L^2-INVARIANTS OF GROUPS UNDER COARSE EQUIVALENCE AND OF GROUPOIDS UNDER MORITA EQUIVALENCE

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Abstract. We prove that triviality of some L^2-invariants of discrete groups is preserved by coarse equivalence, where L^2-invariants are L^2-homologies, L^2-Betti numbers (with 'a mild condition') and Novikov-Shubin invariants. We give definitions of some L^2-invariants of cocompact étale groupoids and prove that their triviality is preserved by Morita equivalence. Also we exhibit basic properties for modules over von Neumann algebras which are not necessarily finite. This paper contains an appendix by Yamashita, where a characterization of finite von Neumann algebras is given.

Keywords: coarse equivalence, discrete groups, étale groupoids, Novikov-Shubin invariants, L^2-Betti numbers, von Neumann algebras.

2010 MSC: Primary: 20F65, Secondary: 22A22, 20J05, 46L10, 37C85.

1. Introduction

Since L^2-Betti numbers of regular coverings of closed manifolds were introduced by Atiyah ([3]), L^2-invariants containing L^2-Betti numbers are defined and studied for some mathematical objects which are not necessarily manifolds (refer to [16]). In all cases, finite von Neumann algebras play an important role. Our paper has two purposes: one is to study behaviors of L^2-invariants of discrete groups under coarse equivalence, and the other is to introduce L^2-invariants of cocompact étale groupoids which are well-behaved under Morita equivalence, where cocompact étale groupoids are étale groupoids with relatively compact open complete transvarsals (see Subsection 3.1). We remark that in this paper von Neumann algebras which are not finite play an important role, too. In this introduction, first we explain a background and one of our main theorems with respect to the former of two purposes, and also in an explanation of an idea of its proof, we write others of our main theorems with respect to the latter. After that, we mention motivations for the latter.

When a discrete group is finite type, that is, its classifying space can be taken as a CW-complex which has a finite number of k-dimensional cells for any non-negative integer k, then its L^2-Betti numbers and Novikov-Shubin invariants are well-defined by using L^2-Betti numbers and Novikov-Shubin invariants of equivariant CW-complexes of finite type (refer to [16, Chapter 1, 2]). For general discrete groups, L^2-Betti numbers originally were introduced by using L^2-Betti numbers of equivariant CW-complexes of finite type and taking some limits in [5]. On the other hand, an equivalent definition of L^2-Betti numbers of discrete groups was given from an algebraic viewpoint in [14, 15]. And also this algebraic viewpoint

can give a definition of Novikov-Shubin invariants of general discrete groups, which were achieved in [17] (also refer to [21]). Now we very briefly recall definitions of L^2-invariants of discrete groups in [14, 15] and [17] (refer to [16, chapter 6]). The k-th L^2-homology of G is defined by the k-th group homology with coefficients in the group von Neumann algebra $\mathcal{N}G$, that is, $H_k(G,\mathcal{N}G) := \mathrm{Tor}_k^{\mathbb{Z}G}(\mathbb{Z},\mathcal{N}G)$ (refer to [4]). The k-th L^2-Betti number of G is defined by $b_k^{(2)}(G) := \dim_{\mathcal{N}G} H_k(G,\mathcal{N}G)$, where $\dim_{\mathcal{N}G}$ can measure a size of a right $\mathcal{N}G$-module ([14]). The $(k+1)$-st Novikov-Shubin invariant of G is defined by $\alpha_{k+1}(G) := \alpha_{\mathcal{N}G}(H_k(G,\mathcal{N}G))$, where $\alpha_{\mathcal{N}G}$ can measure another size of a right $\mathcal{N}G$-module ([17]). In fact the k-th capacity of G is defined in [17], but we denote its inverse by $\alpha_{k+1}(G)$. We remark that $\dim_{\mathcal{N}G}$ and $\alpha_{\mathcal{N}G}$ are defined by using the so-called finite trace of $\mathcal{N}G$. Now we consider two questions:
(1) How do L^2-invariants of discrete groups behave under coarse equivalence?
(2) How do L^2-invariants of discrete groups behave under measure equivalence?
Answers of the second question are well-known. It was proved that L^2-Betti numbers of discrete groups are proportionally invariant under measure equivalence in [7]. Also we can easily confirm that L^2-homologies and Novikov-Shubin invariants of discrete groups have no good behaviors under measure equivalence. Indeed $\mathbb{Z}$ and $\mathbb{Z}^2$ are measure equivalent, but we know that the second Novikov-Shubin invariant and the first L^2-homology of $\mathbb{Z}$ are trivial and them of $\mathbb{Z}^2$ are not trivial. Next we consider the first question (refer to [18] about L^2-torsions). We may have a question whether Novikov-Shubin invariants of discrete groups in a suitable class are preserved by coarse equivalence ([9, 8.A6]). Sauer gave a positive answer for the class of amenable groups in [27], where we remark that he needed 'a mild condition' for amenable groups, but it is removed by using [29, Theorem 7.1] (refer to Remark 2.50). On the other hand it is well-known that L^2-Betti numbers of discrete groups are not proportionally invariant under coarse equivalence (refer to [16, Chapter 7.5]), but whether k-th L^2-Betti numbers of discrete groups are trivial or not is preserved by coarse equivalence if we assume that discrete groups are finite $(k+1)$-type, that is, their classifying spaces can be realized as CW-complexes whose $(k+1)$-st skeletons have only finite cells (see [9, Chapter 8], [8] and [23]). Similar claims about k-th L^2-homologies and $(k+1)$-st Novikov-Shubin invariants of discrete groups of finite $(k+1)$-type are known (see [9, Chapter 8], [8] and [23]). One of our main theorems is a generalization of the above results in [9, Chapter 8], [8] and [23].

Theorem 1.1. *Let G_1 and G_2 be two discrete groups which are coarsely equivalent and k be a non-negative integer. Then we have the following:*
(1) The k-th L^2-homology of G_2 is trivial if the k-th L^2-homology of G_1 is trivial;
(2) The $(k+1)$-st Novikov-Shubin invariant of G_2 is trivial if the $(k+1)$-st Novikov-Shubin invariant of G_1 is trivial;
(3) The k-th L^2-Betti number of G_2 is trivial if the k-th L^2-Betti number of G_1 is trivial and G_1 or G_2 satisfies 'a mild condition'.

'A mild condition' in the above is precisely written in Subsection 4.4, but we remark that a group satisfies 'a mild condition' if there exists a CW-complex which is its classifying space with the only finite number of $(k+1)$-dimensional cells and no conditions about other dimensional cells (Lemma 4.9). At present, we do not know whether we can remove 'a mild condition' in Theorem 1.1 (3) or not.

Before we write an idea of its proof, we recall coarse equivalence of discrete groups, which is important from a viewpoint of the so-called geometric group theory ([9]). Since discrete groups have standard right invariant coarse structures, they are coarsely equivalent if they are coarsely equivalent as the coarse spaces (refer to [24]). We remark that for two finitely generated groups, they (with word metrics) are quasi-isometric if and only if they are coarsely equivalent. There are two equivalent definitions of coarse equivalence of discrete groups at least. First one is well-known: G_1 and G_2 are coarsely equivalent if and only if there exists a uniform embedding from G_1 to G_2 which is essentially surjective (refer to [28]). Second one is also well-known and essentially claimed by Gromov in [9] (also refer to [28] and [27]): G_1 and G_2 are coarsely equivalent if and only if there exists a topological coupling Ω of G_1 and G_2, that is, a locally compact Hausdorff space Ω with a cocompact free proper action of G_1 and a cocompact free proper action of G_2 which mutually commute. When we have a topological coupling Ω of G_1 and G_2, then we can take a G_1-surjection $p_1 : \Omega \to X_1 := G_2\backslash\Omega$ and a G_2-surjection $p_2 : \Omega \to X_2 := G_1\backslash\Omega$. Then p_i is extended to a Morita equivalence map between transformation groupoids $\mathcal{G} := (G_1 \times G_2) \ltimes \Omega$ and $\mathcal{G}_i := G_i \ltimes X_i$ for each $i = 1, 2$ (Subsection 3.1). Conversely when we have a compact Hausdorff space X_1 with a G_1-action and a compact Hausdorff space X_2 with a G_2-action such that transformation groupoids $\mathcal{G}_1 := G_1 \ltimes X_1$ and $\mathcal{G}_2 := G_2 \ltimes X_2$ are Morita equivalent, then we have a topological coupling Ω of G_1 and G_2, a G_1-homeomorphism $X_1 \cong G_2\backslash\Omega$ and a G_2-homeomorphism $X_2 \cong G_1\backslash\Omega$ (see Remark 3.6). Hence we have a modified version of second one: G_1 and G_2 are coarsely equivalent if and only if there exist a compact Hausdorff space X_1 with a G_1-action and a compact Hausdorff space X_2 with a G_2-action such that transformation groupoids $\mathcal{G}_1 := G_1 \ltimes X_1$ and $\mathcal{G}_2 := G_2 \ltimes X_2$ are Morita equivalent. In this paper we use second one or its modified version as a definition of coarse equivalence of discrete groups.

A conceptual hint in order to consider a strategy to prove Theorem 1.1 is the so-called measured group theory, which is introduced as a counter-part of geometric group theory in measure theory by Gromov ([9]). Two discrete groups G_1 and G_2 are measure equivalent if and only if there exist a measurable G_1-space X_1 with an invariant probability measure and a measurable G_2-space X_2 with an invariant probability measure such that two measured transformation groupoids $G_1 \ltimes X_1$ and $G_2 \ltimes X_2$ are weakly orbit equivalent. Based on [5], Gaboriau introduced L^2-Betti numbers of measured groupoids which are proportionally invariant under weak orbit equivalence and also by using it, he proved that L^2-Betti numbers of discrete groups are proportionally preserved by measure equivalence ([7]). We remark that an argument based on [14] was given in [26]. On the other hand we introduce L^2-invariants of cocompact étale groupoids whose triviality is invariant under Morita equivalence and also by using it, we prove that triviality of L^2-invariants of discrete groups is preserved by coarse equivalence. Now we write a general strategy to prove that a property (P) of discrete groups is preserved by coarse equivalence. It is to

find a property $(\mathcal{P})$ of cocompact étale groupoids which satisfies the following:
(1) A property $(\mathcal{P})$ is preserved by Morita equivalence;
(2) For any G-action on a compact Hausdorff space X, G has (P) if and only if $G \ltimes X$ has $(\mathcal{P})$.
Indeed we can prove that (P) of discrete groups is preserved by coarse equivalence by using $(\mathcal{P})$ of cocompact étale groupoids as follows. If two discrete groups G_1 and G_2 are coarsely equivalent, then we have a compact Hausdorff space X_1 with a G_1-action and a compact Hausdorff space X_2 with a G_2-action such that $\mathcal{G}_1 := G_1 \ltimes X_1$ and $\mathcal{G}_2 := G_2 \ltimes X_2$ are Morita equivalent. If G_1 has (P), then $G_1 \ltimes X_1$ has $(\mathcal{P})$ by (2). By (1), $G_2 \ltimes X_2$ has $(\mathcal{P})$. Hence G_2 has (P) by (2). In our case (P) is triviality of L^2-invariants of discrete groups, hence we look for a suitable definition of triviality of L^2-invariants of cocompact étale groupoids as $(\mathcal{P})$. Actually in this paper we succeed it, that is, we have definitions of L^2-invariants of cocompact étale groupoids and the following two theorems, which correspond to the latter of two purposes. Here we do not mention L^2-Betti numbers and Novikov-Shubin invariants because we need more words (precisely see Theorem 4.5 and Theorem 4.7).

