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Dualities for Algebras of Fitting's Many-Valued Modal Logics

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Abstract. Stone-type duality connects logic, algebra, and topology in both conceptual and technical senses. This paper is intended to be a demonstration of this slogan. In this paper we focus on some versions of Fitting's L-valued logic and L-valued modal logic for a finite distributive lattice L. Using the theory of natural dualities, we first obtain a duality for algebras of L-valued logic. Based on this duality, we develop a Jónsson-Tarski-style duality for algebras of L-valued modal logic, which encompasses Jónsson-Tarski duality for modal algebras as the case $L = 2$. We also discuss how the dualities change when the algebras are enriched by truth constants. Topological perspectives following from the dualities provide compactness theorems for the logics and the effective classification of categories of algebras involved, which tells us that Stone-type duality makes it possible to use topology for logic and algebra in significant ways.

Keywords: Stone-type duality; the theory of natural dualities; algebraic logic; Fitting's many-valued modal logic; compactness theorem; classification of categories

1. Introduction

Our slogan is that Stone-type duality connects logic, algebra, and topology in both conceptual and technical senses (note that algebraic or topological structures arising from logics are more or less different from those discussed in ordinary mathematics). This paper is intended to be a demonstration of the above

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slogan and focuses on topological dualities for algebras of many-valued and modal logics with applications of such dualities. From a conceptual point of view, a Stone-type duality connects a logic and the corresponding algebraic structures via the Lindenbaum-algebra construction, and then connects the algebraic structures and the corresponding topological structures via a dual equivalence between the category of the algebraic structures and the category of the topological structures, where usually the topological structures are closely related to the notion of model for the logic and are obtained from the algebraic structures via the spectrum construction (usually the spectrum of an algebra is the space of prime filters or its relatives with possible additional structures and may be seen as the space of models, since models often correspond to prime filters or its relatives). Roughly speaking, the topological structures represent the semantical aspect of the logic, while the algebraic structures represent the syntactical aspect of the logic. For this reason, Stone-type duality is sometimes said to be duality between syntax and semantics. A Stone-type duality also gives technically significant interactions between a logic, the corresponding algebraic and topological structures, which will be explained later in this section.

Fitting [20] introduced L-valued logic and L-valued modal logics for a finite distributive lattice L, where all the elements of L are encoded as truth constants in the languages of Fitting's logics. L-valued modal logics are particularly important for the study of reasoning about knowledge and belief (see [22, 24, 29]). Fitting's logics have been studied from different perspectives. Proof-theoretic aspects are studied in the papers [20, 21, 23]. Model-theoretic aspects are studied in the papers [13, 25, 26]. However, there seems to have been little research on algebraic aspects of them.

Although [13] mentions the difficulty of developing algebraic semantics for Fitting's logics, [27] provides algebraic semantics for Fitting's logics modified by replacing truth constants (except 0 and 1) with unary connectives T_a's for all $a \in L$, where $\mathrm{T}_a(x)$ intuitively means that the truth value of a proposition x is an element a of L (see Definition 2.1). For motivations of the modification, see [27, Section 1]. We remark that Fitting's L-valued logic modified in the above manner can be considered not only as a many-valued logic but also as a modal logic, which is discussed in Section 4. The class of L-**VL**-algebras introduced in [27] is the algebraic counterpart of Fitting's L-valued logic modified in the above manner. The class of L-**ML**-algebras introduced in [27] is the algebraic counterpart of Fitting's L-valued modal logic modified in the above manner.

In this paper, we develop topological dualities for L-**VL**-algebras and L-**ML**-algebras, following Stone's maxim: "One must always topologize" (see [32]). However, why must one always topologize? Some of our answers to this question are as follows.

From a viewpoint of mathematical logic, we give two reasons related to results of this paper: (1) In general, one of the advantages of a topological duality for algebras of a logic is that we can understand the geometric meanings of the logic and its properties. For example, by a topological duality for algebras of a logic, we often notice that the compactness of the logic can be represented by the compactness of the dual space of the Lindenbaum algebra of the logic, which holds true of dualities developed in this paper. Indeed, by using the dualities in this paper, we can actually show compactness theorems for L-valued logic and L-valued modal logic (see Theorem 2.4 and Theorem 3.5). (2) Another advantage is that a duality for algebras of a logic often leads us to a completeness result for the logic, which holds true of dualities developed in this paper, and therefore topological duality contributes to study of completeness, which is one of the most important properties associated with a logic.

From an algebraic point of view, topological dualities make it possible to solve purely algebraic problems more easily by topological methods than by algebraic methods. In this paper, we can indeed obtain the classification of categories of L-**VL**-algebras for finite distributive lattices L by considering the dual topological categories of those algebraic categories via topological dualities for L-**VL**-algebras (see Theorem 2.5). This result implies that it is effectively decidable whether or not the category of L_1-**VL**-algebras is equivalent to the category of L_2-**VL**-algebras for finite distributive lattices L_1 and L_2. Note that in general it is not effectively decidable whether or not two categories are equivalent.

From a viewpoint of computer science, topological dualities provide foundations of the semantics of programming languages (see, e.g., [1, 5, 4]). [1] uses topological dualities to clarify the relationships between denotational semantics and program logics. [5] gives an application of Jónsson-Tarski duality for modal algebras (see [3, 10, 15]) to program semantics.

Finally, we remark that topological dualities play fundamental roles also in other areas of mathematics, such as algebraic geometry, functional analysis and non-commutative geometry (see [14, 11, 9]). In fact, some of them are closely related to Stone-type dualities. For example, the duality between commutative rings and affine schemes can be considered as a generalization of Stone duality for Boolean algebras.

In the following two paragraphs, we briefly explain how to develop dualities for L-**VL**-algebras and for L-**ML**-algebras.

In order to develop a duality for L-**VL**-algebras, we use the theory of natural dualities (see [8]), which is a general theory of a duality for the quasi-variety generated by a finite algebra and therefore may be considered as a general theory of a duality for algebras of M-valued logic for a finite algebra M. Since a finite distributive lattice L equipped with T_a's for all $a \in L$ forms a semi-primal algebra (see Definition 2.9), it follows from the theory of natural dualities that a strong duality holds for L-**VL**-algebras (see Theorem 2.2), which confirms the applicability of the theory of natural dualities. By letting L be the two-element Boolean algebra, we can recover Stone duality for Boolean algebras from the duality for L-**VL**-algebras.

Based on the duality for L-**VL**-algebras, we develop a Jónsson-Tarski-style duality for L-**ML**-algebras (see Theorem 3.3), where "Jónsson-Tarski-style" means that, as in Jónsson-Tarski duality for modal algebras (see [3, 10, 15]), we equip the dual space of an L-**ML**-algebra with a certain binary relation (see Definition 3.4), by which we can define a modal operator $\Box$ on the double dual of an L-**ML**-algebra. The Jónsson-Tarski-style duality for L-**ML**-algebras is an L-valued version of Jónsson-Tarski duality for modal algebras. By letting L be the two-element Boolean algebra, we can recover Jónsson-Tarski duality for modal algebras from the duality for L-**ML**-algebras. We also consider how these dualities change when L-**VL**-algebra and L-**ML**-algebra are enriched by truth constants, based on the theory of natural dualities.

For a related result, we refer to a duality presented in [35]. Whereas the lattice of truth values is $\{0, 1/n, 2/n, ..., 1\}$ for $n \in \omega$ with $n \geq 1$ in [35], the lattice of truth values is an arbitrary (not necessarily totally ordered) finite distributive lattice in this paper. Hence, the duality developed in this paper seems to be more general than that in [35]. Note that [35] considers an Łukasiewicz-style many-valued modal logic, while a Fitting-style many-valued modal logic is considered in this paper. Moreover, [35] does not consider any of classification of categories, enrichment by constants, and compactness theorem. Many other dualities are known for algebras of many-valued and modal logics (see, e.g., [7, 12, 28, 30, 31, 34]).

According to the ideas in [27, Section 4], we could develop different dualities for L-**VL**-algebras and L-**ML**-algebras. The relationships between the dualities developed in this paper and the dualities suggested in [27] are discussed in Section 4. Roughly speaking, the modified L-valued logic is considered as a many-valued logic in the dualities developed in this paper, while it is considered as a modal logic in the dualities suggested in [27] by replacing T_a's with U_a's (see Proposition 2.5), where U_a's are inter-definable with T_a's ([27, Proposition 7]) and can be considered as modalities ([27, Proposition 8]). The modified L-valued logic is "schizophrenic" in this sense.

The paper is organized as follows. In Section 2, we consider a duality for L-**VL**-algebras and its applications. In Section 3, we consider a duality for L-**ML**-algebras and its applications. We remark that we use the notion of prime L-filter throughout the paper and our definition of primeness is different from the usual definition of primeness (i.e., $x \vee y \in F$ implies either $x \in F$ or $y \in F$), which does not work for developing duality theory for L-**VL**-algebras and L-**ML**-algebras.

2. A duality for L-VL-algebras and its applications

In this paper, L denotes a finite distributive lattice with the greatest element 1 and the least element 0, where we assume $0 \neq 1$. Then L forms a finite Heyting algebra. For $a, b \in L$, let $a \to b$ denote the pseudo-complement of a relative to b.

Let **2** denote the two-element Boolean algebra.

