Fundamental Results for Pointfree Convex Geometry*

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Abstract

Inspired by locale theory, we propose "pointfree convex geometry." We introduce the notion of convexity algebra as pointfree convexity space. There are two notions of point for convexity algebra: one is chain-prime meet-complete filter and the other is maximal meet-complete filter. In this paper we show the following: (1) the former notion of point induces a dual equivalence between the category of "spatial" convexity algebras and the category of "sober" convexity spaces as well as a dual adjunction between the category of convexity algebras and the category of convexity spaces; (2) the latter notion of point induces a dual equivalence between the category of "m-spatial" convexity algebras and the category of "m-sober" convexity spaces. We finally argue that the former notion of point is more useful than the latter one from a category theoretic point of view and that the former notion of point actually represents polytope (or generic point) and the latter notion of point properly represents point. We also remark about the close relationships between pointfree convex geometry and domain theory.

Keywords: pointfree geometry; categorical duality; dual adjunction; domain theory; convex geometry; polytope

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1 Introduction

Can you see any "point" in the space? The answer will be no. The notions of topological space and convexity space (explained below) presuppose that of point, which seems to be epistemologically ideal (see [20, 33, 34]). From the viewpoints of duality theory and algebraic geometry (see [22, 18]), we notice that a point amounts to a prime ideal (or a model in logical terms), which is an infinite entity, and we need some indeterministic principle such as (a weaker form of) the axiom of choice in order to show the existence of it and therefore the notion of point is very ideal. On the other hand, we can actually see "regions" of the space in some sense and, from the viewpoints of duality theory and algebraic geometry, a (basic) region can be identified with an algebraic formula, which is a finite entity. Hence, the notion of region seems to be epistemologically more certain than that

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of point (where it is supposed that one notion is epistemologically more certain than another if the former precedes the latter in the human knowledge).

This leads us to the notion of region-based pointfree space. There are several ways to realize this notion in a mathematical form. In mathematics we often encounter the following phenomenon: a space is recovered from the function algebra on it (a space has the same information as the function algebra on it). Such spaces include manifolds, compact Hausdorff spaces, and affine schemes (see [18, 22]). Moreover, geometric notions can often be translated into algebraic ones via correspondence between space and algebra (for example, in algebraic geometry, the dimension of an algebraic variety corresponds to the Krull dimension of the coordinate ring of it). These facts give us the idea "Algebra itself is space." This idea has already been pursued in several areas of mathematics such as non-commutative geometry (see [5]) and we also follow it in this paper.

Locale theory can be considered as an algebraic theory of topological structures which does not presuppose the notion of point and is primarily based on that of region, since locale theory studies the lattice structure of open sets in an algebraic way, i.e., a "space" in locale theory is a join-complete lattice with finite meets that distribute over arbitrary joins, which is called a frame (for locale theory, see [22, 24, 25, 27, 28, 31, 32]). Usually, localic versions of theorems in the ordinary topology do not need non-constructive principles such as the law of excluded middle or the axiom of choice and so locale theory can also be seen as constructive topology (see [1, 7, 9, 10]). There are two fundamental results of locale theory (see [22, 1, 6]): (i) a dual adjunction between the category of frames and the category of topological spaces; (ii) a dual equivalence between the category of spatial frames and the category of sober topological spaces, which is sometimes called Isbell duality (see [4]).

Along with topology, convex geometry has been studied extensively from different perspectives (see [8, 16, 30]). Among many results of convex geometry, Helly-type theorems (see [14]) are important, especially for combinatorial convex geometry. They characterize the dimension of an Euclidean space and so seem to be significant from a philosophical as well as a mathematical point of view. Helly-type theorems can be extended to the case of convexity spaces (see [8]), which are defined as a set S equipped with a subset $\mathcal{C} \subset 2^S$, a convexity, satisfying some conditions (see Definition 2.1). These results contribute to our understanding of the notion of dimension, clarifying the convexity theoretical meaning of it. There have been many more studies on convexity spaces than mentioned above (see [8, 23, 30]). Note that the notion of topological space in topology corresponds to that of convexity space in convex geometry.

Inspired by locale theory, we propose "pointfree convex geometry", toward which this paper takes a first step (for related categorical work, see [21]). Pointfree convex geometry is an algebraic theory of convex structures which does not presuppose the notion of point and is primarily based on that of region. Pointfree convex geometry studies the lattice structure of convex sets in an algebraic way, i.e., a "space" in pointfree convex geometry is a meet-complete poset with joins of chains that distribute over arbitrary meets, which we call a convexity algebra. The notion of point does not appear explicitly, but, if we want, we can consider points of pointfree spaces and recover "all" points under certain assumptions.

We emphasize that there are two ways to recover points. One way is to consider a chain-prime meet-complete filter (cp-mc filter, for short) as a point. The other way is to consider a maximal meet-complete filter (m-mc filter, for short) as a point. These two views on the notion of point induce two kinds of categorical dualities between some convexity algebras and some convexity spaces. The following are the main results in this paper:

- a dual adjunction between the category of convexity algebras and homomorphisms and the category of convexity spaces and convexity preserving maps (Theorem 3.9);
- a dual equivalence between the category of spatial convexity algebras and homomorphisms and the category of sober convexity spaces and convexity preserving maps (Theorem 4.18);
- a dual equivalence between the category of m-spatial convexity algebras and m-homomorphisms and the category of m-sober convexity spaces and convexity preserving maps (Theorem 5.18), where note that convexity preserving maps between m-sober convexity spaces correspond to m-homomorphisms, not homomorphisms.

These results are considered as fundamental for pointfree convex geometry as (i) and (ii) above are for locale theory. These results clarify the categorical relationships between pointfree spaces and pointset spaces, or epistemological and ontological aspects of the notion of space (they might be considered to be almost equivalent from a mathematical point of view). For more discussion on these results, we refer the reader to Section 6.

Our investigation in this paper proceeds as follows. In Section 2, we first review the concept of convexity space and then introduce the concept of convexity algebra as a pointfree analogue of a convexity space and related concepts of filter. In Section 3, we obtain a dual adjunction between the category of convexity algebras and the category of convexity spaces, which is based on the view that a point is a cp-mc filter. In Section 4, by introducing the concepts of spatiality and sobriety, we obtain a duality between the category of spatial convexity algebras and the category of sober convexity spaces, which is based on the view that a point is a cp-mc filter. An algebraic characterization of spatiality is also provided. We remark that Euclidean spaces are sober topological spaces and are not sober convexity spaces (the same thing holds also for the other spaces in Example 2.2), whence the sobriety of convexity seems to be considerably different from the sobriety of topology. We also show that a convexity algebra is filter-closed iff there is no infinite descending chain in it and that an analogue of the prime filter theorem for distributive lattices holds for filter-closed convexity algebras. In Section 5, by introducing the concepts of m-homomorphism, m-spatiality and m-sobriety, we obtain a duality between the category of m-spatial convexity algebras and m-homomorphisms and the category of m-sober convexity spaces and convexity preserving maps, which is based on the view that a point is an m-mc filter. We also give an algebraic characterization of m-spatiality. We remark that many ordinary convexity spaces such as Euclidean spaces are m-sober even if they are not sober. In Section 6, we discuss the questions "Which notion of point is better? Which notion of point is the proper one?" and also the relationships between pointfree geometry and Hilbert's instrumentalism or Husserl's phenomenology. In this section we also remark that pointfree convex geometry is closely related to domain theory (in some sense it coincides with the theory of continuous lattices).

2 Convexity Spaces and Convexity Algebras

In this section we review basics of convexity spaces and introduce the notion of convexity algebra with related basic concepts and propositions.

2.1 Convexity Spaces

We first review the notion of convexity space. For more detailed exposition, see the books [8, 30]. Convexity spaces are sometimes called aligned spaces as in [8].

Let 2^X denote the powerset of X.

Definition 2.1 ([8, 30])**.** *For a set S and a subset $\mathcal{C}$ of 2^S, $(S, \mathcal{C})$ is a convexity space iff $(S, \mathcal{C})$ satisfies the following conditions:*

1. *$\mathcal{C}$ is closed under arbitrary intersections;*
2. *if $\{X_i \in \mathcal{C}\ ;\ i \in I\}$ is totally ordered, then $\bigcup\{X_i\ ;\ i \in I\} \in \mathcal{C}$.*

We call $\mathcal{C}$ the convexity of S and an element of $\mathcal{C}$ a convex set in S. The complement of a convex set in S is called a concave set in S.

Note that $\emptyset, S \in \mathcal{C}$ by letting the index sets be empty in the above conditions.

A convexity space $(S, \mathcal{C})$ is often denoted by its underlying set S.

Let us denote by **2** the two-element distributive lattice $\{0, 1\}$ equipped with the Sierpiński convexity $\{\emptyset, \{1\}, \{0, 1\}\}$.