Theorem 1.2. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two cocompact étale groupoids which are Morita equivalent and k be a non-negative integer. Then we have that the k-th L^2-homology of $\mathcal{G}_1$ is trivial if and only if the k-th L^2-homology of $\mathcal{G}_2$ is trivial.*

Theorem 1.3. *Let G be a discrete group and k be a non-negative integer. Suppose that G has an action on a compact Hausdorff space X. Then we have that the k-th L^2-homology of G is trivial if and only if the k-th L^2-homology of $G \ltimes X$ is trivial, where $G \ltimes X$ is the transformation groupoid of (G, X).*

In Appendix C, we deal with Type FP_n as (P).

Our proof of Theorem 1.1 is strongly inspired by [27], where he proved that Novikov-Shubin invariants of amenable groups are invariant under coarse equivalence by partially practicing the above general strategy (he defined Novikov-Shubin invariants of special groupoids with invariant probability measures, which played an enough role to prove his theorem). If readers are familiar to [27] and interested in Theorem 1.1 about discrete groups, then they can prove it by using Section 2 and referring [27, Section 6]. We note that we need to study some groupoids without invariant probability measures and modules over von Neumann algebras without finite traces in order to prove Theorem 1.1. In Section 2, we exhibit basic properties for modules over von Neumann algebras which are not necessarily finite (refer to [13], [17], [29] and [16, Chapter 6] for the case of finite von Neumann algebras). Now we write two things which we should give attention to in our paper when we consider not only finite von Neumann algebras but also infinite von Neumann algebras. First one is that we can not use Lemma 2.42. Indeed Appendix B by Yamashita gives a characterization of finite von Neumann algebras. Second one is that we can not generally use Appendix A, which causes difficulty of removing 'a mild condition' of Theorem 1.1 (3).

Now we mention motivations for introducing L^2-invariants of cocompact étale groupoids. At least there exist two interesting cocompact étale groupoids: transformation groupoids on compact Hausdorff spaces and fundamental or holonomy groupoids on complete transvarsals coming from closed foliated manifolds. If such groupoids or closed foliated manifolds have invariant probability measures, then their L^2-Betti numbers were introduced in [6], [10], [7] and [26], and also their Novikov-Shubin invariants were introduced for special cases in [11] and [27]. They have some applications to group actions and foliations with invariant measures. On the other hand, many cocompact étale groupoids have no invariant probability measures. Hence it is natural to try to introduce L^2-invariants of such groupoids and to believe that they can have some applications to group actions and foliations. Our paper gives a definition of them (Definition 4.3, Theorem 1.2 and Theorem 4.5). And also for general cocompact étale groupoids with invariant probability measures, we define their L^2-Betti numbers and Novikov-Shubin invariants in Definition 4.6. We remark that even in the case that cocompact étale groupoids have invariant probability measures, their Novikov-Shubin invariants may be interesting. For example, by using them, we can prove that asymptotic behaviors of return probabilities of random walks on compactly generated étale groupoids with invariant probability measures are invariant under Morita equivalence ([22]). In this paper we do not try to compare various definitions of L^2-invariants of cocompact étale groupoids with invariant probability measures, which should be done in the near future (refer to [20]).

Through this paper we suppose that any discrete group is countable and also that for any étale groupoid all fibers by its range map and its state map are countable (see Subsection 3.1). We note that L^2-invariants in this paper mean mainly L^2-homologies, L^2-Betti numbers and Novikov-Shubin invariants.

The author would like to express his gratitude to Professor Guoliang Yu for conversations on some topics discussed herein and also to hospitality of Vanderbilt university. The author would like to express his gratitude to Professor Tsuyoshi Kato for every encouragement to him.

2. von Neumann algebras and modules over them

In this section we study von Neumann algebras which do not necessarily have finite traces and modules over them. Many arguments are parallel to the case of finite von Neumann algebras (refer to [13], [17], [29] and [16, Chapter 6]), but we do not omit details to confirm that we do not use finite traces and also for readers.

2.1. **Two kinds of categories associated to a von Neumann algebra.** Let us consider a von Neumann algebra $\mathcal{A}$ with a faithful normal state ψ through this subsection. Then $\mathcal{A}$ can be represented on $l^2\mathcal{A}$ as the left regular representation by extending the left multiplication on $\mathcal{A}$, where $l^2\mathcal{A}$ is defined by the completion of $\mathcal{A}$ by the Hermitian inner product defined by $\langle a, b\rangle := \psi(a^*b)$ for any $a, b \in \mathcal{A}$. Also we can consider the commutant $\mathcal{A}' \subset B(l^2\mathcal{A})$ of $\mathcal{A} \subset B(l^2\mathcal{A})$, which is also a von Neumann algebra, where $B(l^2\mathcal{A})$ is the set of all bounded operators on $l^2\mathcal{A}$. We have $\mathcal{A}'' = \mathcal{A}$ by the so-called double commutant theorem.

In this subsection we compare two kinds of categories associated to $(\mathcal{A}, \psi)$, namely, categories related to finitely generated projective right $\mathcal{A}$-modules and categories related to finitely generated Hilbert $\mathcal{A}'$-modules, which are defined soon. This subsection is parallel to the case of finite von Neumann algebras (see [13] and [16, Chapter 6.1, 6.2.]).

First we discuss categories related to finitely generated projective right $\mathcal{A}$-modules.

Definition 2.1. Let P be a finitely generated projective right $\mathcal{A}$-module. $\mu : P \times P \to \mathcal{A}$ is an inner product on P if it satisfies the following:
(1) $\mu(x, ya) = \mu(x, y)a$ for any $x, y \in P$ and any $a \in \mathcal{A}$;
(2) $\mu(x, y)^* = \mu(y, x)$ for any $x, y \in P$;
(3) $\mu(x, x) \geq 0$ for any $x \in P$, and $x = 0$ if $\mu(x, x) = 0$;
(4) $\overline{\mu} : P \ni x \mapsto \mu(x, -) \in \mathrm{Hom}_{\mathcal{A}}(P, \mathcal{A})$ is bijective.

When we define $\mu_{st}(x, y) := \sum x_i^* y_i$ for any $x, y \in \mathcal{A}^n$, then it is an inner product on $\mathcal{A}^n$, which is called the standard inner product on $\mathcal{A}^n$.

Definition 2.2. Let (P, μ_P) and (Q, μ_Q) be finitely generated projective right $\mathcal{A}$-modules with inner products. For any $\mathcal{A}$-homomorphism $f : P \to Q$, its involution (or its adjoint) $f^* : Q \to P$ is defined by the composition $Q \xrightarrow{\overline{\mu_Q}} \mathrm{Hom}_{\mathcal{A}}(Q, \mathcal{A}) \xrightarrow{\circ f} \mathrm{Hom}_{\mathcal{A}}(P, \mathcal{A}) \xrightarrow{\overline{\mu_P}^{-1}} P$.

We remark that f^* is an $\mathcal{A}$-homomorphism and depends on μ_P and μ_Q. For example, for the identity homomorphism id between the same finitely generated projective right $\mathcal{A}$-module with different inner products (P, μ_P) and (P, μ'_P), id^* is defined by the composition $P \xrightarrow{\overline{\mu'_P}} \mathrm{Hom}_{\mathcal{A}}(P, \mathcal{A}) \xrightarrow{\overline{\mu_P}^{-1}} P$, which is not the identity homomorphism.

Now we can consider four categories associated to $(\mathcal{A}, \psi)$. First one is a $\mathbb{C}$-category $\mathcal{C}_{proj}$ whose object is a finitely generated projective right $\mathcal{A}$-module and whose morphism is an $\mathcal{A}$-homomorphism. Second one is a $\mathbb{C}$-category with involution $\mathcal{C}^*_{free}$ whose object is a finitely generated free right $\mathcal{A}$-module with the standard inner product and whose morphism is an $\mathcal{A}$-homomorphism. Third one is a $\mathbb{C}$-category with involution $IM(\mathcal{C}^*_{free})$ which is the idempotent completion of $\mathcal{C}^*_{free}$ as a $\mathbb{C}$-category with involution. Fourth one is a $\mathbb{C}$-category with involution $\mathcal{C}^*_{proj}$ whose object is a finitely generated projective right $\mathcal{A}$-module with an inner product and whose morphism is an $\mathcal{A}$-homomorphism.