Definition 2.1. For all $a \in L$, we equip L with a unary operation $\mathrm{T}_a : L \to L$ defined by

$$\mathrm{T}_a(x) = \begin{cases} 1 & (\text{if } x = a) \\ 0 & (\text{if } x \neq a) \end{cases}$$

Then, L-valued logic L-**VL** is defined as follows. The connectives of L-**VL** are $\wedge$, $\vee$, $\to$, 0, 1 and T_a for each $a \in L$, where every T_a is a unary connective, 0 and 1 are nullary connectives, and the others are binary connectives. **PV** denotes the set of propositional variables. The formulas of L-**VL** are recursively defined in the usual way. **Form** denotes the set of formulas of L-**VL**.

Definition 2.2. A function $v : \mathbf{Form} \to L$ is an L-valuation on **Form** iff v satisfies the following:

1. $v(\mathrm{T}_a(x)) = \mathrm{T}_a(v(x))$;
2. $v(x \wedge y) = \inf(v(x), v(y))$;
3. $v(x \vee y) = \sup(v(x), v(y))$;
4. $v(x \to y) = v(x) \to v(y)$;
5. $v(a) = a$ for $a = 0, 1$.

Then, $x \in \mathbf{Form}$ is called valid in L-**VL** iff $v(x) = 1$ for any L-valuation v on **Form**.

We then define L-valued logic L-**VL** as the set of those $x \in \mathbf{Form}$ such that x is valid in L-**VL** in the above sense.

Definition 2.3. Let $X \subset \mathbf{Form}$. Then, X is satisfiable iff there is an L-valuation v on **Form** such that $v(x) = 1$ for any $x \in X$.

A compactness theorem for L-**VL** is proved in Theorem 2.4 below via a topological duality for L-**VL**-algebras, which are defined in the following subsection.

2.1. L-VL-algebras and their spectra

We now introduce L-**VL**-algebras, which provide a sound and complete semantics for L-valued logic L-**VL** as shown in [27]. Although some results in this subsection are contained in [27], they are explained again here for convenience.

Let $x \leq y$ denote $x \wedge y = x$ and $x \leftrightarrow y$ denote $(x \to y) \wedge (y \to x)$.

Definition 2.4. $(A, \wedge, \vee, \to, \mathrm{T}_a\ (a \in L), 0, 1)$ is an L-**VL**-algebra iff it satisfies the following for any $a, b \in L$ and any $x, y \in A$:

1. $(A, \wedge, \vee, \to, 0, 1)$ forms a Heyting algebra;
2. $\mathrm{T}_a(x) \wedge \mathrm{T}_b(y) \leq \mathrm{T}_{a\to b}(x \to y) \wedge \mathrm{T}_{a\wedge b}(x \wedge y) \wedge \mathrm{T}_{a\vee b}(x \vee y)$;
   $\mathrm{T}_b(x) \leq \mathrm{T}_{\mathrm{T}_a(b)}(\mathrm{T}_a(x))$;
3. $\mathrm{T}_0(0) = 1$; $\mathrm{T}_a(0) = 0$ for $a \neq 0$;
   $\mathrm{T}_1(1) = 1$; $\mathrm{T}_a(1) = 0$ for $a \neq 1$;
4. $\bigvee\{\mathrm{T}_a(x)\ ;\ a \in L\} = 1$;
   $\mathrm{T}_a(x) \wedge \mathrm{T}_b(x) = 0$ for $a \neq b$;
   $\mathrm{T}_a(x) \vee (\mathrm{T}_a(x) \to 0) = 1$;
5. $\mathrm{T}_1(\mathrm{T}_a(x)) = \mathrm{T}_a(x)$; $\mathrm{T}_0(\mathrm{T}_a(x)) = \mathrm{T}_a(x) \to 0$;
   $\mathrm{T}_b(\mathrm{T}_a(x)) = 0$ for $b \neq 0, 1$;
6. $\mathrm{T}_1(x) \leq x$; $\mathrm{T}_1(x \wedge y) = \mathrm{T}_1(x) \wedge \mathrm{T}_1(y)$;
7. $\bigwedge_{a\in L}(\mathrm{T}_a(x) \leftrightarrow \mathrm{T}_a(y)) \leq x \leftrightarrow y$.

A homomorphism of L-**VL**-algebras is defined as a function which preserves all the operations $(\wedge, \vee, \to, \mathrm{T}_a(a \in L), 0, 1)$.

The class of L-**VL**-algebras forms a variety. **2**-**VL**-algebras coincide with Boolean algebras (see [27, Proposition 2]).

$\mathrm{T}_a(x) \wedge \mathrm{T}_b(y) \leq \mathrm{T}_{a\vee b}(x \vee y)$ intuitively means that if the truth value of x is a and the truth value of y is b then the truth value of $x \vee y$ is $a \vee b$. The other inequalities following from the item 2 above can be explained in similar ways.

$\bigvee\{\mathrm{T}_a(x); a \in L\} = 1$ in the item 4 above is called the L-valued excluded middle, since the **2**-valued excluded middle coincides with the ordinary excluded middle (see [27, Proposition 2]).

The following notions play important roles in our investigation.

Definition 2.5. Let A be an L-**VL**-algebra. Then, a subset F of A is called an L-filter iff F is a non-empty filter of lattices which is closed under T_1. Let P be an L-filter of A with $A \neq P$.

1. P is a prime L-filter of A iff for any $c \in L$, if $\mathrm{T}_c(x \vee y) \in P$, then there are $a, b \in L$ with $a \vee b = c$ such that $\mathrm{T}_a(x) \in P$ and $\mathrm{T}_b(y) \in P$.
2. P is an ultra L-filter of A iff $\forall x \in A\ \exists a \in L\ \mathrm{T}_a(x) \in P$.

3. P is a maximal L-filter iff P is maximal with respect to inclusion.

The above definition of primeness is different from the usual definition of primeness (i.e., $x \vee y \in P$ implies either $x \in P$ or $y \in P$), which does not work for developing duality theory for L-**VL**-algebras and L-**ML**-algebras. If L is the two-element Boolean algebra, then the above notions coincide with the usual notions of prime filter, ultrafilter and maximal filter for Boolean algebras.

Lemma 2.1. For an L-filter P of an L-**VL**-algebra A, P is a prime L-filter iff P is an ultra L-filter.

Proof:
Let P be a prime L-filter and $x \in A$. By $1 \in P$, $\bigvee\{\mathrm{T}_a(x)\,;\, a \in L\} \in P$. Thus, $\mathrm{T}_1(\bigvee\{\mathrm{T}_a(x)\,;\, a \in L\}) \in P$. Suppose $L = \{a_1, ..., a_n\}$. Since P is prime, there exist $b_1, ..., b_n \in L$ such that $b_1 \vee ... \vee b_n = 1$ and $\mathrm{T}_{b_k}(\mathrm{T}_{a_k}(x)) \in P$ for $k = 1, ..., n$. It follows from $\mathrm{T}_a(\mathrm{T}_b(x)) = 0$ for $a \neq 0, 1$ that all b_k's are either 0 or 1. Since $b_1 \vee ... \vee b_n = 1$, there is $k \in \{1, ..., n\}$ with $b_k = 1$. Then, $\mathrm{T}_1(\mathrm{T}_{a_k}(x)) \in P$. By $\mathrm{T}_1(x) \leq x$, we have $\mathrm{T}_{a_k}(x) \in P$. Hence, P is an ultra L-filter.

Let P be an ultra L-filter and $\mathrm{T}_c(x \vee y) \in P$. There are $a, b \in L$ such that $\mathrm{T}_a(x) \in P$ and $\mathrm{T}_b(y) \in P$. By $\mathrm{T}_a(x) \wedge \mathrm{T}_b(y) \leq \mathrm{T}_{a \vee b}(x \vee y)$, we have $\mathrm{T}_{a \vee b}(x \vee y) \in P$. Since $\mathrm{T}_c(x \vee y) \in P$ and since $\mathrm{T}_a(x) \wedge \mathrm{T}_b(x) = 0$ for $a \neq b$, we have $a \vee b = c$. Hence, P is a prime L-filter. □

Lemma 2.2. For an L-filter P of an L-**VL**-algebra A, P is a maximal L-filter iff P is an ultra L-filter.

Proof:
Let P be a maximal L-filter and $x \in A$. Suppose for contradiction that $\mathrm{T}_a(x) \notin P$ for all $a \in L$. Fix $a \in L$. Since P is maximal, there is $\varphi \in A$ such that $\varphi = 0$ and φ is constructed from $\wedge, \mathrm{T}_1, \mathrm{T}_a(x)$ and the elements of P. Let $\psi = \mathrm{T}_1(\varphi)$. Note $\psi = 0$. From $\mathrm{T}_1(\mathrm{T}_1(x)) = \mathrm{T}_1(x)$ and $\mathrm{T}_1(x \wedge y) = \mathrm{T}_1(x) \wedge \mathrm{T}_1(y)$, it follows that $\psi = \mathrm{T}_1(r_a \wedge \mathrm{T}_a(x))$ for some $r_a \in P$. We have shown that for all $a \in L$ there is $r_a \in P$ such that $\mathrm{T}_1(r_a \wedge \mathrm{T}_a(x)) = 0$. From $\mathrm{T}_1(\mathrm{T}_a(x)) = \mathrm{T}_a(x)$, it follows that $\mathrm{T}_1(r_a \wedge \mathrm{T}_a(x)) = \mathrm{T}_1(r_a) \wedge \mathrm{T}_a(x) = 0$ for all $a \in L$. Therefore, we have $(\bigwedge\{\mathrm{T}_1(r_a)\,;\, a \in L\}) \wedge \mathrm{T}_a(x) = 0$. Thus,

$$\bigwedge\{\mathrm{T}_1(r_a)\,;\, a \in L\} = (\bigwedge\{\mathrm{T}_1(r_a)\,;\, a \in L\}) \wedge (\bigvee\{\mathrm{T}_a(x)\,;\, a \in L\}) = 0.$$

By $r_a \in P$, we have $\bigwedge\{\mathrm{T}_1(r_a)\,;\, a \in L\} \in P$ and so $0 \in P$, which is a contradiction. Hence, P is an ultra L-filter.