Example 2.2. *Consider a vector space V over the real number field $\mathbb{R}$. We can equip V with a natural convexity determined by the condition that $X \subset V$ is convex iff for any $x, y \in X$ and any $t \in [0, 1]$, $tx + (1 - t)y \in X$. In particular, the n-dimensional Euclidean space $\mathbb{R}^n$ for an integer $n \geq 1$ is naturally equipped with a convexity in this way.*

Consider the n-sphere S^n for an integer $n \geq 1$. We can equip S^n with a natural convexity determined by the condition that $X \subset S^n$ is convex iff the following hold: (i) for any $x, y \in X$, the antipodal point of x is not y; (ii) for any $x, y \in X$, the shortest path (i.e., geodesic) between x and y on S^n is a subset of X.

We can also equip the n-dimensional real projective space with a convexity (see [8, Example 6.2.7] or [29]).

By the condition 1 in Definition 2.1, we can define the convex hull of a subset of a convexity space as follows.

Definition 2.3 ([8, 30])**.** *Let $(S, \mathcal{C})$ be a convexity space. For $A \subset S$, define*

$$\mathrm{ch}(A) = \bigcap\{C \in \mathcal{C}\ ;\ A \subset C\}.$$

Then, $\mathrm{ch}(A)$ is called the convex hull of A.

As usual we define a morphism of convexity spaces as follows.

Definition 2.4 ([30, 23])**.** *Let $(S, \mathcal{C})$, $(S', \mathcal{D})$ be convexity spaces. A map $f : S \to S'$ is a convexity preserving map iff, for any $D \in \mathcal{D}$, we have $f^{-1}(D) \in \mathcal{C}$.*

This definition of morphism of convexity spaces seems to be most popular, though other definitions may be possible. Even if a stronger definition of morphism of convexity spaces is employed, our duality results still work by restricting the morphisms parts of the related categories.

Note that the inverse map of a bijective convexity preserving map is not necessarily convexity preserving.

By the following proposition, we can consider the set of convex sets in a convexity space as the hom-set from the convexity space to **2**.

Proposition 2.5. *Let $(S, \mathcal{C})$ be a convexity space. Then, there is a natural bijection between the set $\mathcal{C}$ of all convex sets in S and the set of all convexity preserving maps from S to $\mathbf{2}$.*

Proof. For a convex set C in S, define $f_C : S \to \mathbf{2}$ by

$$f_C(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{otherwise.} \end{cases}$$

Then it is clear that f_C is a convexity preserving map and that if $C \neq D$ for convex sets C and D in S then we have $f_C \neq f_D$. Thus, the map $C \mapsto f_C$ is injective. To show the surjectivity, let g be a convexity preserving map from S to $\mathbf{2}$. Define $C = g^{-1}(\{1\})$. Then it is clear that C is a convex set in S and that $f_C = g$. This completes the proof. □

The notion of polytope is defined as follows.

Definition 2.6 ([30])**.** *Let S be a convexity space. A non-empty subset P of S is called a polytope in S iff P is the convex hull of a finite subset of S.*

By [30, Theorem 1.6], we have the following proposition, which characterize a polytope in a convexity space as a subset satisfying a certain condition similar to compactness in topology.

Proposition 2.7. *For a convexity space $(S, \mathcal{C})$, the following are equivalent.*

1. *$P \in \mathcal{C}$ is a polytope in S;*
2. *if $P = \bigcup_{i \in I} C_i$ for a totally ordered set $\{C_i \in \mathcal{C}\,;\, i \in I\}$, then there is $i \in I$ such that $P = C_i$.*

By [30, Proposition 1.7.1], we have the following proposition.

Proposition 2.8. *Let S be a convexity space. Any convex set in S is the union of a directed set of polytopes in S.*

Thus, the set of all polytopes in a convexity space forms the canonical base of the convexity space. There is no such canonical base of a topological space in general. This is a striking difference between topology and convex geometry.

The notion of polytope shall play a crucial role in our investigation. For instance, the polytopes in a convexity space can be used for "sobrification" of the convexity space as we shall see later.

2.2 Convexity Algebras

Now we define the notion of convexity algebra, which is considered as pointfree convexity space and is analogous to the notion of frame, which is pointfree topological space, in locale theory (for basic concepts of lattice theory, see [12]).

Definition 2.9. *A poset L is a convexity algebra iff it satisfies the following properties:*

1. *L has arbitrary meets;*
2. *if $\{x_i \in L\,;\, i \in I\}$ is totally ordered in L, then $\{x_i\,;\, i \in I\}$ has a join in L;*

3. *for any doubly indexed family* $\{x_{i,j} \in L \,;\, i \in I \text{ and } j \in J_i\}$*, if* $\{x_{i,j} \,;\, j \in J_i\}$ *is totally ordered for every* $i \in I$ *and if* $\{\bigwedge_{i\in I} x_{i,f(i)} \,;\, f \in F\}$ *is totally ordered, then*

$$\bigwedge_{i\in I} \bigvee_{j\in J_i} x_{i,j} = \bigvee_{f\in F} \bigwedge_{i\in I} x_{i,f(i)}$$

where $F = \prod_{i\in I} J_i$ $(= \{f : I \to \bigcup_{i\in I} J_i \,;\, \forall i \in I\ f(i) \in J_i\})$.

Note that a convexity algebra has the least element 0 and the greatest element 1 by letting the index sets be empty in the conditions 1 and 2 above.

We call the condition 3 in the above definition the chain-completely distributive law.

Definition 2.10. *Let* L_1 *and* L_2 *be convexity algebras. A function* $f : L_1 \to L_2$ *is a homomorphism from* L_1 *to* L_2 *iff it satisfies the following properties:*

1. $f(\bigwedge_{i\in I} a_i) = \bigwedge_{i\in I} f(a_i)$ *for any* $\{a_i \,;\, i \in I\} \subset L_1$*;*

2. *if* $\{a_i \in L_1 \,;\, i \in I\}$ *is totally ordered, then* $f(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} f(a_i)$.

Note that for a homomorphism f of convexity algebras, we have $f(0) = 0$ and $f(1) = 1$ by letting the index sets be empty in the above conditions.

We can easily verify the following proposition.

Proposition 2.11. *Let* $(S, \mathcal{C})$ *be a convexity space. Then,* $\mathcal{C}$ *forms a convexity algebra (when equipped with set-theoretical operations).*

Next we define the concepts of meet-complete filter and chain-prime meet-complete filter (cp-mc filter for short), which correspond to filter and completely prime filter respectively in locale theory.

Definition 2.12. *Let* L *be a convexity algebra. A subset* F *of* L *is called a meet-complete filter of* L *iff the following hold:*

1. *if* $a \in F$ *and* $a \le x$ *then* $x \in F$*;*

2. *if* $\{a_i \,;\, i \in I\} \subset F$*, then* $\bigwedge_{i\in I} a_i \in F$.

Note that a meet-complete filter F is non-empty, since we have $1 \in F$ by the condition 2 applied to the empty family.

A meet-complete filter may also be called Moore filter, since the related notion of Moore family is well known.

Definition 2.13. *Let* L *be a convexity algebra. A subset* P *of* L *is called a chain-prime meet-complete filter (cp-mc filter for short) of* L *iff the following hold:*

1. P *is a meet-complete filter of* L*;*

2. *if* $\{a_i \in L \,;\, i \in I\}$ *is totally ordered and* $\bigvee_{i\in I} a_i \in P$*, then there is* $i \in I$ *with* $a_i \in P$.

Note that any cp-mc filter P of L is not L, since we have $0 \notin P$ by the condition 2 applied to the empty family.

For a convexity space $(S, \mathcal{C})$ and $x \in S$, $\{C \in \mathcal{C} \,;\, x \in C\}$ is a cp-mc filter of the convexity algebra $\mathcal{C}$.

The notion of chain-algebraicity is defined as follows.

Definition 2.14. *Let L be a convexity algebra.*

For $a \in L$, a is said to be a chain-compact element in L iff if $a \le \bigvee_{i\in I} a_i$ for a totally ordered subset $\{a_i\ ;\ i \in I\}$ of L then there exists $i \in I$ such that $a \le a_i$.

L is called chain-algebraic iff for any $a \in L$ there is a directed set $\{a_i\ ;\ i \in I\}$ of chain-compact elements in L such that $a = \bigvee\{a_i\ ;\ i \in I\}$.

Note that any chain-compact element is not the least element, which is shown by letting $I = \emptyset$ in its definition.

By Proposition 2.7, we have the following.

Proposition 2.15. *Let $(S, \mathcal{C})$ be a convexity space. Then, $P \in \mathcal{C}$ is a polytope in S iff P is a chain-compact element in the convexity algebra $\mathcal{C}$.*

Proposition 2.16. *Let L be a convexity algebra. Then, there is a natural bijection between the set of all cp-mc filters of L and the set of all chain-compact elements in L.*

Proof. For a cp-mc filter P of L, $\bigwedge P$ is a chain-compact element in L and, for cp-mc filters P, Q of L with $P \neq Q$, we have $\bigwedge P \neq \bigwedge Q$. Thus, the map $P \mapsto \bigwedge P$ is injective. In order to show the surjectivity, let a be a chain-compact element in L. Then, $\{x \in L\ ;\ a \le x\}$ is a cp-mc filter and also we have $\bigwedge\{x \in L\ ;\ a \le x\} = a$. This completes the proof. □

Actually, any meet-complete filter F is principal (i.e., it is generated by an element, namely $\bigwedge F$), though $\bigwedge F$ is not necessarily chain-compact.