Also we have three functors between two of them. First one is the so-called forgetting functor $F : \mathcal{C}^*_{proj} \to \mathcal{C}_{proj}$ which is defined by forgetting inner products. It is an equivalent functor as $\mathbb{C}$-categories. We prove that it is essentially surjective and leave other parts of its proof for readers. For any finitely generated projective right $\mathcal{A}$-module P, there exists an $\mathcal{A}$-surjection $q : \mathcal{A}^n \to P$. Since P is projective, then there exists an $\mathcal{A}$-section $s : P \to \mathcal{A}^n$. Hence P is $\mathcal{A}$-isomorphic to the image of $s \circ q : \mathcal{A}^n \to \mathcal{A}^n$. Since we can consider $s \circ q$ as an $\mathcal{A}$-valued $(n \times n)$-matrix, we have the polar decomposition $s \circ q = u \circ |s \circ q|$. Hence we have an $\mathcal{A}$-projection $p := u \circ u^* : (\mathcal{A}^n, \mu_{st}) \to (\mathcal{A}^n, \mu_{st})$ and an $\mathcal{A}$-isomorphism $s : P \xrightarrow{\cong} p(\mathcal{A}^n)$. Thus P has an inner product $\mu_{st} \circ (s \times s)$, where $\mu_{st} \circ (s \times s)(x, y) := \mu_{st}(s(x), s(y))$ for any

$x, y \in P$. Second one is $i_{proj} : \mathcal{C}^*_{free} \to \mathcal{C}^*_{proj}$ which is defined by regarding $\mathcal{C}^*_{free}$ as a subcategory of $\mathcal{C}^*_{proj}$. Third one is $j_{proj} : IM(\mathcal{C}^*_{free}) \to \mathcal{C}^*_{proj}$ which is defined by taking $(p(\mathcal{A}^n), \mu_{st}|_{p(\mathcal{A}^n)\times p(\mathcal{A}^n)})$ for any object $(\mathcal{A}^n, p, \mu_{st})$ of $IM(\mathcal{C}^*_{free})$, where p is an $\mathcal{A}$-projection on $\mathcal{A}^n$, in other words, an element of $\mathcal{A}$-valued $(n\times n)$-matrix such that $p^2 = p$ and $p^* = p$. It is an equivalent functor as $\mathbb{C}$-categories with involution. We prove that it is essentially surjective and leave other parts of its proof for readers. For any finitely generated projective right $\mathcal{A}$-module P with an inner product μ_P, there exists another inner product $\mu'_P := \mu_{st} \circ (s \times s)$ by the above argument. Since (P, μ_P) and (P, μ'_P) are $\mathcal{A}$-isometric by the composition $P \xrightarrow{\overline{\mu_P}} \mathrm{Hom}_{\mathcal{A}}(P, \mathcal{A}) \xrightarrow{\overline{\mu'_P}^{-1}} P$, then (P, μ_P) and $(p(\mathcal{A}^n), \mu_{st}|_{p(\mathcal{A}^n)\times p(\mathcal{A}^n)})$ are $\mathcal{A}$-isometric.

Definition 2.3. Let P be a finitely generated projective right $\mathcal{A}$-module and K be a submodule of P. The closure $\overline{K}$ of K in P is defined by the following: $x \in \overline{K}$ if and only if $f(x) = 0$ for any $\mathcal{A}$-homomorphism $f : P \to \mathcal{A}$ such that $K \subset \mathrm{Ker} f$. A sequence $P_1 \xrightarrow{f} P_2 \xrightarrow{g} P_3$ of finitely generated projective right $\mathcal{A}$-modules is weakly exact at P_2 if $\overline{\mathrm{Im} f} = \mathrm{Ker} g$.

A sequence $P_1 \xrightarrow{f} P_2 \xrightarrow{g} P_3$ of finitely generated projective right $\mathcal{A}$-modules is weakly exact at P_2 if and only if the following holds: $g \circ f = 0$ and $u \circ v = 0$ for any finitely generated projective right $\mathcal{A}$-modules P and Q and any $\mathcal{A}$-homomorphisms $u : P_2 \to P$ and $v : Q \to P_2$ such that $u \circ f = 0$ and $g \circ v = 0$ (Refer to [16, p.250]).

Next we discuss about categories related to finitely generated Hilbert $\mathcal{A}'$-modules.

Definition 2.4. V is a finitely generated Hilbert $\mathcal{A}'$-module if V is a Hilbert space with an $\mathcal{A}'$-action and there exists an isometric $\mathcal{A}'$-embedding $\iota : V \to (l^2\mathcal{A})^n$.

$(l^2\mathcal{A})^n$ is a finitely generated Hilbert $\mathcal{A}'$-module, which we call a standard Hilbert $\mathcal{A}'$-module.

Definition 2.5. Let V and W be finitely generated Hilbert $\mathcal{A}'$-module. For any bounded $\mathcal{A}'$-operator $f : V \to W$, its involution (or its adjoint) $f^* : W \to V$ is defined by the adjoint of f as a bounded operator between Hilbert spaces.

Now we consider three categories associated to $(\mathcal{A}, \psi)$. First one is a $\mathbb{C}$-category with involution $\mathcal{C}^*_{st}$ whose object is a standard finitely generated Hilbert $\mathcal{A}'$-module and whose morphism is a bounded $\mathcal{A}'$-operator. Second one is a $\mathbb{C}$-category with involution $IM(\mathcal{C}^*_{st})$ which is the idempotent completion of $\mathcal{C}^*_{st}$ as a $\mathbb{C}$-category with involution. Third one is a $\mathbb{C}$-category with involution $\mathcal{C}^*_{Hilb}$ whose object is a finitely generated Hilbert $\mathcal{A}'$-module and whose morphism is a bounded $\mathcal{A}'$-operator.

Also we have two functors between two of them. First one is $i_{Hilb} : \mathcal{C}^*_{st} \to \mathcal{C}^*_{Hilb}$ which is defined by regarding $\mathcal{C}^*_{st}$ as a subcategory of $\mathcal{C}^*_{Hilb}$. Second one is $j_{Hilb} : IM(\mathcal{C}^*_{st}) \to \mathcal{C}^*_{Hilb}$ which is defined by taking $p((l^2\mathcal{A})^n)$ for an object $((l^2\mathcal{A})^n, p)$ of $IM(\mathcal{C}^*_{st})$, where p is an $\mathcal{A}'$-projection on $(l^2\mathcal{A})^n$, in other words, an element of $\mathcal{A}$-valued $(n \times n)$-matrix such that $p^2 = p$ and $p^* = p$. It is an equivalent functor as $\mathbb{C}$-categories with involution. We prove that it is essentially surjective and leave other parts of its proof for readers. Since there exists an isometric $\mathcal{A}'$-embedding $\iota : V \to (l^2\mathcal{A})^n$ for any finitely generated Hilbert $\mathcal{A}'$-module V, we can take the

$\mathcal{A}'$-projection p on $(l^2\mathcal{A})^n$ such that $p((l^2\mathcal{A})^n) = \iota(V)$. Hence V is $\mathcal{A}'$-isometric to $p((l^2\mathcal{A})^n)$.

Definition 2.6. A sequence $V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$ of finitely generated Hilbert $\mathcal{A}'$-modules is weakly exact at V_2 if $\overline{\mathrm{Im} f} = \mathrm{Ker} g$.

A sequence $V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$ of finitely generated Hilbert $\mathcal{A}'$-modules is weakly exact at V_2 if and only if the following holds: $g \circ f = 0$ and $u \circ v = 0$ for any finitely generated Hilbert $\mathcal{A}'$-module V and W and any bounded $\mathcal{A}$-operators $u : V_2 \to V$ and $v : W \to V_2$ such that $u \circ f = 0$ and $g \circ v = 0$ (Refer to [16, p.250]).

Now we compare two kinds of categories associated to $(\mathcal{A}, \psi)$, namely, categories related to finitely generated projective right $\mathcal{A}$-modules and categories related to finitely generated Hilbert $\mathcal{A}'$-modules. For $(\mathcal{A}^n, \mu_{st})$, we consider its completion $\nu(\mathcal{A}^n, \mu_{st})$ by the Hermitian inner product $\psi \circ \mu_{st}$. Then $\nu(\mathcal{A}^n, \mu_{st})$ is regarded as a standard Hilbert $\mathcal{A}'$-module $(l^2\mathcal{A})^n$. Generally when we consider any finitely generated projective right $\mathcal{A}$-module P with an inner product μ_P, its completion $\nu(P, \mu_P)$ by a Hermitian inner product $\psi \circ \mu_P$ is regarded as a Hilbert $\mathcal{A}'$-module. Indeed since there exist a $\mathcal{A}$-projection p on $\mathcal{A}^n$ and an $\mathcal{A}$-isometry $s : (P, \mu_P) \to (p(\mathcal{A}^n), \mu_{st}|_{p(\mathcal{A}^n)\times p(\mathcal{A}^n)})$, which is extended on their completions as an $\mathcal{A}'$-isometry, then $\nu(P, \mu_P)$ is regarded as a Hilbert $\mathcal{A}'$-module by this $\mathcal{A}'$-isometry. Also for any $\mathcal{A}$-homomorphism $f : (P, \mu_P) \to (Q, \mu_Q)$ between finitely generated projective right $\mathcal{A}$-modules with involution, $\nu(f) : \nu(P, \mu_P) \to \nu(Q, \mu_Q)$ is a bounded $\mathcal{A}'$-operator between Hilbert $\mathcal{A}'$-modules, where $\nu(f)$ is the extension of f on completions. Then ν gives a functor from $\mathcal{C}^*_{proj}$ to $\mathcal{C}^*_{Hilb}$. Also by restricting ν on subcategories, we have functors $\nu : \mathcal{C}^*_{free} \to \mathcal{C}^*_{st}$ and $\nu : IM(\mathcal{C}^*_{free}) \to IM(\mathcal{C}^*_{st})$. It is clear that $\nu : \mathcal{C}^*_{free} \to \mathcal{C}^*_{st}$ gives an isomorphism as $\mathbb{C}$-categories with involution. Hence $\nu : IM(\mathcal{C}^*_{free}) \to IM(\mathcal{C}^*_{st})$ gives an isomorphism as $\mathbb{C}$-categories with involution. Since $IM(\mathcal{C}^*_{free})$ are equivalent to $\mathcal{C}^*_{proj}$, and $IM(\mathcal{C}^*_{st})$ are equivalent to $\mathcal{C}^*_{Hilb}$ as $\mathbb{C}$-categories with involution. Thus $\nu : \mathcal{C}^*_{proj} \to \mathcal{C}^*_{Hilb}$ gives an equivalence as $\mathbb{C}$-categories with involution.

It is trivial that $F : \mathcal{C}^*_{proj} \to \mathcal{C}_{proj}$ faithfully preserves exact and weak exact sequences. By using the equivalent definitions of weak exact sequences written after Definition 2.3 and Definition 2.6, we can confirm that $\nu : \mathcal{C}^*_{proj} \to \mathcal{C}^*_{Hilb}$ faithfully preserves exact and weak exact sequences.

From the above arguments we have the following (refer to [16, Theorem 6.24.]).

