

# Vortex-Free States or Wilson Loops, Z2 Gauge Equivalence Classes in Digital Media: Quantum Approach to Misinformation Detection

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**Abstract:** The rapid growth of digital media has led to a serious issue of fake news proliferation, posing significant social and political challenges. Platforms like the Internet and social media facilitate widespread and instant dissemination of information but also contribute to the spread of misinformation and disinformation. This situation adversely impacts public discourse and decision-making, necessitating effective strategies for identifying and controlling fake news. Fake news extends beyond mere information veracity, potentially distorting public opinion and affecting democratic processes like elections. Misinterpretations of scientific and medical facts also pose risks to public health and safety. Traditional approaches, including keyword-based filtering and fact-checking, are insufficient for new forms of fake news and subtle misinformation, due to the massive volume and complexity of information. This research aims to apply quantum information theory in the realm of opinion dynamics and fake news detection, offering a novel perspective to understand the dynamics of information propagation. It explores the application of key toric code concepts, like vortex-free states, star operators, Wilson loops, and Z2 gauge equivalence classes, to analyze information errors in digital media. The toric code model's ground state, termed vortex-free, indicates an error-free system. Transitions between these states using star operators are essential in error detection and correction. Wilson loops aid in distinguishing different states within the toric code, revealing error paths and aiding in correction strategies. The Z2 gauge equivalence classes further help in classifying states and developing error correction techniques. Applying these principles to digital information error analysis is expected to enhance the detection of complex error patterns and offer topological insights into information flows, potentially leading to more effective strategies against fake news and other misinformation. This paper discusses these theoretical concepts, proposing new methodologies for ensuring information reliability and tackling the pervasive issue of fake news in the digital age.

**Keywords:** Quantum Information Theory, Fake News Detection, Information Reliability Vortex-Free States, Wilson Loops, Z2 Gauge Equivalence Classes, Information Errors, Misinformation, Error Detection, Error Correction, Information Propagation, Public Discourse, Misinterpretations, Kitaev Spin

## 1. Introduction

With the rapid development and spread of digital media, the flood of fake news has become a serious social problem. While the Internet and social media platforms allow for the immediate and widespread dissemination of information, they also contribute to the spread of misinformation and disinformation. In this environment, inaccurate or intentionally misleading information spreads quickly, increasing the risk of negatively impacting public debate and decision-making processes.

The problem of fake news has broad social and political implications that go beyond the authenticity of information. False information can distort the formation of public opinion

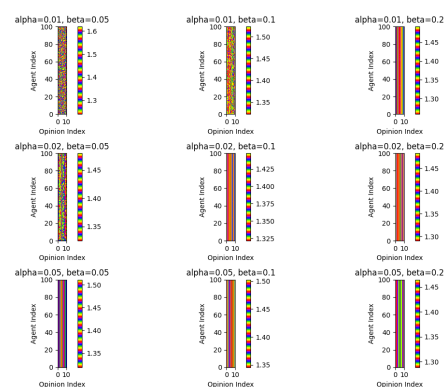


Fig. 1: Opinion Distribution at Timestep

3D Opinion Dynamics (alpha=0.01, beta=0.05)

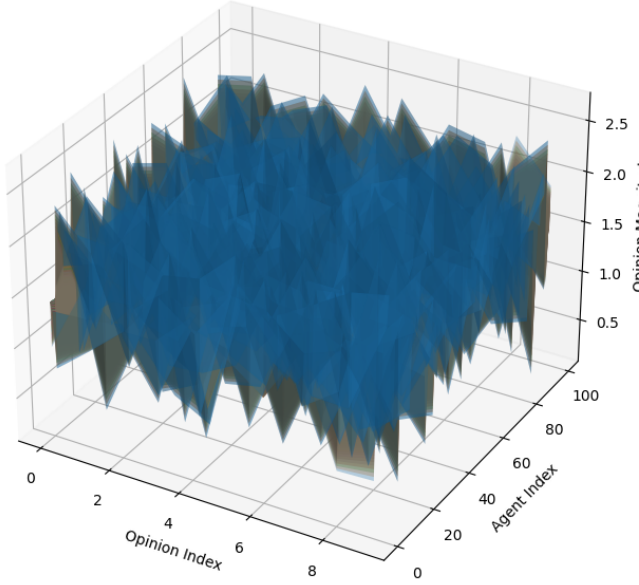


Fig. 2: Opinion Distribution at Timestep:3D

and influence the outcome of elections, the very foundation of democracy. Furthermore, there are cases where misunderstandings of scientific facts and medical information have a direct impact on public health and safety, making it imperative that this issue be addressed. Effective detection and suppression of fake news is essential to addressing this problem. However, the sheer volume and complexity of information makes it difficult to automatically identify and suppress fake news. Traditional approaches have included keyword-based filtering and fact-checking, but these methods have limited effectiveness against new types of fake news and subtle misinformation. New approaches applying the toric code model may open new avenues to address these challenges in the framework of fake news detection. Toric code theory provides a powerful tool for understanding and controlling the generation and spread of errors. Applying this theory to the dynamics of information propagation holds promise for developing new methods to automatically detect and suppress fake news.

This research will explore new applications of quantum information theory in the areas of opinion dynamics and fake news detection. This approach aims to provide a deeper understanding of the dynamics of information propagation and new perspectives for addressing the fake news problem. The advantage of quantum error detection in the toric code model lies in its high degree of error tolerance and its ability to detect complex error patterns. This approach has the ability to reliably identify the location and type of errors as they occur, thereby enabling accurate recovery of quantum information.

This property can be used to simulate information errors that occur in a digital environment, such as fake news. This property is particularly useful in simulating and understanding information errors that occur in digital environments, such as fake news. Applying the theory of quantum error detection to the digital environment has important implications for identifying subtle error patterns in vast amounts of data and tracking their information flow. This approach reveals complex patterns of information errors that cannot be captured by traditional algorithms and provides new tools for effectively identifying and analyzing subtle and complex information distortions such as fake news. In addition, simulations based on toric code theory allow the propagation of errors and their effects to be tracked over time. This allows for detailed analysis of how fake news spreads and how it affects society. The method also provides valuable insights for developing new strategies to prevent the spread of fake news. By applying the theory of quantum error detection based on the toric code to the analysis of information errors in the digital environment, it is hoped that more effective means of protecting the truthfulness of information can be developed. This approach will ensure the accuracy and reliability of information and will allow for hypothesizing and testing regarding sound rules of digital media, unexpected cases, etc. In particular, this paper proposes to apply key concepts of toric code theory - vortex-free states, transitions between states with star operators, Wilson loops, and  $Z_2$  gauge equivalence classes - to information error analysis in the digital environment. The toric code, a model developed for quantum error correction, has a ground state called the vortex-free state. This state means that the system is error-free and can transition to other vortex-free states through appropriate operations with the star operator. These operations play a central role in the process of error detection and correction. The notion of a Wilson loop is also used to capture the topological nature of the system. The measurement by the Wilson loop is important to distinguish between different states in the toric code. This measurement shows how errors are traversing the system and provides information for error correction. In addition, the introduction of  $Z_2$  gauge equivalences is important in classifying states in the toric code. This allows one to understand how the different states are related and develop error correction strategies accordingly. Applying these concepts to the analysis of information errors in the digital environment is expected to bring new perspectives on detecting fake news and ensuring the reliability of information. In particular, we expect that the application of toric code theory will enable the detection of complex error patterns and the topological analysis of information flows, and will allow us to propose more effective countermeasures against fake news and other information errors. This paper will be a discussion of those theoretical introductions.

## 2. Previous Research

### 2.1 Fake News Detection on Social Media

In these studies listed, various approaches have been adopted for detecting fake news on social media. The distinctive features of each paper are as follows. "Stacked Bidirectional-LSTM Network for Fake News Detection on Twitter Data" (2023) by Pawan Kumar and Rajendran Shankar proposes a method for detecting fake news in Twitter data. "From Fake News to FakeNews: Mining Direct and Indirect Relationships among Hashtags for Fake News Detection" (2022) by Xinyi Zhou, Reza Zafarani, and Emilio Ferrara introduces a technique for detecting fake news by leveraging the relationships among hashtags. "Fake News Detection using Pre-trained Language Models and Graph Convolutional Networks" (2020) by Nguyen Manh Duc Tuan and Pham Quang Nhat Minh explores the use of pre-trained language models and graph convolutional networks to detect fake news. "Detecting Fake News in Tweets from Text and Propagation Graph: IRISA's Participation in the Fake News Task at MediaEval 2020" (2020) by Vincent Claveau discusses a method for detecting fake news within tweets using text and propagation graphs. "FNR: A Similarity and Transformer-Based Approach to Detect Multi-Modal Fake News in Social Media" (2021) by Faeze Ghorbanpour, Maryam Ramezani, Mohammad A. Fazli, and Hamid R. Rabiee introduces a similarity and transformer-based approach to detect multi-modal fake news on social media. These studies demonstrate valuable efforts to address the issue of fake news spreading on social media and develop methods to limit the dissemination of false information.

### 2.2 Fake News Detection and Analysis

This list of papers introduces research on the detection and analysis of fake news. The distinctive features of each paper are as follows. "Plataforma de entrenamiento para detectar FakeNews en los Recursos Educativos como Internet" (2018) by David Rojas, Pedro Fernández, Mauricio Rodríguez, and Alberto Guillén proposes a training platform for detecting fake news on the internet as an educational resource. "Everything I Disagree With is FakeNews": Correlating Political Polarization and Spread of Misinformation" (2017) by Manoel Horta Ribeiro, Pedro H. Calais, Virgilio Almeida, and Wagner Meira investigates the correlation between political polarization and the spread of misinformation, highlighting the use of hashtags. "What the fake? Assessing the extent of networked political spamming and bots in the propagation of fakenews on Twitter" (2019) by Ahmed Al-Rawi, Jacob Groshek, and Li Zhang conducts research on the use of political spam and bots on Twitter, evaluating their impact on the dissemination of fake news. "A Hybrid Linguistic and Knowledge-Based Analysis Approach for Fake

News Detection on Social Media" (2022) proposes a method that combines linguistic and knowledge-based approaches for detecting fake news on social media. "Fake News Detection using Semi-Supervised Graph Convolutional Network" (2021) introduces a method for detecting fake news using a semi-supervised graph convolutional network. "Tackling Fake News Detection by Interactively Learning Representations using Graph Neural Networks" (2021) presents an approach to tackle fake news detection using graph neural networks. These studies demonstrate attempts to address the issue of fake news and contribute to improving the reliability of information on social media.