Let P be an ultra L-filter and F an L-filter such that $P \subset F$ and $F \neq A$. Assume $x \in F$. Then, $\mathrm{T}_1(x) \in F$. There exists $a \in L$ with $\mathrm{T}_a(x) \in P$. Then, $\mathrm{T}_a(x) \in F$. Thus, we have $\mathrm{T}_1(x) \wedge \mathrm{T}_a(x) \in F$. If $a \neq 1$, then $0 = \mathrm{T}_1(x) \wedge \mathrm{T}_a(x) \in F$. By $F \neq A$, we have $a = 1$ and so $\mathrm{T}_1(x) \in P$, which implies $x \in P$. Thus, we have $F = P$. Hence, P is a maximal L-filter. □

Thus we obtain the following.

Proposition 2.1. For an L-**VL**-algebra A and a subset P of A, the following are equivalent:

1. P is a prime L-filter;
2. P is an ultra L-filter;
3. P is a maximal L-filter.

An analogue of prime filter theorem is obtained as follows.

Theorem 2.1. Let A be an L-**VL**-algebra. Assume $x \neq y$ for $x, y \in A$. Then, there are $a \in L$ and a prime L-filter P of A such that $\mathrm{T}_a(x) \in P$ and $\mathrm{T}_a(y) \notin P$.

Proof:
By assumption, $x \leftrightarrow y \neq 1$. By 7 in Definition 2.4, we have $\bigwedge_{a \in L}(\mathrm{T}_a(x) \leftrightarrow \mathrm{T}_a(y)) \neq 1$, whence there is $a \in L$ such that $\mathrm{T}_a(x) \neq \mathrm{T}_a(y)$. We may assume $\mathrm{T}_a(x) \nleq \mathrm{T}_a(y)$. Let $F_0 = \{z \in A\,;\, \mathrm{T}_a(x) \leq z\}$. Then, F_0 is an L-filter. Let X be the set of L-filters F such that $\mathrm{T}_a(x) \in F$ and $\mathrm{T}_a(y) \notin F$. By $F_0 \in X$, X is non-empty. Any chain of X has an upper bound in X. By Zorn's lemma, we have a maximal element P in X. Then $\mathrm{T}_a(x) \in P$ and $\mathrm{T}_a(y) \notin P$.

We claim that P is an ultra L-filter. If not, there is $z \in A$ such that $\forall c \in L\ \mathrm{T}_c(z) \notin P$. Fix $c \in L$. Since P is maximal in X, there is $\varphi \in A$ such that $\varphi \leq \mathrm{T}_a(y)$ and φ is constructed from $\wedge, \mathrm{T}_1, \mathrm{T}_c(z)$ and the elements of P. Let $\psi = \mathrm{T}_1(\varphi)$. Then, $\psi \leq \mathrm{T}_1(\mathrm{T}_a(y)) = \mathrm{T}_a(y)$, which leads to a contradiction by arguing as in the first paragraph of the proof of Lemma 2.2. Thus P is an ultra L-filter, whence P is a prime L-filter. □

Prime L-filters provide an internal description of homomorphisms of L-**VL**-algebras into L (i.e., there is a bijective correspondence between prime L-filters and homomorphisms into L), which can be established by using the following construction.

Definition 2.6. Let P be a prime L-filter of an L-**VL**-algebra A. Define $v_P : A \to L$ by

$$v_P(x) = a \Leftrightarrow \mathrm{T}_a(x) \in P.$$

We show that v_P is well defined. Let $x \in A$. Since P is prime and since the L-valued excluded middle holds, there exists $a \in L$ with $\mathrm{T}_a(x) \in P$. If $a \neq b$ then $\mathrm{T}_a(x) \wedge \mathrm{T}_b(x) = 0$. Hence, if $\mathrm{T}_a(x) \in P$ and $\mathrm{T}_b(x) \in P$, then $a = b$.

Proposition 2.2. Let P be a prime L-filter of an L-**VL**-algebra A. Then, v_P is a homomorphism of L-**VL**-algebras.

Proof:
We show that $v_P(\mathrm{T}_a(x)) = \mathrm{T}_a(v_P(x))$. Let $b = v_P(x)$. Then, $\mathrm{T}_b(x) \in P$. By the axiom $\mathrm{T}_b(x) \leq \mathrm{T}_{\mathrm{T}_a(b)}(\mathrm{T}_a(x))$, we have $\mathrm{T}_{\mathrm{T}_a(b)}(\mathrm{T}_a(x)) \in P$, which implies $v_P(\mathrm{T}_a(x)) = \mathrm{T}_a(v_P(x))$.

We show that $v_P(x \to y) = v_P(x) \to v_P(y)$. Let $a = v_P(x)$ and $b = v_P(y)$. Then, $\mathrm{T}_a(x) \in P$ and $\mathrm{T}_b(y) \in P$. Thus, $\mathrm{T}_a(x) \wedge \mathrm{T}_b(y) \in P$. Since $\mathrm{T}_a(x) \wedge \mathrm{T}_b(y) \leq \mathrm{T}_{a \to b}(x \to y)$, we have $\mathrm{T}_{a \to b}(x \to y) \in P$, which implies $v_P(x \to y) = v_P(x) \to v_P(y)$.

The remaining parts of the proof are verified in similar ways. □

The spectrums of L-**VL**-algebras are defined as follows.

Definition 2.7. Let A be an L-**VL**-algebra. For a subalgebra M of L, $\mathrm{Spec}_M(A)$ denotes the set of all homomorphisms of L-**VL**-algebras from A to M. We equip $\mathrm{Spec}_M(A)$ with the topology generated by $\langle x \rangle$ for $x \in A$ where

$$\langle x \rangle = \{v \in \mathrm{Spec}_M(A)\,;\, v(x) = 1\}.$$

Note that $\mathrm{Spec}_M(A)$ is a subspace of $\mathrm{Spec}_L(A)$ for any subalgebra M of L and that $\mathrm{Spec}_{\mathbf{2}}(A)$ consists of all homomorphisms of L-**VL**-algebras from A to **2**.

Definition 2.8. Let A be an L-**VL**-algebra. Define $\mathcal{B}(A) = \{x \in A\,;\, \mathrm{T}_1(x) = x\}$. An element of $\mathcal{B}(A)$ is called a Boolean element of A.

For a homomorphism $f : A \to B$ between L-**VL**-algebras, if $x \in A$ is a Boolean element, then $f(x) \in B$ is also a Boolean element. It is easy to verify the following.

Proposition 2.3. Let A be an L-**VL**-algebra. Then, $\mathcal{B}(A)$ is a Boolean algebra.

Proposition 2.4. For an L-**VL**-algebra A, define $t_1 : \mathrm{Spec}_L(A) \to \mathrm{Spec}_{\mathbf{2}}(\mathcal{B}(A))$ by $t_1(v) = \mathrm{T}_1 \circ v$ for $v \in \mathrm{Spec}_L(A)$. Then, t_1 is a homeomorphism.

Proof:
Note that $t_1(v) \in \mathrm{Spec}_{\mathbf{2}}(\mathcal{B}(A))$ for $v \in \mathrm{Spec}_L(A)$. We show that t_1 is injective. Assume that $v \neq u$ for $v, u \in \mathrm{Spec}_L(A)$. Then $v(x) \neq u(x)$ for some $x \in A$. Let $a = v(x)$. Then $v(\mathrm{T}_a(x)) = 1$ and $u(\mathrm{T}_a(x)) = 0$. Thus we have $t_1(v) \neq t_1(u)$.

We show that t_1 is surjective. Let $u \in \mathrm{Spec}_{\mathbf{2}}(\mathcal{B}(A))$. Define $v : A \to L$ by $v(x) = a \Leftrightarrow u(\mathrm{T}_a(x)) = 1$, where note that $\mathrm{T}_a(x)$ is a Boolean element for any $x \in A$ and that v is well defined as a function. We claim that v is a homomorphism of L-**VL**-algebras. We show only $v(x \to y) = v(x) \to v(y)$ for $x, y \in A$, since the other cases are verified in similar ways. Let $v(x) = a$ and $v(y) = b$. Then $u(\mathrm{T}_a(x)) = 1$ and $u(\mathrm{T}_b(y)) = 1$, whence $u(\mathrm{T}_a(x) \wedge \mathrm{T}_b(y)) = 1$. Thus, by 2 in Definition 2.4, we have $u(\mathrm{T}_{a\to b}(x \to y)) = 1$ and so $v(x \to y) = a \to b = v(x) \to v(y)$. To complete the proof of the surjectivity of t_1, it remains to show $t_1(v) = u$. It is clear that if $u(x) = 1$ for $x \in \mathcal{B}(A)$, then $(t_1(v))(x) = 1$. If $u(x) = 0$ for $x \in \mathcal{B}(A)$, then $u(\mathrm{T}_0(\mathrm{T}_1(x))) = u(\mathrm{T}_1(x) \to 0) = u(x \to 0) = 1$ by 5 in Definition 2.4, whence $v(\mathrm{T}_1(x)) = 0$ and so $(t_1(v))(x) = 0$. Thus we have $t_1(v) = u$.