Theorem 2.7. (1) *The functor $\nu : \mathcal{C}^*_{proj} \to \mathcal{C}^*_{Hilb}$ gives an equivalence as $\mathbb{C}$-categories with involution.*
(2) *The forgetful functor $F : \mathcal{C}^*_{proj} \to \mathcal{C}_{proj}$ gives an equivalence as $\mathbb{C}$-categories.*
(3) *Exact sequences and weak exact sequences are faithfully preserved by two functors $\nu : \mathcal{C}^*_{proj} \to \mathcal{C}^*_{Hilb}$ and $F : \mathcal{C}^*_{proj} \to \mathcal{C}_{proj}$.*

Definition 2.8. Let M be a finitely generated right $\mathcal{A}$-module and K be a submodule of M. The closure $\overline{K}$ of K in M is defined by the following: $x \in \overline{K}$ if and only if $f(x) = 0$ for any $\mathcal{A}$-homomorphism $f : M \to \mathcal{A}$ such that $K \subset \mathrm{Ker} f$. We define $TM := \overline{\{0\}}$ and $\pi_P : M \to PM := M/TM$ by its quotient.

Now we give applications of the above theorem (refer to [16, Theorem 6.7.]).

Theorem 2.9. (1) *Let Q be a finitely generated projective right $\mathcal{A}$-module. For any submodule $K \subset Q$, $Q/\overline{K}$ is projective. In particular $\overline{K}$ is a directed factor of Q and finitely generated projective. Moreover if K is finitely generated, then K and $\overline{K}$ are $\mathcal{A}$-isomorphic.*
(2) *$\mathcal{A}$ is semihereditary, that is, right and left semihereditary.*
(3) *Let M be a finitely generated right $\mathcal{A}$-module and $K \subset M$ be a submodule. Then $M/\overline{K}$ is projective and $\overline{K}$ is finitely generated. In particular PM is projective and TM is finitely generated.*

Proof. We freely use Theorem 2.7. First we prove (1). Let Q be a finitely generated projective right $\mathcal{A}$-module and $M \subset Q$ be a finitely generated submodule. We have an $\mathcal{A}$-homomorphism $f : \mathcal{A}^n \to Q$ such that $\mathrm{Im} f = M$. When we take the standard inner product μ_{st} on $\mathcal{A}^n$ and an inner product μ_Q on Q, then we have $\nu(f) : \nu(\mathcal{A}^n, \mu_{st}) \to \nu(Q, \mu_Q)$. We can take an $\mathcal{A}$-projection $p : (\mathcal{A}^n, \mu_{st}) \to (\mathcal{A}^n, \mu_{st})$ such that $\nu(p) : \nu(\mathcal{A}^n, \mu_{st}) \to \nu(\mathcal{A}^n, \mu_{st})$ is the $\mathcal{A}'$-projection onto $\mathrm{Ker}\nu(f)$. Since $\nu(\mathcal{A}^n) \xrightarrow{\nu(p)} \nu(\mathcal{A}^n) \xrightarrow{\nu(f)} \nu(Q)$ is exact, then $\mathcal{A}^n \xrightarrow{p} \mathcal{A}^n \xrightarrow{f} Q$ is exact. Since we have $\mathcal{A}^n = p(\mathcal{A}^n) \oplus (1-p)(\mathcal{A}^n)$ and f gives an $\mathcal{A}$-isomorphism between $\mathcal{A}^n/\mathrm{Im}p$ and M, then $(1-p)(\mathcal{A}^n)$ and M are $\mathcal{A}$-isomorphic. Hence M is projective. When we write the $\mathcal{A}$-injection $j : M \to Q$ and take an inner product μ_M on M, then we have an $\mathcal{A}'$-injection $\nu(j) : \nu(M, \mu_M) \to \nu(Q, \mu_Q)$. We have an $\mathcal{A}$-homomorphism $u : (M, \mu_M) \to (Q.\mu_Q)$, where $\nu(j) = \nu(u)|\nu(j)|$ is the polar decomposition of $\nu(j)$. In particular $\nu(u) : \nu(M, \mu_M) \to \nu(Q, \mu_Q)$ is an $\mathcal{A}'$-isometry onto $\nu(\overline{M})$ because of $\mathrm{Im}u = \overline{\mathrm{Im}\nu(j)} = \nu(\overline{\mathrm{Im}j}) = \nu(\overline{M})$. Hence $u : (M, \mu_M) \to (Q, \mu_Q)$ is an $\mathcal{A}$-isometry onto $(\overline{M}, \mu_Q|_{\overline{M}\times\overline{M}})$. Thus M and $\overline{M}$ are $\mathcal{A}$-isomorphic.

Let $K \subset Q$ be a submodule. We take a directed set I of all finitely generated submodules P_i of K. Then P_i is projective for any $i \in I$ and we have $K = \bigcup_{i\in I} P_i$. We write the $\mathcal{A}$-injection $j_i : P_i \to Q$ and take an inner product μ_i for any $i \in I$. Then we have the $\mathcal{A}'$-injection $\nu(j_i) : \nu(P_i, \mu_i) \subset \nu(Q, \mu_Q)$ for any $i \in I$. Now we consider an $\mathcal{A}$-projection $\pi : (Q, \mu_Q) \to (Q, \mu_Q)$ such that $\nu(\pi) : \nu(Q, \mu_Q) \to \nu(Q, \mu_Q)$ is an $\mathcal{A}'$-projection onto $\overline{\bigcup_{i\in I} \mathrm{Im}\nu(j_i)}$. Then we have $\overline{K} = \mathrm{Im}\pi$. Hence $\overline{K}$ is finitely generated projective because $\mathrm{Im}\pi$ is finitely generated projective. We prove $\overline{K} = \mathrm{Im}\pi$. For any $i \in I$, we have $\pi \circ j_i = j_i$ because of $\nu(\pi) \circ \nu(j_i) = \nu(j_i)$. Hence we have $P_i \subset \mathrm{Ker}(id - \pi) = \mathrm{Im}(\pi)$ for any $i \in I$. Thus we have $\overline{K} \subset \mathrm{Im}(\pi)$. When we take an $\mathcal{A}$-homomorphism $f : Q \to \mathcal{A}$ whose kernel contains K, then we have $\nu(f) \circ \nu(j_i) = 0$ because of $f \circ j_i = 0$. Since we have $\mathrm{Im}(\nu(j_i)) \subset \mathrm{Ker}(\nu(f))$, then we have $\mathrm{Im}(\nu(\pi)) = \overline{\bigcup_{i\in I} \mathrm{Im}\nu(j_i)} \subset \mathrm{Ker}(\nu(f))$. Hence we have $\mathrm{Im}(\pi) \subset \mathrm{Ker}(f)$. Thus we have $\mathrm{Im}(\pi) \subset \overline{K}$.

We prove (2). We take any projective right $\mathcal{A}$-module Q and any finitely generated submodule $K \subset Q$. Since Q is a directed factor of a free right $\mathcal{A}$-module $\bigoplus_{i\in I} \mathcal{A}$, then we can take an $\mathcal{A}$-injection $j : Q \to \bigoplus_{i\in I} \mathcal{A}$. Then there exists a finite subset $I_0 \subset I$ such that $j(K) \subset \bigoplus_{i\in I_0} \mathcal{A}$ because K is finitely generated. Hence K is projective by (1). Thus $\mathcal{A}$ is right semihereditary.

If M is a right (resp. left) $\mathcal{A}$-module, it can be regarded as a left (resp. right) $\mathcal{A}$-module by using $*$ of $\mathcal{A}$. In this operation preserve whether M is finitely generated or not, and projective or not. Hence $\mathcal{A}$ is also left semihereditary since $\mathcal{A}$ is right semihereditary.

We prove (3). We take an $\mathcal{A}$-surjection $q : \mathcal{A}^n \to M$. Then $\overline{M/\overline{K}}$ and $\mathcal{A}^n/q^{-1}(\overline{K})$ are $\mathcal{A}$-isomorphic by it. Also we can confirm $q^{-1}(\overline{K}) = \overline{q^{-1}(K)}$. Hence $M/\overline{K}$ and $\mathcal{A}^n/\overline{q^{-1}(K)}$ are $\mathcal{A}$-isomorphic. Thus $M/\overline{K}$ is projective by using (1). □

Remark 2.10. We write some properties of semihereditary ([12, Section 2E, 4F, 4G, 4H]). Let R be a right (resp. left) semihereditary ring with a unit. Then we have the following properties (first one is a definition of right (resp. left) semihereditariness):
(1) Any projective right (resp. left) R-module is locally projective, that is, any finitely generated submodule of it is projective;
(2) Any submodule of any flat right or left R-module is also flat;
(3) R is right (resp. left) coherent, namely, any finitely generated submodule of a finitely presented right (resp. left) R-module is finitely presented;
(4) Any direct product of flat left (resp. right) R-modules is flat.

2.2. **Modules over von Neumann algebras.**

Let us consider a von Neumann algebra $\mathcal{A}$ with a faithful normal state ψ through this subsection. We know that it is semihereditary by Theorem 2.9. We give definitions of some modules induced from given right $\mathcal{A}$-modules and also study their behaviors under Morita equivalence (refer to [29] for the case of finite von Neumann algebras).

First we define tM and pM for any right $\mathcal{A}$-module M by using flat modules.

Lemma 2.11. *Let M be a right $\mathcal{A}$-module and F_1 and F_2 be two flat right $\mathcal{A}$-modules. If $f_1 : M \to F_1$ and $f_2 : M \to F_2$ are $\mathcal{A}$-surjections, then there exists a flat right $\mathcal{A}$-module F_3 and $\mathcal{A}$-surjections $f_{13} : F_3 \to F_1$, $f_{23} : F_3 \to F_2$ and $f_3 : M \to F_3$ such that $f_1 = f_{13} \circ f_3$ and $f_2 = f_{23} \circ f_3$. In particular the set $\mathcal{F}(M)$ of all pairs (F, f) of a flat right $\mathcal{A}$-module F and an $\mathcal{A}$-surjection $f : M \to F$ is a directed set.*

Proof. We consider $f_1 \oplus f_2 : M \ni m \mapsto f_1(m) \oplus f_2(m) \in F_1 \oplus F_2$, $p_1 : F_1 \oplus F_2 \ni x_1 \oplus x_2 \mapsto x_1 \in F_1$ and $p_2 : F_1 \oplus F_2 \ni x_1 \oplus x_2 \mapsto x_2 \in F_2$. By restricting the range of $f_1 \oplus f_2$ and the domains of p_1 and p_2 on $F_3 := \mathrm{Im}(f_1 \oplus f_2)$, we have $\mathcal{A}$-surjections $f_3 := f_1 \oplus f_2|^{F_3} : M \ni m \mapsto f_1(m) \oplus f_2(m) \in F_3$, $f_{13} := p_1|_{F_3} : F_3 \ni x_1 \oplus x_2 \mapsto x_1 \in F_1$ and $f_{23} := p_2|_{F_3} : F_3 \ni x_1 \oplus x_2 \mapsto x_2 \in F_2$. Then we have $f_1 = f_{13} \circ f_3$ and $f_2 = f_{23} \circ f_3$. Also F_3 is flat by Remark 2.10. □

Definition 2.12. Let M be a right $\mathcal{A}$-module. Then we define a submodule $tM := \bigcap_{(F,f)\in\mathcal{F}(M)} \mathrm{Ker} f$ of M. Also pM and $\pi_p : M \to pM$ are defined by the quotient of M by tM and its quotient map.