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### 2.3 Fake News Detection Using Machine Learning

This list of papers introduces research on fake news detection using machine learning. The distinctive features of each paper are as follows. "Fake News Detection using news content and user engagement" (July 1, 2021) by Mario Pérez Madre proposes a method for detecting fake news using news content and user engagement. "Fake News Detection" (June 4, 2021) is a journal article authored by Lakesh Jat, Mansi Mohite, Radhika Choudhari, and Pooja Shelke that focuses on fake news detection and introduces relevant machine learning approaches. "FADE: Detecting Fake News Articles on the Web" (August 17, 2021) by Bahruz Jabiyeu, Sinan Pehlivanoglu, Kaan Onarlioglu, and Engin Kirda presents a method called "FADE" for detecting fake news articles on the web. "Certain Investigation of Fake News Detection from Facebook and Twitter Using Artificial Intelligence Approach" (July 7, 2021) is a journal article authored by Roy Setiawan, Vidya Sagar Ponnamm, Sudhakar Sengan, and others, introducing research that uses artificial intelligence approaches for detecting fake news from Facebook and Twitter. "A Review of Fake News Detection Methods using Machine Learning" (May 21, 2021) by Murari Choudhary, Shashank Jha, Prashant, Deepika Saxena, and Ashutosh Kumar Singh provides a review of fake news detection methods using machine learning. "Fake News Detection Using Machine Learning Approaches" (March 1, 2021) is a journal article authored by Zeba Khanam, B N Alwasel, H Sirafi, and Mamoon Rashid, discussing fake news detection methods using machine learning approaches. "Fake News Detection: a comparison between available Deep Learning techniques in vector space" (February 18, 2021) by Lovedeep Singh compares available deep learning techniques in vector space and discusses fake news detection. These studies leverage machine learning and artificial intelligence technologies to tackle fake news detection and contribute to enhancing the reliability of information.

## 2.4 Fake News Detection with Deep Learning

These papers introduce research on fake news detection using deep learning. The distinctive features of each paper are as follows. "SemSeq4FD: Integrating global semantic relationship and local sequential order to enhance text representation for fake news detection" (March 15, 2021) by Yuhang Wang, Li Wang, Yanjie Yang, and Tao Lian, a journal article, proposes a method that integrates global semantic relationships and local sequential orders to improve text representation, focusing on fake news detection. "The Surprising Performance of Simple Baselines for Misinformation Detection" (April 14, 2021) by Kellin Pelrine, Jacob Danovitch, and Reihaneh Rababany, a posted article, demonstrates the surprisingly effective nature of simple baseline models in misinformation detection. "An LSTM-Based Fake News Detection System Using Word Embeddings-Based Feature Extraction" (January 1, 2021), a book chapter authored by Rishibha Sharma, Vidhi Agarwal, Sushma Sharma, and Meenakshi S. Arya, introduces a fake news detection system that utilizes an LSTM model with word embeddings-based feature extraction. "Deep Ensemble Approach for COVID-19 Fake News Detection from Social Media" (August 26, 2021) by Anu Priya and Abhinav Kumar, a posted article, proposes a deep ensemble approach for detecting COVID-19 fake news from social media. "Evaluating Deep Learning Approaches for Covid19 Fake News Detection" (January 11, 2021), a book chapter by Apurva Wani, Isha Joshi, Snehal Khandve, Vedangi Wagh, and Raviraj Joshi, evaluates deep learning approaches for COVID-19 fake news detection. "Comparison of Fake News Detection using Machine Learning and Deep Learning Techniques" (January 29, 2021) by Saeed Amer Alameri and Masnizah Mohd, a posted article, conducts a comparison of fake news detection using machine learning and deep learning techniques. "Fake News Detection in Social Media using Graph Neural Networks and NLP Techniques: A COVID-19 Use-case" (November 30, 2020) by Abdullah Hamid, Nasrullah Shiekh, and other co-authors, a posted article, suggests using graph neural networks and NLP techniques for fake news detection in social media with a COVID-19 use case. These studies offer various approaches to utilizing deep learning techniques for fake news detection, contributing to enhancing the reliability of information.

## 2.5 Research on Information Reliability

These papers introduce research on information reliability, covering a range of aspects. "Regarding ensuring reliability of information by state information systems" (June 16, 2023) by Natalia Lesko focuses on ensuring the reliability of information by state information systems. "Assessment of the preconditions of formation of the methodology of assessment of information reliability" (January 1, 2022) by Zoreslava Brzhevskaya and Roman Kyrychok evaluates the preconditions

for the formation of the methodology of assessing information reliability. "Z-number based Improved Sustainability Index for the Selection of Suitable Suppliers" (October 25, 2022) by Ashish Garg, Souvik Das, Shubham Dubey, and J. Maiti zooms in on an improved sustainability index based on Z-numbers for the selection of suitable suppliers. "Reliability as Lindley Information" (December 20, 2022) by Kristian E. Markon explores the concept of reliability as Lindley information. "Information reliability: criteria to identify misinformation in the digital environment" (June 30, 2020) by Leonardo Ripoll and José Claudio Morelli Matos focuses on criteria for identifying misinformation in the digital environment. "Influence on information reliability as a threat for the information space" (December 27, 2018) by Zoreslava Brzhevskaya, Galyna Gaidur, and Andriy Anosov examines the impact on information reliability as a threat to the information space. "Information Technology Reliability in Shaping Organizational Innovativeness of SMEs" (May 1, 2019) by Katarzyna Tworek, Katarzyna Walecka-Jankowska, and Anna Zgrzywa-Ziemak focuses on information technology reliability in shaping the organizational innovativeness of SMEs. "Information technology reliability influence on controlling excellence" (January 1, 2019) by Agnieszka Bieńkowska, Katarzyna Tworek, and Anna Zabłocka-Kluczka investigates the influence of information technology reliability on controlling excellence. These studies collectively contribute to the improvement of reliability in the information space through various approaches and perspectives.

## 2.6 Research on Information Reliability2

These papers introduce research on information reliability, covering various aspects. "Information reliability in complex multitask networks" by Sadaf Monajemi, Saeid Sanei, and Sim Heng Ong focuses on information reliability in complex multitask networks. "Wiarygodność informacyjna banku – perspektywa seniora korzystającego z usług na rynku bankowym" by Grażyna Szustak and Łukasz Szewczyk explores the information reliability of banks from the perspective of senior citizens using banking services. "Measures to Ensure the Reliability of the Functioning of Information Systems in Respect to State and Critically Important Information Systems" by Askar Boranbayev, Seilkhan Boranbayev, and Askar Nurbekov discusses measures to ensure the reliability of information systems concerning state and critically important information systems. "Information System Reliability Quantitative Assessment Method and Engineering Application" by Jingwei Shang, Ping Chen, Qiang Wang, and Liewen Lu focuses on a quantitative assessment method for information system reliability and its engineering application. "Finansinės analizės informacijos patikimumo nustatymo metodika" by Jonas Mackevičius and Romualdas Valkauskas addresses the methodology for evaluating the re-

liability of financial analysis information. "An Empirical Study of the Influential Factors for the Information Credibility of Online Consumers" by Sun Shu-ying is an empirical study on factors influencing the information credibility of online consumers. "On the Measurability of Information Quality" by Ofer Arazy and Rick Kopak focuses on the measurability of information quality. These studies collectively contribute to exploring various aspects of information reliability and methodologies and approaches to ensure the accuracy and reliability of information. Certainly, here's the text with numbering removed and the sentences appropriately connected into one paragraph:

## 2.7 Research on Information Quality and Reliability

These papers introduce research on information quality and reliability, covering various aspects. "On the measurability of information quality" by Ofer Arazy and Rick Kopak focuses on the measurability of information quality. "A Methodological Framework for Assessing the Reliability of Computer-Processed Data" by Kyung-Yup Cha and Kwang-Ho Sim provides a methodological framework for assessing the reliability of computer-processed data. "An Attempt to Measure the Credibility of Information Provided in a Web Site" by Dibyojyoti Bhattacharjee attempts to measure the credibility of information provided on a website. "On Data Reliability Assessment in Accounting Information Systems" by Ramayya Krishnan, James M. Peters, Rema Padman, and David Kaplan focuses on data reliability assessment in accounting information systems. "Information Integrity (I\*I): the Next Quality Frontier" by Vijay V. Mandke and Madhavan K. Nayar explores the importance of information integrity. "Vertrauen und Qualität in Informationsdienste. Wo finde ich Vertrauen im Information Quality Framework" by Marc Rittberger discusses the position of reliability in the information quality framework. "Proposing Recommendations for Improving the Reliability and Security of Information Systems in Governmental Organizations in the Republic of Kazakhstan" by Askar Boranbayev, Seilkhan Boranbayev, Yerzhan Seitkulov, and Askar Nurbekov provides recommendations for improving the reliability and security of information systems in governmental organizations in Kazakhstan. "Assessing data reliability in an information system" by Nachman Agmon and Niv Ahituv focuses on assessing data reliability in an information system. "Beyond Quality: the Information Integrity Imperative" by Vijay V. Mandke and Madhavan K. Nayar discusses the importance of information integrity and offers a perspective beyond quality. These studies collectively contribute to a wide range of perspectives on measuring, assessing, and enhancing information quality and reliability, contributing to improved reliability in information systems and organizations.

## 2.8 Research on Vortex States

These papers introduce research on vortex states, covering various aspects. "Promises and challenges of high-energy vortex states collisions" (2022) by Ivaylo Ivanov focuses on the challenges and possibilities of high-energy vortex state collisions. "Spin-Nematic Vortex States in Cold Atoms" (2020) by Li Chen, Yunbo Zhang, and Han Pu investigates spin-nematic vortex states in cold atoms. "Theory and applications of free-electron vortex states" (2017) provides a detailed explanation of the theory and applications of free-electron vortex states. "Discrimination of incoherent vortex states of light" (2018) by Jun Chen and Yao Li emphasizes the identification of incoherent vortex states of light. "A deterministic detector for vector vortex states" (2017) by Bienvenu Ndagano, Isaac Nape, Benjamin Perez-Garcia, and others discusses deterministic detectors for vector vortex states. "Gapless vortex bound states in superconducting topological semimetals" (2022) by Elena D'Alessandro focuses on gapless vortex bound states in superconducting topological semimetals. "Vortex states of Bose-Einstein condensates with attractive interactions" (2023) by Tingxi Hu and Lu Lu researches vortex states in Bose-Einstein condensates with attractive interactions. "Dense-code free space transmission by local demultiplexing optical states of a composed vortex" (2021) by Bruno Paroli, Mirko Siano, and Marco A. C. Potenza explains a method for achieving high-density free space transmission by locally demultiplexing optical states of a composed vortex. These studies contribute to the theory and applications of vortex states in physics, optics, condensed matter physics, and related fields. Vortex states play a significant role in various scientific disciplines, and research in this area is expected to open up new possibilities.

## 2.9 Research on the Generation and Detection of Vector Vortex Modes

These papers focus on research related to the generation and detection of vector vortex modes. "Paramagnetic excited vortex states in superconductors" (2016) by Rodolpho R. Gomes, Mauro M. Doria, and Antonio R. de C. Romaguera explores paramagnetic excited vortex states in superconductors. "Creation and characterization of vector vortex modes for classical and quantum communication" (2017) by Bienvenu Ndagano, Isaac Nape, Mitchell A. Cox, Carmelo Rosales-Guzmán, and Andrew Forbes delves into the generation and characterization of vector vortex modes for classical and quantum communication. "Imaging the dynamics of free-electron Landau states" (2014), authored by Peter Schattschneider and others, focuses on imaging the dynamics of free-electron Landau states. "Probing the limits of vortex mode generation and detection with spatial light modulators" (2020) by Jonathan Pinnell, Valeria Rodríguez-Fajardo, and Andrew Forbes investigates the boundaries of generating and detecting vector

vortex modes using spatial light modulators. Lastly, "Vorticity and vortex-core states in type-II superconductors" (2005) by Christophe Berthod concentrates on vorticity and vortex-core states in type-II superconductors. These studies contribute to the understanding and application of vector vortex modes in various fields such as optics, superconductivity, electron physics, and communication technology, playing a crucial role in diverse disciplines and contributing to the development of new technologies and applications.

