The Possibility of Predicative Arithmetic

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0 introduction

0.1 Contents

- Summery of Nelson's Predicative Arithmetic
- The Connection between Predicative Arithmetic and Wittgenstein

Today's purpose is to suggest an interest of Predicative Arithmetic which Nelson himself and others seem to overlook. I want to claim that Predicative Arithmetic can be seen as a formal model of Wittgensitein's conception of proof, and can give an answer to the so-called "paradox of inference" in the case of arithmetic.

0.2 What is Predicative Arithmetic?

- an alternative to classical arithmetic
- weaker than HA, or PRA
- without any induction axiom

1 Summery of Nelson's Predicative Arithmetic

1.1 Nelson's philosophical motivation

Nelson criticizes the validity as an axiom (or axioms) of mathematical induction.

In his essays, there are several lines of arguments to this conclusion. The clearest is the argument from impredicativity.

1.1.1 The argument from impredicativity

a possible justification for induction:

The concept of natural numbers are *defined* as the totality about which mathematical induction can be validly carried out. So, mathematical induction is a kind of definitional truth.

ex.)Frege's construction of the set of natural numbers.

Nelson's criticism[9] :

The above definition contains circularity, so that is inappropriate.

> The reason for mistrusting the induction principle is that it involves an impredicative concept of number. It is not correct to argue that induction only involves the numbers from 0 to n; the property of n being established may be a formula with bound variables that are thought of as ranging over all numbers. That is, the induction principle assumes that the natural number system is given. A number is conceived to

be an object satisfying every inductive formula; for a particular inductive formula, therefore, the bound variables are conceived to range over objects satisfying every inductive formula, including the one in question.[ch.1]

1.1.2 To Predicative Arithmetic

Based on that argument, Nelson thinks it is incorrect to presuppose the totality about which induction can be carried out.

He can't use induction axioms, so he founds his mathematics on Robinson's Arithmetic Q, which has no induction axiom.

1.2 Development of Predicative Arithmetic

1.2.1 The basic idea

Q is too weak in itself. So, when we want to prove some formula $\forall x A(x)$, we takes up a new theory $Q[\forall x A(x)]$, and justify this theory by interpreting it in Q.

1.2.2 The starting point

In the following, we assume classical logic. Our language has the following nonlogical symbols: 0, S, P,+, $\times$, $\leq$.

definition 1 Q is a theory whose axioms are Ax.0-7. Q' is a theory whose axioms are Ax.0-12.

- Ax.0 $Sx \neq 0$
- Ax.1 $Sx = Sy \rightarrow x = y$
- Ax.2 $x + Sy = S(x + y)$
- Ax.3 $x + 0 = x$
- Ax.4 $x \times Sy = x \times y + x$
- Ax.5 $x \times 0 = 0$
- Ax.6 $Px = y \leftrightarrow Sy = x \vee (x = 0 \wedge y = 0)$
- Ax.7 $x \leq y \leftrightarrow \exists z x + z = y$
- Ax.8 $(x + y) + z = x + (y + z)$
- Ax.9 $x \times (y + z) = x \times y + x \times z$
- Ax.10 $(x \times y) \times z = x \times (y \times z)$
- Ax.11 $x + y = y + x$
- Ax.12 $x \times y = y \times x$

Q doesn't have any induction axiom or any axiom justified by induction, so we can accept it on predicative grounds. Q' is interpretable in Q, so we can use it as a starting point for Predicative Arithmetic.

1.2.3 Induction by Relativizasion

Define $F(x)$ by $F(x) \equiv \exists z(2 \times z = x \times Sx)$.

$\forall x F(x)$ seems a very basic truth about numbers, but we can't prove $\forall x F(x)$ in Q. Only $F(0)$ and $\forall x(F(x) \rightarrow F(Sx))$ is provable.

So, instead of using an induction axiom, we try to interpret $Q'[\forall x F(x)]$ in Q' to accept $\forall x F(x)$.

definition 2:Relativization for an unary formula C(x) and a formula A, we define $A_{C(x)}$ as a forumla that can be obtained by replacing in A all the occurances of quantifiers$\forall x B(x), \exists x B(x)$by $\forall x(C(x) \rightarrow B(x)), \exists x(C(x) \wedge B(x))$.

