathcal{C}^{\beta}(Y)$. Hence $\gamma\in W_0^{\beta}(Y)$.

Next let $\beta\leq Y\cap\alpha^{\dagger 1}$. We obtain $Y\cap\beta^{\dagger 1}=\mathcal{W}_1^{\beta}(Y)\cap\beta^{\dagger 1}$, and $\beta^{-1}=\alpha^{-1}\leq\beta<\alpha^{\dagger 1}=\beta^{\dagger 1}$ with $\beta<\beta^{\dagger 0}\leq\alpha<\beta^{\dagger 1}$. On the other hand we have $D_0^{\beta}[\mathcal{W}_1^{\beta}(Y);Y]$ by Lemma 7.8. Therefore $P_0\cap\beta^{\dagger 0}=Y\cap\beta^{\dagger 0}=\mathcal{W}_1^{\beta}(Y)\cap\beta^{\dagger 0}=W_0^{\beta}(\mathcal{W}_1^{\beta}(Y))\cap\beta^{\dagger 0}=W_0^{\beta}(P_0)\cap\beta^{\dagger 0}$ by (50).

Thus $D_i^{\alpha}[P_i;Y]$ is shown. From $\alpha\in P_0$ we see by induction on $i<N$ that $\alpha\in P_i\cap\alpha^{\dagger(i+1)}\subset W_{i+1}^{\alpha}(Y)\cap\alpha^{\dagger(i+1)}=P_{i+1}$ for the maximal i-distinguished set $W_{i+1}^{\alpha}(Y)$ for α and Y. □ of Claim 7.18.

Claim 7.19 $D_N[P_N]$.

Proof of Claim 7.19. Let $\beta\leq P_N=W_N^{\alpha}(Y)\cap\alpha^{\dagger N}$. Then $\beta<\alpha^{\dagger N}$, and $\beta^{-N}\leq\alpha^{-N}<\alpha^{\dagger N}$. We show $W_N^{\beta}(W_N^{\alpha}(Y))\cap\beta^{\dagger N}=W_N^{\alpha}(Y)\cap\beta^{\dagger N}$.
Case 1. $\alpha^{-N}\leq\beta$: By $W_N^{\alpha}(Y)\cap\alpha^{-N}=Y\cap\alpha^{-N}$ with $Wo[Y]$, and $\alpha^{-N}=\beta^{-N}$ we obtain $W_N^{\alpha}(Y)=W_N^{\alpha}(W_N^{\alpha}(Y))=W_N^{\beta}(W_N^{\alpha}(Y))$.
Case 2. $\beta<\alpha^{-N}$ and $\beta^{-N}\leq Y$: We obtain $\beta^{\dagger N}\leq\alpha^{-N}$. Hence $W_N^{\alpha}(Y)\cap\beta^{\dagger N}=Y\cap\beta^{\dagger N}=W_N^{\beta}(Y)\cap\beta^{\dagger N}$ by $D_N[Y]$. Therefore $W_N^{\beta}(Y)=W_N^{\beta}(W_N^{\alpha}(Y))$. We obtain $W_N^{\alpha}(Y)\cap\beta^{\dagger N}=W_N^{\beta}(W_N^{\alpha}(Y))\cap\beta^{\dagger N}$.
Case 3. $\beta<\alpha^{-N}$ and $Y<\beta^{-N}$: Then $\beta^{\dagger N}\leq\alpha^{-N}$. (49) yields $Y\cap\beta^{\dagger N}=W_N^{\beta}(Y)\cap\beta^{\dagger N}$. On the other hand we have $Y\cap\beta^{\dagger N}=W_N^{\alpha}(Y)\cap\beta^{\dagger N}$ and $W_N^{\beta}(Y)\cap\beta^{\dagger N}=W_N^{\beta}(W_N^{\alpha}(Y))\cap\beta^{\dagger N}$. Therefore $W_N^{\beta}(W_N^{\alpha}(Y))\cap\beta^{\dagger N}=W_N^{\alpha}(Y)\cap\beta^{\dagger N}$.
□ of Claim 7.19.

This completes a proof of Lemma 7.16. □

Lemma 7.20 *Assume* $D_N[Y]$, $\mathbb{I}_N > \mathbb{S} \in Y \cap (St_k \cup \{0\})$ *and* $\{0, \Omega\} \subset Y$ *for* $0 < k \leq N$. *Then* $\mathbb{S}^{\dagger k} \in \mathcal{W}_{N+1}$.

Proof. Let us verify the condition (48) in Lemma 7.16 for $\alpha = \mathbb{S}^{\dagger k}$. Let $\alpha^{-1} \leq \beta$. We have $\alpha = \alpha^{-1} \leq \beta$. Hence $\alpha^{\dagger 0} \leq \beta^{\dagger 0}$, and (48) is vacuously fulfilled.

Thus it suffices to show that $\alpha = \mathbb{S}^{\dagger k} \in \mathcal{G}^Y$. $\alpha \in C^\alpha(Y)$ follows from $\mathbb{S} \in Y \cap \alpha$, cf. Definition 7.1.6. We show $\gamma \in C^\alpha(Y) \cap \alpha \Rightarrow \gamma \in Y$ by induction on $\ell\gamma$. By Proposition 7.12 and the assumption $\{0, \Omega\} \subset Y$, we can assume $\mathbb{S} \neq \gamma = \psi_\sigma^f(a) < \alpha = \mathbb{S}^{\dagger k} < \sigma$, cf. Definition 7.1.6. Suppose $\mathbb{S} < \gamma$. Then $\mathbb{S} \in \mathcal{H}_a(\gamma)$, and $\alpha = \mathbb{S}^{\dagger k} \in \mathcal{H}_a(\gamma) \cap \sigma \subset \gamma$. We obtain $\gamma < \mathbb{S}$. Lemma 7.10 with $\mathbb{S} \in Y$ and $D_N[Y]$ yields $\mathbb{S} \in W_0^{\mathbb{S}}(Y) \cap \mathbb{S}^{\dagger 0} = Y \cap \mathbb{S}^{\dagger 0}$ for $W_0^{\mathbb{S}}(Y) = W(C^{\mathbb{S}}(Y))$, where $\forall \delta[\delta \in Y \Rightarrow \delta \in C^\delta(Y)]$. We obtain $\gamma \in C^{\mathbb{S}}(Y)$ by $\gamma \in C^\alpha(Y)$, $\mathbb{S} < \alpha$ and Proposition 7.2.1. Hence $\gamma \in W_0^{\mathbb{S}}(Y) \cap \mathbb{S}^{\dagger 0} \subset Y$ follows. Therefore $\alpha \in \mathcal{G}^Y$. □

Proposition 7.21 $\{0, \Omega\} \subset \mathcal{W}_{N+1}$.

Proof. For each $\alpha \in \{0, \Omega\}$ and any set $Y \subset OT(\mathbb{I}_N)$ we have $\alpha \in C^\alpha(Y)$. First let $\alpha = 0$. We obtain $C^0(\emptyset) \cap \alpha \subset \emptyset$, and $0 \in \mathcal{G}^\emptyset$. Moreover $D_N[\emptyset]$, and there is no β such that $\beta^{\dagger 0} < \alpha^{\dagger 0}$ since $\alpha^{\dagger 0} = \Omega$ is the least in SSt_0. Hence the condition (48) is fulfilled, and we obtain $0 \in X = W_N^0(\emptyset) \cap 0^{\dagger N}$ with $D_N[X]$ by Lemma 7.16.

Next let $\alpha = \Omega$. Let $\gamma \in C^\alpha(X) \cap \alpha$. We show that $\gamma \in X$ by induction on $\ell\gamma$ as follows. We see that each strongly critical number $\gamma \in C^\alpha(X) \cap \alpha$ is in X from Definition 7.1. Otherwise $\gamma \in X$ is seen from IH using Proposition 7.12 and $0 \in X$. Therefore we obtain $\alpha \in \mathcal{G}^X$.

Let $\beta^{\dagger 0} < \alpha^{\dagger 0}$. Then $\beta^{\dagger 0} = \Omega$ and $\beta < \Omega$. Let $\gamma \in W_0^\beta(X) \cap \Omega$. We show $\gamma \in X$. We obtain $D_0^0[X; X]$ by Lemma 7.10, and $\gamma \in W_0^\beta(X) \cap \Omega = W_0^0(X) \cap \Omega = X \cap \Omega$. Hence the condition (48) is fulfilled, and we obtain $\Omega \in \mathcal{W}_{N+1}$ by Lemma 7.16. □

Lemma 7.22 (Σ^1_{N+2}-DC)
If $\alpha \in \mathcal{G}^{\mathcal{W}_{N+1}}$, *then there exists an* N*-distinguished* set Z *such that* $\{0, \Omega\} \subset Z$, $\alpha \in \mathcal{G}^Z$ *and* $\forall k \forall \mathbb{S} \in Z \cap (St_k \cup \{\Omega\})[\mathbb{S}^{\dagger k} \in Z]$.

Proof. Let $\alpha \in \mathcal{G}^{\mathcal{W}_{N+1}}$. We have $\alpha \in C^\alpha(\mathcal{W}_{N+1})$. Pick an N-distinguished set X_0 such that $\alpha \in C^\alpha(X_0)$. We can assume $\{0, \Omega\} \subset X_0$ by Proposition 7.21. On the other hand we have $C^\alpha(\mathcal{W}_{N+1}) \cap \alpha \subset \mathcal{W}_{N+1}$ and $\forall k \forall \mathbb{S} \in \mathcal{W}_{N+1} \cap (St_k \cup \{\Omega\})[\mathbb{S}^{\dagger k} \in \mathcal{W}_{N+1}]$ by Lemma 7.20. We obtain

$$
\begin{aligned}
& \forall n \forall X \exists Y \{D_N[X] \to D_N[Y] \\
\land \quad & \forall \beta \in OT(\mathbb{I}_N) \, (\ell(\beta) \leq n \land \beta \in C^\alpha(X) \cap \alpha \to \beta \in Y) \\
\land \quad & \forall k \forall \mathbb{S} \in (St_k \cup \{\Omega\}) \left(\ell(\mathbb{S}) \leq n \land \mathbb{S} \in X \to \mathbb{S}^{\dagger k} \in Y\right)\}
\end{aligned}
$$

Since $D_N[X]$ is Δ^1_{N+2}, Σ^1_{N+2}-DC yields a set Z such that $Z_0 = X_0$ and

$$
\begin{aligned}
& \forall n \{D_N[Z_n] \to D_N[Z_{n+1}] \\
\land \quad & \forall \beta \in OT(\mathbb{I}_N) \, (\ell(\beta) \leq n \land \beta \in C^\alpha(Z_n) \cap \alpha \to \beta \in Z_{n+1}) \\
\land \quad & \forall k \forall \mathbb{S} \in (St_k \cup \{\Omega\}) \left(\ell(\mathbb{S}) \leq n \land \mathbb{S} \in Z_n \to \mathbb{S}^{\dagger k} \in Z_{n+1}\right)\}
\end{aligned}
$$