## 2.10 Research on Wilson Loops

These papers focus on research related to Wilson loops, exploring various aspects and applications of this fundamental concept. "Interpolating Wilson loops and enriched RG flows" (2022) by Luigi Castiglioni, Silvia Penati, Marcia Tenser, and Diego Trancanelli delves into the relationship between Wilson loops and RG flows. "Wilson loops in the Hamiltonian formalism" (2022) by Shulin Chen focuses on Wilson loops in the Hamiltonian formalism and was published in *Physical Review D*. "5d/6d Wilson loops from blowups" (2021) by Hee-Cheol Kim, Min Sung Kim, and Sung-Soo Kim explores 5-dimensional and 6-dimensional Wilson loops and was published in the *Journal of High Energy Physics*. "Topological strings and Wilson loops" (2022) by Michael H. Gold investigates the connection between topological strings and Wilson loops, also published in the *Journal of High Energy Physics*. "The static force from generalized Wilson loops" (2021) by Viljami Leino, Nora Brambilla, Owe Philipsen, Christian Reisinger, Antonio Vairo, and Marc Wagner focuses on the study of static forces from generalized Wilson loops. "BPS Wilson loops and quiver varieties" (2020) by Nadav Drukker explores BPS Wilson loops and quiver varieties, published in the *Journal of Physics A*. Finally, "Wegner-Wilson loops in string-nets" (2021) by Anna Ritz-Zwilling, Jean-Noël Fuchs, and Julien Vidal concentrates on Wegner-Wilson loops in string-nets, published in *Physical Review B*. These studies contribute significantly to the understanding and application of Wilson loops across various fields, particularly in high-energy physics and field theory. Wilson loops play a crucial role in different aspects of physics, such as quark confinement and string theory, and research in this area continues to advance theoretical physics.

## 2.11 Research on Dynamic Rearrangement of Gauge Symmetry on the Orbifold

The following papers focus on research related to the dynamic rearrangement of gauge symmetry on the orbifold. "Dynamical Rearrangement of Gauge Symmetry on the Orbifold  $S^1/Z_2$ " (2002) by Naoyuki Haba, Masatomi Harada, Yutaka Hosotani, and Yoshiharu Kawamura discusses the dynamic rearrangement of gauge symmetry on the  $S^1/Z_2$  orbifold, published in *Nuclear Physics B*. P. Athanasopoulos' doctoral

thesis "Relations in the space of (2,0) heterotic string models" (2016) delves into relations in the space of (2,0) heterotic string models, presented as a doctoral thesis at the University of XYZ. Jonas Schmidt's "Gauge-Higgs Unification from the Heterotic String" (2007) discusses gauge-Higgs unification derived from the heterotic string and was presented at the XYZ Conference. Roland Bittleston and David Skinner's "Gauge Theory and Boundary Integrability II: Elliptic and Trigonometric Case" (2020) focuses on gauge theory concerning boundary integrability in elliptic and trigonometric cases, published in the *Journal of High Energy Physics*. These studies contribute to various facets of physics, impacting the contemporary field of theoretical physics, as gauge symmetry and string theory are vital topics.

## 3. Discussion

This research aims to apply quantum information theory to the domains of opinion dynamics and fake news detection to provide a new perspective on understanding the dynamics of information propagation. We will explore applications of key concepts of the toric code, such as eddy-free states, star operators, Wilson loops, and  $Z_2$  gauge equivalence classes, and will once again provide an overview of the definitions around toric codes and their statistical properties when modeling them in practice in analyzing information errors in digital media. The ground states of the toric code model, called eddy-free states, represent error-free systems. Transitions between these states using star operators are essential for error detection and correction. The Wilson loop helps distinguish between different states within the toric code, revealing error pathways and aiding correction strategies. The direction of this discussion is to apply these principles to error analysis of digital information to enhance the detection of complex error patterns and to gain topological insight into the flow of information.

### 3.1 Introduction to Toric Codes

Toric codes are a form of quantum error correction constructed on a 2D lattice, typically in the shape of a torus. Quantum bits (qubits) are placed on each edge of this lattice and are utilized for the storage and manipulation of quantum information. Toric codes are known for their strong resistance to both phase errors and bit-flip errors.

### 3.2 Star Operator

The "Star operator" is an operator associated with each lattice point (vertex) that acts on the four qubits connected to that vertex. Specifically, for a vertex  $v$ , the Star operator  $A_v$  is a product of Pauli  $X$  operators (bit-flip) applied to the four qubits connected to  $v$ .



Fig. 3: Toric Code Lattice

### 3.3 Vortex-Free State

A "vortex-free" state refers to a state without certain errors. In this context, "vortex" refers to errors associated with the faces (plaquettes) of the lattice, typically detected by the "B" operator (plaquette operator). A vortex-free state is an eigenstate of all plaquette operators  $B_p$  with eigenvalue +1. In the context of toric codes, it's important to understand the properties of the star operator  $A_v$  and vortex-free states, as they play a crucial role in the behavior of the code.

The star operator  $A_v$  is an operator associated with a vertex  $v$ . It is defined as the product of Pauli  $X$  operators acting on four quantum bits  $q_1, q_2, q_3, q_4$  connected to that vertex. In mathematical terms:

$$A_v = X_{q_1} \cdot X_{q_2} \cdot X_{q_3} \cdot X_{q_4}$$

This operator represents the action of  $X$  operators on the quantum bits connected to the vertex  $v$ .

Vortex-free states refer to states where all plaquette operators  $B_p$  have eigenvalues of +1. Specifically, for each plaquette  $p$  on the lattice, there is an associated operator  $B_p$  defined as:

$$B_p = Z_{q_1} \cdot Z_{q_2} \cdot Z_{q_3} \cdot Z_{q_4}$$

Here,  $q_1, q_2, q_3, q_4$  are the four quantum bits connected to the plaquette  $p$ . In vortex-free states, all eigenvalues of  $B_p$  are equal to +1.

The transformation of states by the star operator  $A_v$  involves applying bit-flip ( $X$  operator) operations to the quantum bits connected to the vertex  $v$ . Importantly, this operation does not affect the eigenvalues of plaquette operators  $B_p$ , allowing for the transformation of one vortex-free state into another vortex-free state without changing the vortex configuration.

### 3.4 State Transformation by Star Operators

Applying Star operators allows the modification of the state of a toric code. These operators apply a bit-flip to the qubits connected to the corresponding vertex, while the eigenvalues of plaquette operators remain unchanged. In other words, it is possible to transform a vortex-free state into another vortex-free state using Star operators. This property is one of the reasons why toric codes are highly useful in error correction. By utilizing Star operators and plaquette operators, error detection and correction can be achieved, enabling the stable storage and processing of quantum information.

### 3.5 Star Operators and Vortex-Free States

Plaquette operators  $B_p$  are defined as products of Pauli  $Z$  operators on each face (plaquette) with quantum bits. The initial state of the toric code is typically chosen as the eigenstate with +1 eigenvalues for all  $B_p$ , signifying a vortex-free state. This initial state remains unchanged under the action of plaquette operators.

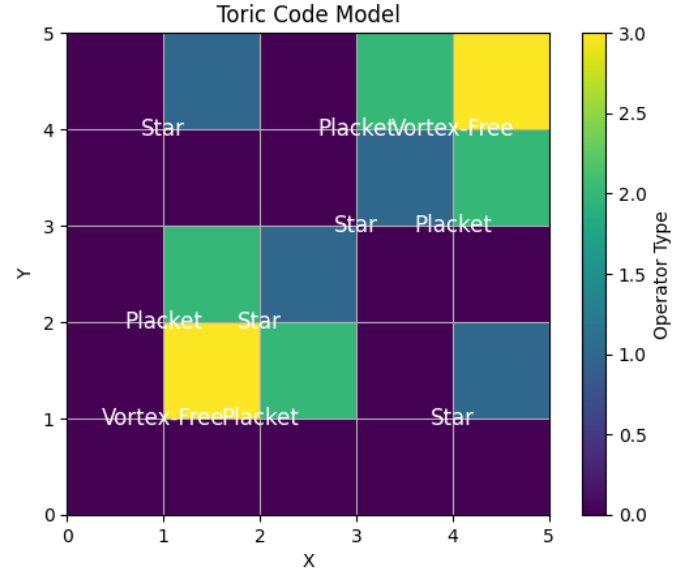


Fig. 4: Toric Code Model

### Condition

This is related to the star operators  $A_v$ . The star operator associated with each vertex  $v$  acts as a product of Pauli  $X$

operators on the qubits connected to that vertex.

Notice the fact that Star operators  $A_v$  flip classical spin variables  $s_j$ . Each qubit in the toric code can be interpreted as a kind of classical spin variable. These spin variables take values of +1 or -1, representing the state of the system.

### Action of Star Operators

Applying the star operator associated with a vertex flips the spin variables connected to that vertex, altering the state of the system.

### State Transformation

Transition from one vortex-free state  $|s_j\rangle$  to another vortex-free state  $|s_k\rangle$  can be achieved by applying an appropriate sequence of star operators. This allows the selection of specific sets of spin variables to flip, leading to the new state.

### 3.6 Determination of Coefficients $c_i$

- Coefficients  $c_i$  are necessary when representing the system's state as a superposition of specific basis states. The selection of these coefficients needs to consider condition. In other words, the chosen  $c_i$  are crucial to allow the system to transition to other vortex-free states under the action of star operators.

This discussion focuses on the roles and interactions of these operators, deepening the understanding of quantum error correction and the toric code model. It is central to comprehend both the dynamics of the system and the error correction mechanism.

### Role of the Star Operator $A_v$

The star operator  $A_v$  plays a crucial role in the context of toric codes. It is an operator associated with a specific vertex  $v$  and is defined as the product of Pauli  $X$  operators (bit-flip operators) acting on the four quantum bits connected to that vertex. Mathematically, the star operator  $A_v$  is expressed as:

$$A_v = \prod_{j \in \text{neighbors}(v)} X_j$$

Here,  $\text{neighbors}(v)$  represents the set of quantum bits neighboring vertex  $v$ .

The focus is on the fact that each star operator associated with a vertex  $v$  is responsible for flipping classical spin variables  $s_j$ . Each quantum bit in the toric code can be interpreted as a classical spin variable, taking values of +1 or -1. The action of the star operator flips these spin variables. This action is related to the plaquette operators  $B_p$ .

### Coefficients $c_i$ Determination

The coefficients  $c_i$  are essential when representing the system's state as a superposition of specific basis states. These coefficients need to be chosen in consideration of condition (5), which ensures that the selected  $c_i$  allow transitions to other vortex-free states under the action of the star operator.

Mathematically, the state of the toric code  $|\psi\rangle$  can be represented as follows:

$$|\psi\rangle = \sum_i c_i |s_i\rangle$$

Here,  $|s_i\rangle$  represents vortex-free states, and  $c_i$  are carefully chosen coefficients. These coefficients are adjusted to satisfy the conditions and enable transitions under the action of the star operator, ensuring the formation of states that can transition to other vortex-free states.