Proposition 2.17. *Let L be a convexity algebra. Then, there is a natural bijection between the set of all cp-mc filters of L and the set of all homomorphisms from L to $\mathbf{2}$.*

Proof. For a cp-mc filter P, define $v_P : L \to \mathbf{2}$ by

$$v_P(x) = \begin{cases} 1 & \text{if } x \in P \\ 0 & \text{otherwise.} \end{cases}$$

Then it is straightforward to verify that v_P is a homomorphism and that if $P \neq Q$ for cp-mc filters P, Q then we have $v_P \neq v_Q$. Therefore, the map $P \mapsto v_P$ is injective. To show the surjectivity, let u be a homomorphism from L to $\mathbf{2}$. Define $P = u^{-1}(\{1\})$. Then it is straightforward to verify that P is a cp-mc filter and that $v_P = u$. This completes the proof. □

By Proposition 2.17 and Proposition 2.16, we do not distinguish between cp-mc filters of L, chain-compact elements in L, and homomorphisms from L to $\mathbf{2}$ for a convexity algebra L.

3 Dual Adjunction between CA and CS

In this section, we show a dual adjunction between categories **CA** and **CS**, which are defined as follows.

Definition 3.1. **CA** *denotes the category of convexity algebras and homomorphisms.*

CS *denotes the category of convexity spaces and convexity preserving maps.*

We then introduce functors Spec and Conv.

Definition 3.2. *We define a contravariant functor* Spec *from* **CA** *to* **CS** *as follows:*

1. *For an object L in* **CA**, $\mathrm{Spec}(L)$ *is defined as the set of all homomorphisms from L to* **2** *equipped with the convexity generated by $\{\langle a\rangle \,;\, a \in L\}$ where*

$$\langle a\rangle = \{v \,;\, v(a) = 1 \text{ and } v : L \to \mathbf{2} \text{ is a homomorphism }\}.$$

2. *For an arrow $f : L_1 \to L_2$ in* **CA**, $\mathrm{Spec}(f) : \mathrm{Spec}(L_2) \to \mathrm{Spec}(L_1)$ *is defined by*

$$\mathrm{Spec}(f)(v) = v \circ f$$

for $v \in \mathrm{Spec}(L_2)$.

By Proposition 2.17, we can consider $\mathrm{Spec}(L)$ as the set of all cp-mc filters of L equipped with the convexity generated by the $\{P \in \mathrm{Spec}(L) \,;\, a \in P\}$'s for $a \in L$.

The well-definedness of the functor Spec is proven by the following lemma.

Lemma 3.3. *Let $f : L_1 \to L_2$ be an arrow in* **CA**. *Then,* $\mathrm{Spec}(f) : \mathrm{Spec}(L_2) \to \mathrm{Spec}(L_1)$ *is a convexity preserving map.*

Proof. It suffices to show that, for any $a \in L_1$, $(\mathrm{Spec}(f))^{-1}(\langle a\rangle)$ is a convex set in $\mathrm{Spec}(L_2)$. Now we have

$$\begin{aligned}(\mathrm{Spec}(f))^{-1}(\langle a\rangle) &= \{v \in \mathrm{Spec}(L_2) \,;\, \mathrm{Spec}(f)(v) \in \langle a\rangle\} \\ &= \{v \in \mathrm{Spec}(L_2) \,;\, v(f(a)) = 1\} \\ &= \langle f(a)\rangle.\end{aligned}$$

This completes the proof. □

The following lemma tells us that $\langle\text{-}\rangle$ preserves the operations of convexity algebras.

Lemma 3.4. *Let L be a convexity algebra. For $\{a_i \,;\, i \in I\} \subset L$, $\langle \bigwedge_{i\in I} a_i\rangle = \bigcap_{i\in I}\langle a_i\rangle$. For a totally ordered subset $\{a_i \,;\, i \in I\}$ of L, $\langle \bigvee_{i\in I} a_i\rangle = \bigcup_{i\in I}\langle a_i\rangle$.*

Proof. This follows immediately from the fact that a homomorphism of convexity algebras preserves arbitrary meets and joins of totally ordered sets. □

The following lemma plays an important role in our duality theory for pointfree convex geometry.

Lemma 3.5. *For a convexity algebra L, the convexity of* $\mathrm{Spec}(L)$ *coincides with $\{\langle a\rangle \,;\, a \in L\}$.*

Proof. By the definition of the convexity of $\mathrm{Spec}(L)$, it suffices to prove that $\{\langle a\rangle \,;\, a \in L\}$ satisfies the two conditions in Definition 2.1.

First, for any subset $\{\langle a_i\rangle \,;\, i \in I\}$ of $\{\langle a\rangle \,;\, a \in L\}$, it follows from Lemma 3.4 that

$$\bigcap_{i\in I}\langle a_i\rangle = \langle \bigwedge_{i\in I} a_i\rangle \in \{\langle a\rangle \,;\, a \in L\},$$

whence $\{\langle a\rangle \,;\, a \in L\}$ is closed under arbitrary intersections.

Second, assume that a subset $\{\langle a_i\rangle \,;\, i \in I\}$ of $\{\langle a\rangle \,;\, a \in L\}$ is totally ordered with respect to inclusion. We show that $\bigcup_{i\in I}\langle a_i\rangle \in \{\langle a\rangle \,;\, a \in L\}$. For each $i \in I$, define

$$b_i = \bigwedge\{a_j \,;\, \langle a_i\rangle \subset \langle a_j\rangle \text{ and } j \in I\}.$$

Note that $\langle b_i\rangle = \langle a_i\rangle$. If $\langle a_k\rangle \subset \langle a_l\rangle$ for $k, l \in I$ then we have $b_k \leq b_l$. Thus, since $\{\langle a_i\rangle \,;\, i \in I\}$ is totally ordered with respect to inclusion, $\{b_i \,;\, i \in I\}$ is totally ordered with respect to the partial order of L. Therefore, by Lemma 3.4, we have

$$\bigcup_{i\in I}\langle a_i\rangle = \bigcup_{i\in I}\langle b_i\rangle = \langle\bigvee_{i\in I} b_i\rangle \in \{\langle a\rangle \,;\, a \in L\}.$$

Hence, $\{\langle a\rangle; a \in L\}$ is closed under unions of totally ordered subsets. This completes the proof. □

Definition 3.6. *We define a contravariant functor* Conv *from* **CS** *to* **CA** *as follows:*

1. *For an object S in* **CS**, $\mathrm{Conv}(S)$ *is defined as the set of all convexity preserving maps from S to* **2** *equipped with the pointwise operations. For instance, given $f_i \in \mathrm{Conv}(S)$ for $i \in I$, $\bigwedge_{i\in I} f_i \in \mathrm{Conv}(S)$ is defined by*

$$(\bigwedge_{i\in I} f_i)(x) = \bigwedge_{i\in I} f_i(x).$$

2. *For an arrow $f : S \to S'$ in* **CS**, $\mathrm{Conv}(f) : \mathrm{Conv}(S') \to \mathrm{Conv}(S)$ *is defined by*

$$\mathrm{Conv}(f)(g) = g \circ f$$

   for $g \in \mathrm{Conv}(S')$.

For a convexity space S, we can consider $\mathrm{Conv}(S)$ as the set of all convex sets equipped with set-theoretical operations by Proposition 2.5. Note that in this case, the arrow part of the functor Conv can be defined by

$$\mathrm{Conv}(f)(C) = f^{-1}(C)$$

for $C \in \mathrm{Conv}(S')$. The two definitions of Conv are essentially equivalent and we do not have to distinguish between them.

The well-definedness of the functor Conv is proven as follows: first, $\mathrm{Conv}(S)$ forms a convexity algebra by Proposition 2.11 and Proposition 2.5; second, $\mathrm{Conv}(f)$ is a homomorphism, since all the operations of $\mathrm{Conv}(S)$ are defined pointwise.

For a category **C**, let $1_{\mathbf{C}}$ denote the identity functor from **C** to **C**.

Definition 3.7. *We define a natural transformation* $\Phi : 1_{\mathbf{CA}} \to \mathrm{Conv} \circ \mathrm{Spec}$ *as follows. For a convexity algebra L, define* $\Phi_L : L \to \mathrm{Conv} \circ \mathrm{Spec}(L)$ *by*

$$\Phi_L(a)(v) = v(a)$$

for $a \in L$ and $v \in \mathrm{Spec}(L)$.

Then, Φ_L is well-defined, since $\Phi_L(a)$ is a convexity preserving map by the following fact

$$\Phi_L(a)^{-1}(\{1\}) = \{v \in \mathrm{Spec}(L)\ ;\ v(a) = 1\} = \langle a \rangle.$$

It is straightforward to verify that Φ_L is a homomorphism. It is proven by direct computation that Φ is a natural transformation.

