Supplementary material for Rigorous numerics of blow-up solutions for ODEs with exponential nonlinearity

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Abstract

This supplementary material provides concrete arguments omitted in the original manuscript; the derivation of desingularized vector fields associated with original ones, the proof of Lemma 2.2 and the Jacobian matrix of desingularized vector fields, which are essential for our numerical validations. Moreover, concrete calculations of blow-up rates of validated blow-up solutions are presented.

A   Transformation of vector fields via directional compactifications

Firstly, we derive the transformed vector field associated with

$$u'_1 = N^2(-2u_1 + u_2) + \lambda e^{u_1^m}, \quad u'_{N-1} = N^2(u_{N-1} - 2u_{N-2}) + \lambda e^{u_{N-1}^m},$$
$$u'_i = N^2(u_{i-1} - 2u_i + u_{i+1}) + \lambda e^{u_i^m}, \quad (i = 2, \cdots, N-2), \tag{A.1}$$

where $' = \frac{d}{dt}$, via the directional compactification

$$u_{N/2} = s^{-1}, \quad u_i = s^{-1}x_i \quad (i = 1, \cdots, N-1, \ i \neq N/2). \tag{A.2}$$

Let

$$h_{k,\alpha;m}(s) := s^{-k}e^{-\alpha/s^m} \tag{A.3}$$

and

$$\begin{cases} \Delta_i := N^2(x_{i-1} - 2x_i + x_{i+1}), \quad (i = 2, \cdots, N-2, i \neq N/2) \\ \Delta_{N/2} := N^2(x_{N/2-1} - 2 + x_{N/2+1}), \quad \Delta_1 := N^2(-2x_1 + x_2), \quad \Delta_{N-1} := N^2(x_{N-2} - 2x_{N-1}). \end{cases}$$

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Direct computations then yield

$$u'_{N/2} \equiv -s^{-2}s' = N^2\left(\frac{x_{\frac{N}{2}-1}}{s} - \frac{2}{s} + \frac{x_{\frac{N}{2}+1}}{s}\right) + \lambda e^{(1/s)^m}$$

$$\Leftrightarrow s' = -s\Delta_{N/2} - \lambda s^2 e^{(1/s)^m} \equiv -s\Delta_{N/2} - \lambda(h_{2;1;m}(s))^{-1},$$

$$u'_i = -s^{-2}x_i s' + s^{-1}x'_i = s^{-1}\Delta_i + \lambda e^{(x_i/s)^m}$$

$$\Leftrightarrow x'_i = s^{-1}x_i s' + \Delta_i + \lambda s e^{(x_i/s)^m}$$

$$= s^{-1}x_i\left\{-s\Delta_{N/2} - \lambda(h_{2;1;m}(s))^{-1}\right\} + \Delta_i + \lambda s e^{(x_i/s)^m}$$

$$= \left\{-x_i\Delta_{N/2} - x_i\lambda(h_{1;1;m}(s))^{-1}\right\} + \Delta_i + \lambda s e^{(x_i/s)^m}$$

$$= -x_i\Delta_{N/2} - x_i\lambda(h_{1;1;m}(s))^{-1} + \Delta_i + \lambda(h_{1,x_i^m;m}(s))^{-1} \quad (i \neq N/2).$$

Therefore, we have the following transformed vector field:

$$s' = -s\Delta_{N/2} - \lambda(h_{2,1;m}(s))^{-1},$$

$$x'_i = -x_i\Delta_{N/2} - x_i\lambda(h_{1,1;m}(s))^{-1} + \Delta_i + \lambda(h_{1,x_i^m;m}(s))^{-1} \quad (i \neq N/2).$$

Introducing the time-scale desingularization

$$\frac{d\tau}{dt} = h_{1,1;m}(s)^{-1}, \tag{A.4}$$

we have

$$\dot{f} \equiv \frac{df}{d\tau} = \frac{df}{dt}h_{1,1;m}(s)$$

for any function f and hence

$$\begin{cases} \dot{s} = -s\Delta_{N/2}h_{1,1;m}(s) - s\lambda = -\Delta_{N/2}e^{-1/s^m} - s\lambda, \\ \dot{x}_i = -x_i\Delta_{N/2}h_{1,1;m}(s) - x_i\lambda + \Delta_i h_{1,1;m}(s) + \lambda h_{0,1-x_i^m;m}(s). \quad (i \neq N/2) \end{cases} \tag{A.5}$$

We leave the calculation of Jacobian matrix later since we need several formulas for $h_{k,\alpha;m}$.