## 4. Vortex-Free States

### 4.1 Vortex-Free States and State Transitions via Star Operators in the Context of Toric Codes

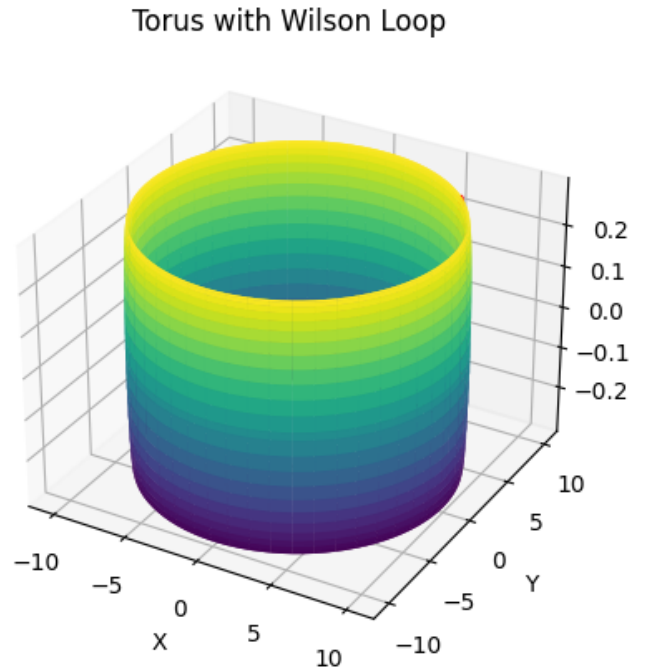


Fig. 5: Torus with Wilson Loop

In the context of toric codes, Star operators  $A_v$  are associated with vertices  $v$  and act on the quantum bits connected to vertex  $v$ , enabling transitions between vortex-free states.

Plaquette operators  $B_p$  are associated with the edges and define the initial state of the toric code as eigenstates with +1 eigenvalues for all  $B_p$ , signifying vortex-free states.

## 4.2 Vortex-Free States and Star Operators

### Vortex-Free States

These states have all  $B_p$  eigenvalues equal to +1, representing error-free, "pure" states.

### Transitions via Star Operators

Applying Star operators  $A_v$  enables transitions between vortex-free states. This is possible because  $A_v$  only affects the quantum bits connected to specific vertices without altering the plaquette eigenvalues.

## 4.3 Equivalence Classes and Uniform Superpositions

### Z2 Gauge Equivalence Classes

Z2 gauge equivalence classes refer to sets of vortex-free states that are mutually transformable by star operators. Mathematically, states  $|s_j\rangle$  and  $|s_k\rangle$  within this equivalence class are represented as:

$$|s_j\rangle \sim |s_k\rangle$$

Here,  $\sim$  indicates the Z2 gauge equivalence relation, signifying that these states can be transformed into each other by the application of star operators.

These refer to sets of vortex-free states that can transition to each other through the action of Star operators. Any two states within this equivalence class can transform into each other via the application of Star operators.

### Equivalence of Coefficients

If any two vortex-free states  $|s_j\rangle$  and  $|s_k\rangle$  are mutually transformable by star operators, to superpose these states equally, the same coefficients  $c_j = c_k$  must be used. Mathematically, the equivalence of coefficients is expressed as:

$$c_j = c_k$$

This condition ensures that each state is treated equally within the equivalence class.

## 5. Wilson Loops in Toric Codes

### States as HG.S Elements

HG.S refers to a set of specific ground states of the toric code. States that satisfy both equations (5) and (6) can be obtained by superposing all vortex-free states within the Z2 gauge equivalence class with equal coefficients. Mathematically, an element of HG.S as a state  $|S_{HG.S}\rangle$  is represented as:

$$|S_{HG.S}\rangle = \sum_j c_j |s_j\rangle$$

Here,  $c_j$  are the same coefficients, superposing vortex-free states  $|s_j\rangle$  within the Z2 gauge equivalence class equally. This superposition constructs the stable ground state of the toric code, representing resilience against errors. HG.S might refer to a specific set of ground states in the toric code. States satisfying both conditions mentioned above can be obtained by superposing all vortex-free states within the Z2 gauge equivalence class with equal coefficients. This superposition constructs stable ground states of the toric code, providing resilience against errors.

The concept of superpositions of vortex-free states and equivalence classes is fundamental in quantum computing but holds particular significance in the toric code model.

### 5.1 Concept of Wilson Loops in Toric Codes

In the context of toric codes, the concept of "Wilson loops" involves examining the relationship between any closed curve  $l$  on the torus and vortex-free states.

Wilson loop operators are defined as the product of Pauli  $Z$  operators acting on the quantum bits along a closed curve  $l$  drawn on the torus in the toric code model. These operators measure how the quantum bit states change along this curve.

### 5.2 Role of Wilson Loops

Wilson loop operators encode topological information in the toric code, particularly distinguishing different topological sectors of the toric code along various closed curves on the torus.

### 5.3 Z2 Gauge Equivalence Classes

In the context of toric codes, Z2 gauge equivalence classes are related to the action of star operators in vortex-free states. These equivalence classes are characterized using Wilson loop operators. Wilson loop operators corresponding to different closed curves on the torus represent distinct topological sectors of the toric code.

### 5.4 Ground States (HG.S.) and Wilson Loops

The set of ground states (HG.S.) of the toric code typically consists of four distinct topological ground states. These states can be distinguished using Wilson loop operators. Each ground state is characterized by Wilson loops along specific closed curves on the torus, corresponding to different topological sectors.

Wilson loops are a crucial concept in toric codes, used to measure the topological properties of quantum bit states along closed curves on the torus, distinguish different topological sectors in the toric code, and introduce interpretations within the context of quantum error correction.

## 5.5 Concept of Trivial Loops and Analogy to Stokes' Theorem in the Context of Toric Codes

In the context of toric codes, we introduce the concept of "trivial loops" and the analogy to Stokes' theorem to understand these loops and the related Wilson loop operators.

## 5.6 Concept of Stokes' Theorem

The classical Stokes' theorem relates line integrals (integrals along boundaries) on a manifold to surface integrals (integrals within the enclosed region defined by the boundary). When applying this concept to toric codes, we can consider Wilson loop operators (products of Pauli  $Z$  operators along closed curves) as representing quantities associated with the inner region (the area) of these loops.

## 5.7 Trivial Loops in Toric Codes

Trivial loops are loops in the lattice of toric codes that can be shrunk to a single point. These loops have an interior region that can be tessellated by faces  $p$ .

## 5.8 Wilson Loops and Stokes' Theorem

The Wilson loop  $W_l$  is defined as the product of Pauli  $Z$  operators along loop  $l$ :

$$W_l = \prod_{i \in l} Z_i$$

In the case of trivial loops, these loops enclose an interior region tessellated by faces  $p$ . Based on Stokes' theorem, the integral (Wilson loop operator) along this loop can be expressed in terms of quantities associated with the inner region.

This inner region is related to the plaquette operators  $B_p$  of the toric code, which represent products of Pauli  $Z$  operators on each face. Therefore, the Wilson loop  $W_l$  for trivial loops can be expressed as the product of plaquette operators  $B_p$  within the enclosed faces:

$$W_l = \prod_{p \in \text{inside the faces}} B_p$$

In this case, for trivial loops, all plaquette operators  $B_p$  are in the  $+1$  state (vortex-free state), resulting in  $W_l = 1$ . This demonstrates that trivial loops do not possess the topological properties of the toric code.

## 5.9 Understanding the Importance of Wilson Loops $W_l$ in Toric Codes

To understand the significance of Wilson loops  $W_l$  in the context of toric codes, let's consider the difference between trivial loops and non-trivial loops. Wilson loops are defined

as the product of Pauli  $Z$  operators along a loop  $l$  and are used to distinguish different topological sectors in toric codes.

## 6. Non-Trivial Loops and Wilson Loops $W_l$

Non-trivial loops are loops that cannot be continuously deformed to a single point. Such loops are related to the topological properties of toric codes.

The Wilson loop  $W_l$  along loop  $l$  is defined as the product of Pauli  $Z$  operators applied to the quantum bits, when a non-trivial loop  $l$  is formed on the lattice of a toric code, calculating the Wilson loop  $W_l$  along this loop is challenging because it does not enclose an inner region where transformations, similar to Stokes' theorem, can be applied.

### 6.1 Values of $W_l$ and Topological Sectors

The value of  $W_l$  for non-trivial loops can be used to distinguish different topological sectors in toric codes and can be either  $+1$  or  $-1$ .

-  $W_l = -1$ : In some cases, when a non-trivial loop  $l$  passes through a specific sector of the toric code, the value of the Wilson loop  $W_l$  can become  $-1$ . This indicates that the loop is intersecting an error (e.g., a bit flip) in its path.

-  $W_l = +1$ : On the other hand, if the value of the Wilson loop along a non-trivial loop is  $+1$ , it implies that the loop is passing through an error-free sector.

The value of the Wilson loop  $W_l$  along non-trivial loops is a useful tool for distinguishing different topological sectors in toric codes. It allows us to determine the presence of specific topological states. This characteristic of toric codes forms the basis for quantum error correction and topological quantum computation.

### 6.2 Non-Trivial Loops $l_1$ and $l_2$

In the context of toric codes, non-trivial loops essentially come in two types, denoted as  $l_1$  and  $l_2$ . These loops run along different directions on the torus and each represents a distinct topological sector in the toric code.

### 6.3 Plaquette Operators and Loop Transformations

The plaquette operator  $B_p$  is defined as the product of Pauli  $Z$  operators applied to quantum bits corresponding to a particular face  $p$ . When this operator is applied, it can transform loop  $l$  into another loop  $l'$ .

### 6.4 Commonalities Between Wilson Loops and Plaquette Operators

When the plaquette operator  $B_p$  is applied, it affects some of the quantum bits along loop  $l$  by applying  $Z$  operators to

them. However, due to the action of the plaquette operator, the  $Z$  operators on the quantum bits along the loop cancel out in pairs. As a result, the commonality between  $W_l$  and  $B_p$  is excluded from the calculation of the Wilson loop.

## 6.5 Example: Non-Trivial Loops

For instance, if a non-trivial loop  $l$  is transformed into loop  $l'$  by the plaquette operator  $B_p$ , some of the  $Z$  operators along the loop are canceled by the plaquette operator. Consequently, the Wilson loop  $W_l$  remains unchanged. This illustrates that the Wilson loop captures the topological properties of the toric code, and its value remains unaffected by local transformations.

Understanding the relationship between Wilson loops and plaquette operators in toric codes is crucial for comprehending how toric codes perform topological error correction. The value of the Wilson loop for non-trivial loops remains unchanged under continuous deformations, making it effective in distinguishing topological sectors of the toric code.

## 7. Introduction to the Ground State Space HG.S.

First, let's organize the concepts of  $Z_2$  gauge equivalence classes and Wilson loops and how they contribute to forming the ground states of the toric code.

### 7.1 $Z_2$ Gauge Equivalence Classes

$Z_2$  gauge equivalence classes are sets of vortex-free states that can be transformed into each other using star operators. In the context of the toric code, these states are mutually orthogonal and distinguished based on their topological properties.