Definition 3.8. *We define a natural transformation* $\Psi : 1_{\mathbf{CS}} \to \mathrm{Spec} \circ \mathrm{Conv}$ *as follows. For a convexity space* S*, define* $\Psi_S : S \to \mathrm{Spec} \circ \mathrm{Conv}(S)$ *by*

$$\Psi_S(x)(f) = f(x)$$

for $x \in S$ *and* $f \in \mathrm{Conv}(S)$*.*

Then, Ψ_S is well-defined, since $\Psi_S(x)$ is a homomorphism by the pointwiseness of the operations of $\mathrm{Conv}(S)$. Moreover, Ψ_S is a convexity preserving map by the following fact

$$\Psi_S^{-1}(\langle f \rangle) = \{x \in S\ ;\ \Psi_S(x) \in \langle f \rangle\} = f^{-1}(\{1\})$$

for $f \in \mathrm{Conv}(S)$. It is proven by direct computation that Ψ is a natural transformation.

Now we show that Spec and Conv give a dual adjunction between **CA** and **CS**.

Theorem 3.9. Spec *is left adjoint to* $\mathrm{Conv}^{\mathrm{op}}$*.*

Proof. Let L be a convexity algebra and S a convexity space. Assume that f is a homomorphism from L to $\mathrm{Conv}(S)$. It suffices to show that there is a unique arrow $g : S \to \mathrm{Spec}(L)$ in **CS** such that $\mathrm{Conv}(g) \circ \Phi_L = f$. Now define a map $g : S \to \mathrm{Spec}(L)$ by

$$g(x)(a) = \Psi_S(x)(f(a))$$

for $x \in S$ and $a \in L$. Then, since f and $\Psi_S(x)$ are homomorphisms, we have $g(x) \in \mathrm{Spec}(L)$. Moreover, g is a convexity preserving map by $f(a) \in \mathrm{Conv}(S)$ and the following fact

$$\begin{aligned} g^{-1}(\langle a \rangle) &= \{x \in S\ ;\ g(x) \in \langle a \rangle\} \\ &= \{x \in S\ ;\ g(x)(a) = 1\} \\ &= \{x \in S\ ;\ f(a)(x) = 1\} \\ &= f(a)^{-1}(\{1\}). \end{aligned}$$

For $a \in L$ and $x \in S$, we have

$$\begin{aligned} (\mathrm{Conv}(g) \circ \Phi_L)(a)(x) &= \Phi_L(a) \circ g(x) \\ &= g(x)(a) \\ &= f(a)(x). \end{aligned}$$

Hence, we have $\mathrm{Conv}(g) \circ \Phi_L = f$.

To show the uniqueness, suppose that h is a convexity preserving map from S to $\mathrm{Spec}(L)$ and that $\mathrm{Conv}(h) \circ \Phi_L = f$. Then, it follows that $\mathrm{Conv}(h) \circ \Phi_L = f = \mathrm{Conv}(g) \circ \Phi_L$. Here, for $a \in L$ and $x \in S$, we have

$$(\mathrm{Conv}(h) \circ \Phi_L)(a)(x) = \Phi_L(a) \circ h(x) = h(x)(a).$$

Similarly we have

$$(\mathrm{Conv}(g) \circ \Phi_L)(a)(x) = g(x)(a).$$

Thus, we conclude that $h = g$. □

Recall that a left adjoint functor preserves colimits and a right adjoint functor preserves limits (see [3]). Thus, some categorical constructions in one category can be transferred into the other category via the above adjunction.

4 Duality between SpCA and SobCS

In this section, we introduce the notions of spatial convexity algebra and sober convexity space. Then, by restricting the dual adjunction between **CA** and **CS**, we shall show a duality between the category **SpCA** of spatial convexity algebras and the category **SobCS** of sober convexity spaces.

4.1 Spatiality

We define the notion of spatiality as the existence of "enough" cp-mc filters:

Definition 4.1. *For a convexity algebra L, L is spatial iff, for any $a, b \in L$ with $a \not\leq b$, there is a cp-mc filter P of L such that $a \in P$ and $b \notin P$.*

We can characterize the spatiality of a convexity algebra L as the injectivity of Φ_L.

Lemma 4.2. *Let L be a convexity algebra. The following are equivalent:*

1. *L is spatial;*
2. *Φ_L is injective, i.e., for any $a, b \in L$ with $a \neq b$ there is $v \in \mathrm{Spec}(L)$ with $v(a) \neq v(b)$;*
3. *if $\langle a \rangle \subset \langle b \rangle$ for $a, b \in L$, then $a \leq b$.*

Proof. By Proposition 2.17, it is straightforward to show that 1 implies 2 and that 3 implies 1. We show that 2 implies 3. Assume 2. To show the contrapositive of 3, assume $a \not\leq b$. Then, since Φ_L is an injective homomorphism, we have $\Phi_L(a) \not\leq \Phi_L(b)$. Therefore, there is $v \in \mathrm{Spec}(L)$ such that

$$v(a) = \Phi_L(a)(v) > \Phi_L(b)(v) = v(b).$$

Hence we have $v \in \langle a \rangle$ and $v \notin \langle b \rangle$. This completes the proof. □

The following proposition provides many natural examples of spatial convexity algebras.

Proposition 4.3. *Let S be a convexity space. Then, $\mathrm{Conv}(S)$ is a spatial convexity algebra.*

Proof. Let $f, g \in \mathrm{Conv}(S)$ with $f \neq g$. Then, we have $f(x) \neq g(x)$ for some $x \in S$. Let $v = \Psi_S(x)$. Then, we have

$$\Phi_{\mathrm{Conv}(S)}(f)(v) = v(f) = f(x).$$

We also have

$$\Phi_{\mathrm{Conv}(S)}(g)(v) = v(g) = g(x).$$

Thus, $\Phi_{\mathrm{Conv}(S)}$ is injective and so $\mathrm{Conv}(S)$ is spatial by Lemma 4.2. □

In fact, Φ_L is always surjective as follows.

Lemma 4.4. *Let L be a convexity algebra. Then, Φ_L is surjective.*

Proof. Based on Proposition 2.5, we can consider $\mathrm{Conv} \circ \mathrm{Spec}(L)$ as the set of all convex sets in $\mathrm{Spec}(L)$. Thus, since $\Phi_L(a)^{-1}(\{1\}) = \langle a \rangle$, we can consider $\Phi_L(a) = \langle a \rangle$. Then, it follows from Lemma 3.5 that

$$\{\langle a \rangle \; ; \; a \in L\} = \mathrm{Conv} \circ \mathrm{Spec}(L).$$

Hence, Φ_L is surjective by $\Phi_L(a) = \langle a \rangle$. □

By Lemma 4.2, Proposition 4.3 and Lemma 4.4, we obtain the following proposition (recall that Φ_L is a homomorphism for any convexity algebra L).

Proposition 4.5. *For a convexity algebra L, L is spatial iff $\Phi_L : L \to \mathrm{Conv} \circ \mathrm{Spec}(L)$ is an isomorphism.*

This proposition implies that any spatial convexity algebra can be represented as the convexity algebra of convex sets in a convexity space.

We can provide an algebraic characterization of spatiality as follows.

Proposition 4.6. *Let L be a convexity algebra. Then, L is spatial iff L is chain-algebraic.*

Proof. Assume that L is spatial. By Proposition 4.5, L is isomorphic to the convexity algebra of all convex sets in a convexity space, which is shown to be chain-algebraic by combining Proposition 2.15 and Proposition 2.8.

Assume that L is chain-algebraic. Let $a, b \in L$ with $a \not\leq b$. Let A be the set of chain-compact elements that are less than or equal to a and B the set of chain-compact elements that are less than or equal to b. Since L is chain-algebraic, we have $a = \bigvee A$ and $b = \bigvee B$. Therefore, it follows from $a \not\leq b$ that there is $c \in A$ such that $c \notin B$. Define $P = \{x \in L \; ; \; c \leq x\}$. Then, since c is a chain-compact element, P is a cp-mc filter by Proposition 2.16 and also we have both $a \in P$ and $b \notin P$. Thus, L is spatial. □

Definition 4.7. *Let L be a convexity algebra and $\mathrm{MCF}(L)$ the set of all meet-complete filters of L. Then, L is filter-closed iff for any non-empty totally ordered subset $\{X_i \; ; \; i \in I\}$ of $\mathrm{MCF}(L)$, $\bigcup_{i \in I} X_i$ is a meet-complete filter.*

For instance, every successor ordinal is a filter-closed convexity algebra and also the finite product of successor ordinals is a filter-closed convexity algebra. More generally, we have the following characterization of filter-closed convexity algebra.