Proposition 2.13. *Let M be a right $\mathcal{A}$-module. Then pM is the universal flat quotient of M, that is, it is flat and for any $(F, f) \in \mathcal{F}(M)$, there exists a unique $\mathcal{A}$-surjection $f_p : pM \to F$ such that $f = f_p \circ \pi_p$.*

Proof. The latter is trivial. Now we give a proof for the former. pM is the image of $\prod_{(F,f)\in\mathcal{F}(M)} f : M \to \prod_{(F,f)\in\mathcal{F}(M)} F$. Since $\prod_{(F,f)\in\mathcal{F}(M)} F$ is flat by Remark 2.10, then $\mathrm{Im}(\prod_{(F,f)\in\mathcal{F}(M)} f)$ is flat by Remark 2.10. □

Second we define TM, PM and qM for any right $\mathcal{A}$-module M by using locally projective modules.

Lemma 2.14. *Let M be a right $\mathcal{A}$-module and P_1 and P_2 be two locally projective right $\mathcal{A}$-modules. If $f_1 : M \to P_1$ and $f_2 : M \to P_2$ are $\mathcal{A}$-surjections, then there exists a locally projective right $\mathcal{A}$-module P_3 and $\mathcal{A}$-surjections $f_{13} : P_3 \to P_1$, $f_{23} : P_3 \to P_2$ and $f_3 : M \to P_3$ such that $f_1 = f_{13} \circ f_3$ and $f_2 = f_{23} \circ f_3$. In particular the set $\mathcal{P}(M)$ of all pairs (P, f) of a locally projective right $\mathcal{A}$-module P and an $\mathcal{A}$-surjection $f : M \to P$ is a directed set.*

Proof. We consider $f_1 \oplus f_2 : M \ni m \mapsto f_1(m) \oplus f_2(m) \in P_1 \oplus P_2$, $p_1 : P_1 \oplus P_2 \ni x_1 \oplus x_2 \mapsto x_1 \in P_1$ and $p_2 : P_1 \oplus P_2 \ni x_1 \oplus x_2 \mapsto x_2 \in P_2$. By restricting the range of $f_1 \oplus f_2$ and the domains of p_1 and p_2 on $P_3 := \mathrm{Im}(f_1 \oplus f_2)$, we have $\mathcal{A}$-surjections $f_3 := f_1 \oplus f_2|^{P_3} : M \ni m \mapsto f_1(m) \oplus f_2(m) \in P_3$, $f_{13} := p_1|_{P_3} : P_3 \ni x_1 \oplus x_2 \mapsto x_1 \in P_1$ and $f_{23} := p_2|_{P_3} : P_3 \ni x_1 \oplus x_2 \mapsto x_2 \in P_2$. Then we have $f_1 = f_{13} \circ f_3$ and $f_2 = f_{23} \circ f_3$. Also trivially P_3 is locally projective. □

Since we know that PM is projective if M is finitely generated (Theorem 2.9), then we have the following.

Lemma 2.15. *Let M be a finitely generated module. Then we have $TM = \bigcap_{(P,f)\in\mathcal{P}(M)} \mathrm{Ker} f$. Moreover PM is the universal finitely generated projective quotient of M, that is, it is projective and for any $(P, f) \in \mathcal{P}(M)$, there exists a unique $\mathcal{A}$-surjection $f_P : PM \to P$ such that $f = f_P \circ \pi_P$.*

We extend Definition 2.8 to general right $\mathcal{A}$-modules.

Definition 2.16. Let M be a right $\mathcal{A}$-module. Then we define a submodule $TM := \bigcap_{(P,f)\in\mathcal{P}(M)} \mathrm{Ker} f$ of M. Also PM and $\pi_P : M \to PM$ are defined by the quotient of M by TM and its quotient map.

The following is trivial.

Lemma 2.17. *Let M be a module. Then for any $(P, f) \in \mathcal{P}(M)$, there exists a unique $\mathcal{A}$-surjection $f_P : PM \to P$ such that $f = f_P \circ \pi_P$.*

Remark 2.18. We do not know whether PM is always locally projective or not for a right $\mathcal{A}$-module M. If $\mathcal{A}$ is a finite von Neumann algebra, then it is correct (Theorem 2.46).

Since locally projective right $\mathcal{A}$-modules are flat right $\mathcal{A}$-modules, then we have the $\mathcal{A}$-inclusion $tM \subset TM$ and the $\mathcal{A}$-surjection $\pi_{Pp} : pM \to PM$ for any right $\mathcal{A}$-module M.

Definition 2.19. Let M be a right $\mathcal{A}$-module. Then we define $qM := TM/tM = \mathrm{Ker}\pi_{Pp}$ by the quotient of TM by tM.

The following is trivial.

Lemma 2.20. *Let M be a right $\mathcal{A}$-module. Then $qM = 0$ and $PM = 0$ if and only if $pM = 0$.*

Next we define t_uM and p_uM for a right $\mathcal{A}$-module M by using measurable modules, where we need to define them without von Neumann dimensions. In the case of finite von Neumann algebras they are defined in [17]. We remark that our t_uM (resp. p_uM) is written by tM (resp. pM) in [17], but it is no problem because we prove $t_uM = tM$ and $p_uM = pM$ in our notations (refer to [29] for the case of finite von Neumann algebras).

Definition 2.21. Let M be a right $\mathcal{A}$-module. M is measurable if there exist a finitely presented right $\mathcal{A}$-module L such that $PL = 0$ and an $\mathcal{A}$-surjection $L \to M$.

Clearly a measurable right $\mathcal{A}$-module is finitely generated.

Lemma 2.22. *Let M be a finitely generated right $\mathcal{A}$-module. M is measurable if and only if $tM = M$.*

Proof. We suppose that $tM \neq M$, that is, there exist a nontrivial flat right $\mathcal{A}$-module F and an $\mathcal{A}$-surjection $f^F : M \to F$. When we take a finitely presented right $\mathcal{A}$-module L and an $\mathcal{A}$-surjection $f_L : L \to M$, then the $\mathcal{A}$-surjection $f^F \circ f_L : L \to F$ factors through some non-trivial finitely generated free right $\mathcal{A}$-module $\mathcal{A}^n$, that is, there exist an $\mathcal{A}$-homomorphism $g_L : L \to \mathcal{A}^n$ and an $\mathcal{A}$-homomorphism $g^F : \mathcal{A}^n \to F$ such that $f^F \circ f_L = g^F \circ g_L$ ([12, (4.32)]). Now $P := g_L(L) \subset \mathcal{A}^n$ is finitely generated projective by Remark 2.10. Because of $PL \neq 0$, then M is not measurable.

We assume that M is not measurable. We take an $\mathcal{A}$-surjection $\pi : \mathcal{A}^n \to M$. Let I be a directed set of all finitely generated submodules $K_i \subset \mathrm{Ker}\pi$. Then $L_i := \mathcal{A}^n/K_i$ is a finitely presented right $\mathcal{A}$-module such that $PL_i \neq 0$ for any $i \in I$ by the assumption. Then we have a flat module $F := \varinjlim_I PL_i \neq 0$ and an $\mathcal{A}$-surjection $M \to F$. Hence we have $tM \neq M$. □

Clearly a right $\mathcal{A}$-module M_1 is measurable if there exist a measurable right $\mathcal{A}$-module M_2 and an $\mathcal{A}$-surjection $M_2 \to M_1$. Let M be a right $\mathcal{A}$-module. If submodules $M_1 \subset M$ and $M_2 \subset M$ are measurable, then $M_1 + M_2 \subset M$ is measurable because $M_1 \oplus M_2$ is measurable. Thus the set of all measurable submodules $\mathcal{M}(M)$ is a directed set. Thus we have the following definition.

Definition 2.23. Let M be a right $\mathcal{A}$-module. t_uM is a directed union of all measurable submodules of M. We define $p_uM := M/t_uM$ by the quotient of M by t_uM.

Lemma 2.24. *For any right $\mathcal{A}$-module M, we have $t_uM \subset tM \subset TM$.*

Proof. $tM \subset TM$ is already proved. If we have $(F, f) \in \mathcal{F}(M)$ and a measurable submodule t of M such that $f|_t \neq 0$, then it contradicts that t has no flat quotients because $\mathrm{Im} f$ is flat by Remark 2.10. □

Lemma 2.25. *For any finitely presented right $\mathcal{A}$-module M, we have $t_uM = tM = TM$ and $p_uM = pM = PM$.*

Proof. Generally we have $t_uM \subset tM \subset TM$. We have $P(TM) = 0$ because PM is the universal projective quotient of M by Theorem 2.9 and Lemma 2.17. Since TM is finitely presented and $P(TM) = 0$, then TM is measurable. Hence $TM \subset t_uM$. □

Lemma 2.26. *Let M be a right $\mathcal{A}$-module. If M is finitely related, namely a quotient of a free right $\mathcal{A}$-module by its finitely generated submodule, then we have $qM = 0$.*

Proof. Since M is finitely related, then we have an $\mathcal{A}$-homomorphism $f : \mathcal{A}^n \to \oplus_I \mathcal{A}$ such that $M \cong \bigoplus_I \mathcal{A}/f(\mathcal{A}^n)$, where $\bigoplus_I \mathcal{A}$ is a free right $\mathcal{A}$-module. Because we can take a finite subset $I_0 \subset I$ such that $f(\mathcal{A}^n) \subset \bigoplus_{I_0} \mathcal{A}$, then we have $M \cong \bigoplus_{I_0} \mathcal{A}/f(\mathcal{A}^n) \oplus \bigoplus_{I \setminus I_0} \mathcal{A}$. Hence we have $qM \cong q(\bigoplus_{I_0} \mathcal{A}/f(\mathcal{A}^n))$. Since $\bigoplus_{I_0} \mathcal{A}/f(\mathcal{A}^n)$ is finitely presented, then we have $qM \cong q(\bigoplus_{I_0} \mathcal{A}/f(\mathcal{A}^n)) = 0$ by the above lemma. □

Theorem 2.27. *Let M be a right $\mathcal{A}$-module. Then we have $t_u M = tM$ and $p_u M = pM$.*

Proof. We know $t_u M \subset tM$. Hence we prove $t_u M \supset tM$. We write $M = \bigcup_{i \in I} M_i$ by the directed union of all finitely generated submodules of M. Moreover for any $i \in I$, we take an $\mathcal{A}$-surjection $\pi_i : L_i \to M_i$ from a finitely presented right $\mathcal{A}$-module L_i, where L_i satisfies $PL_i = 0$ if M_i is measurable. Here we write $\mathrm{Ker}\pi_i = \bigcup_{j_i \in J_i} K_{j_i}$ by the directed union of all finitely generated submodules of $\mathrm{Ker}\pi_i$ for any $i \in I$. Also we define a finitely presented right $\mathcal{A}$-module $L_{j_i} := L_i/K_{j_i}$ for any $i \in I$, which satisfies $PL_i = 0$ if M_i is measurable. Then we have $M = \bigcup_I \varinjlim_{J_i} L_{j_i}$.