Let $Z=\bigcup_n Z_n$. We see by induction on n that $D_N[Z_n]$ for every n. Lemma 7.9.3 yields $D_N[Z]$. Let $\beta\in C^\alpha(Z)\cap\alpha$. Pick an n such that $\beta\in C^\alpha(Z_n)$ and $\ell\beta\leq n$. We obtain $\beta\in Z_{n+1}\subset Z$. Therefore $\alpha\in\mathcal{G}^Z$. Furthermore let $\mathbb{S}\in Z\cap(St_k\cup\{\Omega\})$. Pick an n such that $\mathbb{S}\in Z_n$ and $\ell(\mathbb{S})\leq n$. We obtain $\mathbb{S}^{\dagger k}\in Z_{n+1}\subset Z$. □

Proposition 7.23 *Let $D_N[Y]$ and $\alpha\in C^\beta(Y)$. Assume $Y\cap\beta<\delta$. Then $F_\delta(\alpha)\subset C^\beta(Y)$.*

Proof. By induction on $\ell\alpha$. Let $\{0,\Omega,\mathbb{I}_N\}\not\ni\alpha\in C^\beta(Y)$. We have $\mathcal{E}(\alpha)\leq\alpha$ First consider the case $\alpha\not\in\mathcal{E}(\alpha)$. If $\alpha\in Y\cap\beta\subset\mathcal{G}^Y$ by Proposition 7.15, then $\mathcal{E}(\alpha)\subset C^\alpha(Y)\cap\alpha\subset Y\subset C^\beta(Y)$ by Proposition 7.5. Otherwise we have $\alpha\not\in\mathcal{E}(\alpha)\subset C^\beta(Y)$. In each case IH yields $F_\delta(\alpha)=F_\delta(\mathcal{E}(\alpha))\subset C^\beta(Y)$.

Let $\alpha=\psi_\pi^f(a)$ for some π,f,a. If $\alpha<\delta$, then $F_\delta(\alpha)=\{\alpha\}$, and there is nothing to prove. Let $\alpha\geq\delta$. Then $F_\delta(\alpha)=F_\delta(\{\pi,a\}\cup SC(f))$. On the other side we see $\{\pi,a\}\cup SC(f)\subset C^\beta(Y)$ from $\alpha\in C^\beta(Y)$ and the assumption. IH yields $F_\delta(\alpha)\subset C^\beta(Y)$.

Finally let $\alpha\in N(\rho)$. Then $F_\delta(\alpha)=F_\delta(\rho)$. If $\rho\in C^\beta(Y)$, then IH yields $F_\delta(\rho)\subset C^\beta(Y)$. Otherwise we have $\alpha\in Y$, and $\alpha\in C^\alpha(Y)$. Hence $\rho\in C^\alpha(Y)\cap\alpha\subset Y\subset C^\beta(Y)$. □

Proposition 7.24 *Let $\gamma<\beta$. Assume $\alpha\in C^\gamma(Y)$ and $G_\beta(\alpha)<\gamma$. Moreover assume $\forall\delta[\ell\delta\leq\ell\alpha\,\&\,\delta\in C^\gamma(Y)\cap\gamma\Rightarrow\delta\in C^\beta(Y)]$. Then $\alpha\in C^\beta(Y)$.*

Proof. By induction on $\ell\alpha$. If $\alpha<\gamma$, then $\alpha\in C^\gamma(Y)\cap\gamma$. The third assumption yields $\alpha\in C^\beta(Y)$. Assume $\alpha\geq\gamma$. Consider the case $\alpha=\psi_\pi^f(a)$ for some $\{\pi,a\}\cup SC(f)\subset C^\gamma(Y)$ and $\pi>\gamma$. If $\pi\leq\beta$, then $\{\alpha\}=G_\beta(\alpha)<\gamma$ by the second assumption. Hence this is not the case, and we obtain $\pi>\beta$. Then $G_\beta(\{\pi,a\}\cup SC(f))=G_\beta(\alpha)<\gamma$. IH yields $\{\pi,a\}\cup SC(f)\subset C^\beta(Y)$. We conclude $\alpha\in C^\beta(Y)$ from $\pi>\beta$.

Next let $\gamma\leq\alpha\in N(\rho)$ with $\rho\in C^\gamma(Y)$. If $\alpha<\beta$, then $\{\alpha\}=G_\beta(\alpha)<\gamma$, and this is not the case. Let $\alpha\geq\beta$. Then $G_\beta(\alpha)=G_\beta(\rho)$. IH yields $\rho\in C^\beta(Y)$, and $\alpha\in C^\beta(Y)$ by $\alpha\geq\beta$. □

The following Lemma 7.25 is a key result on distinguished classes.

Lemma 7.25 *Suppose $D_N[Y]$ with $\{0,\Omega\}\subset Y$ and $\forall k\forall\mathbb{U}\in Y\cap(St_k\cup\{\Omega\})[\mathbb{U}^{\dagger k}\in Y]$. For $\eta\in\Psi_{\mathbb{I}_N}\cup\bigcup_{\mathbb{S}\in SSt}L(\mathbb{S})$, cf. Definition 6.12,*

$$\eta\in\mathcal{G}^Y \tag{52}$$

$$R(\eta)\cap\{\gamma\in OT(\mathbb{I}_N)\cap\mathbb{I}_N:Y\cap\eta^{\dagger 1}<\gamma\}\cap\mathcal{G}^Y\subset Y \tag{53}$$

and

$$\forall\mathbb{T}\in\{\Omega\}\cup(LSt\cap\Psi)\forall\vec{k}(\eta\in L(\mathbb{T}^{\dagger\vec{k}})\Rightarrow\mathbb{T}\in Y) \tag{54}$$

Then $\eta\in\mathcal{W}_{N+1}$. Moreover if there exists a set Z and an ordinal γ such that $Y=W_N^\gamma(Z)$ and $\eta^{-N}=\gamma^{-N}$, then $\eta\in Y$ holds.

Proof. By Lemma 7.16 and the hypothesis (52) it suffices to show (48)

$$\forall\beta \geq \eta^{-1} \left(Y \cap \eta^{\dagger 1} < \beta \,\&\, \beta^{\dagger 0} < \eta^{\dagger 0} \to W_0^{\beta}(Y) \cap \beta^{\dagger 0} \subset Y\right).$$

Assume $Y \cap \eta^{\dagger 1} < \beta$ and $\beta^{\dagger 0} < \eta^{\dagger 0}$. We have to show $W_0^{\beta}(Y) \cap \beta^{\dagger 0} \subset Y$. We prove this by induction on $\gamma \in W_0^{\beta}(Y) \cap \beta^{\dagger 0}$. Suppose $\gamma \in C^{\beta}(Y) \cap \beta^{\dagger 0}$ and

$$\text{MIH} : C^{\beta}(Y) \cap \gamma \subset Y.$$

We show $\gamma \in Y$. We can assume that

$$Y \cap \eta^{\dagger 1} < \gamma \tag{55}$$

since if $\gamma \leq \delta$ for some $\delta \in Y \cap \eta^{\dagger 1}$, then by $Y \cap \eta^{\dagger 1} < \beta$ and $\gamma \in C^{\beta}(Y)$ we obtain $\delta < \beta$, $\gamma \in C^{\delta}(Y)$ and $\delta \in W(C^{\delta}(Y)) \cap \delta^{\dagger 0} = Y \cap \delta^{\dagger 0}$ by Lemma 7.10. Hence $\gamma \in W(C^{\delta}(Y)) \cap \delta^{\dagger 0} \subset Y$.

Moreover we can assume $\gamma \not\in (Reg_0 \setminus \{\Omega, \mathbb{I}_N\}) \cap \beta$ with $Reg_0 = (Reg \setminus \Psi)$. For otherwise $\gamma \in Y$ by Definition 7.1.6 and $\gamma \in C^{\beta}(Y) \cap \beta$.

We show first

$$\gamma \in \mathcal{G}^Y \tag{56}$$

First $\gamma \in C^{\gamma}(Y)$ by $\gamma \in C^{\beta}(Y) \cap \beta^{\dagger 0}$ and Proposition 7.2. Second we show the following claim by induction on $\ell\alpha$:

$$\alpha \in C^{\gamma}(Y) \cap \gamma \Rightarrow \alpha \in Y \tag{57}$$

Proof of (57). Assume $\alpha \in C^{\gamma}(Y) \cap \gamma$. We can assume $\gamma^{\dagger 0} \leq \beta$ for otherwise we have $\alpha \in C^{\gamma}(Y) \cap \gamma = C^{\beta}(Y) \cap \gamma \subset Y$ by MIH.

By induction hypothesis on lengths, Proposition 7.12, and $\{0, \Omega\} \subset Y$, we can assume that $\alpha = \psi_{\pi}^{f}(a)$ for some $\pi > \gamma$ such that $\{\pi, a\} \cup SC(f) \subset C^{\gamma}(Y)$.

Case 1. $\beta < \pi$: Then $G_{\beta}(\{\pi, a\} \cup SC(f)) = G_{\beta}(\alpha) < \alpha < \gamma$ by Proposition 6.19.1. Proposition 7.24 with induction hypothesis on lengths yields $\{\pi, a\} \cup SC(f) \subset C^{\beta}(Y)$. Hence $\alpha \in C^{\beta}(Y) \cap \gamma$ by $\pi > \beta$. MIH yields $\alpha \in Y$.

Case 2. $\beta \geq \pi$: We have $\alpha < \gamma < \pi \leq \beta$. It suffices to show that $\alpha \leq Y \cap \eta^{\dagger 1}$. Then by (55) we have $\alpha \leq \delta \in Y \cap \eta^{\dagger 1}$ for some $\delta < \gamma$. $C^{\delta}(Y) \ni \alpha \leq \delta \in Y \cap \delta^{\dagger 0} = W(C^{\delta}(Y)) \cap \delta^{\dagger 0}$ yields $\alpha \in W(C^{\delta}(Y)) \cap \delta^{\dagger 0} \subset Y$.

Consider first the case $\gamma \not\in \mathcal{E}(\gamma)$. By $\alpha = \psi_{\pi}^{f}(a) < \gamma < \pi$, we can assume that $\gamma \not\in \{0, \Omega, \mathbb{I}_N\}$. Then let $\delta = \max S(\gamma)$ denote the largest immediate subterm of γ. Then $\delta \in C^{\gamma}(Y) \cap \gamma$, and by (55), $Y \cap \eta^{\dagger 1} < \gamma \in C^{\beta}(Y)$ we have $\delta \in C^{\beta}(Y) \cap \gamma$. Hence $\delta \in Y \cap \eta^{\dagger 1}$ by MIH. Also by $\alpha < \gamma$, we obtain $\alpha \leq \delta$, i.e., $\alpha \leq Y \cap \eta^{\dagger 1}$, and we are done.

Next let $\gamma \not\in (Reg_0 \setminus \{\Omega, \mathbb{I}_N\})$ and $\gamma \in \mathcal{E}(\gamma)$. This means that $\gamma \in \Psi$. Let $\gamma = \psi_{\kappa}^{g}(b)$ for some b, g and $\kappa > \beta$ by (55) and $\gamma \in C^{\beta}(Y)$. We have $\alpha < \gamma < \pi \leq \beta < \kappa$. Let $\pi \preceq \rho$ and $\kappa \preceq \tau$ with $\{\rho, \tau\} \subset Reg_0$. We obtain $\rho = \tau$ by Proposition 3.39.