Proposition 4.8. *For a convexity algebra L, L is filter-closed iff there is no infinite descending chain in L.*

Proof. We first show that filter-closedness implies the non-existence of an infinite descending chain. In order to prove the contrapositive, assume that there is an infinite descending chain $\{a_i \in L \, ; \, i \in I\}$. Then, we have $\bigwedge_{i \in I} a_i < a_k$ for any $k \in I$, since if not, then there is $k \in I$ such that $a_k \leq a_i$ for any $i \in I$, i.e., $\{a_i \in L \, ; \, i \in I\}$ is not an infinite descending chain. Define

$$A_i = \{x \in L \, ; \, a_i \leq x\},$$

which is a meet-complete filter. Clearly, $\{A_i \; ; \; i \in I\}$ is totally ordered. Moreover, $\bigcup_{i \in I} A_i$ is not a meet-complete filter, since we have both $a_i \in \bigcup_{i \in I} A_i$ for any $i \in I$ and $\bigwedge_{i \in I} a_i \notin \bigcup_{i \in I} A_i$ by the fact that $\bigwedge_{i \in I} a_i < a_k$ for any $k \in I$. Therefore, L is not filter-closed.

To show the converse, assume that there is no infinite descending chain in L. Let $\{X_i\,;\, i \in I\}$ be a non-empty totally ordered subset of $\mathrm{MCF}(L)$. Since any meet-complete filter X is generated by $\bigwedge X$, $\{\bigwedge X_i\,;\, i \in I\}$ is totally ordered in L. However, it follows from assumption that $\{\bigwedge X_i\,;\, i \in I\}$ is not an infinite descending chain. Thus, $\{X_i\,;\, i \in I\}$ is not an infinite ascending chain. Then there is $j \in I$ such that

$$X_j = \bigcup_{i \in I} X_i.$$

Hence, $\bigcup_{i \in I} X_i$ is a meet-complete filter. Thus, L is filter-closed. □

An analogue of the prime filter theorem for distributive lattices holds for filter-closed convexity algebras.

Proposition 4.9. *Let L be a filter-closed convexity algebra. Then, L is spatial.*

Proof. Let $a, b \in L$ with $a \not\leq b$. Let $\mathcal{H}$ be the set of meet-complete filters F of L such that $a \in F$ and $b \notin F$. Since $\{x \in L\,;\, a \leq x\} \in \mathcal{H}$, $\mathcal{H}$ is not empty. Since L is filter-closed, every totally ordered subset $\{F_i\,;\, i \in I\}$ of $\mathcal{H}$ has an upper bound $\bigcup_{i \in I} F_i$ in $\mathcal{H}$. Thus, by Zorn's lemma, we have a maximal element P in $\mathcal{H}$. Clearly, $a \in P$ and $b \notin P$.

In order to complete the proof, we show that P is a cp-mc filter of L. Let $\{a_i\,;\, i \in I\}$ be a totally ordered subset of L and $\bigvee_{i \in I} a_i \in P$. Suppose for contradiction that $a_i \notin P$ for any $i \in I$. Then, it follows from the maximality of M that for every $i \in I$ there exists $p_i \in P$ such that $a_i \wedge p_i \leq b$. Let

$$p = \bigwedge_{i \in I} p_i.$$

Since P is a meet-complete filter, we have $p \in P$. Clearly, $a_i \wedge p \leq b$. Hence, we have

$$\bigvee_{i \in I} (a_i \wedge p) \leq b.$$

It follows from the chain-completely distributive law (i.e., the item 3 in Definition 2.9) that

$$\bigvee_{i \in I} (a_i \wedge p) = (\bigvee_{i \in I} a_i) \wedge p.$$

Since $\bigvee_{i \in I} a_i \in P$ and $p \in P$, we have $(\bigvee_{i \in I} a_i) \wedge p \in P$ and so $\bigvee_{i \in I} (a_i \wedge p) \in P$. By $\bigvee_{i \in I} (a_i \wedge p) \leq b$, we have $b \in P$, which is a contradiction. Thus, P is a cp-mc filter. Hence, L is spatial. □

4.2 Sobriety

In order to define sober convexity space, we first define chain-irreducible convex set, whose role in our duality theory is analogous to that of irreducible closed set in Isbell duality.

Definition 4.10. *Let $(S, \mathcal{C})$ be a convexity space. A convex set C in $\mathcal{C}$ is said to be chain-irreducible iff if $C = \bigcup_{i \in I} C_i$ for a totally ordered subset $\{C_i\,;\, i \in I\}$ of $\mathcal{C}$ then there exists $i \in I$ such that $C = C_i$.*

Note that a convex set in a convexity space $(S, \mathcal{C})$ is chain-irreducible iff it is a chain-compact element in the convexity algebra $\mathcal{C}$. By Proposition 2.7, we have the following lemma.

Lemma 4.11. *Let S be a convexity space. Then, a convex subset of S is chain-irreducible iff it is a polytope in the convexity space.*

Now we introduce the notion of sober convexity space.

Definition 4.12. *A convexity space S is said to be sober iff, for every chain-irreducible convex set C in S, there is a unique point $x \in S$ such that $C = \mathrm{ch}(\{x\})$.*

By Lemma 4.11, we obtain the following alternative definition of sobriety, which clarifies the convexity theoretical meaning of sobriety.

Proposition 4.13. *A convexity space is sober iff every polytope in it is the convex hull of a unique point.*

We remark that not all natural examples of convexity spaces are sober. For example, by the above proposition, $\mathbb{R}^n$ with the usual convexity (see Example 2.2) is not a sober convexity space, though it is a sober topological space.

Example 4.14. *Consider $\mathbf{2}^\omega$, i.e., the set of all functions from the set ω of all non-negative integers to $\mathbf{2}$ $(=\{0,1\})$. Let $C_0 = \mathbf{2}^\omega$. For $k \in \omega$ with $k \geq 1$ and $n_1, ..., n_k \in \omega$, let*

$$C_k(n_1, ..., n_k) = \{f \in \mathbf{2}^\omega \; ; \; f(n_1) = f(n_2) = ... = f(n_k) = 1\}.$$

Equip $\mathbf{2}^\omega$ with the convexity generated by

$$\{C_k(n_1, ..., n_k) \; ; \; k \in \omega \text{ and } n_1, ..., n_k \in \omega\}.$$

Then, $\mathbf{2}^\omega$ forms a sober convexity space.

The next proposition provides many natural examples of sober convexity spaces.

Proposition 4.15. *Let L be a convexity algebra. Then, $\mathrm{Spec}(L)$ is a sober convexity space.*

Proof. Assume that C is a chain-irreducible convex set in $\mathrm{Spec}(L)$. Define

$$a = \bigwedge\{x \in L \; ; \; C = \langle x \rangle\},$$

where $\{x \in L \, ; \, C = \langle x \rangle\}$ is not empty, since any convex set in $\mathrm{Spec}(L)$ is of the form $\langle x \rangle$ for $x \in L$ by Lemma 3.5. Then, we have $C = \langle a \rangle$ by Lemma 3.4. We claim that a is a chain-compact element in L. Suppose that $a \leq \bigvee_{i \in I} a_i$ for a totally ordered subset $\{a_i \; ; \; i \in I\}$ of L. Then, by Lemma 3.4, we have

$$C = \langle a \rangle = \langle \bigvee_{i \in I} a_i \rangle \cap \langle a \rangle = \bigcup_{i \in I} \langle a_i \wedge a \rangle.$$

Since C is chain-irreducible and since $\{\langle a_i \wedge a \rangle \; ; \; i \in I\}$ is totally ordered, there exists $i \in I$ such that

$$C = \langle a \rangle = \langle a \wedge a_i \rangle.$$

Thus, it follows from the definition of a that $a \leq a \wedge a_i$, whence we have $a \leq a_i$. Therefore, a is a chain-compact element in L. Let $P_a = \{x \in L \, ; \, a \leq x\}$. Then, P_a is a cp-mc filter. In the following, we do not distinguish between cp-mc filters and homomorphisms into **2**, based on Proposition 2.17. Then we have

$$C = \langle a \rangle = \bigcap\{\langle x \rangle \ ; \ x \in P_a\} = \bigcap\{\langle x \rangle \ ; \ P_a \in \langle x \rangle\} = \mathrm{ch}(\{P_a\}).$$

To show the uniqueness, assume that, for $P, Q \in \mathrm{Spec}(L)$, $\mathrm{ch}(\{P\}) = C = \mathrm{ch}(\{Q\})$. Suppose for contradiction that $P \neq Q$. Then, we may assume that there is $b \in L$ such that $b \in P$ and $b \notin Q$. Therefore, we have $\mathrm{ch}(\{P\}) \subset \langle b \rangle$ and $\neg(\mathrm{ch}(\{Q\}) \subset \langle b \rangle)$, which is a contradiction. □

By letting L be the convexity algebra of convex sets in a convexity space, $\mathrm{Spec}(L)$ can be considered as the space of polytopes in the convexity space. Spaces of polytopes in convexity spaces seem to be natural examples of sober convexity spaces.

Proposition 4.16. *Let S be a sober convexity space and $\mathcal{C}$ its convexity. Then, $\Psi_S : S \to \mathrm{Spec} \circ \mathrm{Conv}(S)$ is an isomorphism in* **CS**.