It is straightforward to verify the remaining part of the proof. □

We define unary operations U_a as in the next proposition (for its proof, see [27, Proposition 6]). $\mathrm{U}_a(x)$ intuitively means: The truth value of x is more than or equal to a. We remark that U_a shall be used in the definition of L-**ML**-algebras below.

Proposition 2.5. For $a \in L$, define

$$\mathrm{U}_a(x) = \bigvee\{\mathrm{T}_b(x)\,;\, a \leq b\}.$$

For $x \in L$, we have both $\mathrm{U}_a(x) = 1$ for $a \leq x$ and $\mathrm{U}_a(x) = 0$ for $a \not\leq x$.

Note that U_a is order-preserving and that $\mathrm{U}_a(x \wedge y) = \mathrm{U}_a(x) \wedge \mathrm{U}_a(y)$.

In fact, U_a's are inter-definable with T_a's (see [27, Proposition 7]).

2.2. A duality for L-VL-algebras

As in universal algebra, we mean by an algebra a set A equipped with a collection of operations on A (see [6, 8]). For an algebra A, $\mathbb{ISP}(A)$ denotes the class of all isomorphic copies of subalgebras of direct powers of A (see [6, 8]).

Definition 2.9. Let A be a finite algebra. Define the ternary discriminator function $t : A^3 \to A$ on A as follows:

$$t(x, y, z) = \begin{cases} x & \text{if } x \neq y \\ z & \text{if } x = y. \end{cases}$$

A is called a semi-primal algebra iff t is a term function of A and there is no isomorphism between non-empty subalgebras of A other than the identity maps.

Lemma 2.3. $(L, \wedge, \vee, \to, \mathrm{T}_a(a \in L), 0, 1)$ forms a semi-primal algebra.

Proof:
Define $t : L^3 \to L$ by

$$t(x, y, z) = ((\mathrm{T}_1(x \leftrightarrow y) \to 0) \to x) \wedge (\mathrm{T}_1(x \leftrightarrow y) \to z).$$

It is easily verified that t is actually the ternary discriminator function on L.

Suppose for contradiction that there exists an isomorphism $f : L_1 \to L_2$ such that both L_1 and L_2 are non-empty subalgebras of L and that $f(a) \neq a$ for some $a \in L_1$. Then, we have $1 = f(\mathrm{T}_a(a)) = \mathrm{T}_a(f(a)) = 0$, which is a contradiction. □

Definition 2.10. L-VA denotes the category of L-**VL**-algebras and homomorphisms of L-**VL**-algebras.

A Boolean space is defined as a zero-dimensional compact Hausdorff space.

Definition 2.11. $\mathrm{SubAlg}(L)$ denotes the set of all subalgebras of L. For a Boolean space S, $\mathrm{SubSp}(S)$ denotes the set of all closed subspaces of S.

Note that a closed subspace of a Boolean space is also a Boolean space.

Definition 2.12. We define a category L-BS as follows.

An object in L-BS is a tuple (S, α) such that S is a Boolean space and that a function $\alpha : \mathrm{SubAlg}(L) \to \mathrm{SubSp}(S)$ satisfies the following:

1. $S = \alpha(L)$;
2. if L_1 is a subalgebra of L_2, then $\alpha(L_1) \subset \alpha(L_2)$;
3. if $L_3 = L_1 \cap L_2$, then $\alpha(L_3) = \alpha(L_1) \cap \alpha(L_2)$.

An arrow $f : (S, \alpha) \to (T, \beta)$ in L-BS is a continuous map $f : S \to T$ which satisfies the condition that, for any $M \in \mathrm{SubAlg}(L)$, if $x \in \alpha(M)$ then $f(x) \in \beta(M)$. We call a map satisfying the condition a subspace-preserving map.

To obtain a duality between the categories L-VA and L-BS, we introduce two contravariant functors Spec and Cont as follows.

Definition 2.13. We define a contravariant functor $\mathrm{Spec} : L\text{-VA} \to L\text{-BS}$.

For an object A in L-VA, define $\mathrm{Spec}(A) = (\mathrm{Spec}_L(A), \alpha_A)$, where α_A is defined by $\alpha_A(M) = \mathrm{Spec}_M(A)$ for $M \in \mathrm{SubAlg}(L)$.

For an arrow $f : A \to B$ in L-VA, $\mathrm{Spec}(f) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is defined by $\mathrm{Spec}(f)(v) = v \circ f$ for $v \in \mathrm{Spec}_L(B)$.

We equip L and its subalgebras with the discrete topologies. Define $\alpha_L : \mathrm{SubAlg}(L) \to \mathrm{SubSp}(L)$ by $\alpha_L(M) = M$ for $M \in \mathrm{SubAlg}(L)$. Then, (L, α_L) is an object in L-BS.

Definition 2.14. We define a contravariant functor $\mathrm{Cont} : L\text{-BS} \to L\text{-VA}$.

For an object (S, α) in L-BS, $\mathrm{Cont}(S, \alpha)$ is defined as the set of all subspace-preserving continuous maps from (S, α) to (L, α_L) equipped with the operations $(\wedge, \vee, \to, \mathrm{T}_a (a \in L), 0, 1)$ defined pointwise: For $f, g \in \mathrm{Cont}(S, \alpha)$, define $(f@g)(x) = f(x)@g(x)$ for $@ = \wedge, \vee, \to$. Define $(\mathrm{T}_a(f))(x) = \mathrm{T}_a(f(x))$. Finally 0 (resp. 1) is defined as the constant function whose value is always 0 (resp. 1).

For an arrow $f : (S_1, \alpha_1) \to (S_2, \alpha_2)$ in L-BS, $\mathrm{Cont}(f) : \mathrm{Cont}(S_2, \alpha_2) \to \mathrm{Cont}(S_1, \alpha_1)$ is defined by $\mathrm{Cont}(f)(g) = g \circ f$ for $g \in \mathrm{Cont}(S_2, \alpha_2)$.

We then obtain the following duality for L-**VL**-algebras.

Theorem 2.2. L-VA is dually equivalent to L-BS via the functors Spec and Cont.

Proof:
Let Id_1 denote the identity functor on L-VA and Id_2 denote the identity functor on L-BS. It suffices to show that there are two natural isomorphisms $\epsilon : \mathrm{Id}_1 \to \mathrm{Cont} \circ \mathrm{Spec}$ and $\eta : \mathrm{Id}_2 \to \mathrm{Spec} \circ \mathrm{Cont}$.

For an L-**VL**-algebra A, define $\epsilon_A : A \to \mathrm{Cont} \circ \mathrm{Spec}(A)$ by $\epsilon_A(x)(v) = v(x)$, where $x \in A$ and $v \in \mathrm{Spec}_L(A)$. For an object (S, α) in L-BS, define $\eta_{(S,\alpha)} : (S, \alpha) \to \mathrm{Spec} \circ \mathrm{Cont}(S, \alpha)$ by $\eta_{(S,\alpha)}(x)(f) = f(x)$, where $x \in S$ and $f \in \mathrm{Cont}(S, \alpha)$.

For an L-**VL**-algebra A, ϵ_A is injective by Theorem 2.1 and Proposition 2.2. Hence the class of L-**VL**-algebras coincides with $\mathbb{ISP}(L)$. Thus, since L forms a semi-primal algebra by Lemma 2.3, it follows from [8, Theorem 3.3.14] that η and ϵ are natural isomorphisms. □

The above theorem can also be shown using a duality result in [17], though such proof is essentially the same as the above one.

2.3. Applications of the duality for L-VL-algebras

We first consider what duality holds for L-**VL**-algebras enriched by truth constants.

Definition 2.15. $(A, \wedge, \vee, \to, \mathrm{T}_b\ (b \in L), a\ (a \in L))$ is called an L-**VL**-algebra with constants iff $(A, \wedge, \vee, \to, \mathrm{T}_b\ (b \in L), 0, 1)$ is an L-**VL**-algebra and $(\{a\ ;\ a \in L\}, \wedge, \vee, \to, \mathrm{T}_b\ (b \in L))$ is an L-**VL**-algebra isomorphic to L.

A homomorphism of L-**VL**-algebras with constants is defined as a homomorphism of L-**VL**-algebras which additionally preserves all constants $a \in L$.

Theorem 2.3. The category of L-**VL**-algebras with constants and their homomorphisms is dually equivalent to the category of Boolean spaces and continuous maps.

Proof:
By reformulating Theorem 2.1 and Proposition 2.2 in the case of L-**VL**-algebras with constants, which is straightforward, we can show that the class of L-**VL**-algebras with constants coincides with the quasi-variety generated by L equipped with all constants $a \in L$. Since L with constants is a primal algebra, we obtain this theorem by Hu's general duality theorem (see [8, Theorem 4.1.1] or [16]). □

By the above theorem, we notice that α in the definition of L-BS disappears when L-**VL**-algebras are enriched by truth constants. It seems that equipping truth constants makes a duality simpler.

We next show a compactness theorem for L-**VL**.

Theorem 2.4. Let $X \subset \mathbf{Form}$. Assume that any finite subset of X is satisfiable. Then, X is satisfiable.