Now we prove $t_u M = \bigcup_I \varinjlim_{J_i} TL_{j_i}$. For $x \in \varinjlim_{J_i} TL_{j_i} \setminus \{0\}$, we can take j_i and $y \in TL_{j_i}$ such that $TL_{j_i} \ni y \mapsto x \in \varinjlim_{J_i} TL_{j_i}$. Hence x belongs to a measurable submodule of M because the image of TL_{j_i} is measurable. Thus we have $t_u M \supset \bigcup_I \varinjlim_{J_i} TL_{j_i}$. Since any measurable submodule $t \subset M$ is finitely generated, then we have $i \in I$ such that $M_i = t$. Hence we have $t_u M \subset \bigcup_I \varinjlim_{J_i} TL_{j_i}$ by $t = M_i = \varinjlim_{J_i} L_{j_i} = \varinjlim_{J_i} TL_{j_i}$. Thus $p_u M$ is flat because we have $p_u M = \varinjlim_I \varinjlim_{J_i} PL_{j_i}$ and an inductive limit of projective right $\mathcal{A}$-modules is flat. Since we have an $\mathcal{A}$-surjection $p_u M \to pM$ by $t_u M \subset tM$ and pM is the universal flat quotient module of M, then we have $pM = p_u M$. □

Finally we define $T_u M$ and $P_u M$ for any right $\mathcal{A}$-module M by using finitely generated right $\mathcal{A}$-modules which have no non-trivial finitely generated projective quotients and also compare them with TM and PM.

Definition 2.28. Let M be a right $\mathcal{A}$-module. Then we define $T_u M := \bigcup_{i \in I} M_i$, where $\{M_i\}_{i \in I}$ is the directed set of all finitely generated submodules which are $TM_i = M_i$ of M, namely, which have no non-trivial finitely generated projective quotients. Also we define $P_u M$ by the quotient of M by $T_u M$.

If M is a finitely generated right $\mathcal{A}$-module, then $TM = T_u M$ and $PM = P_u M$.

Lemma 2.29. *Let M be a right $\mathcal{A}$-module. Then we have $t_u M = tM \subset T_u M \subset TM$.*

Proof. $t_u M = tM$ is already proved and $t_u M \subset T_u M$ is trivial. If $T_u M \subset TM$ is not correct, then we have $(P, f) \in \mathcal{P}(M)$ and a finitely generated submodule M' of M which has no non-trivial finitely generated projective quotients and $f|_{M'} \neq 0$. However $\mathrm{Im} f$ is finitely generated projective. Hence it contradicts that M' has no non-trivial finitely generated projective quotients. □

Definition 2.30. Let M be a right $\mathcal{A}$-module. Then we define $q_uM := T_uM/t_uM$ by the quotient of T_uM by t_uM.

Remark 2.31. We do not know whether PM is always equal to P_uM or not for a right $\mathcal{A}$-module M. If $\mathcal{A}$ is a finite von Neumann algebra, then it is correct (Theorem 2.46).

Before we consider behaviors of tM, pM and so on defined for a right $\mathcal{A}$-module M under Morita equivalence, we recall general properties of Morita equivalence. Let R be a unital ring and $\chi \in R$ be an idempotent, that is, satisfy $\chi^2 = \chi$. When we define $\chi R\chi := \{r \in R \mid r = \chi r\chi\}$, $R\chi R := \{r_1\chi r_2 \in R \mid r_1, r_2 \in R\}$ and $\chi R := \{r \in R \mid r = \chi r\}$ (resp. $R\chi := \{r \in R \mid r = r\chi\}$), then $\chi R\chi$ is a subring in R with a unit χ, $R\chi R$ is a left and right R-module, and also χR (resp. $R\chi$) is a left (resp. right) $\chi R\chi$-module and a right (resp. left) R-module. We write a category of right R-modules (resp. a category of left R-modules) $\mathcal{M}_R$ (resp. ${}_R\mathcal{M}$). The following is well-known (refer to [12, Section 18]).

Theorem 2.32. *Let R be a unital ring and $\chi \in R$ be a full idempotent, that is, satisfy $\chi^2 = \chi$ and $R = R\chi R$. Then R and $\chi R\chi$ are Morita equivalent. Specifically we have mutually inverse category equivalences:*

$$\otimes_R R\chi : \mathcal{M}_R \ni M \mapsto M \otimes_R R\chi \in \mathcal{M}_{\chi R\chi},\ \otimes_{\chi R\chi}\chi R : \mathcal{M}_{\chi R\chi} \ni M \mapsto M \otimes_{\chi R\chi} \chi R \in \mathcal{M}_R$$

and

$$\chi R \otimes_R : {}_R\mathcal{M} \ni M \mapsto \chi R \otimes_R M \in {}_{\chi R\chi}\mathcal{M},\ R\chi \otimes_{\chi R\chi} : {}_{\chi R\chi}\mathcal{M} \ni M \mapsto R\chi \otimes_{\chi R\chi} M \in {}_R\mathcal{M}.$$

Moreover they are faithfully flat functors. Also finitely generated, finitely presented, projective, locally projective and flat modules are preserved by them, respectively.

We remark that for a von Neumann algebra with a faithful normal state $(\mathcal{A}, \psi)$ and a projection $\chi \in \mathcal{A}$, $(\chi\mathcal{A}\chi, \frac{1}{\psi(\chi)}\psi)$ is also a von Neumann algebra with a faithful normal state. We call $\chi \in \mathcal{A}$ a full projection if χ is a full idempotent such that $\chi^* = \chi$.

Proposition 2.33. *Let $\chi \in \mathcal{A}$ be a full projection. Then two short exact sequences:*

$$0 \to t(M \otimes_{\mathcal{A}} \mathcal{A}\chi) \to M \otimes_{\mathcal{A}} \mathcal{A}\chi \to p(M \otimes_{\mathcal{A}} \mathcal{A}\chi) \to 0$$

and

$$0 \to (tM) \otimes_{\mathcal{A}} \mathcal{A}\chi \to M \otimes_{\mathcal{A}} \mathcal{A}\chi \to (pM) \otimes_{\mathcal{A}} \mathcal{A}\chi \to 0$$

are $\chi\mathcal{A}\chi$-isomorphic for any right $\mathcal{A}$-module M. Also two short exact sequences:

$$0 \to t(M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A}) \to M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \to p(M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A}) \to 0$$

and

$$0 \to (tM) \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \to M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \to (pM) \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \to 0$$

are $\mathcal{A}$-isomorphic for any right $\chi\mathcal{A}\chi$-module M.

Proof. We freely use Theorem 2.32 everywhere.

For any right $\mathcal{A}$-module M, $(pM) \otimes_{\mathcal{A}} \mathcal{A}\chi$ is flat since pM is flat. Hence we have $(tM) \otimes_{\mathcal{A}} \mathcal{A}\chi \supset t(M \otimes_{\mathcal{A}} \mathcal{A}\chi)$ since $p(M \otimes_{\mathcal{A}} \mathcal{A}\chi)$ is the universal flat quotient of $M \otimes_{\mathcal{A}} \mathcal{A}\chi$. We prove $(tM) \otimes_{\mathcal{A}} \mathcal{A}\chi \subset t(M \otimes_{\mathcal{A}} \mathcal{A}\chi)$. We assume $(tM) \otimes_{\mathcal{A}} \mathcal{A}\chi \not\subset t(M \otimes_{\mathcal{A}} \mathcal{A}\chi)$. Then we have an $m \in tM$ and a $\chi\mathcal{A}\chi$-surjection $f : M \otimes_{\mathcal{A}} \mathcal{A}\chi \to F$ onto a non-trivial flat right $\chi\mathcal{A}\chi$-module such that $f(m \otimes \chi) \neq 0$ and $m\chi = m$. Then $f \otimes 1 : M \otimes_{\mathcal{A}} \mathcal{A}\chi \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \to F \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A}$ is a non-trivial $\chi\mathcal{A}\chi$-surjection onto a non-trivial flat right $\mathcal{A}$-module such that $(f \otimes 1)(m \otimes \chi \otimes \chi) \neq 0$. Hence $f' : tM \hookrightarrow M \cong M \otimes_{\mathcal{A}} \mathcal{A}\chi \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \to F \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A}$ is not trivial because $f'(m) = f'(m\chi) = (f \otimes 1)(m \otimes \chi \otimes \chi) \neq 0$. It contradicts that tM has no non-trivial flat quotients.