B   Derivations around $h_{k,\alpha;m}$

In this section, we calculate differentials of $h_{k,\alpha;m}$ defined in (A.3) with several basic properties stated in Lemma 2.2.

Obviously, for fixed positive integers $k, m > 0$ and positive number $\alpha > 0$, $h_{k,\alpha;m}(s)$ is C^1 (in particular, C^∞) with respect to $s > 0$. The limit $\lim_{s\to 0+} h_{k,\alpha;m}(s) = 0$ follows from the following argument. Let

$$h_{k,\alpha;m}(s) = \frac{f(s)}{g(s)}, \quad f(s) = e^{-\alpha/s^m}, \quad g(s) = s^k.$$

Lemma B.1. *For $\alpha > 0$ and positive integer $m > 0$, the function*

$$f(s) = \begin{cases} e^{-\alpha/s^m} & s > 0 \\ 0 & s \leq 0 \end{cases}$$

is C^1 on $\mathbb{R}$. In particular, $\lim_{s\to 0} f'(s)$ exists and is equal to 0.

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Proof. Continuity of f on $\mathbb{R}$ and smoothness of f on $\mathbb{R} \setminus \{0\}$ are obvious. The remaining issue is the smoothness of f at $s = 0$. The smoothness with $m = 1$ is well-known, and hence we assume $m = 1$ in the remaining argument. For $s > 0$, $f'(s) = \alpha m s^{-(m+1)} f(s)$. Introducing $t = s^m$, the function $f'(s)$ is rewritten by

$$f'(s) = \alpha m s^{-(m+1)} e^{-\alpha/s^m} = \alpha m s^{m-1} s^{-2m} e^{-\alpha/s^m} = \alpha m t^{\frac{m-1}{m}} \frac{e^{-\alpha/t}}{t^2}.$$

Using the well-known results that $\lim_{t \to +0} t^{-n} e^{-\alpha/t} = 0$ for any nonnegative integer n and $\lim_{t \to +0} t^c = 0$ for $c > 0$, we have

$$\lim_{t \to +0} t^{\frac{m-1}{m}} \frac{e^{-\alpha/t}}{t^2} = \left(\lim_{t \to +0} t^{\frac{m-1}{m}} \right) \left(\lim_{t \to +0} \frac{e^{-\alpha/t}}{t^2} \right) = 0.$$

Obviously $t \to +0$ corresponds one-to-one to $s \to 0$ and hence $f'(s) \to 0$ as $s \to +0$, which implies that f is C^1 at $s = 0$. $\qquad\square$

The above proof gives an explicit form of $f'(s)$ via the transformation $s = t^m$. Now we have

$$\frac{d}{ds} h_{k,\alpha;m}(s) = \frac{d}{ds} \left(\frac{f(s)}{g(s)} \right)$$
$$= \alpha m s^{-(m+1+k)} e^{-\alpha/s^m} - k s^{-(k+1)} e^{-\alpha/s^m}.$$

Here, there are positive integers $\tilde{m}_1, \tilde{m}_2$ such that $0 \le r_1 \equiv \tilde{m}_1 m - (m+1+k) < m$ and that $0 \le r_2 \equiv \tilde{m}_2 m - (k+1) < m$. Therefore

$$\alpha m s^{-(m+1+k)} e^{-\alpha/s^m} - k s^{-(k+1)} e^{-\alpha/s^m} = \alpha m t^{r_1/m} \left(\frac{e^{-\alpha/t}}{t^{\tilde{m}_1}} \right) - k t^{r_2/m} \left(\frac{e^{-\alpha/t}}{t^{\tilde{m}_2}} \right).$$

By the same argument as the proof of Lemma B.1, we know that $\lim_{s \to +0} \frac{d}{ds} h_{k,\alpha;m}(s)$ exists and is equal to 0. Consequently, the function

$$\overline{h_{k,\alpha;m}}(s) := \begin{cases} h_{k,\alpha;m}(s), & s > 0, \\ 0, & s \le 0 \end{cases}$$

is a C^1-extension of $h_{k,\alpha;m}$ over $\mathbb{R}$ for any nonnegative integer k, positive integer m and positive number α.