Proof:
Let A be the Lindenbaum algebra of L-**VL**. We may consider $X \subset A$. Then $\{\langle x \rangle \; ; \; x \in X\}$ consists of clopen subsets of $\mathrm{Spec}_L(A)$. By assumption, $\{\langle x \rangle \; ; \; x \in X\}$ has finite intersection property, since an L-valuation on $\mathbf{Form}$ can be seen as a homomorphism of L-**VL**-algebras from A to L. Since $\mathrm{Spec}_L(A)$ is compact by Theorem 2.2, there is $v \in \bigcap\{\langle x \rangle \; ; \; x \in X\}$. Hence, X is satisfiable. □

By Theorem 2.2, we can also classify the categories L-VA's for finite distributive lattices L as follows (for definitions from category theory, see [2]).

Theorem 2.5. Let L_1 and L_2 be finite distributive lattices. Then, the following are equivalent:

1. L_1-VA and L_2-VA are categorically equivalent;
2. $(\mathrm{SubAlg}(L_1), \subset)$ and $(\mathrm{SubAlg}(L_2), \subset)$ are order isomorphic.

Proof:
We show that (2) implies (1). Assume (2). Then, L_1-BS and L_2-BS are categorically equivalent by the definition of L-BS. Thus, by Theorem 2.2, L_1-VA and L_2-VA are categorically equivalent.

We show that (1) implies (2). Assume (1). Then, by Theorem 2.2, L_1-BS and L_2-BS are categorically equivalent. For $i = 1, 2$, equip $\{i\}$ with the discrete topology and define $\alpha_i : \mathrm{SubAlg}(L_i) \to \mathrm{SubSp}(\{i\})$ by $\alpha_i(M) = \{i\}$ for any $M \in \mathrm{SubAlg}(L_i)$. Then, $(\{i\}, \alpha_i)$ is a terminal object of L_i-BS for $i = 1, 2$. Let $\mathrm{SubOb}(\{i\}, \alpha_i)$ denote the category of non-empty subobjects of $(\{i\}, \alpha_i)$ in L_i-BS (for the definition of the category of subobjects, see [2, Definition 5.1]). Since L_1-BS and L_2-BS are categorically equivalent, $\mathrm{SubOb}(\{1\}, \alpha_1)$ and $\mathrm{SubOb}(\{2\}, \alpha_2)$ are categorically equivalent. We consider $\mathrm{SubOb}(\{1\}, \alpha_1)$ and $\mathrm{SubOb}(\{2\}, \alpha_2)$ as partially ordered sets in the usual way. Then, $\mathrm{SubOb}(\{1\}, \alpha_1)$ and $\mathrm{SubOb}(\{2\}, \alpha_2)$ are order isomorphic.

In order to complete the proof, it suffices to show that $\mathrm{SubOb}(\{i\}, \alpha_i)^{\mathrm{op}}$ is order isomorphic to $(\mathrm{SubAlg}(L_i), \subset)$ for $i = 1, 2$. For $(X, \alpha_X) \in \mathrm{SubOb}(\{i\}, \alpha_i)$, define

$$h_i(X, \alpha_X) = \bigcap\{M \in \mathrm{SubAlg}(L_i) \; ; \; \alpha_X(M) = \{i\}\},$$

where we may assume $X = \{i\}$. Then we claim that h_i is an order isomorphism from $\mathrm{SubOb}(\{i\}, \alpha_i)^{\mathrm{op}}$ to $(\mathrm{SubAlg}(L_i), \subset)$. Since $\{M \,;\, \alpha_X(M) = \{i\}\}$ is upward closed with respect to $\subset$ by the item 2 of Definition 2.12 and since $\alpha_X(h_i(X, \alpha_X)) = \{i\}$ by the item 3 of Definition 2.12, h_i is injective. It is straightforward to verify the remaining part of the claim. □

By the above theorem and the finiteness of L_1 and L_2, it is effectively decidable whether or not the categories L_1-VA and L_2-VA are equivalent. Note that in general it is not decidable whether or not two categories are equivalent.

3. A duality for L-ML-algebras and its applications

L-valued modal logic L-**ML** is defined by L-valued Kripke semantics as follows. The connectives of L-**ML** are a unary connective $\Box$ and the connectives of L-**VL**. Then, $\mathbf{Form}_\Box$ denotes the set of formulas of L-**ML**.

Definition 3.1. Let (M, R) be a Kripke frame, i.e., R is a binary relation on a set M. Then, e is a Kripke L-valuation on (M, R) iff e is a function from $M \times \mathbf{Form}_\Box$ to L and satisfies the following for each $w \in M$ and $x \in \mathbf{Form}_\Box$:

1. $e(w, \Box x) = \bigwedge\{e(w', x) \,;\, wRw'\}$;
2. $e(w, \mathrm{T}_a(x)) = \mathrm{T}_a(e(w, x))$;
3. $e(w, x@y) = e(w, x)@e(w, y)$ for $@ = \wedge, \vee, \rightarrow$;
4. $e(w, a) = a$ for $a = 0, 1$.

We call (M, R, e) an L-valued Kripke model. Then, $x \in \mathbf{Form}_\Box$ is said to be valid in L-**ML** iff $e(w, x) = 1$ for any L-valued Kripke model (M, R, e) and any $w \in M$.

We then define L-valued modal logic L-**ML** as the set of those $x \in \mathbf{Form}_\Box$ such that x is valid in L-**ML** in the above sense.

Definition 3.2. Let $X \subset \mathbf{Form}_\Box$. X is Kripke satisfiable iff there are an L-valued Kripke model (M, R, e) and $w \in M$ such that $e(w, x) = 1$ for any $x \in X$.

A compactness theorem for L-**ML** is shown in Theorem 3.5 below.

3.1. L-ML-algebras and their relational spectra

We introduce L-**ML**-algebras, which provide a sound and complete semantics for L-valued modal logic L-**ML** as shown in [27]. Recall that $\mathrm{U}_a(x)$ is the abbreviation of $\bigvee\{\mathrm{T}_b(x) \,;\, a \leq b\}$.

Definition 3.3. $(A, \wedge, \vee, \rightarrow, \mathrm{T}_a (a \in L), \Box, 0, 1)$ is an L-**ML**-algebra iff it satisfies the following:

1. $(A, \wedge, \vee, \rightarrow, \mathrm{T}_a (a \in L), 0, 1)$ forms an L-**VL**-algebra;
2. $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box 1 = 1$;

3. $\Box \mathrm{U}_a(x) = \mathrm{U}_a(\Box x)$ for all $a \in L$.

A homomorphism of L-**ML**-algebras is defined as a homomorphism of L-**VL**-algebras which additionally preserves the operation $\Box$.

The class of L-**ML**-algebras forms a variety. Note that **2**-**ML**-algebras coincide with modal algebras.

Let A be an L-**ML**-algebra. Since $\mathrm{T}_1(x) = \mathrm{U}_1(x)$ for any $x \in A$, we have $\Box \mathrm{T}_1(x) = \mathrm{T}_1(\Box x)$ by the item 3 in the above definition.

The notion of L-valued canonical model is defined as follows.

Definition 3.4. Let A be an L-**ML**-algebra. Define a binary relation $R_\Box$ on $\mathrm{Spec}_L(A)$ as follows: $vR_\Box u$ iff

$$\forall a \in L\, \forall x \in A\ (v(\Box x) \geq a \text{ implies } u(x) \geq a).$$

Define $e : \mathrm{Spec}_L(A) \times A \to L$ by $e(v, x) = v(x)$ for $v \in \mathrm{Spec}_L(A)$ and $x \in A$. Then we call $(\mathrm{Spec}_L(A), R_\Box, e)$ the L-valued canonical model of A.

The following lemma is crucial in order to show that $(\mathrm{Spec}_L(A), R_\Box, e)$ actually forms an L-valued Kripke model.

Lemma 3.1. Let A be an L-**ML**-algebra and $v \in \mathrm{Spec}_L(A)$. For $a \in L$ and $x \in A$, the following are equivalent:

1. $v(\Box x) \geq a$;
2. $\forall u \in \mathrm{Spec}_L(A)\ (vR_\Box u \text{ implies } u(x) \geq a)$.

Proof:
It is straightforward to see that 1 implies 2. We show that 2 implies 1. To prove the contrapositive, assume that $v(\Box x) \not\geq a$. Let $P_v = v^{-1}(\{1\})$. Note $\mathrm{U}_a(\Box x) \notin P_v$. By Proposition 2.2, it suffices to show that there is a prime L-filter Q of A such that for any $b \in L$ and $y \in A$, $\mathrm{U}_b(\Box y) \in P_v$ implies $\mathrm{U}_b(y) \in Q$ and that $\mathrm{U}_a(x) \notin Q$. Let F be the L-filter generated by

$$\{\mathrm{U}_b(y)\ ;\ b \in L,\ y \in A \text{ and } \mathrm{U}_b(\Box y) \in P_v\}.$$

We claim $\mathrm{U}_a(x) \notin F$. Suppose for contradiction that $\mathrm{U}_a(x) \in F$. Then there is $\varphi \in A$ such that $\varphi \leq \mathrm{U}_a(x)$ and φ is constructed from $\wedge$, T_1 and the elements of $\{\mathrm{U}_b(y)\ ;\ \mathrm{U}_b(\Box y) \in P_v\}$. Since $\mathrm{T}_1(\mathrm{U}_b(y)) = \mathrm{U}_b(y)$ and since both T_1 and U_b distribute over $\wedge$, we may assume that

$$\varphi = \bigwedge\{\mathrm{U}_b(y_b)\ ;\ b \in L\}$$

for some $\mathrm{U}_b(y_b) \in \{\mathrm{U}_b(y)\ ;\ \mathrm{U}_b(\Box y) \in P_v\}$. By $\varphi \leq \mathrm{U}_a(x)$, we have $\Box\varphi \leq \Box\mathrm{U}_a(x)$. It holds that $\Box\varphi = \bigwedge\{\mathrm{U}_b(\Box y_b)\,;\, b \in L\}$. It follows from $\mathrm{U}_b(\Box y_b) \in P_v$ that $\Box\mathrm{U}_a(x) \in P_v$, which is a contradiction. Thus, we have $\mathrm{U}_a(x) \notin F$.