$\pi \not\in \mathcal{H}_b(\gamma)$ since otherwise by $\pi < \kappa$ we would have $\pi < \gamma$. Then by Proposition 3.27 we have $a \geq b$ and $SC(g) \cup \{\kappa, b\} \not\subset \mathcal{H}_a(\alpha)$. On the other hand we have $K_\gamma(SC(g)\cup\{\kappa,b\}) < b \leq a$, i.e., $SC(g)\cup\{\kappa,b\}\subset\mathcal{H}_a(\gamma)$. By Proposition 6.20 pick a $\delta\in F_\gamma(SC(g)\cup\{\kappa,b\})$ such that $\mathcal{H}_a(\alpha)\not\ni\delta\in\gamma$. In particular $\delta<\gamma$. Also we have $SC(g)\cup\{\kappa,b\}\subset C^\beta(Y)$, $Y\subset\mathcal{G}^Y$ by Proposition 7.15, and $Y\cap\eta^{\dagger 1}<\gamma$ by (55). Therefore by Proposition 7.23 with MIH we obtain $\alpha\leq\delta\in C^\beta(Y)\cap\gamma\subset Y$.

□ of (57) and (56).

Hence we obtain $\gamma\in\mathcal{G}^Y$. We have $\gamma<\beta^{\dagger 0}\leq\eta$ and $\gamma\in C^\gamma(Y)$. If $\gamma\in R(\eta)$, then the hypothesis (53) yields $\gamma\in Y$. In what follows assume $\gamma\not\in R(\eta)$.

If $G_\eta(\gamma)<\gamma$, then Proposition 7.24 yields $\gamma\in C^\eta(Y)\cap\eta\subset Y$ by $\eta\in\mathcal{G}^Y$.

In what follows suppose $G_\eta(\gamma)=\{\gamma\}$. This means $\gamma\in\Psi$ by $\gamma\not\in(Reg_0\setminus\{\Omega,\mathbb{I}_N\})$, and $\gamma\prec\tau$ for a $\tau<\eta$ by $\gamma\not\prec\eta$ and Definition 6.17.3. If $\eta\prec\mathbb{I}_N$, then $\gamma\prec\mathbb{I}_N$ by $\gamma\not\in R(\eta)$. Hence this is not the case.

Let $\eta\in L(\mathbb{T}^{\dagger\vec{k}})$ with $\mathbb{T}\in\{\Omega\}\cup(LSt\cap\Psi)$. By (54) we obtain $\mathbb{T}\in Y$. On the other hand we have $Y\cap\eta^{\dagger 1}<\gamma$ by (55), and $\mathbb{T}^{\dagger\vec{i}}\in Y$ since Y is closed under $\mathbb{U}\mapsto\mathbb{U}^{\dagger i}$. Hence $\mathbb{T}^{\dagger\vec{i}}<\gamma$ as long as $\mathbb{T}^{\dagger\vec{i}}<\eta$. We obtain $\gamma\in L(\mathbb{T}^{\dagger\vec{k}})$ by Definition 3.35.4.

Let τ be maximal such that $\gamma\prec\tau<\eta$. We obtain $\tau\in\Psi$ by $\gamma\in L(\mathbb{T}^{\dagger\vec{k}})\setminus R(\eta)$ and Proposition 6.13. From $\gamma\in C^\gamma(Y)$ we see $\tau\in C^\gamma(Y)$.

Next we show that

$$G_\eta(\tau)<\gamma \tag{58}$$

Let $\tau=\psi^f_\kappa(b)$ and $\gamma\preceq\gamma_1=\psi^g_\tau(a_1)$. Then $\eta<\kappa$ by the maximality of τ, and $G_\eta(\tau)=G_\eta(\{\kappa,b\}\cup SC(f))<\tau$ by Proposition 6.19.1. On the other hand we have $\tau\in\mathcal{H}_{a_1}(\gamma_1)$. Proposition 6.19.2 yields $G_\eta(\tau)\subset\mathcal{H}_{a_1}(\gamma_1)\cap\tau\subset\gamma_1$. We see $G_\eta(\tau)<\gamma$ inductively.

(58) is shown. Proposition 7.24 yields $\tau\in C^\eta(Y)$, and $\tau\in C^\eta(Y)\cap\eta\subset Y$ by $\eta\in\mathcal{G}^Y$. Therefore $Y\cap\eta^{\dagger 1}<\gamma<\tau\in Y$. This is not the case by (55). We are done. □

Proposition 7.26 *For $\alpha_1=\psi_{\mathbb{I}_N}(a)$, $\alpha_1\in\mathcal{G}^{\mathcal{W}_{N+1}}\Rightarrow\alpha_1\in\mathcal{W}_{N+1}$.*

Proof. Let $\alpha_1\in\mathcal{G}^{\mathcal{W}_{N+1}}$. By Lemma 7.22 pick an N-distinguished set Z such that $\{0,\Omega\}\subset Z$, $\alpha_1\in\mathcal{G}^Z$ and $\forall k\forall\mathbb{S}\in Z\cap(St_k\cup\{\Omega\})[\mathbb{S}^{\dagger k}\in Z]$.

Claim 7.27 *Let $SSt\ni\mathbb{T}<\alpha_1$ and $\gamma\in\mathcal{G}^Z\cap L(\mathbb{T})\cap\Psi$. Then $\gamma<Z\cap\alpha_1$.*

Proof of Claim 7.27. Let $\rho\prec\mathbb{S}^{\dagger\vec{k}}=\mathbb{T}<\alpha_1$ for an $\mathbb{S}\in LSt\cup\{\Omega\}$ and a $\vec{k}\neq\emptyset$.

First let $\gamma=\rho$. We obtain $\mathbb{T}\in C^\gamma(Z)$ by $\gamma\in C^\gamma(Z)$, and $\mathbb{S}\in C^\gamma(Z)\cap\gamma\subset Z$. Hence $\gamma<\mathbb{T}=\mathbb{S}^{\dagger\vec{k}}\in Z\cap\alpha_1$ since Z is closed under $\mathbb{U}\mapsto\mathbb{U}^{\dagger i}$.

Second let $\gamma\prec^R\kappa\in N(\rho)$ for a κ. We show $\rho\in Z$ by induction on $\ell\gamma$. First let $\gamma=\psi_{\mathbb{I}_N[\sigma]}(b)$ for some b and $\sigma\preceq^R\kappa$. Then we obtain $\mathbb{I}_N[\sigma]\in C^\gamma(Z)$ by $\gamma\in C^\gamma(Z)$, and $\sigma\in C^\gamma(Z)\cap\gamma\subset Z$. Proposition 7.15 yields $\sigma\in\mathcal{G}^Z$. If $\sigma=\kappa=\mathbb{I}_N[\rho]$, then $\sigma\in C^\sigma(Z)$ yields $\rho\in C^\sigma(Z)\cap\sigma\subset Z$. Otherwise IH yields $\rho\in Z$. Second let $\gamma=\psi^f_{\sigma^{\dagger\vec{i}}}(b)\in C^\gamma(Z)$ for some f, b and $\sigma^{\dagger\vec{i}}\preceq^R\kappa$. We obtain $\sigma\in C^\gamma(Z)\cap\gamma\subset Z$, and $\sigma\in\mathcal{G}^Z$. We obtain $\sigma\prec^R\kappa$. IH yields $\rho\in Z$.

Third let $\gamma = \psi_\tau^f(a)$ with $\tau = \mathbb{W}^{\dagger\vec{j}}[\sigma/\mathbb{W}]$. We obtain $\gamma < \tau \in C^\gamma(Z)$, and $\sigma \in C^\gamma(Z) \cap \gamma$. Hence $\sigma \in \mathcal{G}^Z$. If $\tau = \mathbb{W}^{\dagger\vec{j}}[\sigma/\mathbb{W}] = \mathbb{U}^{\dagger\vec{i}}[\rho/\mathbb{S}]$, then $\sigma \in C^\sigma(Z)$ yields $\rho \in C^\sigma(Z) \cap \sigma \subset Z$. Otherwise IH yields $\rho \in Z$.

Now $\rho \in Z$ yields $\rho \in C^\rho(Z)$, and this yields $\mathbb{S} \in C^\rho(Z) \cap \rho \subset Z$. Since Z is closed under $\mathbb{U} \mapsto \mathbb{U}^{\dagger i}$, we obtain $\gamma < \mathbb{S}^{\dagger\vec{k}} \in Z \cap \alpha_1$. □ of Claim 7.27.

Since there is no $\gamma \prec \alpha_1$, if $\gamma \in R(\alpha_1)$, then $\gamma \in L(\mathbb{T}) \cap \Psi$ for a $SSt \ni \mathbb{T} < \alpha_1$ by Definition 6.12.1. Also $\alpha_1 \notin LmS)$ for any $\mathbb{S} \in SSt$, and we have (53) by Claim 7.27. We conclude $\alpha_1 \in \mathcal{W}_{N+1}$ by Lemma 7.25. □

Lemma 7.28 *For* each $n < \omega$, *the following holds:*
Let $a \in \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap \omega_n(\mathbb{I}_N + 1)$. *Then* $\psi_{\mathbb{I}_N}(a) \in \mathcal{W}_{N+1}$ *holds.*

Proof. For each $n < \omega$, we have $\mathrm{TI}[\mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap (\omega_n(\mathbb{I}_N + 1))]$ by Lemma 7.13.2. We show the lemma by induction on $a \in \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap \omega_n(\mathbb{I}_N + 1)$. Assume

$$\mathrm{IH} :\Leftrightarrow \forall b \in \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap a\, (\psi_{\mathbb{I}_N}(b) \in OT(\mathbb{I}_N) \Rightarrow \psi_{\mathbb{I}_N}(b) \in \mathcal{W}_{N+1})\,.$$

Let $\alpha_1 = \psi_{\mathbb{I}_N}(a) \in OT(\mathbb{I}_N)$ with $a \in \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap \omega_n(\mathbb{I}_N + 1)$. By Proposition 7.26 it suffices to show $\alpha_1 \in \mathcal{G}^{\mathcal{W}_{N+1}}$.

From $a \in \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1})$ with $\alpha_1 < \mathbb{I}_N$ we see $\alpha_1 \in \mathcal{C}^{\alpha_1}(\mathcal{W}_{N+1})$. It suffices to show the following (59) by induction on $\ell\beta_1$.

$$\forall \beta_1 \in \mathcal{C}^{\alpha_1}(\mathcal{W}_{N+1}) \cap \alpha_1 [\beta_1 \in \mathcal{W}_{N+1}]. \tag{59}$$

Proof of (59). Assume $\beta_1 \in \mathcal{C}^{\alpha_1}(\mathcal{W}_{N+1}) \cap \alpha_1$ and let

$$\mathrm{LIH} :\Leftrightarrow \forall \gamma \in \mathcal{C}^{\alpha_1}(\mathcal{W}_{N+1}) \cap \alpha_1 [\ell\gamma < \ell\beta_1 \Rightarrow \gamma \in \mathcal{W}_{N+1}].$$

We show $\beta_1 \in \mathcal{W}_{N+1}$. We can assume $\beta_1 \notin \{0, \Omega\}$ by Proposition 7.21.