Proof. We first show that Ψ_S is injective. Assume that $\Psi_S(x) = \Psi_S(y)$ for $x, y \in S$. Then we have $\Psi_S(x)^{-1}(\{1\}) = \Psi_S(y)^{-1}(\{1\})$, i.e.,

$$\{f \in \mathrm{Conv}(S) \ ; \ f(x) = 1\} = \{f \in \mathrm{Conv}(S) \ ; \ f(y) = 1\}.$$

By Proposition 2.5, we have $\{C \in \mathcal{C} \ ; \ x \in C\} = \{C \in \mathcal{C} \ ; \ y \in C\}$. By taking the intersections, it follows from the definition of convex hull that

$$\mathrm{ch}(\{x\}) = \bigcap\{C \in \mathcal{C} \ ; \ x \in C\} = \bigcap\{C \in \mathcal{C} \ ; \ y \in C\} = \mathrm{ch}(\{y\}).$$

Since S is sober and since $\mathrm{ch}(\{x\})$ is a chain-irreducible convex set, we have $x = y$. Thus, Ψ_S is injective.

We next show that Ψ_S is surjective. Let $v \in \mathrm{Spec} \circ \mathrm{Conv}(S)$. By Proposition 2.17, $v^{-1}(\{1\})$ is a cp-mc filter of $\mathrm{Conv}(S)$. By Proposition 2.16, $\bigwedge v^{-1}(\{1\})$ is a chain-compact element in $\mathrm{Conv}(S)$. Since $\mathrm{Conv}(S)$ is isomorphic to the convexity algebra $\mathcal{C}$ via the map $f \mapsto f^{-1}(\{1\})$, it follows that

$$\bigcap\{f^{-1}(\{1\}) \ ; \ f \in v^{-1}(\{1\})\}$$

is a chain-compact element in $\mathcal{C}$ and is thus a chain-irreducible convex set in S. Since S is sober, there is $x \in S$ such that

$$\bigcap\{f^{-1}(\{1\}) \ ; \ f \in v^{-1}(\{1\})\} = \mathrm{ch}(\{x\}).$$

We claim that $\Psi_S(x) = v$. Let $g \in \mathrm{Conv}(S)$. We first assume that $v(g) = 1$. Then we have $g \in v^{-1}(\{1\})$. By the choice of x, we have $x \in \mathrm{ch}(\{x\}) \subset g^{-1}(\{1\})$. Thus it follows that $\Psi_S(x)(g) = 1 = v(g)$. We next assume that $v(g) = 0$. Suppose for contradiction that $\Psi_S(x)(g) = 1$, i.e., $g(x) = 1$. Since $g^{-1}(\{1\})$ is a convex set in S and $x \in g^{-1}(\{1\})$, we have

$$\bigcap\{f^{-1}(\{1\}) \ ; \ f \in v^{-1}(\{1\})\} = \mathrm{ch}(\{x\}) \subset g^{-1}(\{1\}).$$

Thus we have $\bigwedge v^{-1}(\{1\}) \leq g$ in $\mathrm{Conv}(S)$. Since $v^{-1}(\{1\})$ is a cp-mc filter and $\bigwedge v^{-1}(\{1\}) \in v^{-1}(\{1\})$, we have $g \in v^{-1}(\{1\})$, which contradicts $v(g) = 0$. Therefore we have $\Psi_S(x)(g) = 0 = v(g)$. Thus we obtain $\Psi_S(x) = v$. Hence, Ψ_S is surjective.

It has already been shown that Ψ_S is a convexity preserving map. To complete the proof, we show that Ψ_S^{-1} is a convexity preserving map. Let C be a convex set in S. Define $f_C : S \to \mathbf{2}$ as in the proof of Proposition 2.5. We claim that $\Psi_S(C) = \langle f_C \rangle$. Suppose $v \in \Psi_S(C)$. Then, $v = \Psi_S(x)$ for some $x \in C$, whence we have

$$v(f_C) = \Psi_S(x)(f_C) = f_C(x) = 1.$$

Hence, we have $v \in \langle f_C \rangle$. Conversely, suppose $v \in \langle f_C \rangle$. Since Ψ_S is surjective, there exists $x \in S$ such that $\Psi_S(x) = v$. By $v \in \langle f_C \rangle$, we have $\Psi_S(x)(f_C) = f_C(x) = 1$, i.e., $x \in C$. Hence, we have $v \in \Psi_S(C)$. □

In this way, we can recover the points of a sober convexity space from the convexity algebra of convex sets in it. The above proposition implies that any sober convexity space can be represented as Spec(L) for a convexity algebra L.

By Proposition 4.15 and Proposition 4.16, we have the following characterization of sobriety.

Proposition 4.17. *For a convexity space S, S is sober iff Ψ_S is an isomorphism in* **CS**.

4.3 Duality between SpCA and SobCS

SpCA denotes the category of spatial convexity algebras and homomorphisms. **SobCS** denotes the category of sober convexity spaces and convexity preserving maps. Finally we obtain the following duality between spatial convexity algebras and sober convexity spaces.

Theorem 4.18. **SpCA** *and* **SobCS** *are dually equivalent via the functors* Spec *and* Conv.

Proof. By Proposition 4.15, (the restriction of) Spec is well-defined. By Proposition 4.3, (the restriction of) Conv is well-defined. By Proposition 4.5, $\Phi : 1_{\mathbf{SpCA}} \to \mathrm{Conv} \circ \mathrm{Spec}$ is a natural isomorphism. By Proposition 4.16, $\Psi : 1_{\mathbf{SobCS}} \to \mathrm{Spec} \circ \mathrm{Conv}$ is a natural isomorphism. □

This is a convexity-theoretical analogue of Isbell duality between spatial frames and sober topological spaces. However, there is a big difference between the above duality and Isbell duality, especially between the notion of sobriety for convexity spaces and the notion of sobriety for topological spaces. That is, most of ordinary topological spaces such as $\mathbb{R}^n$ are sober and so fall into Isbell duality, while most of ordinary convexity spaces such as $\mathbb{R}^n$ are not sober and so do not fall into the above duality. In the next section, we consider another duality into which most of ordinary convexity spaces do fall.

5 Duality between mSpCA and mSobCS

In this section, by introducing the notions of m-spatiality, m-homomorphism and m-sobriety, we shall show a duality between the category of m-spatial convexity algebras and m-homomorphisms and the category of m-sober convexity spaces and convexity preserving maps.

5.1 m-spatiality and m-sobriety

For a convexity algebra L, we mean by an m-mc filter of L a maximal meet-complete filter of L where maximality means that with respect to inclusion.

Consider a convexity space $(S, \mathcal{C})$ such that $\{x\}$ is convex for any $x \in S$. Then, for $x \in S$, $\{C \in \mathcal{C} \,;\, x \in C\}$ is an m-mc filter of the convexity algebra $\mathcal{C}$.

Lemma 5.1. *Let M be an m-mc filter of a convexity algebra L. Then, M is a cp-mc filter of L.*

Proof. Assume that $\bigvee_{i \in I} a_i \in M$ for a totally ordered subset $\{a_i \,;\, i \in I\}$ of L. Suppose for contradiction that for any $i \in I$, $a_i \notin M$. Since M is an m-mc filter, we have: For any $i \in I$ there is $b_i \in M$ such that $a_i \wedge b_i = 0$. Then we have $\bigwedge_{i \in I} b_i \in M$ by $b_i \in M$. We also have $a_i \wedge (\bigwedge_{i \in I} b_i) = 0$, whence it follows that

$$\bigvee_{i \in I} (a_i \wedge (\bigwedge_{i \in I} b_i)) = 0.$$

Since $\{a_i \,;\, i \in I\}$ is totally ordered, it follows from the chain-completely distributive law (i.e., the item 3 in Definition 2.9) that

$$(\bigvee_{i \in I} a_i) \wedge (\bigwedge_{i \in I} b_i)) = 0.$$

Since $\bigwedge_{i \in I} b_i \in M$ and $\bigvee_{i \in I} a_i \in M$, we have $0 \in M$, which is a contradiction. Thus there is $i \in I$ such that $a_i \in M$. Hence, M is a cp-mc filter of L. □

Then, m-spatiality is defined as follows.

Definition 5.2. *A convexity algebra L is called m-spatial iff for any $a, b \in L$ with $a \not\leq b$ there is an m-mc filter M of L such that $a \in M$ and $b \notin M$.*

By Lemma 5.1, we obtain the following proposition.

Proposition 5.3. *Let L be a convexity algebra. If L is m-spatial then L is spatial.*

We remark that although m-spatiality implies spatiality, m-sobriety defined below does not imply sobriety.

We next introduce the notion of m-homomorphism. A similar notion is used also in the context of duality theory for distributive semilattices (see [15, 17]).

Definition 5.4. *An m-homomorphism $f : L_1 \to L_2$ between convexity algebras L_1 and L_2 is defined as a homomorphism of convexity algebras such that for any m-mc filter M of L_2, $f^{-1}(M)$ is an m-mc filter of L_1.*

It shall be shown that the dual notion of convexity preserving map between m-sober (defined below) convexity spaces is m-homomorphism and is not homomorphism.

Let us review the concept of atomistic poset (see [12]). Recall that an atom in a poset P with a least element 0 is an element of P that is minimal in $P \setminus \{0\}$.