Next we check the monotonous behavior of $h_{k,\alpha;m}(s)$. Direct calculations yield the following alternative formula of the derivative of $h_{k,\alpha;m}$:

$$\frac{d}{ds} h_{k,\alpha;m}(s) = \frac{d}{ds} \left(s^{-k} e^{-\alpha/s^m} \right)$$
$$= -k s^{-(k+1)} e^{-\alpha/s^m} + s^{-k} e^{-\alpha/s^m} \frac{d}{ds}(-\alpha s^{-m})$$
$$= -k s^{-(k+1)} e^{-\alpha/s^m} + m \alpha s^{-(k+m+1)} e^{-\alpha/s^m}$$
$$= s^{-(k+1)} e^{-\alpha/s^m} \left\{ -k + m \alpha s^{-m} \right\}$$
$$= \left\{ \frac{m\alpha}{s^m} - k \right\} h_{k+1,\alpha;m}(s).$$

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We see that, for sufficiently small $s > 0$, $h_{k,\alpha;m}(s)$ and $\frac{d}{ds}h_{k,\alpha;m}(s)$ are positive. Thus $h_{k,\alpha;m}(s)$ increases monotonously with respect to $s \in [0,\bar{s}]$ as long as $\frac{d}{ds}h_{k,\alpha;m}(s) \geq 0$ over $[0,\bar{s}]$. Since $h_{k,\alpha;m}(s) > 0$ for all $s > 0$ and any $k \geq 0$, then the monotonous property of $h_{k,\alpha;m}(s)$ can break at $s = \bar{s}$ such that $\frac{d}{ds}h_{k,\alpha;m}(\bar{s}) = 0$, which is equivalent to $m\alpha\bar{s}^{-m} - k = 0$. Therefore we have

$$\bar{s} = \left(\frac{m\alpha}{k}\right)^{1/m}$$

for real $\bar{s}$. We easily know that $\frac{m\alpha}{s^m} - k$ is positive for $s \in (0,\bar{s})$, and hence we conclude that $h_{k,\alpha;m}(s)$ is monotonously increasing over $(0, (m\alpha/k)^{1/m})$.

C   Jacobian matrix for (A.5)

Once we obtain differentials of $h_{k,\alpha;m}$, we can compute the Jacobian matrix of (A.5). Direct computations with Part 1. of Lemma 2.2 yield

$$\begin{aligned}
\frac{\partial f_{N/2}}{\partial s} &= \frac{\partial}{\partial s}\left\{-e^{-1/s^m}\Delta_{N/2} - s\lambda\right\} \\
&= \frac{\partial}{\partial s}\left\{-h_{0,1;m}(s)\Delta_{N/2} - s\lambda\right\} = -\left\{\frac{m}{s^m}\right\}h_{1,1;m}\Delta_{N/2} - \lambda \\
&= -mh_{m+1,1;m}(s)\Delta_{N/2} - \lambda, \\
\frac{\partial f_{N/2}}{\partial x_j} &= \frac{\partial}{\partial x_j}\left\{-e^{-1/s^m}\Delta_{N/2} - s\lambda\right\} \\
&= -e^{-1/s^m}\frac{\partial}{\partial x_j}\Delta_{N/2}.
\end{aligned}$$

Here we note that

$$\frac{\partial}{\partial x_j}\Delta_{N/2} = N^2\frac{\partial}{\partial x_j}(x_{N/2-1} - 2 + x_{N/2+1}) = N^2(\delta_{j,N/2-1} + \delta_{j,N/2+1}),$$

where $\delta_{i,j}$ is the Kronecker delta. In particular, we have

$$\frac{\partial f_{N/2}}{\partial x_j} = -e^{-1/s^m}\frac{\partial}{\partial x_j}\Delta_{N/2} = -(\delta_{j,N/2-1} + \delta_{j,N/2+1})N^2 h_{0,1;m}(s) \quad (j \neq N/2).$$