Case 1. $\beta_1 \notin \mathcal{E}(\beta_1)$: Assume $\beta_1 \notin \mathcal{W}_{N+1}$. Then $\beta_1 \notin N(\rho)$ for any ρ by $\beta_1 \in \mathcal{C}^{\alpha_1}(\mathcal{W}_{N+1}) \cap \alpha_1$ and Definition 7.1. We obtain $S(\beta_1) \subset \mathcal{C}^{\alpha_1}(\mathcal{W}_{N+1}) \cap \alpha_1$. LIH yields $S(\beta_1) \subset \mathcal{W}_{N+1}$. Hence we conclude $\beta_1 \in \mathcal{W}_{N+1}$ from Proposition 7.12.

Case 2. In what follows consider the cases when $\beta_1 = \psi_\pi^g(b)$ for some π, b, g. We can assume $\pi > \alpha_1$. Then we see $\pi = \mathbb{I}_N$ and $\beta_1 = \psi_{\mathbb{I}_N}(b)$ with $b \in \mathcal{C}^{\alpha_1}(\mathcal{W}_{N+1})$. We obtain $b < a$ by Proposition 3.17.1, and $b \in \mathcal{H}_b(\beta_1)$. By IH it suffices to show $b \in \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1})$.

By induction on ℓc we see that $c \in \mathcal{H}_b(\beta_1) \Rightarrow G_{\mathbb{I}_N}(c) < \beta_1$. For example let $c = \gamma_1^{\dagger\vec{i}}$ with $\gamma_1 \in LSt_N \cup \{\Omega\}$ and $\vec{i} \neq \emptyset$. Suppose $c > \beta_1$. Then $\gamma_1 \in \mathcal{H}_b(\beta_1)$. The induction hypothesis on ℓc yields $\{\gamma_1\} = G_{\mathbb{I}_N}(\gamma_1) < \beta_1 \in LSt_N$, and hence $\{c\} = G_{\mathbb{I}_N}(c) < \beta_1$.

In particular we obtain $G_{\mathbb{I}_N}(b) < \beta_1$. Proposition 7.24 with LIH yields $b \in \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1})$. This shows (59). □

7.3 Layers of stable ordinals

In this subsection we examine ordinals in layers $L(\mathbb{S}) = \{\alpha \in OT(\mathbb{I}_N) : \alpha \prec^R \mathbb{S}\}$ for $\mathbb{S} \in SSt$. We show that there is no infinite descending chain in $L(\mathbb{S})$, cf. Lemma 7.32. Here we need the condition (12) and the fact that $\alpha \in M_\rho$ if α is in the domain of the Mostowski collapsing $\alpha \mapsto \alpha[\rho/\mathbb{S}]$, cf. Definition 3.33 and Proposition 7.31.

Let $k(\psi^f_\kappa(a)) = \{\kappa, a\} \cup SC(f)$ and $\mathtt{h}(\psi^f_\kappa(a)) = \{a, \mathtt{g}^*_0(\psi^f_\kappa(a))\}$.

Proposition 7.29 *Let Z be an N-distinguished set such that $\{0, \Omega\} \subset Z$ and $\forall k \forall \mathbb{S} \in Z \cap (St_k \cup \{\Omega\})[\mathbb{S}^{\dagger k} \in Z]$. Assume $\psi_{\mathbb{I}_N}(b) \in Z$, and let*

$$\begin{aligned} \mathrm{MIH}(b; Z) :\Leftrightarrow & \forall \mathbb{T} \in (St \cup \{\Omega\}) \cap Z \forall k \forall \gamma \in L(\mathbb{T}^{\dagger k}) \cap \Psi \\ & \left[k(\gamma) \subset \mathcal{C}^{\mathbb{I}_N}(Z) \,\&\, \mathtt{h}(\gamma) \subset \mathcal{C}^{\mathbb{I}_N}(Z) \cap b \Rightarrow \{\gamma\} \cup N(\gamma) \subset Z\right]. \end{aligned}$$

Then for any $\Theta \subset Z$, $\mathcal{H}_b(\Theta) \subset \mathcal{C}^{\mathbb{I}_N}(Z)$ holds.

Proof. Let $\Theta \subset Z$. Assuming $\gamma \in \mathcal{H}_b(\Theta)$, we show $\gamma \in \mathcal{C}^{\mathbb{I}_N}(Z)$ by induction on $\ell\gamma$. Let $\gamma \notin \Theta$. By IH and Proposition 7.12, we can assume $\gamma \in \Psi \cup (Reg_0 \setminus \{\Omega, \mathbb{I}_N\})$.

Case 1. $\gamma = \psi^f_\kappa(a)$ with $k(\gamma) \subset \mathcal{H}_b(\Theta)$: We show $\{\gamma\} \cup N(\gamma) \subset Z$. IH yields $\{\kappa, a\} \subset k(\gamma) \subset \mathcal{C}^{\mathbb{I}_N}(Z)$.

Case 1.1. $\kappa = \mathbb{I}_N$: Then we obtain $f = \emptyset$ and $\gamma = \psi_{\mathbb{I}_N}(a) < \psi_{\mathbb{I}_N}(b) = \delta \in Z$ and $N(\gamma) = \emptyset$. $a \in \mathcal{C}^{\mathbb{I}_N}(Z) \subset \mathcal{C}^\delta(Z)$ yields $\gamma \in \mathcal{C}^\delta(Z) \cap \delta \subset Z$.

Case 1.2. $\kappa < \mathbb{I}_N$: Let $\gamma \in L(\mathbb{S})$ with $\mathbb{S} = \mathbb{T}^{\dagger k}$ and $\mathbb{T} \in St \cup \{\Omega\}$. We claim that $\mathbb{T} \in Z$ and $\mathtt{h}(\gamma) \subset \mathcal{C}^{\mathbb{I}_N}(Z) \cap b$. We have $\kappa \in \mathcal{C}^{\mathbb{I}_N}(Z) \cap \mathbb{I}_N \subset Z$. We obtain $\kappa \in \mathcal{G}^Z$. Let $\rho \prec \mathbb{S}$ be such that either $\rho = \kappa$ or $\kappa \prec^R \sigma \in N(\rho)$. In the latter case we obtain $\rho \in \mathcal{C}^\kappa(Z) \cap \kappa \subset Z$. We obtain $\rho \in Z$ and $\rho \in \mathcal{G}^Z$, from which we see $\mathbb{S} \in \mathcal{C}^\rho(Z)$ and $\mathbb{T} \in \mathcal{C}^\rho(Z) \cap \rho \subset Z$.

On the other, IH yields $a \in \mathcal{C}^{\mathbb{I}_N}(Z) \cap b$. We show $\mathtt{g}^*_0(\gamma) \in \mathcal{C}^{\mathbb{I}_N}(Z) \cap b$.

Case 1.2.1. $\gamma \prec \mathbb{S}$: Then $\mathtt{g}^*_0(\gamma) = \mathtt{p}_{\mathbb{S}}(\gamma_0)$ for $\gamma \preceq \gamma_0 = \psi^g_{\mathbb{S}}(c)$. IH with $\mathtt{p}_{\mathbb{S}}(\gamma_0) \in \mathcal{H}_b(\Theta)$ yields $\mathtt{p}_{\mathbb{S}}(\gamma_0) \in \mathcal{C}^{\mathbb{I}_N}(Z)$. On the other hand we have $\mathtt{p}_{\mathbb{S}}(\gamma_0) < b$ by $\gamma_0 \in \mathcal{H}_b(\Theta)$.

Case 1.2.2. $\rho \prec \mathbb{S}$ and $\gamma \prec^R \sigma \in N(\rho)$ for some ρ and σ: Then $\mathtt{g}^*_0(\gamma) = \mathtt{g}^*_0(\rho)$. We obtain $\rho \in \mathcal{H}_b(\Theta)$. From **Case 1.2.1** with IH we see $\mathtt{g}^*_0(\rho) \in \mathcal{C}^{\mathbb{I}_N}(Z) \cap b$.

Therefore $\mathrm{MIH}(b; Z)$ yields $\{\gamma\} \cup N(\gamma) \subset Z$.

Case 2. $\gamma \in N(\gamma_1)$ for a $\gamma_1 \in \Psi$: Then $\gamma_1 \in \mathcal{H}_b(\Theta)$, and **Case 1** yields $\gamma \in N(\gamma_1) \subset Z$. □

Proposition 7.30 *1. Let $\gamma_1 = \gamma[\rho/\mathbb{S}]^{-1}$ be the Mostowski uncollapsing, and $\{\mathbb{S}, \gamma\} \subset C^\rho(Z)$. Then $\gamma_1 \in C^\rho(Z)$.*

2. $\gamma \in \mathcal{H}_b(\rho) \cap C^\rho(Z) \Rightarrow \gamma \in \mathcal{H}_b(C^\rho(Z) \cap \rho)$.

Proof. Each is seen by induction on $\ell\gamma$. For Proposition 7.30.1, use the fact $\gamma_1 = \gamma[\rho/\mathbb{S}]^{-1} \geq \gamma$. □

Proposition 7.31 *Let $\mathbb{S} \in SSt$, $\eta \in L(\mathbb{S})$ and Z be an N-distinguished set such that $\{0, \Omega\} \subset Z$, $\forall k \forall \mathbb{S} \in Z \cap (St_k \cup \{\Omega\})[\mathbb{S}^{\dagger k} \in Z]$. Assume $\eta \in \mathcal{G}^Z$, $\psi_{\mathbb{I}_N}(b) \in Z$ and* MIH$(b; Z)$ *in Proposition 7.29 for a $b \geq \mathtt{g}_0^*(\eta)$. Then the following holds.*

1. $\mathtt{g}_0(\eta) \in \mathcal{C}^{\mathbb{I}_N}(Z)$.
2. $\mathtt{g}_1(\eta) \in \mathcal{C}^{\mathbb{I}_N}(Z)$.
3. $\mathtt{g}_2(\eta) \in \mathcal{C}^{\mathbb{I}_N}(Z)$.

Proof. Proposition 7.31.2 is seen from Proposition 7.31.1 by induction on $\ell\eta$ as follows. Let $\eta > \rho \in L(\mathbb{S}) \cap \Psi$ be in the trail to η. We see $\mathtt{g}_0^*(\rho) = \mathtt{g}_0^*(\eta)$ from Definition 6.7. Moreover we see $\rho \in \mathcal{C}^\eta(Z) \cap \eta \subset Z$ from $\eta \in \mathcal{G}^Z$. In particular $\rho \in \mathcal{G}^Z$. By IH we obtain $\mathtt{g}_0(\rho) \in \mathcal{C}^{\mathbb{I}_N}(Z)$. On the other side, we see $\mathtt{g}_1(\eta) \in \mathcal{C}^{\mathbb{I}_N}(Z)$ if $\mathtt{g}_0(\rho) \in \mathcal{C}^{\mathbb{I}_N}(Z)$ for every $\eta > \rho \in L(\mathbb{S}) \cap \Psi$ in the trail to η from Definition 6.14.