For any right $\chi\mathcal{A}\chi$-module M, $(pM) \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A}$ is flat since pM is flat. Hence we have $(tM) \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \supset t(M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A})$ since $p(M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A})$ is the universal flat quotient of $M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A}$. We prove $(tM) \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \subset t(M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A})$. We assume $(tM) \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \not\subset t(M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A})$. Then we have an $m \in tM$ and an $\mathcal{A}$-surjection $f : M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \to F$ onto a non-trivial flat right $\mathcal{A}$-module such that $f(m \otimes \chi) \neq 0$. Then $f \otimes 1 : M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A}\chi \to F \otimes_{\mathcal{A}} \mathcal{A}\chi$ is a non-trivial $\chi\mathcal{A}\chi$-surjection onto a non-trivial flat right $\chi\mathcal{A}\chi$-module such that $(f \otimes 1)(m \otimes \chi \otimes \chi) \neq 0$. Hence $f' : tM \hookrightarrow M \cong M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A}\chi \to F \otimes_{\mathcal{A}} \mathcal{A}\chi$ is not trivial because $f'(m) = (f \otimes 1)(m \otimes \chi \otimes \chi) \neq 0$. It contradicts that tM has no non-trivial flat quotients. □

Proposition 2.34. *Let $\chi \in \mathcal{A}$ be a full projection. Then two short exact sequences:*

$$0 \to T(M \otimes_{\mathcal{A}} \mathcal{A}\chi) \to M \otimes_{\mathcal{A}} \mathcal{A}\chi \to P(M \otimes_{\mathcal{A}} \mathcal{A}\chi) \to 0$$

and

$$0 \to (TM) \otimes_{\mathcal{A}} \mathcal{A}\chi \to M \otimes_{\mathcal{A}} \mathcal{A}\chi \to (PM) \otimes_{\mathcal{A}} \mathcal{A}\chi \to 0$$

are $\chi\mathcal{A}\chi$-isomorphic for any right $\mathcal{A}$-module M. Also two short exact sequences:

$$0 \to T(M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A}) \to M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \to P(M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A}) \to 0$$

and

$$0 \to (TM) \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \to M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \to (PM) \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \to 0$$

are $\mathcal{A}$-isomorphic for any right $\chi\mathcal{A}\chi$-module M.

Proof. We freely use Theorem 2.32 everywhere.

For any right $\mathcal{A}$-module M, we have $(TM) \otimes_{\mathcal{A}} \mathcal{A}\chi \supset T(M \otimes_{\mathcal{A}} \mathcal{A}\chi)$ by Lemma 2.17. We prove $(TM) \otimes_{\mathcal{A}} \mathcal{A}\chi \subset T(M \otimes_{\mathcal{A}} \mathcal{A}\chi)$. We assume $(TM) \otimes_{\mathcal{A}} \mathcal{A}\chi \not\subset T(M \otimes_{\mathcal{A}} \mathcal{A}\chi)$. Then we have an $m \in TM$ and a $\chi\mathcal{A}\chi$-surjection $f : M \otimes_{\mathcal{A}} \mathcal{A}\chi \to P$ onto a non-trivial locally projective right $\chi\mathcal{A}\chi$-module such that $f(m \otimes \chi) \neq 0$ and $m\chi = m$. Then $f \otimes 1 : M \otimes_{\mathcal{A}} \mathcal{A}\chi \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \to P \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A}$ is a non-trivial $\chi\mathcal{A}\chi$-surjection onto a non-trivial locally projective right $\mathcal{A}$-module such that $(f \otimes 1)(m \otimes \chi \otimes \chi) \neq 0$. Hence $f' : TM \hookrightarrow M \cong M \otimes_{\mathcal{A}} \mathcal{A}\chi \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \to P \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A}$ is not trivial because $f'(m) = f'(m\chi) = (f \otimes 1)(m \otimes \chi \otimes \chi) \neq 0$. It contradicts that TM has no non-trivial locally projective quotients.

For any right $\chi\mathcal{A}\chi$-module M, we have $TM \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \supset T(M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A})$ by Lemma 2.17. We prove $TM \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \subset T(M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A})$. We assume $TM \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \not\subset$

$T(M\otimes_{\chi\mathcal{A}\chi}\chi\mathcal{A})$. Then we have an $m \in TM$ and an $\mathcal{A}$-surjection $f : M\otimes_{\chi\mathcal{A}\chi}\chi\mathcal{A} \to P$ onto a non-trivial locally projective right $\mathcal{A}$-module such that $f(m \otimes \chi) \neq 0$. Then $f \otimes 1 : M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A}\chi \to P \otimes_{\mathcal{A}} \mathcal{A}\chi$ is a non-trivial $\chi\mathcal{A}\chi$-surjection onto a non-trivial flat right $\chi\mathcal{A}\chi$-module such that $f(m \otimes \chi \otimes \chi) \neq 0$. Hence $f' : TM \hookrightarrow M \cong M\otimes_{\chi\mathcal{A}\chi}\chi\mathcal{A}\otimes_{\mathcal{A}}\mathcal{A}\chi \to P\otimes_{\mathcal{A}}\mathcal{A}\chi$ is not trivial because $f'(m) = (f\otimes 1)(m\otimes\chi\otimes\chi) \neq 0$. It contradicts that TM has no non-trivial locally projective quotients. □

By using Proposition 2.33 and Proposition 2.34, we have the following.

Corollary 2.35. *Let $\chi \in \mathcal{A}$ be a full projection. Then then we have $q(M \otimes_{\mathcal{A}} \mathcal{A}\chi) \cong (qM) \otimes_{\mathcal{A}} \mathcal{A}\chi$ for any right $\mathcal{A}$-module M and also $q(M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A}) \cong (qM) \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A}$ for any right $\chi\mathcal{A}\chi$-module M.*

We prove Proposition 2.36 and Corollary 2.37, but they are not used in proofs of main theorems. Hence readers can skip them.

Proposition 2.36. *Let $\chi \in \mathcal{A}$ be a full projection. Then two short exact sequences:*

$$0 \to T_u(M \otimes_{\mathcal{A}} \mathcal{A}\chi) \to M \otimes_{\mathcal{A}} \mathcal{A}\chi \to P_u(M \otimes_{\mathcal{A}} \mathcal{A}\chi) \to 0$$

and

$$0 \to (T_uM) \otimes_{\mathcal{A}} \mathcal{A}\chi \to M \otimes_{\mathcal{A}} \mathcal{A}\chi \to (P_uM) \otimes_{\mathcal{A}} \mathcal{A}\chi \to 0$$

are $\chi\mathcal{A}\chi$-isomorphic for any right $\mathcal{A}$-module M. Also two short exact sequences:

$$0 \to T_u(M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A}) \to M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \to P_u(M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A}) \to 0$$

and

$$0 \to (T_uM) \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \to M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \to (P_uM) \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \to 0$$

are $\mathcal{A}$-isomorphic for any right $\chi\mathcal{A}\chi$-module M.

Proof. We freely use Theorem 2.32 everywhere.

For any right $\mathcal{A}$-module M, we have $(T_uM)\otimes_{\mathcal{A}}\mathcal{A}\chi \subset T_u(M\otimes_{\mathcal{A}}\mathcal{A}\chi)$ by definition. Also for any right $\chi\mathcal{A}\chi$-module M, we have $(T_uM) \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \subset T_u(M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A})$ by definition.

For any right $\mathcal{A}$-module M, we take a finitely generated submodule of $K \subset M \otimes_{\mathcal{A}} \mathcal{A}\chi$ which has no projective quotients. Because of $K \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \subset T_u(M\otimes_{\mathcal{A}}\mathcal{A}\chi)\otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \subset T_u(M\otimes_{\mathcal{A}}\mathcal{A}\chi\otimes_{\chi\mathcal{A}\chi}\chi\mathcal{A}) \cong T_uM$, $K\otimes_{\chi\mathcal{A}\chi}\chi\mathcal{A}$ is a finitely generated submodules which has no projective quotients. Hence by $K \cong K\otimes_{\chi\mathcal{A}\chi}\chi\mathcal{A}\otimes_{\mathcal{A}}\mathcal{A}\chi \subset (T_uM)\otimes_{\mathcal{A}}\mathcal{A}\chi$, we have $T_u(M \otimes_{\mathcal{A}} \mathcal{A}\chi) \subset (T_uM) \otimes_{\mathcal{A}} \mathcal{A}\chi$.

For any right $\chi\mathcal{A}\chi$-module M, we take a finitely generated submodule $K \subset M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A}$ which has no projective quotients. Because of $K \otimes_{\mathcal{A}} \mathcal{A}\chi \subset T_u(M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{A}\chi \subset T_u(M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A}\chi) \cong T_uM$, $K \otimes_{\mathcal{A}} \mathcal{A}\chi$ is a finitely generated submodules which has no projective quotients. Hence by $K \cong K \otimes_{\mathcal{A}} \mathcal{A}\chi \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A} \subset (T_uM) \otimes_{\mathcal{A}} \mathcal{A}\chi$, we have $T_u(M \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A}) \subset (T_uM) \otimes_{\chi\mathcal{A}\chi} \chi\mathcal{A}$. □

By using Theorem 2.27, Proposition 2.33 and Proposition 2.36, we have the following.

Corollary 2.37. *Let $\chi \in \mathcal{A}$ be a full projection. Then then we have $q_u(M\otimes_{\mathcal{A}}\mathcal{A}\chi) \cong (q_uM)\otimes_{\mathcal{A}}\mathcal{A}\chi$ for any right $\mathcal{A}$-module M and also $q_u(M\otimes_{\chi\mathcal{A}\chi}\chi\mathcal{A}) \cong (q_uM)\otimes_{\chi\mathcal{A}\chi}\chi\mathcal{A}$ for any right $\chi\mathcal{A}\chi$-module M.*

2.3. **Extensions of von Neumann algebras.** Let $(\mathcal{A}, \psi_{\mathcal{A}})$ and $(\mathcal{B}, \psi_{\mathcal{B}})$ be von Neumann algebras with faithful normal states, respectively and $\iota : \mathcal{A} \hookrightarrow \mathcal{B}$ be an injective $*$-homomorphism preserving states through this subsection.

The following is proved by the same argument in his proof of [16, Theorem 6.29 (1)] if we pay attention to that we do not use finite traces.

Theorem 2.38. *$\mathcal{B}$ is a faithfully flat right $\mathcal{A}$-module by ι.*

Proof. We freely use Subsection 2.1 everywhere.

We prove that for any module M,
(1) $\mathrm{Tor}_n^{\mathcal{A}}(M, \mathcal{B}) = 0$ for any $n \leq 1$,
(2) If we have $M \otimes_{\mathcal{A}} \mathcal{B} = 0$, then $M = 0$.
We consider finitely generated projective right $\mathcal{A}$-modules, finitely presented right $\mathcal{A}$-modules, finitely generated right $\mathcal{A}$-modules and general right $\mathcal{A}$-modules in order.

Let M be a finitely generated projective right $\mathcal{A}$-module. Then we have a projective resolution $0 \to M \to M \to 0$ of M. Hence (1) is trivial. Since there exists a projection $p \in M_{n,n}(\mathcal{A})$ such that $M \cong p\mathcal{A}^n$, we have $M \otimes_{\mathcal{A}} \mathcal{B} \cong p\mathcal{A}^n \otimes_{\mathcal{A}} \mathcal{B} \cong p\mathcal{B}^n$ by $\mathcal{A}^n \otimes_{\mathcal{A}} \mathcal{B} \cong \mathcal{B}^n$. Since $p\mathcal{B}^n = 0$ if and only if $p = 0$ if and only if $p\mathcal{A}^n = 0$, (2) is clear.