Next,

$$\begin{aligned}
\frac{\partial f_i}{\partial s} &= \left\{-x_i h_{1,1;m}(s)\Delta_{N/2} - x_i\lambda + h_{1,1;m}(s)\Delta_i + \lambda h_{0,1-x_i^m;m}(s)\right\} \\
&= -x_i\Delta_{N/2}\frac{\partial}{\partial s}h_{1,1;m}(s) + \Delta_i\frac{\partial}{\partial s}h_{1,1;m}(s) + \lambda\frac{\partial}{\partial s}h_{0,1-x_i^m;m}(s) \\
&= \{\Delta_i - x_i\Delta_{N/2}\}\left\{\frac{m}{s^m} - 1\right\}h_{2,1;m}(s) + \lambda\left\{\frac{m(1-x_i^m)}{s^m}\right\}h_{1,1-x_i^m;m}(s) \\
&= (1 - ms^{-m})h_{2,1;m}(s)(x_i\Delta_{N/2} - \Delta_i) - \lambda m(x_i^m - 1)h_{m+1,1-x_i^m;m}(s), \quad (i \neq N/2).
\end{aligned}$$

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Finally,

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left\{-x_i h_{1,1;m}(s)\Delta_{N/2} - x_i\lambda + h_{1,1;m}(s)\Delta_i + \lambda h_{0,1-x_i^m;m}(s)\right\}$$

$$= -\left\{\frac{\partial x_i}{\partial x_j}h_{1,1;m}(s)\Delta_{N/2} + x_i h_{1,1;m}(s)\frac{\partial}{\partial x_j}\Delta_{N/2}\right\} - \lambda\frac{\partial x_i}{\partial x_j}$$

$$+ h_{1,1;m}(s)\frac{\partial}{\partial x_j}\Delta_i + \lambda\frac{\partial}{\partial x_j}h_{0,1-x_i^m;m}(s)$$

$$= -h_{1,1;m}(s)\Delta_{N/2}\delta_{i,j} + N^2 x_i h_{1,1;m}(s)(\delta_{j,N/2-1} + \delta_{j,N/2+1}) - \lambda\delta_{i,j}$$

$$+ h_{1,1;m}(s)\frac{\partial}{\partial x_j}\Delta_i + \lambda\frac{\partial}{\partial x_j}e^{(1-x_i^m)/s^m}$$

$$= -h_{1,1;m}(s)\Delta_{N/2}\delta_{i,j} + N^2 x_i h_{1,1;m}(s)(\delta_{j,N/2-1} + \delta_{j,N/2+1}) - \lambda\delta_{i,j}$$

$$+ h_{1,1;m}(s)\frac{\partial}{\partial x_j}\Delta_i - \lambda m x_i^{m-1}s^{-m}\delta_{i,j}e^{(1-x_i^m)/s^m}$$

$$= -h_{1,1;m}(s)\Delta_{N/2}\delta_{i,j} + N^2 x_i h_{1,1;m}(s)(\delta_{j,N/2-1} + \delta_{j,N/2+1}) - \lambda\delta_{i,j}$$

$$+ h_{1,1;m}(s)\frac{\partial}{\partial x_j}\Delta_i - \lambda m x_i^{m-1}\delta_{i,j}h_{m,1-x_i^m;m}(s).$$

Now we consider $\frac{\partial}{\partial x_j}\Delta_i$ in detail. Typically we have

$$\frac{\partial}{\partial x_j}\Delta_i = N^2(\delta_{i-1,j} - 2\delta_{i,j} + \delta_{i+1,j}).$$

However, if $i = N/2 \pm 1$, Δ_i contains the term corresponding to $x_{N/2}$, which is identically set as 1 in the present case. Hence $\frac{\partial}{\partial x_{N/2}}\Delta_i$ must be identically zero. Note that the case $i = N/2$ is eliminated since we have already treated above. Moreover, if $i = 1$ and $N - 1$, then $\delta_{i-1,j}$ and $\delta_{i+1,j}$ are eliminated, respectively, since we have formally set as $x_0 = x_N \equiv 0$. Therefore we have

$$\frac{\partial}{\partial x_j}\Delta_i = N^2(\delta_{i-1,j}(1 - \delta_{i-1,N/2})(1 - \delta_{i-1,0}) - 2\delta_{i,j} + \delta_{i+1,j}(1 - \delta_{i+1,N/2})(1 - \delta_{i-1,N})).$$