7.31.1. Let $\eta \in \Psi$.

Case 1. $\eta = \rho$ or $\eta = \psi_{\mathbb{I}_N[\rho]}(c)$ for a $\rho \prec \mathbb{S}$ and a c: Then $\mathtt{p}_0(\rho) \leq \mathtt{g}_0(\rho) = \mathtt{g}_0(\eta) = \mathtt{g}_0^*(\eta) \leq b$. We show $\mathtt{p}_\mathbb{S}(\rho) = \mathtt{g}_0(\rho) \in \mathcal{C}^{\mathbb{I}_N}(Z)$. By (12) in Definition 3.31.6 we have $\mathtt{p}_0(\rho) \in \mathcal{H}_b(\rho)$, and $\mathtt{p}_\mathbb{S}(\rho) \in \mathcal{H}_b(\rho)$. On the other hand we have $\rho \in \mathcal{G}^Z$. We obtain $\mathtt{p}_\mathbb{S}(\rho) \in \mathcal{C}^\rho(Z)$ by $\rho \in \mathcal{C}^\rho(Z)$, and $\mathtt{p}_\mathbb{S}(\rho) \in \mathcal{H}_b(\mathcal{C}^\rho(Z) \cap \rho)$ by Proposition 7.30.2. Moreover we have $\mathcal{C}^\rho(Z) \cap \rho \subset Z$. Proposition 7.29 with MIH$(b; Z)$ yields $\mathtt{p}_\mathbb{S}(\rho) \in \mathcal{H}_b(\mathcal{C}^\rho(Z) \cap \rho) \subset \mathcal{C}^{\mathbb{I}_N}(Z)$.

Case 2. Otherwise: Let $\rho \prec \mathbb{S}$ be such that $\eta \prec^R \tau \in N(\rho)$. Let $\eta_1 \in M_\rho$ be such that $\eta = \eta_1[\rho/\mathbb{S}]$. Then $\mathtt{g}_0(\eta) = \mathtt{g}_0(\eta_1)$ and $\mathtt{p}_0(\rho) \leq \mathtt{g}_0(\rho) = \mathtt{g}_0^*(\eta) \leq b$.

On the other hand we have $\eta \in \mathcal{G}^Z$. $\eta \in \mathcal{C}^\eta(Z)$ yields $\rho \in \mathcal{C}^\eta(Z) \cap \eta \subset Z$. Hence $\rho \in Z$. We obtain $\rho \in \mathcal{G}^Z$. We see $\mathbb{S} \in \mathcal{C}^\rho(Z)$ from $\rho \in \mathcal{C}^\rho(Z)$. Hence $\{\mathbb{S}, \eta\} \subset \mathcal{C}^\rho(Z)$. Proposition 7.30.1 yields $\eta_1 \in \mathcal{C}^\rho(Z)$, and $\mathtt{g}_0(\eta_1) \in \mathcal{C}^\rho(Z)$ by $\eta_1 > \rho$. $\eta_1 \in M_\rho \subset \mathcal{H}_b(\rho)$ yields $\mathtt{g}_0(\eta_1) \in \mathcal{H}_b(\rho)$. We obtain $\mathtt{g}_0(\eta_1) \in \mathcal{H}_a(\mathcal{C}^\rho(Z) \cap \rho)$ by Proposition 7.30.2. $\rho \in \mathcal{G}^Z$ yields $\mathcal{C}^\rho(Z) \cap \rho \subset Z$. Hence Proposition 7.29 yields $\mathtt{g}_0(\eta_1) \in \mathcal{C}^{\mathbb{I}_N}(Z)$.

7.31.3. By induction on $\ell\eta$. Let $\eta \in \Psi$.

Case 1. $\eta = \rho \prec \mathbb{S}$: Then $\mathtt{p}_0(\rho) = \mathtt{p}_0(\eta)$. Let $f = m(\rho)$. Then $\mathtt{g}_2(\rho) = o_\mathbb{S}(f) + 1$ and $SC(f) \subset \mathcal{H}_b(\rho)$ for $b \geq \mathtt{p}_0(\eta)$ by (12) in Definition 3.31.6. Moreover $SC(f) \subset \mathcal{C}^\rho(Z)$ by $\rho \in \mathcal{C}^\rho(Z)$. Hence we obtain $SC(f) \subset \mathcal{H}_b(\mathcal{C}^\rho(Z) \cap \rho)$ by Proposition 7.30.2, where $\mathcal{C}^\rho(Z) \cap \rho \subset Z$. Proposition 7.29 with MIH$(b; Z)$ yields $SC(f) \subset \mathcal{C}^{\mathbb{I}_N}(Z)$, and $o_\mathbb{S}(f) \in \mathcal{C}^{\mathbb{I}_N}(Z)$.

Case 2. Otherwise: Let $\rho \prec \mathbb{S}$ be such that $\eta = \eta_1[\rho/\mathbb{S}]$ with $\eta_1 \in M_\rho \cap L(\mathbb{S}_1)$ and $\mathtt{g}_0(\eta_1) = \mathtt{g}_0(\eta)$, where $\eta \prec^R (\mathbb{S}_1[\rho/\mathbb{S}])$ and $M_\rho \subset \mathcal{H}_b(\rho)$ for $\mathtt{p}_0(\rho) \leq \mathtt{g}_0(\rho) = \mathtt{g}_0^*(\eta) \leq b$. Then $\ell\eta_1 < \ell\eta$. $\eta \in \mathcal{C}^\eta(Z)$ with $\rho < \eta$ yields $\eta \in \mathcal{C}^\rho(Z)$ and $\rho \in \mathcal{C}^\eta(Z) \cap \eta \subset Z$. We see $\mathbb{S} \in \mathcal{C}^\rho(Z)$ from $\rho \in \mathcal{C}^\rho(Z)$. We obtain $\eta_1 \in \mathcal{C}^\rho(Z)$ by Proposition 7.30.1 and $\eta \in \mathcal{C}^\rho(Z)$, and $\eta_1 \in \mathcal{H}_b(\mathcal{C}^\rho(Z) \cap \rho)$ by Proposition 7.30.2. On the other hand we have $\mathcal{C}^\rho(Z) \cap \rho \subset Z$. By Proposition 7.29 we obtain $\eta_1 \in \mathcal{C}^{\mathbb{I}_N}(Z) \cap \mathbb{I}_N \subset Z$. Hence $\eta_1 \in \mathcal{G}^Z$. Moreover we see $\mathtt{g}_0^*(\eta_1) < b = \mathtt{p}_0(\rho) \leq \mathtt{g}_0^*(\eta)$ from $\eta_1 \in \mathcal{H}_b(\rho)$. IH yields $\mathtt{g}_2(\eta) = \mathtt{g}_2(\eta_1) \in \mathcal{C}^{\mathbb{I}_N}(Z)$. □

Lemma 7.32 *Let* $\mathbb{S} = \mathbb{T}^{\dagger\vec{k}} \in SSt$ *with* $\mathbb{T} \in \{\Omega\} \cup (LSt \cap \Psi)$, $\mathbb{T} < \eta \in L(\mathbb{S})$, *and* Z *be an* N*-distinguished set such that* $\{0, \Omega, \mathbb{T}\} \subset Z$, $\forall k \forall \mathbb{U} \in Z \cap (St_k \cup \{\Omega\})[\mathbb{U}^{\dagger k} \in Z]$. *Assume* $\eta \in \mathcal{G}^Z$, $b \geq \mathsf{g}_0^*(\eta)$, $\Lambda = \psi_{\mathbb{I}_N}(b) \in Z$, *and* MIH$(b; Z)$ *in Proposition 7.29. Then* $\eta \in Z$.

Proof. By Lemma 6.15 we obtain $SC(\mathsf{g}_2(\eta)) \subset \Lambda = \psi_{\mathbb{I}_N}(b)$. An ordinal $\mathsf{g}_2^\Lambda(\eta) = o_\Lambda(f) + 1 < \mathbb{I}_N$ is obtained from $\mathsf{g}_2(\eta) = o_{\mathbb{I}_N}(f) + 1$ in Definition 6.1.2 by changing the base $\mathbb{I}_N$ to Λ. Then for $SC(\mathsf{g}_2(\gamma)) \cup SC(\mathsf{g}_2(\delta)) \subset \Lambda$, $\mathsf{g}_2(\delta) < \mathsf{g}_2(\gamma) \Leftrightarrow \mathsf{g}_2^\Lambda(\delta) < \mathsf{g}_2^\Lambda(\gamma)$ by Proposition 3.3, and $\mathsf{g}_2(\gamma) \in \mathcal{C}^{\mathbb{I}_N}(Z) \Leftrightarrow \mathsf{g}_2^\Lambda(\gamma) \in Z$ by the assumption $\Lambda \in Z$.

On the other side, we see $W_N^{\mathbb{T}}(Z) \cap \mathbb{S} = Z \cap \mathbb{S}$ from $\mathbb{T} \in Z$ and $D_N[Z]$. Hence $\mathcal{G}^Y \cap \mathbb{S} = \mathcal{G}^Z \cap \mathbb{S}$ for $Y = W_N^{\mathbb{T}}(Z) \cap \mathbb{S}$.

We see $Wo[\mathcal{C}^{\mathbb{I}_N}(Z)]$ from $\mathcal{C}^{\mathbb{I}_N}(Z) \cap \mathbb{I}_N = Z \cap \mathbb{I}_N$ as in Lemma 7.13. We show $\eta \in Z \cap \mathbb{S} = W_N^{\mathbb{T}}(Z) \cap \mathbb{S}$ by induction on $\mathsf{g}^\Lambda(\eta) = (\mathsf{g}_1(\eta), \mathsf{g}_2^\Lambda(\eta))$ with respect to the lexicographic order $<_{lx}$ on $\mathcal{C}^{\mathbb{I}_N}(Z) \times Z$.

Let $\gamma \in R(\eta)$ be such that $\gamma \in \mathcal{G}^Z$. Then $\gamma \in R(\eta) \subset L(\mathbb{S})$, $\mathbb{T}^{-N} = \gamma^{-N} = \eta^{-N}$ and $\mathbb{T} < \gamma < \eta < \mathbb{S}$. By Lemma 6.15 we obtain $\mathsf{g}_0^*(\gamma) \leq \mathsf{g}_0^*(\eta)$, $\mathsf{g}(\gamma) <_{lx} \mathsf{g}(\eta)$ and $SC(\mathsf{g}_2(\gamma)) \subset \Lambda = \psi_{\mathbb{I}_N}(b)$. Proposition 7.31 yields $\{\mathsf{g}_1(\gamma), \mathsf{g}_2(\gamma), \mathsf{g}_1(\eta), \mathsf{g}_2(\eta)\} \subset C^{\mathbb{I}_N}(Z)$. We obtain $\mathsf{g}^\Lambda(\gamma) <_{lx} \mathsf{g}^\Lambda(\eta)$. IH yields $\gamma \in Z$, and (53) is shown. On the other hand we have $\mathbb{T} \in Z$ for (54).