In order to interpret $Q'[\forall x F(x)]$, our first idea may be to relativize the domain of Q' to $F(x)$, but it is unclear whether $F^3(x)$ is closed under the functions $S, P, +, \times$, and whether axioms of Q' holds even under that relativizaion (especially, Ax.7 is not open). So we introduce Relativization Scheme.

definition 3:Relativization Scheme for an unary formula $C(x)$, $C^1(x), C^2(x), C^3(x)$ are defined in the following way.

- $C^1(x) \equiv \forall y(y \leq x \to C(y))$
- $C^2(x) \equiv \forall y(C^1(y) \to C^1(y+x))$
- $C^3(x) \equiv \forall y(C^2(y) \to C^2(y \times x))$

theorem 1 if $Q' \vdash C(0) \land \forall x(C(x) \to C(Sx))$, then the following are theorems in Q'

- R0) $C^3(x) \to C(x)$
- R1) $(C^3(x) \land y \leq x) \to C^3(y)$
- R2) $C^3(0) \land \forall x(C^3(x) \to C^3(Sx))$
- R3) $C^3(x) \to C^3(Px)$
- R4) $(C^3(x) \land C^3(y)) \to (C^3(x+y) \land C^3(x \times y))$
- R5) $(C^3(x) \land C^3(y)) \to (x \leq y \leftrightarrow \exists z(C^3(z) \land x + z = y))$

It is clear from the theorem 1 that $F^3(x)$ contains 0, and is closed under $S, P, +, \times$, and that axioms of Q' holds even when we relativize the domain to $F^3(x)$. Also, $\forall x F(x)$ holds under that relativization.

theorem 2 $Q' \vdash F^3(x) \to F(x)_{F^3(x)}$

Proof $F(x)_{F^3(x)} \equiv \exists z(F^3(z) \land 2 \times z = x \times Sx)$. Assume $F^3(x)$. Then from R0, $F(x)$. In other words, there exists z such that $2 \times z = x \times Sx$. From $F^3(x)$, R2, R4, we can see $F^3(Sx)$,and $F^3(x \times Sx)$. From this, R1, and $z \leq x \times Sx$, $F^3(z)$. So, there exists z such that $(F^3(z) \land 2 \times z = x \times Sx)$. in other words, $F(x)_{F^3(x)}$. Q.E.D.

Apply universal generalisation to this, we get the relativization by $F^3(x)$ of $\forall x F(x)$. So, all the axioms of $Q'[\forall x F(x)]$ hold under that relativization.

definition 4 $C(freeA)$ is the conjunction of all the $C(v)$, for any variable v occuring freely in A.

theorem 3: Interpretablity for any formula A, $Q'[\forall x F(x)] \vdash A \Rightarrow Q' \vdash F^3(freeA) \to A_{F^3(x)}$

In this way, $Q'[\forall x F(x)]$is interpretable in Q'. So we can accept $\forall x F(x)$ on predicative grounds.

We can generalise the process of accepting $\forall x F(x)$ in this way. Assume the current theory is T;

1. There is an inductive formula $C(x)$.
2. We strengthen $C(x)$ into $C^3(x)$.
3. For any axiom A in $T[\forall x C(x)]$, we prove $T \vdash C^3(freeA) \to A_{C^3(x)}$. Whether this is possible depends on the form of $C(x)$
4. We interpret $T[\forall x C(x)]$ in T by relativization to $C(x)$.
5. We replace T by $T[\forall x C(x)]$, and accept $\forall x C(x)$.

According to Nelson[9], this process can be seen as a refinement of the concept of numbers. We refine the original concept of number to the new concept $F^3(x)$, which enables us to accept $\forall x F(x)$ as a truth about all numbers.