Let X be the set of L-filters G of A such that $\mathrm{U}_a(x) \notin G$ and $F \subset G$. Note $F \in X$. By Zorn's lemma, we have a maximal element Q in X. By arguing as in the second paragraph of the proof of Theorem 2.1, Q is shown to be a prime L-filter. Clearly, $\mathrm{U}_a(x) \notin Q$. It follows from $F \subset Q$ that for any $b \in L$ and $y \in A$, $\mathrm{U}_b(\Box y) \in P_v$ implies $\mathrm{U}_b(y) \in Q$. This completes the proof. $\Box$

By the above lemma, we obtain the following proposition.

Proposition 3.1. Let A be an L-**ML**-algebra. Then, the L-valued canonical model $(\mathrm{Spec}_L(A), R_\Box, e)$ of A is an L-valued Kripke model. In particular, for $x \in A$ and $v \in \mathrm{Spec}_L(A)$,

$$e(v, \Box x) = v(\Box x) = \bigwedge\{u(x) \; ; \; vR_\Box u\}.$$

Proposition 3.2. Let A be an L-**ML**-algebra. Then $\mathcal{B}(A)$ is a modal algebra.

Proof:
By Proposition 2.3, it suffices to show that $\mathcal{B}(A)$ is closed under $\Box$. For $x \in \mathcal{B}(A)$, we have $\mathrm{T}_1(\Box x) = \Box \mathrm{T}_1(x) = \Box x$. $\Box$

Proposition 3.3. Let A be an L-**ML**-algebra and $v, u \in \mathrm{Spec}_L(A)$. Then the following holds: $vR_\Box u$ iff $t_1(v)R_\Box t_1(u)$ where $R_\Box$ in the right-hand side is supposed to be defined only on $\mathrm{Spec}_{\mathbf{2}}(\mathcal{B}(A))$ (for the definition of t_1, see Proposition 2.4).

Proof:
From $\Box \mathrm{T}_1(x) = \mathrm{T}_1(\Box x)$, it follows that $vR_\Box u$ implies $t_1(v)R_\Box t_1(u)$. We show the converse. Assume $t_1(v)R_\Box t_1(u)$. In order to show $vR_\Box u$, it suffices to show that, for any $a \in L$ and any $x \in A$, $v(\Box \mathrm{U}_a(x)) \geq 1$ implies $u(\mathrm{U}_a(x)) \geq 1$, which follows from the assumption, since we have $\mathrm{U}_a(x) \in \mathcal{B}(A)$ and $\mathrm{T}_1(\mathrm{U}_a(x)) = \mathrm{U}_a(x)$. $\Box$

By Proposition 2.4 and Proposition 3.3, we notice that t_1 is an "isomorphism" from $(\mathrm{Spec}_L(A), R_\Box)$ to $(\mathrm{Spec}_{\mathbf{2}}(\mathcal{B}(A)), R_\Box)$ for an L-**ML**-algebra A.

3.2. A Jónsson-Tarski-style duality for L-ML-algebras

In this subsection, we shall show a Jónsson-Tarski-style duality for L-**ML**-algebras.

Definition 3.5. L-MA denotes the category of L-**ML**-algebras and homomorphisms of L-**ML**-algebras.

Given a Kripke frame (S, R), we can define a unary operation $\Box_R$ on the "L-valued powerset algebra" L^S of S.

Definition 3.6. Let (S, R) be a Kripke frame and f a function from S to L. Define $\Box_R f : S \to L$ by $(\Box_R f)(x) = \bigwedge\{f(y) \; ; \; xRy\}$.

For a Kripke frame (S, R) and $X \subset S$, let $R^{-1}[X] = \{y \in S \; ; \; \exists x \in X \; yRx\}$. For $x \in S$, let $R[x] = \{y \in S \; ; \; xRy\}$.

Definition 3.7. We define a category L-RS as follows.

An object in L-RS is a triple (S, α, R) such that (S, α) is an object in L-BS and a binary relation R on S satisfies the following conditions:

1. if $\forall f \in \mathrm{Cont}(S, \alpha)((\Box_R f)(x) = 1 \Rightarrow f(y) = 1)$ then xRy;

2. if $X \subset S$ is clopen in S, then $R^{-1}[X]$ is clopen in S;

3. for any $M \in \mathrm{SubAlg}(L)$, if $x \in \alpha(M)$ then $R[x] \subset \alpha(M)$.

An arrow $f : (S_1, \alpha_1, R_1) \to (S_2, \alpha_2, R_2)$ in L-RS is an arrow $f : (S_1, \alpha_1) \to (S_2, \alpha_2)$ in L-BS satisfying the following conditions:

1. if xR_1y then $f(x)R_2f(y)$;

2. if $f(x_1)R_2x_2$ then there is $y_1 \in S_1$ such that $x_1R_1y_1$ and $f(y_1) = x_2$.

The item 1 in the object part of Definition 3.7 is an L-valued version of the tightness condition of descriptive general frames in classical modal logic (for the definition of the tightness condition, see [10]).

Definition 3.8. We define a contravariant functor $\mathrm{RSpec} : L\text{-MA} \to L\text{-RS}$. For an object A in L-MA, define

$$\mathrm{RSpec}(A) = (\mathrm{Spec}_L(A), \alpha_A, R_\Box).$$

For an arrow $f : A \to B$ in L-MA, define $\mathrm{RSpec}(f)$ by $\mathrm{RSpec}(f)(v) = v \circ f$ for $v \in \mathrm{Spec}_L(B)$.

The well-definedness of RSpec is shown by the following two lemmas.

Lemma 3.2. Let A be an L-**ML**-algebra. Then, $\mathrm{RSpec}(A)$ is an object in L-RS.

Proof:
First, we show that $\mathrm{RSpec}(A)$ satisfies the condition 1 in the object part of Definition 3.7. We show the contrapositive. Assume $(v_1, v_2) \notin R_\Box$ for $v_1, v_2 \in \mathrm{Spec}_L(A)$. Then there are $a \in L$ and $x \in A$ such that $v_1(\Box x) \geq a$ and $v_2(x) \not\geq a$, whence we have $v_1(\mathrm{U}_a(\Box x)) = 1$ and $v_2(\mathrm{U}_a(x)) \neq 1$. Define $f : \mathrm{Spec}_L(A) \to L$ by $f(v) = v(\mathrm{U}_a(x))$. Then, by Proposition 3.1 and $\Box\mathrm{U}_a(x) = \mathrm{U}_a(\Box x)$, we have

$$(\Box_R f)(v_1) = \bigwedge\{f(v) \,;\, v_1R_\Box v\} = \bigwedge\{v(\mathrm{U}_a(x)) \,;\, v_1R_\Box v\} = v_1(\Box\mathrm{U}_a(x)) = 1.$$

By the definition of f, $f(v_2) \neq 1$. It is easy to verify that f is continuous.

Second, we show that $\mathrm{RSpec}(A)$ satisfies the condition 2. It is enough to show that $R_\Box^{-1}(\langle x\rangle)$ is clopen in S for any $x \in A$. We claim that

$$R_\Box^{-1}(\langle x\rangle) = \langle\neg\Box\neg\mathrm{T}_1(x)\rangle,$$

where $\neg x$ is the abbreviation of $x \to 0$. Note that the right-hand side is clopen. Assume that $v \in \langle\neg\Box\neg\mathrm{T}_1(x)\rangle$. Then, $v(\neg\Box\neg\mathrm{T}_1(x)) = 1$ and so $v(\Box\neg\mathrm{T}_1(x)) = 0$. By Proposition 3.1, we have

$$0 = v(\Box\neg\mathrm{T}_1(x)) = \bigwedge\{u(\neg\mathrm{T}_1(x)) \,;\, vR_\Box u\}.$$

Since $u(\neg\mathrm{T}_1(x))$ is either 0 or 1, there exists $u \in \mathrm{Spec}_L(A)$ with $vR_\Box u$ such that $u(\neg\mathrm{T}_1(x)) = 0$, i.e., $u(x) = 1$. Therefore we have $v \in R_\Box^{-1}(\langle x\rangle)$. The converse is proved in a similar way.

Third, we show that $\mathrm{RSpec}(A)$ satisfies the condition 3. Suppose for contradiction that $u \in \mathrm{Spec}_M(A)$ and $R[u] \setminus \mathrm{Spec}_M(A) \neq \emptyset$ for some $M \in \mathrm{SubAlg}(L)$. Then there is $v \in R[u] \setminus \mathrm{Spec}_M(A)$ and so there is $x_0 \in A$ with $v(x_0) \notin M$. Define $a = v(x_0)$. Then we have: For $w \in \mathrm{Spec}_L(A)$,

$$w(\mathrm{T}_a(x_0) \to x_0) = \begin{cases} 1 & \text{if } w(x_0) \neq a \\ a & \text{if } w(x_0) = a. \end{cases}$$

Therefore, it follows from Proposition 3.1 and uRv that

$$u(\Box(\mathrm{T}_a(x_0) \to x_0)) = \bigwedge\{w(\mathrm{T}_a(x_0) \to x_0)\ ;\ uRw\} = a = v(x_0).$$

This contradicts $u \in \mathrm{Spec}_M(A)$ by $v(x_0) \notin M$, which completes the proof. □

Lemma 3.3. For L-**ML**-algebras A_1 and A_2, let $f : A_1 \to A_2$ be a homomorphism of L-**ML**-algebras. Then, $\mathrm{RSpec}(f)$ is an arrow in L-RS.