Lemma 7.25 yields $\eta \in Z$. □

Proposition 7.33 *Let* $D_N[Z]$ *and* $\rho \in L(\mathbb{S}) \cap Z \cap \Psi$ *with an* $\mathbb{S} \in SSt$. *Then* $N(\rho) \subset \mathcal{G}^Z$.

Proof. Let $\alpha \in N(\rho)$. We obtain $\alpha \in \mathcal{C}^\alpha(Z)$ by $\rho \in Z \cap \alpha$. We show $\beta \in \mathcal{C}^\alpha(Z) \cap \alpha \Rightarrow \alpha \in Z$ by induction on $\ell\beta$. Let $\rho \neq \beta \in \mathcal{C}^\alpha(Z) \cap \alpha$. If $\beta < \rho$, then $\beta \in \mathcal{C}^\rho(Z) \cap \rho \subset Z$ by Propositions 7.2.1 and 7.15. Let $\rho < \beta < \alpha$. By IH, Proposition 7.12 and Definition 7.1 we may assume that $\beta = \psi_\sigma^f(c)$ with $\sigma > \alpha$. Then $\beta < \rho$ by Proposition 3.39. □

Corollary 7.34 *For* each $\zeta \in \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1})$, *the following holds:*

Let $\mathbb{S} = \mathbb{T}^{\dagger\vec{k}} \in SSt$ *with* $\mathbb{T} \in \{\Omega\} \cup (LSt \cap \Psi)$, $\eta \in N(\rho)$ *with* $\rho \in L(\mathbb{S})$, *and* $\{\mathbb{T}, \rho\} \subset \mathcal{W}_{N+1}$. *Assume* $\zeta \geq \mathsf{g}_0^*(\eta)$ *and* MIH$(\zeta; \mathcal{W}_{N+1})$ *in Proposition 7.29. Then* $\eta \in \mathcal{W}_{N+1}$.

Proof. By $\mathsf{g}_0^*(\eta) \leq \zeta \in \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1})$ and Lemma 7.28 we obtain $\psi_{\mathbb{I}_N}(\zeta) \in \mathcal{W}_{N+1}$. As in the proof of Lemma 7.22 we see that there exists an N-distinguished set Z such that $\{0, \Omega, \mathbb{T}, \rho\} \subset Z$, $\forall k \forall \mathbb{U} \in Z \cap (St_k \cup \{\Omega\})[\mathbb{U}^{\dagger k} \in Z]$, $\psi_{\mathbb{I}_N}(\zeta) \in Z$, and MIH$(\zeta; Z)$. Then $\eta \in Z \subset \mathcal{W}_{N+1}$ follows from Lemma 7.32 and Proposition 7.33. □

Definition 7.35 For irreducible functions f let

$$f \in J :\Leftrightarrow SC(f) \subset \mathcal{W}_{N+1}.$$

For $a \in OT(\mathbb{I}_N)$ and irreducible functions f, define:

$$\begin{array}{rcl} A(\zeta, a, f) & :\Leftrightarrow & \forall\sigma \in \mathcal{W}_{N+1} \cap \mathbb{I}_N[\mathsf{g}_0^*(\psi_\sigma^f(a)) \leq \zeta \Rightarrow \psi_\sigma^f(a) \in \mathcal{W}_{N+1}] \\ \mathrm{SIH}(\zeta, a) & :\Leftrightarrow & \forall b \in C^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap a \forall f \in J\, A(\zeta, b, f). \\ \mathrm{SSIH}(\zeta, a, f) & :\Leftrightarrow & \forall g \in J[g <_{lx}^0 f \Rightarrow A(\zeta, a, g)]. \end{array}$$

Lemma 7.36 *For* each $\zeta \in \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1})$, *the following holds:*

Assume $a \in \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap (\zeta + 1)$, $f \in J$, $\mathrm{SIH}(\zeta, a)$, $\mathrm{SSIH}(\zeta, a, f)$ *in Definition 7.35. Moreover assume* $\mathrm{MIH}(\zeta; \mathcal{W}_{N+1})$ *in Proposition 7.29. Then for any* $\mathbb{S} = \mathbb{T}^{\dagger\vec{k}} \in SSt$ *with* $\mathbb{T} \in (\{\Omega\} \cup (LSt \cap \Psi)) \cap \mathcal{W}_{N+1}$ *and any* $\kappa \in \mathcal{W}_{N+1} \cap (L(\mathbb{S}) \cup \{\mathbb{S}\})$ *the following holds:*

$$\mathsf{g}_0^*(\psi_\kappa^f(a)) \leq \zeta \Rightarrow \psi_\kappa^f(a) \in \mathcal{W}_{N+1}.$$

Proof. Let $\alpha_1 = \psi_\kappa^f(a) \in OT(\mathbb{I}_N)$ with $a \in \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap (\zeta + 1)$, $\kappa \in \mathcal{W}_{N+1} \cap (L(\mathbb{S}) \cup \{\mathbb{S}\})$ and $f \in J$ such that $\mathbb{S} = \mathbb{T}^{\dagger\vec{k}}$ with $\mathbb{T} \in \mathcal{W}_{N+1}$, and $\mathsf{g}_0^*(\alpha_1) \leq \zeta$. By Lemma 7.28 we have $\psi_{\mathbb{I}_N}(\zeta) \in \mathcal{W}_{N+1}$. By Lemma 7.32 and the assumption $\mathrm{MIH}(\zeta; \mathcal{W}_{N+1})$ it suffices to show $\alpha_1 \in \mathcal{G}^{\mathcal{W}_{N+1}}$.

By Lemma 7.10 we have $\{\kappa, a\} \cup SC(f) \subset C^{\alpha_1}(\mathcal{W}_{N+1})$, and hence $\alpha_1 \in C^{\alpha_1}(\mathcal{W}_{N+1})$. It suffices to show the following claim by induction on $\ell\beta_1$.

Claim 7.37 $\forall\beta_1 \in C^{\alpha_1}(\mathcal{W}_{N+1}) \cap \alpha_1[\beta_1 \in \mathcal{W}_{N+1}]$.

Proof of Claim 7.37. Assume $\beta_1 \in C^{\alpha_1}(\mathcal{W}_{N+1}) \cap \alpha_1$ and let

$$\mathrm{LIH} :\Leftrightarrow \forall\gamma \in C^{\alpha_1}(\mathcal{W}_{N+1}) \cap \alpha_1[\ell\gamma < \ell\beta_1 \Rightarrow \gamma \in \mathcal{W}_{N+1}].$$

We show $\beta_1 \in \mathcal{W}_{N+1}$. We can assume $\beta_1 \notin \{0, \Omega\}$ by Proposition 7.21.

Case 1. $\beta_1 \notin \mathcal{E}(\beta_1)$: Assume $\beta_1 \notin \mathcal{W}_{N+1}$. Then $\beta_1 \notin N(\rho)$ for any ρ by $\beta_1 \in \mathcal{C}^{\alpha_1}(\mathcal{W}_{N+1}) \cap \alpha_1$ and Definition 7.1. We obtain $S(\beta_1) \subset C^{\alpha_1}(\mathcal{W}_{N+1}) \cap \alpha_1$. LIH yields $S(\beta_1) \subset \mathcal{W}_{N+1}$. Hence we conclude $\beta_1 \in \mathcal{W}_{N+1}$ from Proposition 7.12.

In what follows consider the cases when $\beta_1 = \psi_\pi^g(b)$ for some π, b, g. We can assume $\pi > \alpha_1$ and $\{\pi, b\} \cup SC(g) \subset C^{\alpha_1}(\mathcal{W}_{N+1})$. Then either $\pi = \mathbb{I}_N$ or $\beta_1 \in L(\mathbb{S})$ for $\alpha_1 \in L(\mathbb{S})$.

Case 2. $\pi = \mathbb{I}_N$ and $b < a$: As in the proof of Lemma 7.28 we see $b \in \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1})$. We obtain $\beta_1 = \psi_{\mathbb{I}_N}(b) \in \mathcal{W}_{N+1}$ by $b < a \leq \zeta$ and Lemma 7.28.

Case 3. $\pi < \mathbb{I}_N$, $b < a$, $\beta_1 < \kappa$ and $\{\pi, b\} \cup SC(g) \subset \mathcal{H}_a(\alpha_1)$: Then $\beta_1 \in L(\mathbb{S})$.

Let B denote a set of subterms of β_1 defined recursively as follows. First $\{\pi, b\} \cup SC(g) \subset B$. Let $\alpha_1 \leq \beta \in B$. If $\beta =_{NF} \gamma_m + \cdots + \gamma_0$, then $\{\gamma_i : i \leq m\} \subset B$. If $\beta =_{NF} \varphi\gamma\delta$, then $\{\gamma, \delta\} \subset B$. If $\beta = \psi_\sigma^h(c)$, then $\{\sigma, c\} \cup SC(h) \subset B$. If $\beta \in N(\tau)$, then $\tau \in B$.

Then from $\{\pi, b\} \cup SC(g) \subset C^{\alpha_1}(\mathcal{W}_{N+1})$ we see inductively that $B \subset C^{\alpha_1}(\mathcal{W}_{N+1})$. Hence by LIH we obtain $B \cap \alpha_1 \subset \mathcal{W}_{N+1}$. Moreover if $\alpha_1 \leq \psi_\sigma^h(c) \in B$, then $c \in K_{\alpha_1}(\{\pi, b\} \cup SC(g)) < a$.

We claim that

$$\forall\beta \in B(\beta \in C^{\mathbb{I}_N}(\mathcal{W}_{N+1})) \tag{60}$$

Proof of (60) by induction on $\ell\beta$. Let $\beta \in B$. We may assume that $\alpha_1 \leq \beta$ is a strongly critical number such that $\beta \notin \{\Omega, \mathbb{I}_N\} \cup SSt$ by induction hypothesis on the lengths. First consider the case when $\alpha_1 \leq \beta = \psi_\sigma^h(c)$. By induction hypothesis we have $\{\sigma, c\} \cup SC(h) \subset \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1})$. On the other hand we have $c < a$ and $\mathsf{g}_0^*(\beta) \leq \mathsf{g}_0^*(\alpha_1)$ by Proposition 6.16. SIH(ζ, a) yields $\beta \in \mathcal{W}_{N+1}$.

Second let $\alpha_1 \leq \beta \in N(\tau)$ for a $\tau \in L(\mathbb{S})$. By IH we obtain $\tau \in \mathcal{W}_{N+1}$. We claim that $\mathsf{g}_0^*(\tau) \leq \mathsf{g}_0^*(\alpha_1) \leq \zeta$. If $\tau \leq \alpha_1$, then we obtain $\mathsf{g}_0^*(\tau) = \mathsf{g}_0^*(\alpha_1)$. Otherwise $\alpha_1 < \tau = \psi_\sigma^h(c) \in B \subset \mathcal{H}_a(\alpha_1)$ for some σ, h, c. We obtain $\mathsf{g}_0^*(\tau) \leq \mathsf{g}_0^*(\alpha_1)$ by Proposition 6.16. On the other hand we have $\mathbb{T} \in \mathcal{W}_{N+1}$ by one of the assumptions. Corollary 7.34 yields $\beta \in \mathcal{W}_{N+1}$.