We also note that, since the 0-Dirichlet boundary condition is imposed, we do not have to pay extra attentions to the cases $i = 1, N - 1$ in the present setting[1]. Consequently, we have

$$\frac{\partial f_i}{\partial x_j} = -h_{1,1;m}(s)\Delta_{N/2}\delta_{i,j} + N^2 x_i h_{1,1;m}(s)(\delta_{j,N/2-1} + \delta_{j,N/2+1}) - \lambda\delta_{i,j}$$

$$+ h_{1,1;m}(s)N^2(\delta_{i-1,j}(1 - \delta_{i-1,N/2})(1 - \delta_{i-1,0}) - 2\delta_{i,j} + \delta_{i+1,j}(1 - \delta_{i+1,N/2})(1 - \delta_{i-1,N}))$$

$$- \lambda m x_i^{m-1}\delta_{i,j}h_{m,1-x_i^m;m}(s). \quad (i \neq N/2)$$

D Blow-up behavior : theoretical study

Following arguments of asymptotic behavior [1, 2], we can discuss blow-up rates of validated solutions. Arguments of blow-up rates begin with asymptotic behavior of solutions of (A.5) tending

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[1]If we consider other boundary conditions such as 0-Neumann boundary, non-trivial treatments involving boundary conditions are necessary.

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to equilibria on the horizon $\{s = 0\}$. Let p_* be a hyperbolic equilibrium for (A.5). Then, the s-component of solutions asymptotic to p_* is written by

$$s(\tau) = Ce^{\lambda_s \tau}(1 + o(1)), \quad \text{as } \tau \to \infty,$$

where $C > 0$ denotes a generic constant which can change in each calculation and $\lambda_s < 0$ is a negative number such that $\operatorname{Re}\mu \leq \lambda_s < 0$ holds for any eigenvalues μ of the Jacobian matrix at $p_*{}^2$. Then we have

$$t_{\max} = \int_0^\infty \frac{e^{-1/s(\eta)^m}}{s(\eta)} d\eta = C \int_0^\infty e^{-\lambda_s \eta} e^{-e^{-m\lambda_s \eta}}(1 + o(1))d\eta.$$

Let $\mu = e^{-\lambda_s \eta}$. Then, $\eta = \frac{1}{-\lambda_s} \log \mu$, $\mu : 1 \to \infty$ holds as $\eta : 0 \to \infty$, $s(\mu) = C\mu^{-1}(1 + o(1))$ as $\mu \to \infty$ and $d\mu = -\lambda_s \mu d\eta$. Thus

$$t_{\max} = C \int_0^\infty e^{-\lambda_s \eta} e^{-e^{-m\lambda_s \eta}} d\eta = \frac{C}{-\lambda_s} \int_1^\infty e^{-\mu^m}(1 + o(1))d\mu. \tag{D.1}$$

Remark D.1. *As mentioned in the end of Section 3, we can directly prove that $t_{\max} < \infty$ from the convergence of (D.1). However, the whole arguments in this section do not tell us the concrete value of $t_{\max}$.*

The same argument yields the following asymptotic behavior of $t_{\max} - t$ as $t \to t_{\max}$:

$$t_{\max} - t = \int_\tau^\infty e^{-\lambda_s \eta} e^{-e^{-m\lambda_s \eta}} d\eta = C \int_{e^{-\lambda_s \tau}}^\infty e^{-\mu^m}(1 + o(1))d\mu$$

where $t = t(\tau)$ given by

$$t = \int_0^\tau \frac{e^{-1/s(\eta)^m}}{s(\eta)} d\eta.$$

In particular, we have

$$e^{-\lambda_s \tau} \sim C \left[\ln(t_{\max} - t)^{-1}\right]^{1/m} \quad \text{as } t \to t_{\max}.$$

Summarizing the argument, we finally have

$$\frac{1}{s(\mu)} = \mu = C \left[\ln\{(t_{\max} - t)^{-1}\}\right]^{1/m}(1 + o(1)) \quad \text{as } \mu \to \infty.$$

References

[1] K. Matsue. Geometric treatments and a common mechanism in finite-time singularities for autonomous ODEs. *arXiv:1806.08487*, 2018.

[2] K. Matsue. On blow-up solutions of differential equations with Poincaré-type compactifications. *SIAM Journal on Applied Dynamical Systems*, 17(3):2249 - –2288, 2018.

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^{2}In [2], eigenvalues with non-trivial multiplicity such that algebraic multiplicities are different from geometric ones are also taken into account. However, the concrete form of the Jacobian matrix in Section C shows that the matrix is diagonalizable, and hence the present consideration becomes simpler.