> Let C be an inductive formula; our intuitive feeling is that if x is a number, then C[x] should hold. Now the formula C^3 respects all of the function symbols of Q'_1[our Q'] and the defining axiom of $\leq$, by

REL[our theorem 1]. All of the other nonlogical axioms of Q_1' are open, so it automatically respects them all as well. In other words, the entire theory Q_1' can be relativized by C^3. We can replace our concept of number (any x) by a more refined concept of number (any x such that $C^3[x]$). We can read $C^3[x]$ as "x is a number" (leaving open the possiblity of formalizing an even more refined concept of number at some time in the future).[ch.5]

In summery, in Predicative Arithmetic mathematical induction is not an axiom, but a process of concept refinement. When we want to prove $\forall x A(x)$, we refine our concept of numbers to $A^3(x)$, and strengthen our theory. Newer theories should always be interpretable in older ones, especially in Q, and whether it is possible depends on the exact form of $A(x)$. If it isn't possible, we can't understand the new concept of number in an predicative (or non-circular) way.

The principles we can accept in this way;

1. induction for bounded formulae
2. tautalogical consistency of Q

The principles we can't accept in this way;

1. totality of exponentiation
2. consistency of Q

2 Predicative Arithmetic and Wittensteinan proof

2.1 The paradox of inference

"paradox of inference"[4]
how to explain both of the two aspects, validity and utility, of logical inference? They seem to compete with each other.

- For validity, there should be no conceptual distance between premises and the conclusion.
- For utility, there should be some conceptual distance between premises and the conclusion.

The paradox of inference is the problem of reconciling these two competing aspects of inference.
Apply this to mathematical proof.

- For validity of proof, there should be no conceptual distance between axioms and the theorem.
- For utility of proof, there should be some conceptual distance between axioms and the theorem.

Let me call this application of the paradox "paradox of proof".
We can think Wittgenstein conception of proof as an answer to this problem.

2.2 Wittegensteinan proof: proof as concept-formation

According to some interpretations (especially by Dummett and Wright), Wittgenstein thinks proofs are transformation of our concept and introduce new criteria for application of the concept, not expressed by axioms.

> When I said that a proof introduces a new concept, I meant something like: the proof puts a new paradigm among the paradigms of the language; like when someone mixes a special reddish blue, somehow settles the special mixture of the colours and gives it a name. But even if we are inclined to call a proof such a new paradigm[it gives a new criterion for application] – what is the exact similarity of the proof to

such a concept model? One would like to say: the proof changes the grammar of our language, changes our concepts. It makes new connexions, and it creates the concept of these connexions. (It does not establish that they are there; they do not exist until it makes them.)([19],III,S31)

The idea that proof creates a new concept might also be roughly put as follows:a proof is not its foundations[axioms?] plus the rules of inference, but a *new* building – although it is an example of such and such a style. a proof is a *new* paradigm. The concept which the proof creates may for example be a new concept of inference, a new concept of correct inferring. ([19],III,S41)

Dummett criticizes :

- Based on this conception, we can explain the utility of proof. Proof gives us something completely new; new applications of the concept.
- We can't explain the validity. Proofs are themselves a type of application of concepts. If proofs change the criterion of application freely, doesn't the norm of proof evaporates completely? Isn't the result "anything goes"?

2.3 Predicative Arithmetic as an concept formation

The treatment of mathematical induction in Predicative Arithmetic can be viewed as a formal model for Wittgenstein's conception of proof, and importantly, in a way that avoids the Dummettian criticism.

表 1 Predicative Arithmetic and Wittgenstein's conception

Wit's conception of proof	Mathematical Induction in PredA
concept-transformation	refinement of the concept of numbers
new criteria of application	adding new axioms
how to explain the validity of proof?	interpretablitity in older theories

In these points, Predicative Arithmetic clearly embodies Wittgenstein's idea.
In Predicative Arithmetic, the validity of Mathematical Induction as concept-refinement is judged by the interpretablity in older theories (if a new theory isn't interpretable in the older one, we can't understand the new concept of numbers in a non-circular way). So, Dummettian criticism doesn't apply.
So, Predicative Arithmetic

- can give a formal model for Wittgenstein's conception
- can rebut Dummett's criticism on Wittgenstein by giving a Wittgensteinian model that avoids his criticism.
- can give a model for reconciling the two aspects of mathematical proof. It explains the utility of proof by new criteria of application, and the validity of proof by interpretablity.

Reservations: if my opinion that Predicative Arithmetic can answer the paradox of proof is correct, this answer has only the limited application, because of the weakness of this theory.

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