Definition 5.5. *A poset with a least element 0 is called atomistic iff any element of the poset is the join of a set of atoms of the poset.*

Note that in general, being atomistic is not equivalent to being atomic.

We can provide an algebraic characterization of m-spatiality as follows.

Proposition 5.6. *For a convexity algebra L, the following are equivalent:*

1. *L is m-spatial;*
2. *L is atomistic.*

Proof. We first show that 1 implies 2. Let $a \in L$. If $a = 0$ then a is the join of $\emptyset$. Assume $a > 0$. Let a' be the join of those atoms $x \in L$ such that $x \leq a$. It suffices to show that $a = a'$. Since if there is no atom $x \in L$ with $x \leq a$ then we have $a' = 0$, it follows from the choice of a' that $a \geq a'$. Suppose for contradiction that $a > a'$. Since L is m-spatial, there is an m-mc filter M of L such that $a \in M$ and $a' \notin M$. Since M is an m-mc filter of L, $\bigwedge M$ is an atom of L. By $a \in M$, we also have $\bigwedge M \leq a$. Then it follows from the definition of a' that

$$\bigwedge M \leq a'.$$

Since M is a meet-complete filter, we have $\bigwedge M \in M$ and so $a' \in M$, which contradicts $a' \notin M$. Hence $a = a'$.

We next show that 2 implies 1. Let $a, b \in L$ with $a \not\leq b$. Since L is atomistic, there is a set A of atoms of L such that $\bigvee A = a$. Similarly, there is a set B of atoms of L such that $\bigvee B = b$. Then we may assume that B is the set of those atoms $x \in L$ such that $x \leq b$. By $a \not\leq b$, there is $c \in A$ such that $c \notin B$. Define

$$M = \{x \in L \,;\, c \leq x\}.$$

Then, we have both $a \in M$ and $b \notin M$, since B is the set of those atoms $x \in L$ such that $x \leq b$. Now it remains to show that M is an m-mc filter of L, which follows from the fact that c is an atom of L. □

We next introduce the notion of m-sobriety.

Definition 5.7. *A convexity space S is called m-sober iff $\{x\}$ is convex for any $x \in S$.*

Many ordinary convexity spaces are m-sober, including those in Example 2.2.

An m-sober convexity space is not necessarily sober. For example, the n-dimensional Euclidean space $\mathbb{R}^n$ with the usual convexity (see Example 2.2) is not sober and is m-sober. The same thing holds also for other convexity spaces such as those in Example 2.2. Thus we may consider that the notion of m-sobriety is more natural than that of sobriety.

5.2 Duality between mSpCA and mSobCS

In this subsection we show a dual equivalence between categories **mSpCA** and **mSobCS**, which are defined as follows.

Definition 5.8. mSobCS *denotes the category of m-sober convexity spaces and convexity preserving maps.* **mSpCA** *denotes the category of m-spatial convexity algebras and m-homomorphisms.*

We introduce a functor mSpec based on the view that a point is an m-mc filter.

Definition 5.9. *We define a contravariant functor* mSpec *from* **mSpCA** *to* **mSobCS** *as follows:*

1. *For an object L in* **mSpCA**, mSpec(L) *is defined as the set of all m-mc filters of L equipped with the convexity generated by* $\{\langle a\rangle_m \,;\, a \in L\}$ *where*

$$\langle a\rangle_m = \{M \in \mathrm{mSpec}(L)\,;\, a \in M\}.$$

2. *For an arrow $f : L_1 \to L_2$ in* **mSpCA**, $\mathrm{mSpec}(f) : \mathrm{mSpec}(L_2) \to \mathrm{mSpec}(L_1)$ *is defined by*

$$\mathrm{mSpec}(f)(M) = f^{-1}(M)$$

for $M \in \mathrm{mSpec}(L_2)$.

The well-definedness of the functor mSpec is shown by the following proposition.

Proposition 5.10. *For a convexity algebra L,* mSpec(L) *is m-sober.*

Proof. Let $M \in \mathrm{mSpec}(L)$. We claim that $\langle \bigwedge M\rangle_m = \{M\}$. Since $\bigwedge M \in M$, we have

$$\{M\} \subset \langle \bigwedge M\rangle_m.$$

Assume that $\bigwedge M \in N$ for $N \in \mathrm{mSpec}(L)$. It follows from $M, N \in \mathrm{mSpec}(L)$ that $M \subset N$ and so $M = N$ by maximality. Thus we have

$$\langle \bigwedge M\rangle_m \subset \{M\}.$$

Therefore we have $\langle \bigwedge M\rangle_m = \{M\}$. Hence, $\{M\}$ is convex. □

Remark 5.11. *Throughout this section, based on Proposition 2.5, we consider* Conv *as a functor from* **mSobCS** *to* **mSpCA** *defined as follows. For an object S in* **mSobCS**, Conv(S) *is defined as the convexity algebra of all convex subsets of S. For an arrow $f : S_1 \to S_2$ in* **mSobCS**, $\mathrm{Conv}(f) : \mathrm{Conv}(S_2) \to \mathrm{Conv}(S_1)$ *is defined by* $\mathrm{Conv}(f)(C) = f^{-1}(C)$ *for* $C \in \mathrm{Conv}(S_2)$.

Then the well-definedness of the functor Conv : **mSobCS** $\to$ **mSpCA** is shown by the following two propositions.

Proposition 5.12. *Let S be an object in* **mSobCS**. *Then,* Conv(S) *is an m-spatial convexity algebra.*

Proof. Let $C_1, C_2 \in \mathrm{Conv}(S)$ such that C_1 is not a subset of C_2. Then, there is $x \in C_1$ with $x \notin C_2$. Define

$$M = \{C \in \mathrm{Conv}(S)\,;\, x \in C\}.$$

Then we have both $C_1 \in M$ and $C_2 \notin M$. Now it suffices to show that M is an m-mc filter of Conv(S). It is straightforward to verify that M is a meet-complete filter. Since S is m-sober, $\{x\}$ is convex and so $\{x\} \in M$. If $C \notin M$ for $C \in \mathrm{Conv}(S)$ then $C \cap \{x\} = \emptyset$. Thus, M is maximal. □

Since an arrow in **mSpCA** is an m-homomorphism, not a homomorphism, it is important to verify that the arrow part of Conv is well-defined.

Proposition 5.13. *Let $f : S_1 \to S_2$ be an arrow in* **mSobCS**. *Then,* $\mathrm{Conv}(f) : \mathrm{Conv}(S_2) \to \mathrm{Conv}(S_1)$ *is an m-homomorphism.*

Proof. Clearly, $\mathrm{Conv}(f)$ is a homomorphism. Let M be an m-mc filter of $\mathrm{Conv}(S_1)$. Since $\bigcap M \in M$ and $M \neq \mathrm{Conv}(S_1)$, we have $\bigcap M \neq \emptyset$ and so there is $m \in \bigcap M$. Then, $M \subset \{C \in \mathrm{Conv}(S_1)\,;\, m \in C\}$. Since $\{C \in \mathrm{Conv}(S_1)\,;\, m \in C\}$ is a proper meet-complete filter, it follows from the maximality of M that

$$M = \{C \in \mathrm{Conv}(S_1)\,;\, m \in C\}.$$

Thus it follows that

$$\begin{aligned} \mathrm{Conv}(f)^{-1}(M) &= \{C \in \mathrm{Conv}(S_2)\,;\, f^{-1}(C) \in M\} \\ &= \{C \in \mathrm{Conv}(S_2)\,;\, m \in f^{-1}(C)\} \\ &= \{C \in \mathrm{Conv}(S_2)\,;\, f(m) \in C\}. \end{aligned}$$

Since S_2 is m-sober, $\{f(m)\}$ is convex and so $\{C \in \mathrm{Conv}(S_2)\,;\, f(m) \in C\}$ is an m-mc filter of $\mathrm{Conv}(S_2)$. This completes the proof. □

We next define two natural transformations.

Let Id_1 denote the identity functor on **mSpCA** and Id_2 the identity functor on **mSobCS**.

Definition 5.14. *We define a natural transformation* $\alpha : \mathrm{Id}_1 \to \mathrm{Conv} \circ \mathrm{mSpec}$ *as follows. For an m-spatial convexity algebra* L*, define* $\alpha_L : L \to \mathrm{Conv} \circ \mathrm{mSpec}(L)$ *by*

$$\alpha_L(a) = \{M \in \mathrm{mSpec}(L)\,;\, a \in M\} = \langle a \rangle_m.$$

It is straightforward to verify that α is actually a natural transformation.