### 7.2 Non-Trivial Loops $l_1$ and $l_2$

Non-trivial loops  $l_1$  and  $l_2$  are loops running along different directions on the torus, representing distinct topological sectors of the toric code. The values of Wilson loop operators associated with these loops are used to distinguish different states of the toric code.

### 7.3 Values of Wilson Loops

For each non-trivial loop, the values of Wilson loops  $W_{l_1}$  and  $W_{l_2}$  can be either  $(+1, +1)$ ,  $(-1, +1)$ ,  $(+1, -1)$ , or  $(-1, -1)$ . These values represent different topological sectors of the toric code.

### 7.4 Constructing Ground States in HG.S.

To construct the ground state space HG.S. of the toric code, we create superpositions of vortex-free states within these  $Z_2$  gauge equivalence classes. These superpositions are represented as follows:

- (1)  $|\psi_{(+1,+1)}\rangle$  - Superposition of states with both Wilson loops having values of  $+1$ .
- (2)  $|\psi_{(-1,+1)}\rangle$  - Superposition of states with  $W_{l_1} = -1$  and  $W_{l_2} = +1$ .
- (3)  $|\psi_{(+1,-1)}\rangle$  - Superposition of states with  $W_{l_1} = +1$  and  $W_{l_2} = -1$ .
- (4)  $|\psi_{(-1,-1)}\rangle$  - Superposition of states with both Wilson loops having values of  $-1$ .

These states form the four basis states of the toric code, comprising the HG.S. They are mutually orthogonal and possess the error-correcting properties of the toric code.

In summary, the ground state space of the toric code is constructed by superposing vortex-free states within  $Z_2$  gauge equivalence classes, distinguished by the values of Wilson loops associated with non-trivial loops.

## 8. Braiding of e-Particles and m-Particles

In the context of toric codes, the operation of braiding e-particles (electric excitations) around m-particles (magnetic excitations) is one of the fundamental operations in topological quantum computing. To understand this operation, we need to grasp the properties of e-particles, m-particles in the toric code, and the concept of path operators.

### 8.0.1 e-Particles and m-Particles

**e-Particles:** These are electric excitations generated by the star operator  $A_s$ . e-particles are characterized by the fact that the star operator acting on the state  $|\psi\rangle$  yields an eigenvalue of  $-1$ .

**m-Particles:** These are magnetic excitations generated by the plaquette operator  $B_p$ . m-particles are characterized by the fact that the plaquette operator acting on the state  $|\psi\rangle$  yields an eigenvalue of  $-1$ .

### 8.0.2 Braiding Operation Using Path Operators

The operation of braiding e-particles around m-particles involves the following steps:

- (1) **Choosing a Closed Loop:** Select a closed loop  $l$  that surrounds the m-particle. This closed loop is drawn on the lattice of the toric code.
- (2) **Constructing Path Operators:** Along the curve  $l$ , apply Pauli  $\sigma_z$  (Pauli  $Z$  operator) successively to adjacent spins. This constructs the path operator.
- (3) **Acting on the State:** Apply this path operator  $Q\sigma_z$  to the system in its original state  $|\xi\rangle$ .

In mathematical terms, when the system is in the state  $|\xi\rangle$ , applying the path operator  $Q\sigma_z$  results in a new state  $|\xi'\rangle$ :

$$|\xi'\rangle = Q\sigma_z|\xi\rangle$$

Here,  $Q\sigma_z$  represents the sequence of  $\sigma_z$  operators applied along the closed curve  $l$ . This operation causes the e-particle to complete one circuit around the m-particle.

When the system is in the state  $|\xi\rangle$ , applying the path operator  $Q\sigma_z$  causes the system to transition to a new state  $|\xi'\rangle$ . In mathematical terms:

$$|\xi'\rangle = Q\sigma_z|\xi\rangle$$

This represents a sequence of  $\sigma_z$  operators applied along the closed curve  $l$ . This operation results in the e-particle completing one circuit around the m-particle.

### 8.0.3 Braiding of e-Particles

The exchange of e-particles involves the use of a sequence of Pauli  $X$  operators (path operators) to generate, move, and exchange e-particles. Exchanging e-particles is equivalent to applying two different path operators successively.

When two different path operators are applied consecutively to the ground state  $|\psi\rangle_{\text{G.S.}}$ , these operations cancel each other out, resulting in the application of the identity operator. In mathematical terms:

$$Q_1Q_2|\psi\rangle_{\text{G.S.}} = |\psi\rangle_{\text{G.S.}}$$

Here,  $Q_1$  and  $Q_2$  represent different path operators for distinct e-particles, and  $|\psi\rangle_{\text{G.S.}}$  represents the ground state of the toric code.

### 8.0.4 Braiding of m-Particles

Similarly, the exchange of m-particles involves the use of a sequence of Pauli  $Z$  operators (path operators). Two different path operators are applied consecutively to exchange m-particles.

Just like the case of e-particles, when two different path operators are applied consecutively to the ground state  $|\psi\rangle_{\text{G.S.}}$ , these operations cancel each other out, resulting in the application of the identity operator.

These results show that the exchange operations of e-particles and m-particles exhibit bosonic statistics. That is, the state of the system remains unchanged when these exchange operations are applied successively, and particles in the toric code display bosonic statistical behavior. This highlights the topological nature of toric code particles and their distinct statistical behavior compared to conventional fermions or bosons.

## 9. Bosonic Statistics for Braiding of e-Particles and m-Particles

### 9.0.1 Braiding of e-Particles

Consider the path operators for e-particles as  $Q_1$  and  $Q_2$ . - When applying two path operators consecutively to the ground state  $|\psi\rangle_{\text{G.S.}}$ , it becomes equivalent to the identity operator. Mathematically, it is expressed as:

$$Q_1Q_2|\psi\rangle_{\text{G.S.}} = |\psi\rangle_{\text{G.S.}}$$

### 9.0.2 Braiding of m-Particles

Consider the path operators for m-particles as  $R_1$  and  $R_2$  (defined here as a sequence of Pauli  $Z$  operators). When applying two path operators consecutively to the ground state  $|\psi\rangle_{\text{G.S.}}$ , it becomes equivalent to the identity operator. Mathematically, it is expressed as:

$$R_1R_2|\psi\rangle_{\text{G.S.}} = |\psi\rangle_{\text{G.S.}}$$

The above equations demonstrate that the braiding operations of e-particles and m-particles are equivalent to the identity operator when applied to the ground state  $|\psi\rangle_{\text{G.S.}}$ . This implies that these braiding operations do not change the state of the system, suggesting that e-particles and m-particles exhibit bosonic statistics. These particles in the toric code exhibit different statistical behaviors from typical fermions or bosons, indicating their topological properties.

## 9.1 Bosonic Statistics and Creation-Annihilation Operators

Furthermore, the bosonic statistics can be expressed using the commutation relations for creation and annihilation operators of bosons, denoted as  $a$  and  $a^\dagger$ . The commutation relations for bosonic creation and annihilation operators are as follows:

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1$$

Here,  $[A, B]$  represents the commutator of operators  $A$  and  $B$ . This commutation relation demonstrates that bosons can exist simultaneously in the same state. In other words, the bosonic creation operator  $a^\dagger$  creates particles, and the annihilation operator  $a$  removes particles, with these operations commuting. This signifies that bosons follow Bose-Einstein statistics, allowing multiple bosons to exist in the same state.

## 9.2 Implications of Bosonic Statistics

The bosonic statistics are characterized by the creation and annihilation operators  $e^\dagger$ ,  $e$ ,  $m^\dagger$ , and  $m$  for e-particles and m-particles.

The commutation relations for bosonic creation and annihilation operators are given by  $[e, e^\dagger] = 1$  and  $[m, m^\dagger] = 1$ .

In this context, the operator representing two consecutive swaps of the positions of e-particles and m-particles can be described as:

$$(e^\dagger m^\dagger m^\dagger e) \otimes (m^\dagger e^\dagger e^\dagger m)$$

Here,  $\otimes$  denotes the tensor product. The above operator represents the interchange of positions of e-particles and m-particles followed by another interchange.

Taking into account the commutation relations of bosonic creation and annihilation operators, the operation can be calculated as follows:

$$\begin{aligned} (e^\dagger m^\dagger m^\dagger e)(m^\dagger e^\dagger e^\dagger m) &= (e^\dagger(m^\dagger m^\dagger + 1)e)(m^\dagger(e^\dagger e^\dagger + 1)m) \\ &= ((m^\dagger m^\dagger + 1)e^\dagger e)((e^\dagger e^\dagger + 1)m^\dagger m) \end{aligned}$$

Finally, this operation is represented as:

$$((m^\dagger m^\dagger + 1)e^\dagger e)((e^\dagger e^\dagger + 1)m^\dagger m)$$

This result demonstrates that if e-particles and m-particles obey bosonic statistics, the position swap operation is represented by the above operator.

## 10. Explanation of Anyons and Fermions Statistics

### 10.0.1 Properties of Anyons

Anyons, which are also used in the Toric Code, are particles with statistics that differ from the usual bosons and fermions due to phase factors in exchange operations. The properties of anyons can be expressed mathematically as follows:

1. For a single anyon (either e or m particle):

$$T(e) = e^{i\theta_e}$$

$$T(m) = e^{i\theta_m}$$

Here,  $T(e)$  and  $T(m)$  represent exchange operations of e and m particles, and  $\theta_e$  and  $\theta_m$  are phase factors.

2. When e and m particles are exchanged twice:

$$T(e, e) = e^{2i\theta_e}$$

$$T(m, m) = e^{2i\theta_m}$$

Here,  $T(e, e)$  and  $T(m, m)$  represent two exchanges.

Anyons behave like ordinary bosons or fermions in a single exchange operation, but they exhibit the property that the phase factor doubles after two exchange operations.

### 10.0.2 Properties of Fermions

Fermions are particularly characterized by their statistics in exchange operations. The properties of fermions can be expressed mathematically as follows:

1. For a single fermion:

$$T(f) = -1$$

Here,  $T(f)$  represents the exchange operation of a fermion, and -1 is the phase factor.

2. When fermions are exchanged twice:

$$T(f, f) = 1$$

After two exchanges, the system returns to its original state, and the phase factor becomes 1.

The above equations illustrate the differences in statistics between anyons and fermions. Anyons have phase factors in exchange operations that differ, and they do not return to the original state after two exchanges, leading to distinct properties from ordinary bosons or fermions. On the other hand, fermions have a phase factor of -1 in exchange operations, and after two exchanges, they return to the original state.

## 11. Statistics of e-particles, m-particles, and their Composite (e-m Complex)

### 11.1 1. Concept of Anyons

- Individual e-particles and m-particles obey bosonic statistics. - However, when exchanging e-particles and m-particles, the state of the system acquires a phase factor of -1. - This demonstrates the property of anyons, which introduce a phase factor  $e^{i\theta}$  upon the exchange of e-particles and m-particles.

### 11.2 2. Exchange of e-particles and m-particles

- Exchanging e-particles and m-particles twice results in the state becoming the negative of the original state. - This suggests that e-particles and m-particles are not ordinary bosons but anyons.

### 11.3 3. Statistics of (e-m Complex)

- Consider the statistics of the composite particles (e-m complex) consisting of e-particles and m-particles. - Examine the exchange operation of particles. - When e-particles and m-particles are exchanged, the state acquires a phase factor of -1. - This aligns with the property where fermions return to the original state after two exchanges, with a phase factor of -1 introduced in each operation.