Proof:
By Theorem 2.2, $\mathrm{RSpec}(f)$ is an arrow in L-BS. Define $f_* : \mathcal{B}(A_1) \to \mathcal{B}(A_2)$ by $f_*(x) = f(x)$ for $x \in \mathcal{B}(A_1)$. By Proposition 3.2, f_* is a homomorphism between **2**-**ML**-algebras (i.e., modal algebras). Consider

$$\mathrm{RSpec}(f_*) : \mathrm{RSpec}(\mathcal{B}(A_2)) \to \mathrm{RSpec}(\mathcal{B}(A_1)).$$

It follows from Proposition 2.4, Proposition 3.3 and Jónsson-Tarski duality for modal algebras (see [15, 3]) that $\mathrm{RSpec}(f_*)$ is an arrow in **2**-RS. We also have $\mathrm{RSpec}(f) = \mathrm{RSpec}(f_*)$ on $\mathrm{RSpec}(\mathcal{B}(A_2))$. By using these facts and Proposition 3.3, it is verified that $\mathrm{RSpec}(f)$ is an arrow in L-RS. □

We remark that we can also give a direct proof of the above lemma without using the original Jónsson-Tarski duality for modal algebras (the direct proof uses Zorn's lemma).

Definition 3.9. We define a contravariant functor $\mathrm{MCont} : L\text{-RS} \to L\text{-MA}$. For an object (S, α, R) in L-RS, define

$$\mathrm{MCont}(S, \alpha, R) = (\mathrm{Cont}(S, \alpha), \Box_R).$$

For an arrow $f : (S_1, \alpha_1, R_1) \to (S_2, \alpha_2, R_2)$ in L-RS, define $\mathrm{MCont}(f)$ by $\mathrm{MCont}(f)(g) = g \circ f$ for $g \in \mathrm{Cont}(S_2, \alpha_2)$.

The well-definedness of MCont is shown by the following two lemmas.

Lemma 3.4. Let (S, α, R) be an object in L-RS. Then, $\mathrm{MCont}(S, \alpha, R)$ is an L-**ML**-algebra.

Proof:
We first show that if $f \in \mathrm{MCont}(S, \alpha, R)$ then $\Box_R f \in \mathrm{MCont}(S, \alpha, R)$. Let $f \in \mathrm{MCont}(S, \alpha, R)$. Now we have the following: For $a \in L$,

$$(\Box_R f)^{-1}(a) = R^{-1}[(\mathrm{T}_a(f))^{-1}(1)] \cap (S \setminus R^{-1}[(\mathrm{U}_a(f))^{-1}(0)]),$$

where note: $x \in R^{-1}[(\mathrm{T}_a(f))^{-1}(1)]$ means that there is $y \in S$ such that xRy and $f(y) = a$; $x \in S \setminus R^{-1}[(\mathrm{U}_a(f))^{-1}(0)]$ means that there is no $y \in S$ such that xRy and $f(y) \not\geq a$. Since

$R^{-1}[(\mathrm{T}_a(f))^{-1}(1)] \cap (S \setminus R^{-1}[(\mathrm{U}_a(f))^{-1}(0)])$ is clopen in S, $\Box_R f$ is a continuous map from S to L. It follows from the condition 3 in Definition 3.7 that $\Box_R f$ is subspace-preserving. Thus $\Box_R f \in \mathrm{MCont}(S, \alpha, R)$.

Next we show that $\mathrm{U}_a(\Box f) = \Box \mathrm{U}_a(f)$. It suffices to show that

$$\mathrm{U}_a(\bigwedge\{f(y)\ ;\ xRy\}) = \bigwedge\{\mathrm{U}_a(f(y))\ ;\ xRy\},$$

which is easily verified. The remaining part of the proof is also easy to check. □

Lemma 3.5. Let $f : (S_1, \alpha_1, R_1) \to (S_2, \alpha_2, R_2)$ be an arrow in L-RS. Then, $\mathrm{MCont}(f)$ is a homomorphism of L-**ML**-algebras.

Proof:
By Theorem 2.2, $\mathrm{MCont}(f)$ is an arrow in L-VA. It remains to show that

$$\mathrm{MCont}(f)(\Box g_2) = \Box(\mathrm{MCont}(f)(g_2))$$

for $g_2 \in \mathrm{Cont}(S_2, \alpha_2)$. Let $x_1 \in S_1$. Then,

$$(\mathrm{MCont}(f)(\Box g_2))(x_1) = \Box g_2 \circ f(x_1) = \bigwedge\{g_2(y_2)\ ;\ f(x_1)R_2 y_2\}.$$

Let a denote the rightmost side of the above equation. We also have

$$(\Box(\mathrm{MCont}(f)(g_2)))(x_1) = (\Box(g_2 \circ f))(x_1) = \bigwedge\{g_2(f(y_1))\ ;\ x_1 R_1 y_1\}.$$

Let b denote the rightmost side of the above equation. Since $x_1 R_1 y_1$ implies $f(x_1) R_1 f(y_1)$, we have $a \le b$. Since f satisfies the condition 2 in the arrow part of Definition 3.7, we have $a \ge b$. Hence $a = b$, which completes the proof. □

Theorem 3.1. For an L-**ML**-algebra A, A is isomorphic to $\mathrm{MCont} \circ \mathrm{RSpec}(A)$ in the category L-MA.

Proof:
Define $\epsilon'_A : A \to \mathrm{MCont} \circ \mathrm{RSpec}(A)$ by $\epsilon'_A(x)(v) = v(x)$ for $x \in A$ and $v \in \mathrm{Spec}_L(A)$. Note that ϵ'_A is almost the same as ϵ_A in the proof of Theorem 2.2. By Theorem 2.2, ϵ'_A is an isomorphism of L-**VL**-algebras.

Therefore, it remains to show that ϵ'_A preserves $\Box$, i.e., $\epsilon'_A(\Box x) = \Box_{R_\Box} \epsilon'_A(x)$ for $x \in A$. For $v \in \mathrm{RSpec}(A)$, we have the following:

$$\begin{aligned}
(\Box_{R_\Box} \epsilon'_A(x))(v) &= \bigwedge\{\epsilon'_A(x)(u)\ ;\ vR_\Box u\} \\
&= \bigwedge\{u(x)\ ;\ vR_\Box u\} \\
&= v(\Box x) \quad \text{(by Proposition 3.1)} \\
&= \epsilon'_A(\Box x)(v).
\end{aligned}$$

This completes the proof. □

Theorem 3.2. For an object (S, α, R) in L-RS, (S, α, R) is isomorphic to $\mathrm{RSpec} \circ \mathrm{MCont}(S, \alpha, R)$ in the category L-RS.

Proof:
Define $\eta'_{(S,\alpha,R)} : (S, \alpha, R) \to \mathrm{RSpec} \circ \mathrm{MCont}(S, \alpha, R)$ by $\eta'_{(S,\alpha,R)}(x)(f) = f(x)$ for $x \in S$ and $f \in \mathrm{Cont}(S, \alpha)$. Note that $\eta'_{(S,\alpha,R)}$ is almost the same as $\eta_{(S,\alpha)}$ in the proof of Theorem 2.2. By Theorem 2.2, $\eta'_{(S,\alpha,R)}$ is an isomorphism in the category L-BS. In the below, we denote $\eta'_{(S,\alpha,R)}$ by η'_S.

We show: For any $x, y \in S$, xRy iff $\eta'_S(x)R_{\Box_R}\eta'_S(y)$. Note that the right-hand side holds iff the following holds:

$$\forall a \in L\, \forall f \in \mathrm{Cont}(S, \alpha)\ (\eta'_S(x)(\Box_R f) \geq a \text{ implies } \eta'_S(y)(f) \geq a).$$

Assume xRy. Let $a \in L$ and $f \in \mathrm{Cont}(S, \alpha)$ with $\eta'_S(x)(\Box_R f) \geq a$. Since

$$a \leq \eta'_S(x)(\Box_R f) = (\Box_R f)(x) = \bigwedge\{f(z)\, ;\, xRz\},$$

we have $\eta'_S(y)(f) = f(y) \geq a$. Next we show the converse. To prove the contrapositive, assume that $(x, y) \notin R$. By Definition 3.7, there is $f \in \mathrm{Cont}(S, \alpha)$ such that $(\Box_R f)(x) = 1$ and $f(y) \neq 1$. Then, $\eta'_S(x)(\Box_R f) \geq 1$ and $\eta'_S(y)(f) \not\geq 1$.

Now it remains to prove that η_S and η_S^{-1} satisfy the condition 2 in the arrow part of Definition 3.7, which follows immediately from the above facts. □

By the above two theorems, we can obtain the following Jónsson-Tarski-style duality for L-**ML**-algebras, which is our main theorem.

Theorem 3.3. L-MA and L-RS are dually equivalent via the functors RSpec and MCont.