Thus (60) is shown. □

In particular we obtain $\{\pi, b\} \cup SC(g) \subset C^{\mathbb{I}_N}(\mathcal{W}_{N+1})$. Moreover we have $b < a$ and $\mathsf{g}_0^*(\beta_1) \leq \mathsf{g}_0^*(\alpha_1)$ by Proposition 6.9. Therefore once again SIH(ζ, a) yields $\beta_1 \in \mathcal{W}_{N+1}$.

Case 4. $b = a$, $\pi = \kappa$, $\forall\delta \in SC(g)(K_{\alpha_1}(\delta) < a)$ and $g <_{lx}^0 f$: Obviously $\mathsf{g}_0^*(\beta_1) = \mathsf{g}_0^*(\alpha_1)$. As in (60) we see that $SC(g) \subset \mathcal{W}_{N+1}$ from SIH(ζ, a). SSIH(ζ, a, f) yields $\beta_1 \in \mathcal{W}_{N+1}$.

Case 5. $a \leq b \leq K_{\beta_1}(\delta)$ for some $\delta \in SC(f) \cup \{\kappa, a\}$: It suffices to find a γ such that $\beta_1 \leq \gamma \in \mathcal{W}_{N+1} \cap \alpha_1$. Then $\beta_1 \in \mathcal{W}_{N+1}$ follows from $\beta_1 \in C^{\alpha_1}(\mathcal{W}_{N+1})$ and Propositions 7.2.1 and 7.11.

$k_X(\alpha)$ denotes the set in Definition 6.17. In general we see that $a \in K_X(\alpha)$ iff $\psi_\sigma^h(a) \in k_X(\alpha)$ for some σ, h, and for each $\psi_\sigma^h(a) \in k_X(\psi_{\sigma_0}^{h_0}(a_0))$ there exists a sequence $\{\alpha_i\}_{i \leq m}$ of subterms of $\alpha_0 = \psi_{\sigma_0}^{h_0}(a_0)$ such that $\alpha_m = \psi_\sigma^h(a)$, $\alpha_i = \psi_{\sigma_i}^{h_i}(a_i)$ for some σ_i, a_i, h_i, and for each $i < m$, $X \not\ni \alpha_{i+1} \in \mathcal{E}(C_i)$ for $C_i = \{\sigma_i, a_i\} \cup SC(h_i)$.

Let $\delta \in SC(f) \cup \{\kappa, a\}$ such that $b \leq \gamma$ for a $\gamma \in K_{\beta_1}(\delta)$. Pick an $\alpha_2 = \psi_{\sigma_2}^{h_2}(a_2) \in \mathcal{E}(\delta)$ such that $\gamma \in K_{\beta_1}(\alpha_2)$, and an $\alpha_m = \psi_{\sigma_m}^{h_m}(a_m) \in k_{\beta_1}(\alpha_2)$ for some σ_m, h_m and $a_m \geq b \geq a$. We have $\alpha_2 \in \mathcal{W}_{N+1}$ by $\delta \in \mathcal{W}_{N+1}$. If $\alpha_2 < \alpha_1$, then $\beta_1 \leq \alpha_2 \in \mathcal{W}_{N+1} \cap \alpha_1$, and we are done. Assume $\alpha_2 \geq \alpha_1$, i.e., $\alpha_2 \notin \alpha_1$. Then $a_2 \in K_{\alpha_1}(\alpha_2) < a \leq b$, and $m > 2$.

Let $\{\alpha_i\}_{2 \leq i \leq m}$ be the sequence of subterms of α_2 such that $\alpha_i = \psi_{\sigma_i}^{h_i}(a_i)$ for some σ_i, a_i, h_i, and for each $i < m$, $\beta_1 \leq \alpha_{i+1} \in \mathcal{E}(C_i)$ for $C_i = \{\sigma_i, a_i\} \cup SC_{\mathbb{I}}(h_i)$.

Let $\{n_j\}_{0 \leq j \leq k}$ $(0 < k \leq m-2)$ be the increasing sequence $n_0 < n_1 < \cdots < n_k \leq m$ defined recursively by $n_0 = 2$, and assuming n_j has been defined so that $n_j < m$ and $\alpha_{n_j} \geq \alpha_1$, n_{j+1} is defined by $n_{j+1} = \min(\{i : n_j < i < m : \alpha_i < \alpha_{n_j}\} \cup \{m\})$. If either $n_j = m$ or $\alpha_{n_j} < \alpha_1$, then $k = j$ and n_{j+1} is undefined. Then we claim that

$$\forall j \leq k(\alpha_{n_j} \in \mathcal{W}_{N+1}) \,\&\, \alpha_{n_k} < \alpha_1 \tag{61}$$

Proof of (61). By induction on $j \leq k$ we show first that $\forall j \leq k(\alpha_{n_j} \in \mathcal{W}_{N+1})$. We have $\alpha_{n_0} = \alpha_2 \in \mathcal{W}_{N+1}$. Assume $\alpha_{n_j} \in \mathcal{W}_{N+1}$ and $j < k$.

Then $n_j < m$, i.e., $\alpha_{n_{j+1}} < \alpha_{n_j}$, and by $\alpha_{n_j} \in C^{\alpha_{n_j}}(\mathcal{W})$, we have $C_{n_j} \subset C^{\alpha_{n_j}}(\mathcal{W}_{N+1})$, and hence $\alpha_{n_j+1} \in \mathcal{E}(C_{n_j}) \subset C^{\alpha_{n_j}}(\mathcal{W}_{N+1})$. We see inductively that $\alpha_i \in C^{\alpha_{n_j}}(\mathcal{W}_{N+1})$ for any i with $n_j \leq i \leq n_{j+1}$. Therefore $\alpha_{n_{j+1}} \in C^{\alpha_{n_j}}(\mathcal{W}_{N+1}) \cap \alpha_{n_j} \subset \mathcal{W}_{N+1}$ by Propositions 7.2.1 and 7.11.

Next we show that $\alpha_{n_k} < \alpha_1$. We can assume that $n_k = m$. This means that $\forall i(n_{k-1} \leq i < m \Rightarrow \alpha_i \geq \alpha_{n_{k-1}})$. We have $\alpha_2 = \alpha_{n_0} > \alpha_{n_1} > \cdots > \alpha_{n_{k-1}} \geq \alpha_1$, and $\forall i < m(\alpha_i \geq \alpha_1)$. Therefore $\alpha_m \in k_{\alpha_1}(\alpha_2) \subset k_{\alpha_1}(\{\kappa, a\} \cup SC(h))$, i.e., $a_m \in K_{\alpha_1}(\{\kappa, a\} \cup SC(h))$ for $\alpha_m = \psi_{\sigma_m}^{h_m}(a_m)$. On the other hand we have $K_{\alpha_1}(\{\kappa, a\} \cup SC(h)) < a$ for $\alpha_1 = \psi_{\sigma}^{h}(a)$. Thus $a \leq a_m < a$, a contradiction.

(61) is shown, and we obtain $\beta_1 \leq \alpha_{n_k} \in \mathcal{W}_{N+1} \cap \alpha_1$.

This completes a proof of Claim 7.37 and of the lemma. □

Corollary 7.38 *For* each $\zeta \in \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1})$*,* $\mathrm{MIH}(\zeta; \mathcal{W}_{N+1})$ *holds.*

Proof. For each $n < \omega$, we have $\mathrm{TI}[\mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap \omega_n(\mathbb{I}_N + 1)]$ by Lemma 7.13.2. We show $\mathrm{MIH}(\zeta; \mathcal{W}_{N+1})$ by induction on $\zeta \in \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1})$. Assume $\forall \xi \in \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap \zeta\, \mathrm{MIH}(\xi; \mathcal{W}_{N+1})$.

Let $\mathbb{S} = \mathbb{T}^{\dagger \vec{k}}$ with $\mathbb{T} \in \mathcal{W}_{N+1}$, and $\gamma = \psi_{\kappa}^{f}(a) \in L(\mathbb{S})$ be such that $k(\gamma) = \{\kappa, a\} \cup SC(f) \subset \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1})$ and $\mathtt{h}(\gamma) = \{a, \mathtt{g}_0^*(\gamma)\} \subset \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap \zeta$. We obtain $\mathrm{MIH}(\xi; \mathcal{W}_{N+1})$ by IH for $\xi = \max\{a, \mathtt{g}_0^*(\gamma)\}$.

We obtain $\gamma \in \mathcal{W}_{N+1}$ by Lemma 7.36 and $\mathrm{MIH}(\xi; \mathcal{W}_{N+1})$ with subsidiary induction on $a \in \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap (\xi + 1)$ and sub-subsidiary induction on $f \in J$. Then Corollary 7.34 yields $N(\gamma) \subset \mathcal{W}_{N+1}$.

Here by induction on $f \in J$ we mean by induction along $g <_{lx}^{0} f$. In the proof of Lemma 7.36, $\mathrm{SSIH}(\zeta, a, f)$ is invoked in **Case 4**, i.e., only when $\psi_{\kappa}^{g}(a) < \psi_{\kappa}^{f}(a)$ with $\kappa < \mathbb{I}_N$. Then Lemma 6.3 yields $o_{\mathbb{I}_N}(g) < o_{\mathbb{I}_N}(f) \in \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1})$ for $SC(f) \subset \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap \Lambda$, where $\Lambda = \psi_{\mathbb{I}_N}(b)$ and $b = \mathtt{g}_0^*(\psi_{\kappa}^{f}(a)) \geq \mathtt{p}_0(\psi_{\kappa}^{f}(a))$. Hence $o_{\Lambda}(g) < o_{\Lambda}(f) \in \mathcal{W}_{N+1}$ by $\Lambda \in \mathcal{W}_{N+1}$. □

Lemma 7.39 *For* each $n < \omega$*, the following holds:*

If one of the followings holds, then $\alpha \in \mathcal{W}_{N+1}$ *for* $\alpha \in OT(\mathbb{I}_N)$*.*

1. $\alpha = \mathbb{S}^{\dagger k}$ *with* $\mathbb{S} \in \mathcal{W}_{N+1} \cap (St_k \cup \{\Omega\})$*.*
2. $\alpha = \psi_{\mathbb{I}_N}(a)$ *with* $a \in C^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap \omega_n(\mathbb{I}_N + 1)$*.*
3. $\alpha = \psi_{\kappa}^{f}(a) \in L(\mathbb{S})$ *for* $\mathbb{S} = \mathbb{T}^{\dagger k}$ *with* $\mathbb{T} \in \mathcal{W}_{N+1}$ *and* $k(\alpha) \cup \mathtt{h}(\alpha) = \{\kappa, a, \mathtt{g}_0^*(\alpha)\} \cup SC(f) \subset \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap \omega_n(\mathbb{I}_N + 1)$*.*
4. $\alpha \in N(\rho)$ *for* $\rho \in \mathcal{W}_{N+1} \cap L(\mathbb{S})$ *with* $\mathbb{S} = \mathbb{T}^{\dagger k}$ *such that* $\mathbb{T} \in \mathcal{W}_{N+1}$ *and* $\mathtt{g}_0^*(\rho) < \omega_n(\mathbb{I}_N + 1)$*.*

Proof. 7.39.1 is seen from Lemma 7.20.
7.39.2 follows from Lemma 7.28.
7.39.3 follows from Lemma 7.36 and Corollary 7.38.
7.39.4 follows from Corollaries 7.34 and 7.38. □

Let us conclude Theorem 1.2. For each $\alpha \in OT(\mathbb{I}_N)$, $\alpha \in \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1})$ is seen by metainduction on the lengths $\ell\alpha$ using Propositions 7.12, 7.21 and Lemma 7.39. Note that $\ell(\mathsf{g}_0^*(\psi_\kappa^f(a))) < \ell(\psi_\kappa^f(a))$ and $\ell(\mathbb{T}) < \ell(\rho)$ for $\rho \in L(\mathbb{S})$ and $\mathbb{S} = \mathbb{T}^{\dagger k}$. Therefore we obtain Σ^1_{N+2}-DC+BI $\vdash \alpha \in \mathcal{C}^{\mathbb{I}_N}(\mathcal{W}_{N+1}) \cap \Omega = \mathcal{W}_{N+1} \cap \Omega = W(\mathcal{C}^0(\mathcal{W}_{N+1})) \cap \Omega = W(OT(\mathbb{I}_N)) \cap \Omega$, and Σ^1_{N+2}-DC+BI $\vdash Wo[\alpha]$ for each $\alpha < \psi_\Omega(\varepsilon_{\mathbb{I}_N+1})$.