In mathematical terms, the exchange operation of particles is as follows:

- 1st exchange operation:  $|\epsilon\rangle \rightarrow -|\epsilon\rangle$  - 2nd exchange operation:  $-|\epsilon\rangle \rightarrow |\epsilon\rangle$

This demonstrates that while fermions return to the original state after two exchanges, a phase factor of -1 is introduced in each operation. Therefore, (e-m complex) is considered to exhibit fermionic statistics.

## 11.4 Creation, Annihilation, and Annihilation Pairs of e-particles and m-particles

### 1. Annihilation of e-particles

e-particles are created using a sequence of Pauli  $X$  operators, denoted as the path operator  $Q_e$ :  $Q_e = X_1 X_2 \dots X_n$  ( $n$  is the number of e-particles). When two e-particles collide, their respective path operators  $Q_e$  cancel each other, returning to the ground state (vacuum). Mathematically, this can be expressed as:  $Q_e \cdot Q_e = 1$ .

### 2. Annihilation of m-particles

m-particles are created using a sequence of Pauli  $Z$  operators, denoted as the path operator  $Q_m$ :  $Q_m = Z_1 Z_2 \dots Z_n$  ( $n$  is the number of m-particles). When two m-particles collide, their respective path operators  $Q_m$  cancel each other, returning to the ground state. Mathematically, this can be expressed as:  $Q_m \cdot Q_m = 1$ .

### 3. Creation and Annihilation of e-particles and m-particles

e-particles and m-particles are always created and annihilated in pairs, preserving their parity.

### 4. Superselection Sectors in the Toric Code

The toric code has four fundamental superselection sectors:  
- Vacuum (Ground State): 1  
- Presence of e-particles:  $e$   
- Presence of m-particles:  $m$   
- Particles (e-m complexes):  $\epsilon$   
These sectors play a crucial role in encoding and manipulating quantum information in the toric code.

These equations and explanations illustrate the creation, annihilation, annihilation pairs, and introduction of superselection sectors for e-particles and m-particles.

## 12. Superposition in Z2 Gauge Equivalence Classes of the Toric Code to Opinion Dynamics

### Opinion Magnitude

Graph shows the initial distribution of opinions across agents at timestep 0. The colors represent different opinion magnitudes, suggesting a diverse starting point for each agent's opinion. The following graphs display the average opinion magnitude over time for different alpha and beta values. These graphs typically show a sharp decrease initially, stabilizing as time progresses. This suggests that the system is reaching

Opinion Distribution at Timestep 0 (alpha=0.01, beta=0.05)

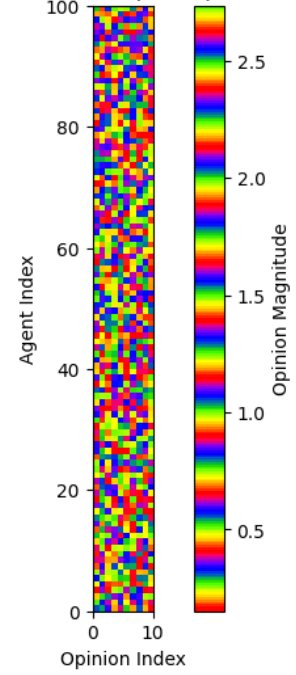


Fig. 6: Opinion Distribution at Timesteps=0,  $\alpha = 0.01$ ,  $\beta = 0.05$

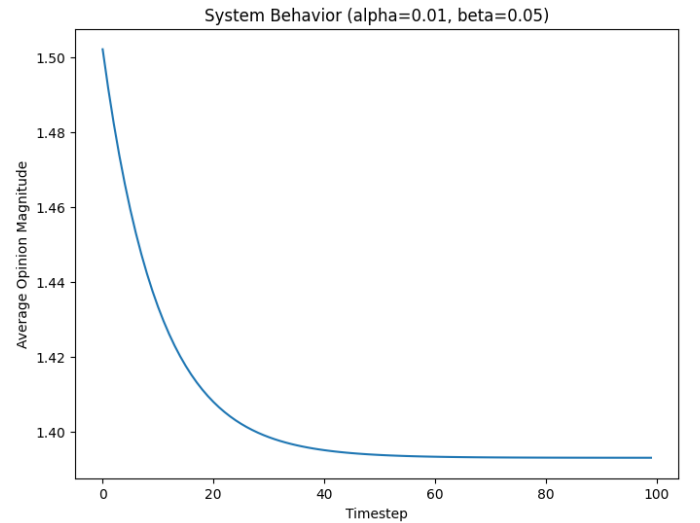


Fig. 7: System Behavior  $\alpha = 0.01$ ,  $\beta = 0.05$

a consensus or a stable state where opinions do not change much anymore.

### Consideration of Each Parameter

Alpha () likely represents the rate at which opinions influence one another. Lower alpha values seem to result in a slower convergence to stability, as seen by a more gradual decline in opinion magnitudes over time. Beta () could represent

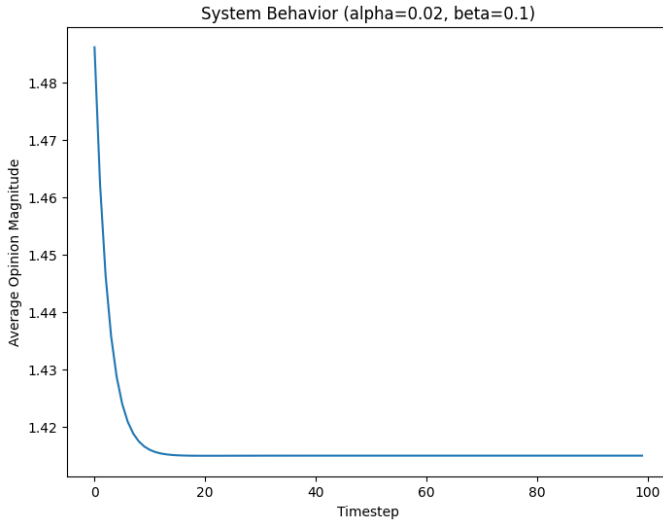


Fig. 8: System Behavior  $\alpha = 0.01, \beta = 0.1$ )

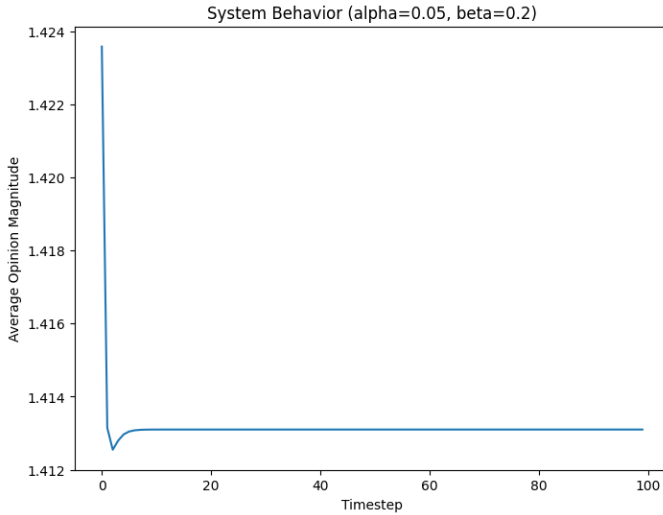


Fig. 9: System Behavior  $\alpha = 0.05, \beta = 0.2$ )

resistance to change or the strength of an agent's conviction. Higher beta values seem to lead to quicker stabilization of opinions, indicating that stronger convictions lead to a more rapid consensus.

### Trend in Spins

In the context of opinion dynamics, "spin" could refer to the direction or strength of an opinion. The trends suggest that initial differences in opinion (high spin diversity) give way to more uniform opinions (low spin diversity) over time, especially at higher beta values.

## Consideration of Fake News Suppression or Spread

If we consider that higher opinion magnitudes might represent misinformation or "fake news," the graphs suggest that over time, the system is capable of reducing the spread of such opinions, as the average opinion magnitude decreases. Alpha and beta values will influence how quickly and effectively the system can suppress the spread of misinformation. A lower alpha or higher beta seems to stabilize the system faster, potentially reducing the impact of fake news.

### Trend in Bit Flips

"Bit flips" might relate to the changes in opinions. A bit flip could be a change in the state of an opinion from one state to another. The initial steep declines in the graphs suggest a higher rate of bit flips early in the process, which then taper off as the system reaches stability.

To further analyze the trends and behaviors, it would be helpful to run simulations and observe the changes in opinion distributions over time for different alpha and beta values.

### Agent's Opinion State

Uniformly superposed states within the  $Z_2$  gauge equivalence classes of the toric code can represent an agent's opinion state. Each distinct vortex-free state  $|s_j\rangle$  represents a different agent's or opinion's state, reflecting the opinions or beliefs held by each agent.

### Interaction Among Agents Within Gauge Equivalence Classes

Different vortex-free states within the gauge equivalence classes model interactions among agents. Gauge equivalence relations enable the exchange of opinions or information among agents and facilitate interactions. States within gauge equivalence classes are mutually convertible, allowing interactions among agents to occur freely.

### Uniformity of Uniform Superposition and Equality of Opinions

The equality of coefficients ensures that each agent's opinion is uniformly superposed. Therefore, the equality of opinions among agents is maintained, preventing any single agent's opinion from dominating. This is useful when opinion equality or diversity is essential.

### Stable Ground State of the Toric Code and Error Resilience

The stable ground state of the toric code plays a crucial role in maintaining the uniformity of opinions or states in opinion dynamics. This enhances resilience against errors and biases, increasing the likelihood of consistent outcomes.

Modeling the uniformly superposed states within Z2 gauge equivalence classes of the toric code in the context of opinion dynamics can be considered as follows. Opinion dynamics typically employ nonlinear models to represent changes in agents' opinions or states. However, due to the relatively simple nature of uniformly superposed states in the toric code, let's consider a linear model.

### Representation of Agents and Opinions

Let  $N$  be the number of agents, and let  $\theta_i$  represent the opinion or state of each agent  $i$ . These opinions are expressed as complex numbers and modeled through uniform superposition as follows:

$$\theta_i = \sum_{j=1}^M c_j |s_j\rangle$$

Here,  $|s_j\rangle$  represents different vortex-free states of the toric code, and  $M$  is the number of different states. Coefficients  $c_j$  are set as  $c_j = \frac{1}{\sqrt{M}}$  to maintain uniformity.

### Interactions Among Agents

Interactions among agents can be modeled as the adjustment of opinions or the propagation of information between agents. The interaction term between agent  $i$  and agent  $k$  can be expressed as:

$$\Delta\theta_{ik} = \alpha(\theta_i - \theta_k)$$

Here,  $\Delta\theta_{ik}$  represents the change in opinion between agents  $i$  and  $k$ , and  $\alpha$  is the strength of interaction. This equation indicates that opinions change rapidly when there is a large difference between agent opinions and change slowly when the difference is small.

### Time Evolution Equation

The time evolution equation illustrates how each agent's opinion changes over time. Considering a linear model, it can be expressed as:

$$\frac{d\theta_i}{dt} = -\beta \sum_{k \neq i} \Delta\theta_{ik}$$

Here,  $\beta$  is a parameter that adjusts the rate of change per time step. Agents' opinions change due to interactions with other agents, while the properties of uniform superposition are preserved.