Proof:
Let $\mathrm{Id}_{1'}$ denote the identity functor on L-MA and $\mathrm{Id}_{2'}$ denote the identity functor on L-RS. It suffices to show that there are natural isomorphisms $\epsilon' : \mathrm{Id}_{1'} \to \mathrm{MCont} \circ \mathrm{RSpec}$ and $\eta' : \mathrm{Id}_{2'} \to \mathrm{RSpec} \circ \mathrm{MCont}$. For an L-**ML**-algebra A, define ϵ'_A as in the proof of Theorem 3.1. For an object (S, α, R) in L-RS, define $\eta'_{(S,\alpha,R)}$ as in the proof of Theorem 3.2. It is straightforward to verify that η' and ϵ' are natural transformations. By Theorem 3.1 and Theorem 3.2, η' and ϵ' are natural isomorphisms. □

3.3. Applications of the duality for L-ML-algebras

We first consider what duality holds for L-**ML**-algebras enriched by truth constants.

Definition 3.10. $(A, \wedge, \vee, \to, \mathrm{T}_b\ (b \in L), \Box, a\ (a \in L))$ is called an L-**ML**-algebra with constants iff (i) $(A, \wedge, \vee, \to, \mathrm{T}_b (b \in L), \Box, 0, 1)$ is an L-**ML**-algebra; (ii) $(\{a\, ;\, a \in L\}, \wedge, \vee, \to, \mathrm{T}_b\ (b \in L))$ is an L-**VL**-algebra isomorphic to L.

A homomorphism of L-**ML**-algebras with constants is a homomorphism of L-**ML**-algebras which additionally preserves all constants $a \in L$.

Theorem 3.4. The category of L-**ML**-algebras with constants and their homomorphisms is dually equivalent to **2**-RS where **2** denotes the two-element Boolean algebra.

Proof:
Just as the duality for L-**ML**-algebras has been established based on the duality for L-**VL**-algebras, so this theorem is established based on the duality for L-**VL**-algebras with constants in a similar way. Note that the proof of this theorem is much simpler than that of the duality for L-**ML**-algebras, since α in the definition of L-RS disappears in the case of the duality for L-**ML**-algebras with constants. $\Box$

The following is a compactness theorem for L-**ML**, which we prove using the compactness of the spectrum of an L-**ML**-algebra.

Theorem 3.5. Let $X \subset \mathbf{Form}_{\Box}$. Assume that any finite subset of X is Kripke satisfiable. Then, X is Kripke satisfiable.

Proof:
Let A be the Lindenbaum algebra of L-**ML**. We may consider $X \subset A$. Then, $\{\langle x \rangle \,;\, x \in X\}$ consists of clopen subsets of $\mathrm{Spec}_L(A)$.

We show that $\{\langle x \rangle ; x \in X\}$ has the finite intersection property. Since $\langle x \rangle \cap \langle y \rangle = \langle x \wedge y \rangle$ for $x, y \in A$, it suffices to show that $\langle x \rangle \neq \emptyset$ for any $x \in X$. Since $\{x\}$ is Kripke satisfiable by assumption, we have $\mathrm{T}_1(x) \neq 0$. Then it follows from Theorem 2.1 that there are $a \in L$ and a prime L-filter P of A such that $\mathrm{T}_a(\mathrm{T}_1(x)) \in P$ and $\mathrm{T}_a(0) \notin P$. Since $0 \notin P$, we have $a = 1$ and so $\mathrm{T}_1(x) = \mathrm{T}_1(\mathrm{T}_1(x)) \in P$. Define $v_P : A \to L$ by $v_P(z) = a \Leftrightarrow \mathrm{T}_a(z) \in P$ for $z \in A$. By Proposition 2.2, v_P is a homomorphism of L-**VL**-algebras with $v_P(x) = 1$, whence $v_P \in \langle x \rangle$. Thus, $\{\langle x \rangle \,;\, x \in X\}$ has the finite intersection property.

Since $\mathrm{Spec}_L(A)$ is compact and $\langle x \rangle$ is closed, we have $v \in \bigcap\{\langle x \rangle \,;\, x \in X\}$. Consider the L-valued canonical model $(\mathrm{Spec}_L(A), R_{\Box}, e)$ of A. Then, $e(v, x) = v(x) = 1$ for any $x \in X$. Thus, X is Kripke satisfiable. $\Box$

Proposition 3.4. Let (S, α) be an object in L-BS and R a binary relation on S. The following are equivalent:

1. $R[x]$ is closed for any $x \in S$;
2. if $\forall f \in \mathrm{Cont}(S, \alpha)((\Box_R f)(x) = 1 \Rightarrow f(y) = 1)$ then xRy.

Proof:
Assume 1. In order to show the contrapositive of 2, suppose $(x, y) \notin R$ for $x, y \in S$. Since $y \notin R[x]$, it follows from 1 and $(S, \alpha) \in L$-BS that there is a clopen subset O of S such that $y \in O$ and $O \cap R[x] = \emptyset$. Define $f \in \mathrm{Cont}(S, \alpha)$ by $f(z) = 1$ for $z \notin O$ and $f(z) = 0$ for $z \in O$. Then, we have both $(\Box_R f)(x) = 1$ by $O \cap R[x] = \emptyset$ and $f(y) = 0$ by $y \in O$.

Assume 2. Let $x \in S$. In order to show 1, it suffices to show that for any $y \notin R[x]$ where $y \in S$, there is an open subset O of S such that $y \in O$ and $O \cap R[x] = \emptyset$. If $y \notin R[x]$ for $y \in S$, then it follows from 2 that there is $f \in \mathrm{Cont}(S, \alpha)$ such that $(\Box_R f)(x) = 1$ and $f(y) < 1$. Then, $f^{-1}(\{a \in L; a < 1\})$ is open by $f \in \mathrm{Cont}(S, \alpha)$. It follows from $(\Box_R f)(x) = 1$ that $f^{-1}(\{a \in L \,;\, a < 1\}) \cap R[x] = \emptyset$. By $f(y) < 1$, we have $y \in f^{-1}(\{a \in L \,;\, a < 1\})$. This completes the proof. $\Box$

Since the condition 1 in Proposition 3.4 does not depend on L, we have the following fact: For finite distributive lattices L_1 and L_2, if $(\mathrm{SubAlg}(L_1), \subset)$ and $(\mathrm{SubAlg}(L_2), \subset)$ are order isomorphic, then L_1-RS and L_2-RS are categorically equivalent. We do not know whether or not the converse holds.

4. Conclusions and future work

The theory of natural dualities led us to a duality for L-**VL**-algebras, based on which we showed a Jónsson-Tarski-style duality for L-**ML**-algebras, which is the main result in this paper. Topological perspectives following from the dualities gave compactness theorems for L-**VL** and L-**ML**, and the effective classification of categories L-VA's for finite distributive lattices L. We also obtained dualities for L-**VL**-algebras and L-**ML**-algebras enriched by truth constants. Throughout the paper, we have intended to demonstrate the slogan that Stone-type duality connects logic, algebra, and topology in both conceptual and technical senses. We hope that the reader has understood the meaning of the slogan and the significance of Stone-type duality.

The dualities developed in this paper are essentially different from those mentioned in [27, Section 4]. The dualities mentioned in [27] are based on the theory of canonical extensions ([19, 18]), while the dualities in this paper are based on the theory of natural dualities ([8]). In other words, L-**VL** is considered as a many-valued logic in the dualities in this paper, while L-**VL** is considered as a modal logic in the dualities mentioned in [27], where note that the theory of canonical extensions can be considered as a general theory of dualities for modal-like logics (or lattices with operators). Let us explain the difference in more details in the following three paragraphs.

In the dualities mentioned in [27], we consider U_a's as modalities and equip the dual space of an L-**VL**-algebra with canonical binary relations corresponding to U_a's for $a \in L$ (see [19, Subsection 2.3]), where recall that U_a's are inter-definable with T_a's. On the other hand, in the dualities presented in this paper, U_a's (or T_a's) are considered as the same kind of operations as the other operations of L-**VL**-algebras and the dual space $\mathrm{Spec}_L(A)$ of an L-**VL**-algebra A is equipped with no binary relation. This seems to be the most striking difference between the dualities mentioned in [27] and the dualities presented in this paper.

One of the most significant aspects of topological dualities is that we can understand the geometric meanings of logics or algebras by them. Since a duality becomes less geometric when a dual space is equipped with a relation, we may consider from the geometric point of view that it is better to equip a dual space with as less relations as possible and therefore the dualities developed in this paper are superior to the dualities mentioned in [27].

However, it also seems to be significant to see U_a's as modalities, i.e., L-**VL** as a multi-modal logic, and develop dualities for L-**VL**-algebras and L-**ML**-algebras in the way described above. The reasons are as follows. It seems interesting that a many-valued logic can be seen also as a modal logic. From the mathematical point of view, it confirms the applicability of the theory of canonical extensions, which is one of the most important duality theories along with the theory of natural dualities. From the philosophical point of view, it would contribute to our understanding of the notion of modality. Thus, our future work will be to develop dualities for L-**VL**-algebras and for L-**ML**-algebras by considering U_a's as modalities and using the theory of canonical extensions.

Finally, there is a philosophical remark on Theorem 2.5. Let L_1 and L_2 be finite distributive lattices. Theorem 2.5 implies that, even if L_1 and L_2 are not isomorphic, L_1-VA and L_2-VA can be categorically equivalent. This means that, even if L_1-**VL** and L_2-**VL** are not equivalent as logics, L_1-VA and L_2-VA can be categorically equivalent, which might contradict our intuition. By these facts, the identity of logics seems to be strictly weaker (as a relation) than the equivalence of categories of Lindenbaum algebras of logics (for a discussion on the identity of logics, see [33]).

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