8 Outcomes on $\mathbf{Z}_2$

In this final section let us conclude some standard outcomes of an ordinal analysis of the theory $\mathbf{Z}_2$.

Let $\mathrm{TI}[\Pi^{1-}_0, \psi_\Omega(\varepsilon_{\mathbb{I}_N+1})]$ denote a schema of transfinite induction $\forall \alpha \in OT(\mathbb{I}_N) \cap \Omega\, (\mathrm{Prg}[OT(\mathbb{I}_N), A] \to OT(\mathbb{I}_N) \cap \alpha \subset A)$ up to $\psi_\Omega(\varepsilon_{\mathbb{I}_N+1})$ in $OT(\mathbb{I}_N)$ applied to arithmetic formulas $A \in \Pi^{1-}_0$ in the language of the first-order arithmetic PA. Let $T_0 = \mathsf{PA} + \bigcup\{\mathrm{TI}[\Pi^{1-}_0, \psi_\Omega(\varepsilon_{\mathbb{I}_N+1})] : N < \omega\}$, and $T_1 = \mathrm{FiX}^i(T_0)$ denote the intuitionistic fixed point theory over T_0. The language of the theory T_1 is expanded by unary predicate symbols I for each operator $\Phi(X, x)$, in which every occurrence of a unary predicate symbol X is strictly positive. The axioms in T_1 are obtained from T_0 by adding the axioms $\forall x[I(x) \leftrightarrow \Phi(I, x)]$ for a fixed point I. The axiom schema $\mathrm{TI}[\Pi^{1-}_0, \psi_\Omega(\varepsilon_{\mathbb{I}_N+1})]$ of transfinite induction as well as schema of complete induction may be applied to arbitrary first-order formulas in the expanded language with the predicates I. The underlying logic in T_1 is the intuitionistic first-order logic with the axiom $\forall x, y(x = y \to I(x) \to I(y))$. The excluded middle $\forall x(\neg I(x) \lor I(x))$ for the predicate I is not available in T_1.

Lemma 8.1 $\mathrm{FiX}^i(T_0)$ *is a conservative extension of* T_0*. Moreover the fact is provable in the fragment* $I\Sigma^0_1$ *of the first-order arithmetic:* $I\Sigma^0_1 \vdash \mathrm{Pr}_{T_1}(\lceil\varphi\rceil) \to \mathrm{Pr}_{T_0}(\lceil\varphi\rceil)$*, where* $\mathrm{Pr}_T(x)$ *is a standard provability predicate for a theory* T*.*

Proof. The fact is seen as in [1, 3]. To formalize a proof of the fact in $I\Sigma^0_1$, follow a finitary analysis in section 4.4 of [3]. □

Theorem 8.2 $\mathbf{Z}_2$ *is a conservative extension of* $\mathsf{PA} + \bigcup\{\mathrm{TI}[\Pi^{1-}_0, \psi_\Omega(\varepsilon_{\mathbb{I}_N+1})] : N < \omega\}$*. Moreover the fact is provable in the fragment* $I\Sigma^0_1$*.*

Proof. Assume that $\mathbf{Z}_2 \vdash A$ for an arithmetic sentence $A \in \Pi^{1-}_0$. Pick an $N < \omega$ such that Σ^1_{N+2}-DC+BI $\vdash A$. By Lemma 2.3 we obtain $\mathsf{KP}\omega+\Pi_N$-Collection$+(V = L) \vdash A^{set}$, and hence $\mathsf{KP}\omega + \Pi_N$-Collection $\vdash A^{set}$. Then by Lemma 2.5 we obtain $S_{\mathbb{I}_N} \vdash A^{set}$.

Now we see that the proof of Theorem 1.1 in sections 4 and 5 is formalizable in the intuitionistic fixed point theory $T_1 = \mathrm{FiX}^i(T_0)$ over T_0. Let us regard each of the relations $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_{c,\gamma_0} \Gamma; \Pi^{\{\cdot\}}$ and $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_{c,d,e,\beta,\gamma_0} \Gamma$ as a fixed point of a strictly positive operator. Then by applying transfinite induction to first-order formulas with the fixed point predicates, Theorem 1.1 is proved. Therefore we obtain $\mathrm{FiX}^i(T_0) \vdash A$, and $T_0 \vdash A$ by Lemma 8.1. □

We see readily that the transfinite induction $\mathrm{TI}(\psi_\Omega(\mathbb{I}_\omega))$ up to $\psi_\Omega(\mathbb{I}_\omega)$ is equivalent to the Π^1_1-soundness $\mathrm{RFN}_{\Pi^1_1}(\mathbf{Z}_2)$ of $\mathbf{Z}_2$ over RCA_0, where $\mathrm{TI}(\psi_\Omega(\mathbb{I}_\omega))$ denotes a Π^1_1-sentence $\forall N \forall \alpha \in OT(\mathbb{I}_N) \cap \Omega \forall Y\, (\mathrm{Prg}[OT(\mathbb{I}_N), Y] \to OT(\mathbb{I}_N) \cap \alpha \subset Y)$.

Definition 8.3 Let $\alpha \in OT(\mathbb{I}_N)$ be an ordinal term.

1. DS_α denotes a Π^0_2-sentence saying that 'there is no primitive recursive and descending sequence $\{f(n)\}_n$ of ordinals with $f(0) < \alpha$'. This means that $f(0) < \alpha \Rightarrow \exists n(f(n+1) \not< f(n))$.

2. WDS_α denotes a Π^0_3-sentence saying that 'for every primitive recursive and weakly descending sequence $\{f(n)\}_n$ of ordinals with $f(0) < \alpha$, there exists an n such that $\forall m \geq n(f(m) = f(n))$'. This is equivalent to the principle that 'for every primitive recursive sequence $\{f(n)\}_n$ of ordinals, there exists an n such that $\forall m(f(n) \leq f(m))$.

3. $DS_{\psi_\Omega(\varepsilon_{\mathbb{I}_N+1})} :\Leftrightarrow \forall \alpha \in OT(\mathbb{I}_N) \cap \Omega\, DS_\alpha$ and $DS_{\psi_\Omega(\mathbb{I}_\omega)} :\Leftrightarrow \forall N > 0\, DS_{\psi_\Omega(\varepsilon_{\mathbb{I}_N+1})}$. Also $WDS_{\psi_\Omega(\varepsilon_{\mathbb{I}_N+1})} :\Leftrightarrow \forall \alpha \in OT(\mathbb{I}_N) \cap \Omega\, WDS_\alpha$ and $WDS_{\psi_\Omega(\mathbb{I}_\omega)} :\Leftrightarrow \forall N > 0\, WDS_{\psi_\Omega(\varepsilon_{\mathbb{I}_N+1})}$.

4. A computable (total) function f on integers is said to be $\psi_\Omega(\varepsilon_{\mathbb{I}_N+1})$-*recursive* if f is defined from α-recursive functions g, r, h by $\psi_\Omega(\varepsilon_{\mathbb{I}_N+1})$-recursion:
$$f(y,x) = \begin{cases} g(y,x,f(y,r(y,x))) & \text{if } r(y,x) < x < \Omega \text{ in } OT(\mathbb{I}_N) \\ h(y,x) & \text{otherwise} \end{cases}$$

5. $\mathrm{RFN}_{\Sigma^0_n}(\mathbf{Z}_2)$ denotes the uniform reflection principle of $\mathbf{Z}_2$ for Σ^0_n-formulas.

Corollary 8.4
1. *The 2-consistency $\mathrm{RFN}_{\Sigma^0_2}(\mathbf{Z}_2)$ of $\mathbf{Z}_2$ is equivalent to $WDS_{\psi_\Omega(\mathbb{I}_\omega)}$ over $I\Sigma^0_1$.*

2. *$\mathbf{Z}_2$ is Π^0_3-conservative over $I\Sigma^0_1 + \{WDS_{\psi_\Omega(\varepsilon_{\mathbb{I}_N+1})} : 0 < N < \omega\}$.*

3. *The 1-consistency $\mathrm{RFN}_{\Sigma^0_1}(\mathbf{Z}_2)$ of $\mathbf{Z}_2$ is equivalent to $DS_{\psi_\Omega(\mathbb{I}_\omega)}$ over $I\Sigma^0_1$.*

4. *$\mathbf{Z}_2$ is Π^0_2-conservative over $I\Sigma^0_1 + \{DS_{\psi_\Omega(\varepsilon_{\mathbb{I}_N+1})} : 0 < N < \omega\}$.*

5. *For computable total function f on $\mathbb{N}$, f is provably computable in $\mathbf{Z}_2$ iff f is $\psi_\Omega(\varepsilon_{\mathbb{I}_N+1})$-recursive for an $N < \omega$.*

Proof. Each follows from Theorem 8.2 as in chapter 4 of [3]. □

For the consistency $\mathrm{Con}(\mathbf{Z}_2)$ of $\mathbf{Z}_2$ we obtain the following.

Corollary 8.5 *There are primitive recursive predicate B and primitive recursive function f such that both of $\forall N > 0 \forall \alpha \in OT(\mathbb{I}_N) \cap \Omega(f(N,\alpha) < \alpha \to B(N, f(N,\alpha)) \to B(N,\alpha))$ and $\forall N > 0 \forall \alpha \in OT(\mathbb{I}_N) \cap \Omega\, B(N,\alpha) \to \mathrm{Con}(\mathbf{Z}_2)$ is provable in $I\Sigma^0_1$.*

Proof. This is seen from Theorem 8.2 as in section 4.3 of [3]. □

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