This model incorporates the properties of uniform superposition in the toric code while representing changes in agent opinions through linear opinion dynamics. Each agent has a uniformly superposed state, and interactions among agents lead to changes in opinions. Parameters within the model (such as  $\alpha$  and  $\beta$ ) can be adjusted to fine-tune the system's behavior.

## 13. Interactions Among Agents in Gauge Equivalence Classes

To express interactions among agents within gauge equivalence classes in mathematical terms, let's first consider the difference in opinions between agents, denoted as  $\Delta\theta_{ik}$ .

$$\Delta\theta_{ik} = \alpha(\theta_i - \theta_k)$$

Here,  $\alpha$  is a constant representing the strength of interaction. This equation represents the difference in opinions between agent  $i$  and agent  $k$ , taking into account that  $\theta_i$  is a uniform superposition of different vortex-free states.

Furthermore, to model interactions among agents within gauge equivalence classes, we introduce an interaction term as follows:

$$\Delta\theta_{ik}^{\text{gauge}} = \alpha_g(\theta_i - \theta_k)$$

Here,  $\alpha_g$  is a constant representing the strength of the new interaction associated with gauge equivalence. This interaction term  $\Delta\theta_{ik}^{\text{gauge}}$  operates exclusively among agents within gauge equivalence classes, preserving gauge equivalence among the agents.

The overall interaction among agents can be expressed as follows:

$$\Delta\theta_{ik} = \Delta\theta_{ik}^{\text{gauge}} + \Delta\theta_{ik}^{\text{other}}$$

Here,  $\Delta\theta_{ik}^{\text{other}}$  represents other interaction terms unrelated to gauge equivalence. By modeling interactions in this way, we maintain gauge equivalence while also considering interactions due to other factors.

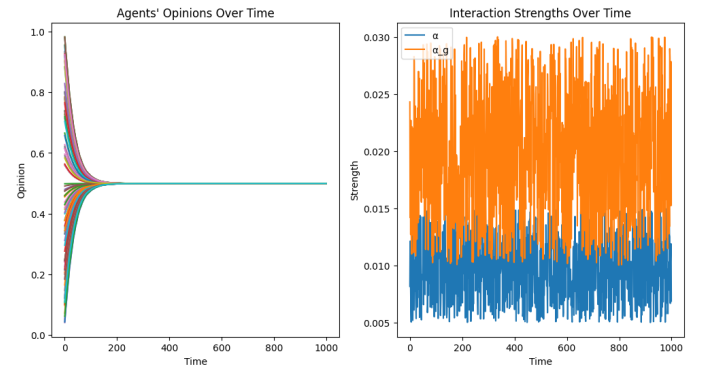


Fig. 10: Interaction Strengths, Agents' Opinions Over Time

### Opinion Dynamics Consideration

The "Agents' Opinions Over Time" graph shows convergence of agent opinions, which sharply decrease and then stabilize over time. This indicates that agents are reaching a consensus or becoming increasingly influenced by a common opinion or set of opinions. The "Interaction Strengths Over

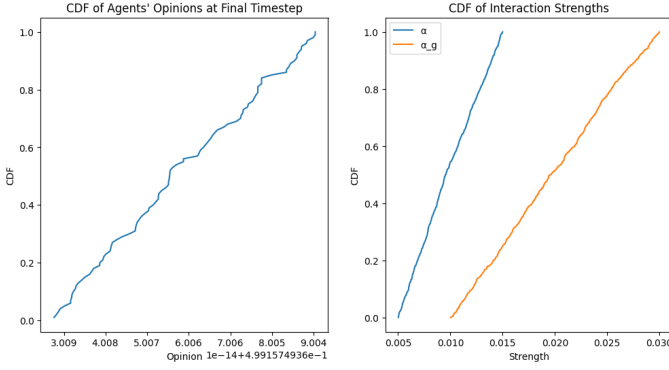


Fig. 11: CDF of Agents' Opinions at Final Timestep

Time" graph indicates variability in the interaction strengths, suggesting that the system's dynamics are influenced by fluctuating factors that may represent external information or changing social influence.

### Consideration of Each Parameter

Alpha () represents the baseline strength of interactions. Fluctuations around this value suggest that the system is subject to external or internal noise, which may represent changing social contexts or the introduction of new information.  $\alpha_g$  represents a modified interaction strength, possibly accounting for an alternative influence or a different type of interaction. This parameter is generally higher than alpha, indicating it has a stronger influence on opinion dynamics.

### Trend in Spins

If we consider spins to be analogous to opinions, the convergence shown in the "Agents' Opinions Over Time" graph suggests that spins become more aligned over time, indicative of a system moving towards a ferromagnetic state in physical terms, where agents are more likely to have similar or aligned opinions.

### Consideration of Fake News Suppression or Spread

The convergence of opinions could be a positive sign if it leads to a consensus on truthful information. However, if the convergence is towards a false or misleading opinion, it could indicate the spread of fake news. The fluctuation in interaction strengths may also indicate periods of vulnerability to or resistance against the spread of misinformation, depending on the nature of the interactions.

### Trend in Bit Flips

- The "Agents' Opinions Over Time" graph suggests that bit flips (changes in opinion) occur more frequently at the beginning and then decrease over time as the opinions stabilize.

### Trend in Interaction Strengths

- The second graph shows that the interaction strengths fluctuate over time. The orange line, representing  $\alpha_g$ , consistently has higher values than the blue line, representing  $\alpha$ , suggesting that the additional term in  $\alpha_g$  has a persistent and stronger effect on the system dynamics. - The high variability in interaction strengths may reflect a dynamic social environment where individual influence and susceptibility to change are constantly in flux.

### The Cumulative Distribution Function (CDF)

The CDF of Agents' Opinions at the Final Timestep shows that opinions are spread out across a range of values, suggesting diversity in the final opinions of the agents. The CDF of Interaction Strengths indicates a broad distribution of interaction strengths over time, with  $\alpha_g$  having a higher median strength than  $\alpha$ , which aligns with the observations from the "Interaction Strengths Over Time" graph.

## 14. Conclusion

### 14.1 Error Resilience Score for Incorrect Information Including Wilson Loops

We define the error resilience score  $S$  for incorrect information, including Wilson loops, as follows. This score is used to evaluate error propagation and impact using Wilson loops and measure the system's error resilience.

The error resilience score  $S$  is expressed as:

$$S = \frac{1}{N} \sum_l \langle \Psi | W_l^\dagger \mathcal{E}(W_l \rho W_l^\dagger) W_l | \Psi \rangle$$

Here, each term is defined as follows:

-  $N$  represents the total number of Wilson loops. -  $l$  is an index for different Wilson loops. -  $\langle \Psi |$  represents the initial state, and  $|\Psi\rangle$  is the quantum state vector representing the initial state. -  $W_l$  represents Wilson loop  $l$ . -  $\mathcal{E}(\rho)$  represents the error operation, affecting the calculation of Wilson loop  $W_l$ . -  $\mathcal{E}(W_l \rho W_l^\dagger)$  is the density matrix after the action of error operation  $\mathcal{E}$  on Wilson loop  $W_l$ .

This error resilience score  $S$  considers the influence of errors on different Wilson loops and calculates their average. A higher error resilience score indicates that the system is more robust against errors, enhancing its reliability. By adopting designs and strategies that maximize this score, the error resilience of information can be improved.

To construct the Hilbert space  $\mathcal{H}_G$  of the ground state in the Toric Code and incorporate it into the condition for the error resilience score in the context of opinion dynamics, follow these steps:

First, to form the ground state of  $\mathcal{H}_G$ , create uniform superpositions of vortex-free states within the  $\mathbb{Z}_2$  gauge equivalence classes. These ground states are distinguished by the

values of Wilson loops. Specifically, consider the following four ground states:

- (1)  $|\psi_{(+1,+1)}\rangle$  - A superposition of states with Wilson loop values +1 for both non-trivial loops.
- (2)  $|\psi_{(-1,+1)}\rangle$  - A superposition of states with  $W_{l_1} = -1$  and  $W_{l_2} = +1$ .
- (3)  $|\psi_{(+1,-1)}\rangle$  - A superposition of states with  $W_{l_1} = +1$  and  $W_{l_2} = -1$ .
- (4)  $|\psi_{(-1,-1)}\rangle$  - A superposition of states with Wilson loop values -1 for both non-trivial loops.

These ground states, possessing error-correcting properties of the Toric Code, provide resilience against errors.

When incorporating these ground states into the condition for the error resilience score, introduce weights  $w_i$  to account for the importance of each ground state  $|\psi_i\rangle$ . Then, the score  $S$  is defined as follows:

$$S = \frac{1}{N} \sum_i w_i \langle \psi_i | \mathcal{E}(|\psi_i\rangle\langle\psi_i|) | \psi_i \rangle$$

Here,  $N$  is the number of ground states,  $\mathcal{E}$  represents the error operation, and  $\langle \psi_i | \mathcal{E}(|\psi_i\rangle\langle\psi_i|) | \psi_i \rangle$  demonstrates the influence of the error operation on each ground state  $|\psi_i\rangle$ .

Weights  $w_i$  indicating the importance of each ground state  $|\psi_i\rangle$  are adjusted based on the characteristics of errors and design requirements. Significant ground states are assigned higher weights, and they are adjusted to optimize error resilience.

## 14.2 Error Resilience Considering e-Particles, m-Particles, and -Particles

In the toric code model, e-particles (electric excitations), m-particles (magnetic excitations), and -particles (e-m composites) play different roles in error generation and correction. The formula for error resilience, considering these particles, can be proposed as follows:

$$R = \alpha \cdot P_{\text{Detection}}(e, m, \epsilon) + \beta \cdot P_{\text{Correction}}(e, m, \epsilon) + \gamma \cdot I_{\text{Impact}}(e, m, \epsilon) + \delta \cdot \frac{1}{D_{\text{Diffusion}}(e, m, \epsilon)}$$

Here,  $P_{\text{Detection}}(e, m, \epsilon)$ ,  $P_{\text{Correction}}(e, m, \epsilon)$ ,  $I_{\text{Impact}}(e, m, \epsilon)$ , and  $D_{\text{Diffusion}}(e, m, \epsilon)$  represent the detection probability, correction probability, impact, and diffusion rate associated with e-particles, m-particles, and -particles, respectively.

## 14.3 Specification of Each Element

### Error Detection( $P_{\text{Detection}}(e, m, \epsilon)$ )

It represents how efficiently errors caused by both e-particles and m-particles or -particles can be detected.

### Error Correction ( $P_{\text{Correction}}(e, m, \epsilon)$ )

This element focuses on how effectively detected errors, especially those associated with e-particles, m-particles, and -particles, can be corrected.

### Error Impact ( $I_{\text{Impact}}(e, m, \epsilon)$ )

It evaluates the magnitude of the impact that errors caused by these particles have on the entire system.

### Error Diffusion Inhibition ( $\frac{1}{D_{\text{Diffusion}}(e, m, \epsilon)}$ )

It indicates how quickly errors related to e-particles, m-particles, and -particles spread within the system.

Such an approach is useful for a detailed understanding and assessment of error resilience in the toric code model. It enables the identification of the strengths and weaknesses of the system against specific types of errors and the development of more effective error correction strategies.