Proposition 5.15. *For an m-spatial convexity algebra* L*,* $\alpha_L : L \to \mathrm{Conv} \circ \mathrm{mSpec}(L)$ *is an isomorphism in* **mSpCA***.*

Proof. Since an isomorphism in **CA** is always an isomorphism in **mSpCA**, it suffices to show that α_L is an isomorphism in **CA**. We first show that α_L is a homomorphism. By Lemma 5.1, an m-mc filter is a cp-mc filter. Thus we have

$$\langle \bigvee_{i \in I} a_i \rangle_m = \bigcup_{i \in I} \langle a_i \rangle_m$$

for a totally ordered subset $\{a_i\,;\, i \in I\}$ of L. We also have

$$\langle \bigwedge_{i \in I} a_i \rangle_m = \bigcap_{i \in I} \langle a_i \rangle_m$$

for a subset $\{a_i\,;\, i \in I\}$ of L. Thus, α_L is a homomorphism. It is straightforward to see that α_L is injective by the m-spatiality of L. By arguing as in the proof of Lemma 3.5, it is shown that $\{\langle a \rangle_m\,;\, a \in L\}$ coincides with the convexity of $\mathrm{mSpec}(L)$. Thus, α_L is surjective. This completes the proof. □

Definition 5.16. *We define a natural transformation* $\beta : \mathrm{Id}_2 \to \mathrm{mSpec} \circ \mathrm{Conv}$ *as follows. For an m-sober convexity space* S*, define* $\beta_S : S \to \mathrm{mSpec} \circ \mathrm{Conv}(S)$ *by*

$$\beta_S(x) = \{C \in \mathrm{Conv}(S)\,;\, x \in C\}.$$

It is straightforward to verify that β is actually a natural transformation.

Proposition 5.17. *For an m-sober convexity space S, $\beta_S : S \to \mathrm{mSpec} \circ \mathrm{Conv}(S)$ is an isomorphism in* **mSobCS**.

Proof. Since S is m-sober, $\beta_S(x)$ is an m-mc filter for $x \in S$ and so β_S is well-defined. Clearly, β_S is injective. Since S is m-sober, an m-mc filter of $\mathrm{Conv}(S)$ is of the form

$$\{C \in \mathrm{Conv}(S)\ ;\ x \in C\}$$

for some $x \in S$. Thus, β_S is surjective. Since $\beta_S^{-1}(\langle C\rangle_m) = C$ for $C \in \mathrm{Conv}(S)$, β_S is convexity preserving. It is easily verified that

$$\beta_S(C) = \langle C\rangle_m$$

for $C \in \mathrm{Conv}(S)$, whence β_S^{-1} is convexity preserving. Hence, β_S is an isomorphism in **mSobCS**. □

In this way, we can recover the points of an m-sober convexity space from the convexity algebra of convex sets in it. This proposition implies that any m-sober convexity space can be represented as $\mathrm{mSpec}(L)$ for a convexity algebra L, where note that most of ordinary convexity spaces are m-sober.

By Proposition 5.15 and Proposition 5.17, α and β are natural isomorphisms and thus we obtain the following duality between m-spatial convexity algebras and m-sober convexity spaces.

Theorem 5.18. mSpCA *and* **mSobCS** *are dually equivalent via the functors* mSpec *and* Conv.

Most of ordinary convexity spaces are m-sober (recall that a singleton is usually convex) and thus fall into the above duality.

We remark that convexity preserving maps between m-sober convexity spaces correspond to m-homomorphisms between m-spatial convexity algebras and do not correspond to homomorphisms.

6 Concluding Remarks

In this work we have obtained the following main results with other modest ones: (1) Spec and Conv give a dual adjunction between **CA** and **CS**; (2) **SpCA** and **SobCS** are dually equivalent via Spec and Conv; (3) **mSpCA** and **mSobCS** are dually equivalent via mSpec and Conv. Note that many ordinary convexity spaces are not sober but m-sober, while most of ordinary topological spaces are sober, which is a striking difference between topology and convex geometry. Now, (1) and (2) are based on the view that a point is a cp-mc filter, while (3) is based on the view that a point is an m-mc filter. Then natural questions arise. Which view is better? Which notion of point is the proper one? Our answers are as follows.

It seems difficult to obtain a dual adjunction between the category of all convexity algebras and the category of all convexity spaces based on the view that a point is an m-mc filter. Some of the reasons are as follows: (1) for a convexity space S and $x \in S$, $\beta_S(x)$ (see Definition 5.17) is not always an m-mc filter of $\mathrm{Conv}(S)$; (2) the left adjoint functor of $\mathrm{Conv}^{\mathrm{op}}$ is uniquely determined up to isomorphism (see [3]) and it is Spec, not mSpec. Thus we may consider that the view that a point is a cp-mc filter is superior to the view that a point is an m-mc filter from a category theoretic standpoint.

However, the proper notion of point seems to be m-mc filter. Consider the n-dimensional Euclidean space $\mathbb{R}^n$ equipped with the usual convexity. Then, Spec $\circ$ Conv($\mathbb{R}^n$) does not coincide with $\mathbb{R}^n$ (i.e., $\mathbb{R}^n$ is not a sober convexity space, though it is a sober topological space) but coincides with the set of polytopes in $\mathbb{R}^n$ by Proposition 2.15. The same thing holds true not only for $\mathbb{R}^n$ but also for many other ordinary convexity spaces such as vector spaces over $\mathbb{R}$ and manifolds including the n-sphere and the n-dimensional real projective space (for their convexities, see Example 2.2). Actually, for any convexity space S, the space of polytopes in S coincides with Spec $\circ$ Conv(S) by Proposition 2.15 and is thus sober by Proposition 4.15, whence we can notice that the space of polytopes in a convexity space is the sobrification of the convexity space. Note that conversely any sober convexity space can be represented as the space of polytopes in a convexity space by Proposition 4.16.

Therefore, we conclude that a cp-mc filter (or a homomorphism into **2**) actually represents a polytope, not a point, and an m-mc filter properly represents a point in many ordinary cases. In a nutshell, an m-mc filter is a point and a cp-mc filter is a "generic" point, which makes it possible to generate a polytope as the convex hull of a unique point, as in algebraic geometry a prime ideal is considered as a generic point, which makes it possible to generate an irreducible algebraic variety as the closure of a unique point (see [18]). In this sense, the notion of polytope in convex geometry corresponds to that of irreducible algebraic variety in algebraic geometry.

We remark that pointfree convex geometry is closely related to domain theory (for domain theory, see [6]). In this paper, the notions of convexity space and convexity algebra are defined in terms of chains. However, it is possible to define them in terms of directed sets instead of chains and most of arguments in this paper work well even if we replace chains with directed sets. Interestingly, the notion of continuous lattice, which is well known in domain theory, coincides with the notion of convexity algebra defined in terms of directed sets, which follows from [6, Theorem I-2.7]. Moreover, under this reformulation of related notions, the duality between spatial convexity algebras and sober convexity spaces turns out to reveal the convexity-theoretical nature of Hoffman-Mislove-Stralka duality between algebraic (continuous) lattices and join-semilattices with the least elements (for this duality, see [6, Theorem IV-1.16] and [19]), though there is a minor difference between the morphism parts of the two dualities. Here note that algebraic lattices coincide with spatial convexity algebras. By combining the two dualities, we notice that sober convexity spaces are equivalent to join-semilattices with the least elements. A consequence of this observation is that the set of polytopes in any convexity space forms a join-semilattice with the least element and conversely any join-semilattice with the least element can be represented as the join-semilattice of polytopes in a convexity space. Another consequence of it is that the set of ideals of any join-semilattice forms a sober convexity on the join-semilattice and conversely any sober convexity space can be represented as a join-semilattice equipped with the convexity consisting of ideals of the join-semilattice. In some sense, pointfree convex geometry formulated in terms of directed sets is nothing but the theory of continuous lattices.

We next discuss the relationships between pointfree convex geometry and Hilbert's philosophy. In the introduction of [11], Coquand states that Hilbert's program may be reformulated using pointfree topology. According to the results in this paper and the idea in [11], we notice that pointfree convex geometry may also be considered to be in harmony with Hilbert's philosophy, especially his instrumentalism (see [13]). We remark that this view seems to hold true of locale theory, but it is not clear whether or not the view on locale theory is the same as Coquand's one. This harmony may be explained as follows. Convexity algebras correspond to real objects in Hilbert's sense, which actually exists. On the other hand, the points of convexity algebras correspond to ideal objects in his sense, which do not actually exist, and are mere instruments for the study of the real objects. We may work in the category of convexity spaces by using the functors Spec and mSpec, which corresponds to the introduction of ideal objects in his sense. However, results obtained by using the ideal objects can (sometimes) be pulled back to the category of convexity algebras via the functor Conv, which corresponds to the elimination of ideal objects in his sense.

Finally we discuss relationships between our view on pointfree mathematics and Husserl's phenomenology (for Husserl's phenomenology, see [20, 2]; for the relationships between Husserl's phenomenology and Brouwer's notion of the continuum, which may be thought of as an origin of pointfree topology, see [2]). In [26], Husserl's phenomenology of space is summarized as follows: "Epistemology of space before ontology of space." According to our view in the first paragraph of Section 1, this may be paraphrased as follows: "Region before point" in terms of pointfree geometry, "Theory before model" in logical terms and "Algebra before spectrum" in terms of duality theory and algebraic geometry. We conclude the paper by emphasizing that region is the epistemological ingredient of the notion of space and point is the ontological or metaphysical ingredient of it, whence duality theory clarifies the relationships between epistemological and ontological aspects of the notion of space.

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