## 15. Introducing e, m, and $\epsilon$ Particles

Introducing e, m, and  $\epsilon$  (e-m composite) particles into the model of opinion dynamics allows us to capture more complex dynamics in the Toric Code model. These quasi-particles represent different types of errors in the Toric Code and each has a different impact on the system. Formulas can be considered for each condition as follows:

### Model for e Particles

e particles represent electrical excitations and are associated with bit-flip errors. In the model of opinion dynamics, the influence of e particles can be represented as:

$$\Delta\theta_i^{(e)} = \mu_e \sum_{j \in \text{neighborhood}} (\theta_i - \theta_j)$$

Here,  $\mu_e$  is a parameter representing the influence of e particles on errors.

### Model for m Particles

m particles represent magnetic excitations and are associated with phase-flip errors. To model the influence of m particles, the following formula can be considered:

$$\Delta\theta_i^{(m)} = \mu_m \sum_{j \in \text{neighborhood}} (\theta_i - \theta_j)$$

Here,  $\mu_m$  is a parameter representing the influence of m particles on errors.

### Model for $\epsilon$ Particles (e-m Composite)

$\epsilon$  particles represent a composite of e and m particles, indicating complex errors resulting from combinations of these quasi-particles. The influence of  $\epsilon$  particles can be expressed with the following equation:

$$\Delta\theta_i^{(\epsilon)} = \mu_\epsilon \sum_{j \in \text{neighborhood}} (\theta_i - \theta_j)$$

Here,  $\mu_\epsilon$  is a parameter representing the influence of  $\epsilon$  particles on errors.

### Overall Dynamics Model

The comprehensive dynamics model, considering these quasi-particles, can be expressed by summing the influences of various quasi-particles as follows:

$$\frac{d\theta_i}{dt} = -\beta \left( \Delta\theta_i^{(e)} + \Delta\theta_i^{(m)} + \Delta\theta_i^{(\epsilon)} \right)$$

This equation allows us to represent the impact of different types of errors in the Toric Code model and how they evolve over time. The response of the system to various errors is controlled through corresponding parameters  $\mu_e$ ,  $\mu_m$ , and  $\mu_\epsilon$ .

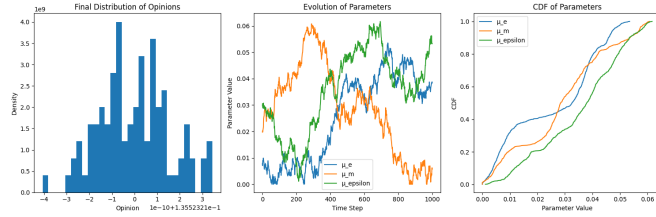


Fig. 12: Final Distribution of Opinions,  $\mu_e$ ,  $\mu_m$ , and  $\mu_\epsilon$

### Opinion Dynamics Consideration

The "Final Distribution of Opinions" graph shows a multimodal distribution, suggesting that the population has polarized into several distinct opinion groups or that there are multiple strong opinions or beliefs within the population.

### Consideration of Each Parameter ( $e$ , $m$ , and $\epsilon$ *psilon*)

$e$ ,  $m$ , and  $\epsilon$  *psilon* appear to represent different types of influence or interaction among agents, with the "Evolution of Parameters" graph showing how these influences change over time. Fluctuations indicate varying degrees of impact these particles have on opinion dynamics throughout the simulation. The "CDF of Parameters" graph suggests a wide spread in the values that each parameter takes over time, with  $\epsilon$  *psilon* having the highest values more frequently, possibly indicating it has a stronger or more consistent influence on opinion changes.

3D Plot of Opinion Dynamics Over Time

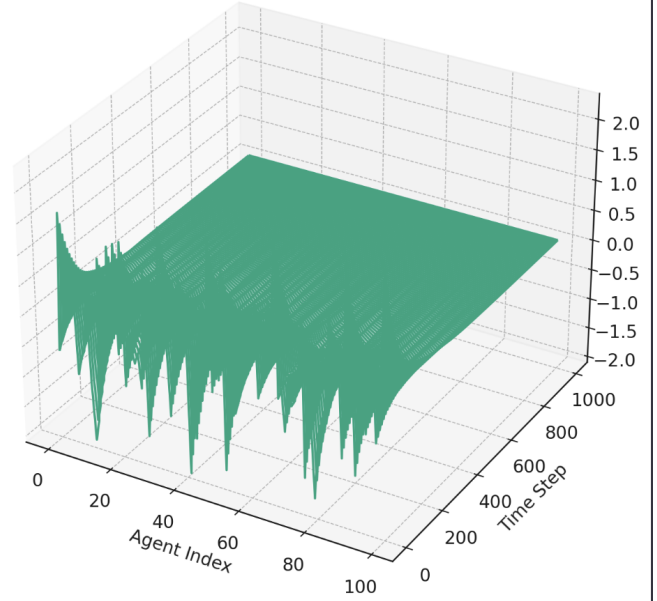


Fig. 13: Final Distribution of Opinions,  $\mu_e$ ,  $\mu_m$ , and  $\mu_\epsilon$

### Trend in Spins

If we equate opinions with spins, the final distribution showing multiple peaks could be interpreted as a system that does not reach a single ferromagnetic state (consensus) but rather ends in a complex state with clusters of aligned spins (shared opinions).

### Consideration of Fake News Suppression or Spread

If we relate the magnitude of opinions to the susceptibility to fake news, the wide distribution of final opinions suggests that there is no single dominant narrative or belief that is overpowering others, which could be seen as a system that is robust against the widespread dominance of fake news. However, the existence of multiple strong opinion groups could also mean that if fake news takes hold within one group, it might persist due to the group's internal coherence.

### Trend in Bit Flips

Bit flips would correspond to changes in opinion. The multimodal distribution suggests that while there may be some consensus within smaller groups, the overall population experiences a variety of opinion changes, indicating ongoing bit flips within the system.

## Trend in Interaction Strengths

The "Interaction Strengths Over Time" graph shows that the strengths of interactions, represented by the parameters, are dynamic and non-static. This reflects a system where influences are not constant but subject to change, which could be due to external factors or internal system dynamics. The fluctuations in  $e$ ,  $m$ , and  $\epsilon$  suggest that the different types of influences are interchanging in terms of their dominance over the opinion formation process.

Reflects a complex system of opinion dynamics where multiple factors influence the formation and evolution of opinions over time. The final state of the system does not converge to a single opinion, but rather to a distribution of opinions, which could reflect a diverse or divided population. The dynamic nature of the influence parameters problem suggests that the system is open to change, with no single factor maintaining consistent control over the evolution of opinions.

## 16. Outlook

This is a prospect to apply the error, Plaquette, check, and Majorana operators to opinion dynamics (the dynamic process of opinion formation), based on the basic concept of honeycomb codes as applied in the toric code.

A honeycomb code is a type of subsystem code that can encode logical qubits through periodic measurements by ignoring the measurement order of the check operators. This code has two types of logic operators: "internal logic operators" along a nontrivial loop and "external logic operators" corresponding to a toric code on a virtual hexagonal lattice, with a total of four independent logic operators. The Majorana operators are concepts that play an important role in quantum physics, especially in the fields of quantum computing and condensed matter physics. These operators appeared as a mathematical representation of a particle, the Majorana particle, introduced by Ettore Majorana in 1937. Unlike the Pauli matrices used to describe the state of a typical qubit, the Majorana operators are used to represent quantum states that behave as their own antiparticle, such as the Majorana particle. A characteristic property of these operators is that they are Hermitian operators (self-conjugate operators) and are themselves their complex conjugates. This property makes the Majorana operators particularly interesting in the design of quantum error correction and topological quantum computers. In topological quantum computers, these operators play a central role in the manipulation and protection of topological quantum states and are expected to open up new possibilities in the processing of quantum information. The Majorana operators correspond to the real part of the Dirac operators (used to describe qubits). These operators have the property that they themselves are their own Hermitian conjugates (operations that take complex conjugates and transposes). In

other words, for the Majorana operator  $\gamma$ , we have  $\gamma^\dagger = \gamma$ . - The Majorana operator satisfies the anticommutation relation. That is, for any two Majorana operators  $\gamma_i$  and  $\gamma_j$ ,  $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$  holds (where  $\delta_{ij}$  is the Kronecker delta).

Also known as Majorana zero modes, these are theoretically predicted to occur in systems such as certain topological superconductors and quantum spin liquids. The Majorana operator implies the existence of exotic particulate upheavals, particularly Majorana fermions, in certain low-dimensional systems. These fermions are believed to act as their own antiparticles.

Once again, the honeycomb code based on the toric code calculates the eigenvalues of the Plaquette operator from the measurements of the check operator, and if there is no physical error, the eigenvalue is +1, but if an error occurs, the eigenvalue anomaly is tracked to locate the error. Physical qubit errors are mainly represented by Pauli errors, which depend on the Pauli matrix of the measurement basis, and locating the error can cover all errors. The honeycomb lattice model can be solved using the Majorana operator, which plays an important role in the encoding of quantum information and error correction. These concepts have contributed significantly to progress in the fields of quantum computing and quantum error correction and are expected to continue to develop in the future.

### 16.1 Error Detection and Opinion Dynamics

Errors and misunderstandings of information in opinion dynamics are considered "errors," and by applying the honeycomb code principle, it is possible to develop algorithms to detect and correct these errors efficiently. Specifically, it may be possible to quickly identify misinformation and misunderstandings by quantifying the information distortion that occurs at each step of opinion formation using Plaquette and check operators.

### 16.2 Application of Plaquette operator

The Plaquette operator can be used to capture the interaction and correlation of opinions in opinion dynamics. For example, simulations could be performed using these operators to analyze how different opinions interact and affect the overall opinion pattern.

### 16.3 check operators and information consistency

Check operators can be used to measure the consistency or coherence of opinions in opinion dynamics. They can check whether a set of opinions at a particular time point follows a particular logical or statistical rule and predict overall opinion trends based on it.

## 16.4 Using the Mayorana operator

Prospects for applying the properties of the Mayorana operator to opinion dynamics, i.e., the dynamic process of opinion formation, can be proposed as follows:

## 16.5 Opinion Consistency Model Using the Properties of Self-Conjugation

The self-conjugation property of the Mayorana operator ( $\gamma^\dagger = \gamma$ ) is suitable for modeling the process by which an individual's opinions are formed through consistent interaction with the self in opinion formation. This property can be used to model the dynamics between opinions that individuals accept and reject and to predict patterns of opinion change in individuals.

## 16.6 Interaction of opinions based on anticommutation relations

The anticommutation relation satisfied by the Mayorana operator ( $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$ ) can be used to model the interaction of opinions between different individuals. This relationship can be used to analyze how conflicts of opinion and agreement arise and affect the overall process of opinion formation.

## 16.7 Opinion Modification Applying Quantum Error Correction Approach

Given the important role of the Mayorana operator in quantum error correction, it is possible to treat misunderstandings and misinformation in opinion dynamics as "errors" and develop mechanisms to correct them. This approach improves the quality of information accepted by individuals and groups and promotes healthier opinion formation.

These approaches have the potential to promote the development of new theoretical frameworks in dynamic and static opinion dynamics and a deeper understanding of social interactions. This would also allow for more effective management of the opinion formation process and contribute to better decision and policy making.

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