Let M be a finitely presented right $\mathcal{A}$-module. When we take a finite presentation $\mathcal{A}^m \xrightarrow{f} \mathcal{A}^n \xrightarrow{f_0} M \to 0$, we have a finite presentation $\mathcal{B}^m \xrightarrow{f\otimes 1} \mathcal{B}^n \xrightarrow{f_0 \otimes 1} M \otimes_{\mathcal{A}} \mathcal{B} \to 0$ since $\otimes_{\mathcal{A}}\mathcal{B}$ is a right exact functor. First we prove (2). If $M \otimes_{\mathcal{A}} \mathcal{B} = 0$, then $\mathcal{B}^m \to \mathcal{B}^n$ is surjective. Since $\nu(f \otimes 1) : \nu(\mathcal{B}^m, \mu_{st}) \to \nu(\mathcal{B}^n, \mu_{st})$ is also surjective, $\nu(f \otimes 1)\nu(f \otimes 1)^*$ is invertible. Hence $f \circ f^* \otimes 1$ is also invertible. Because $\iota_n : M_n(\mathcal{A}) \hookrightarrow M_n(\mathcal{B})$ induced by $\iota : \mathcal{A} \hookrightarrow \mathcal{B}$ preserves invertibility, then $f \circ f^*$ is also invertible. Hence f is surjective. Thus we have $M = 0$. Next we prove (1). We have a short exact sequence $0 \to P \xrightarrow{i} \mathcal{A}^n \xrightarrow{f_0} M \to 0$, where $P := \mathrm{Im} f = \mathrm{Ker} f_0$ which is finitely generated projective by Theorem 2.9. We prove that $P \otimes_{\mathcal{A}} \mathcal{B} \xrightarrow{i \otimes 1} \mathcal{B}^n$ is injective. Now we consider a weakly short exact sequence of finitely generated projective right $\mathcal{A}$-modules $0 \to P \xrightarrow{i} \mathcal{A}^n \xrightarrow{q} Q \to 0$, where $Q := \mathcal{A}^n/\overline{\mathrm{Im} i}$ is finitely generated projective by Theorem 2.9. Hence when we take inner products μ_P and μ_Q, we have a weakly short exact sequence $0 \to \nu(P, \mu_P) \xrightarrow{\nu(i)} \nu(\mathcal{A}^n, \mu_{st}) \xrightarrow{\nu(q)} \nu(Q, \mu_Q) \to 0$. Since $\nu(i)$ is injective, then $\nu(i \otimes 1)$ is also injective because the spectral decomposition of $|\nu(i)|$ induces the spectral decomposition of $|\nu(i \otimes 1)|$. Hence $i \otimes 1$ is injective.

Let M be a finitely generated right $\mathcal{A}$-module. When we take a finite generation $f : \mathcal{A}^n \to M \to 0$, then we take the directed set J of all finitely generated submodules of $K := \mathrm{Ker} f$. We have $K = \bigcup_{j \in J} K_j$ and $\varinjlim_{j \in J} M_j$, where $M_j := M/K_j$ for any $j \in J$. More precisely M is considered as an inductive limit of $\mathcal{A}$-surjections $f_{j_0}^{j_1} : M_{j_0} \to M_{j_1}$ for any $j_0, j_1 \in J$ such that $K_{j_0} \subset K_{j_1}$. Then we have $M \otimes_{\mathcal{A}} \mathcal{B} = \varinjlim_{j \in J}(M_j \otimes_{\mathcal{A}} \mathcal{B})$. We remark that $M \otimes_{\mathcal{A}} \mathcal{B} = 0$ if and only if for any j_0 and any $x \in M_{j_0} \otimes_{\mathcal{A}} \mathcal{B}$, there exists $j_1 \in J$ such that $K_{j_0} \subset K_{j_1}$ and $(f_{j_0}^{j_1} \otimes 1)(x) = 0$. Since M is finitely generated, if we take a finitely generating set $\{m_1, \ldots, m_n\}$, then $\{m_1 \otimes 1, \ldots, m_n \otimes 1\}$ is a finitely generating set of $M \otimes_{\mathcal{A}} \mathcal{B}$. Then we have that $M \otimes_{\mathcal{A}} \mathcal{B} = 0$ if and only if $m_1 \otimes 1 = 0$ for any $i = 1, \ldots, n$ if and only if there exists

$j \in J$ such that $M_j \otimes_{\mathcal{A}} \mathcal{B} = 0$ because n is finite. Since M_j is finitely presented, then $M_j = 0$. Hence we have $M = 0$. Now we proved (2). Since M_j for any $j \in J$ is finitely presented, $\mathrm{Tor}_n^{\mathcal{A}}(M, \mathcal{B}) = \mathrm{Tor}_n^{\mathcal{A}}(\varinjlim_{j\in J} M_j, \mathcal{B}) = \varinjlim_{j\in J} \mathrm{Tor}_n^{\mathcal{A}}(M_j, \mathcal{B}) = 0$ for any $n \leq 1$.

Now we remark that for a short exact sequence of right $\mathcal{A}$-modules $0 \to M_1 \to M_2 \to M_3 \to 0$, if M_3 is a finitely generated modules, then $0 \to M_1 \otimes_{\mathcal{A}} \mathcal{B} \to M_2 \otimes_{\mathcal{A}} \mathcal{B} \to M_3 \otimes_{\mathcal{A}} \mathcal{B} \to 0$ is exact. Actually $0 \to M_1 \to M_2 \to M_3 \to 0$ induce a long exact sequence: $\cdots \to \mathrm{Tor}_1^{\mathcal{A}}(M_3, \mathcal{B}) \to M_1 \otimes_{\mathcal{A}} \mathcal{B} \to M_2 \otimes_{\mathcal{A}} \mathcal{B} \to M_3 \otimes_{\mathcal{A}} \mathcal{B} \to 0$ and we have $\mathrm{Tor}_1^{\mathcal{A}}(M_3, \mathcal{B}) = 0$ because M_3 is finitely generated.

Let M be a right $\mathcal{A}$-module. We take a directed set I of all finitely generated submodules M_i of M. Since $M = \bigcup_{i\in I} M_i$, we have $M = \bigcup_{i\in I}(M_i \otimes_{\mathcal{A}} \mathcal{B})$. Actually because for any $\phi_i^j : M_i \hookrightarrow M_j$ the cokernel is finitely generated, we have the $\mathcal{B}$-injection $\phi_i^j \otimes 1 : M_i \otimes_{\mathcal{A}} \mathcal{B} \hookrightarrow M_j \otimes_{\mathcal{A}} \mathcal{B}$ by the above remark. Hence for any $i \in I$, we have the $\mathcal{B}$-injection $M_i \otimes_{\mathcal{A}} \mathcal{B} \hookrightarrow M \otimes_{\mathcal{A}} \mathcal{B}$. Because of $M = \bigcup_{i\in I}(M_i \otimes_{\mathcal{A}} \mathcal{B})$, $M \otimes_{\mathcal{A}} \mathcal{B} = 0$ if and only if $M_i \otimes_{\mathcal{A}} \mathcal{B} = 0$ for any $i \in I$. For any $i \in I$, $M_i \otimes_{\mathcal{A}} \mathcal{B} = 0$ if and only if $M_i = 0$ because M_i is finitely generated. Since $M_i = 0$ for any $i \in I$ if and only if $M = 0$, we have that $M \otimes_{\mathcal{A}} \mathcal{B} = 0$ if and only if $M = 0$. Since M_i for any $i \in I$ is finitely generated, $\mathrm{Tor}_n^{\mathcal{A}}(M, \mathcal{B}) = \mathrm{Tor}_n^{\mathcal{A}}(\varinjlim_{i\in I} M_i, \mathcal{B}) = \varinjlim_{i\in I} \mathrm{Tor}_n^{\mathcal{A}}(M_i, \mathcal{B}) = 0$ for any $n \leq 1$. □

Lemma 2.39. *Let M be a flat right $\mathcal{B}$-module. When M is regarded as a right $\mathcal{A}$-module by using ι, then M is a flat right $\mathcal{A}$-module.*

Proof. $\mathcal{B}^n$ is flat as a right $\mathcal{A}$-module by Theorem 2.38. Hence any finitely generated right $\mathcal{B}$-module P is flat as a right $\mathcal{A}$-module by Remark 2.10. For any flat right $\mathcal{B}$-module M, we have a directed family of finitely generated projective right $\mathcal{B}$-modules $\{M_i\}_{i\in I}$ such that $M = \varinjlim_{i\in I} M_i$. Hence M is regarded as an inductive limit of flat right $\mathcal{A}$-modules, since M_i is flat as a right $\mathcal{A}$-module for any $i \in I$. Hence M is flat as a right $\mathcal{A}$-module. □

Proposition 2.40. *Let M be a right $\mathcal{A}$-module. Then we have the following:*
(1) *M is finitely generated if and only if $M \otimes_{\mathcal{A}} \mathcal{B}$ is finitely generated;*
(2) *M is finitely presented if and only if $M \otimes_{\mathcal{A}} \mathcal{B}$ is finitely presented;*
(3) *M is flat if and only if $M \otimes_{\mathcal{A}} \mathcal{B}$ is flat;*
(4) *M is finitely generated projective if and only if $M \otimes_{\mathcal{A}} \mathcal{B}$ is finitely generated projective;*
(5) *M is locally projective if and only if $M \otimes_{\mathcal{A}} \mathcal{B}$ is locally projective;*
(6) *M is measurable if and only if $M \otimes_{\mathcal{A}} \mathcal{B}$ is measurable;*
(7) *M has no flat quotient if and only if $M \otimes_{\mathcal{A}} \mathcal{B}$ no flat quotient.*

Proof. See [12, (4.79)] about proofs of (1) and (2).

If M is finitely generated projective, then we have $\mathcal{A}^n \cong M \oplus Q$. Hence $M \otimes_{\mathcal{A}} \mathcal{B}$ is finitely generated projective since we have $\mathcal{B}^n \cong M \otimes_{\mathcal{A}} \mathcal{B} \oplus Q \otimes_{\mathcal{A}} \mathcal{B}$. Thus if M is flat, then $M \otimes_{\mathcal{A}} \mathcal{B}$ is flat. Also if M is locally projective, then $M \otimes_{\mathcal{A}} \mathcal{B}$ is locally projective.

If $M \otimes_{\mathcal{A}} \mathcal{B}$ is flat, then it is flat as a right $\mathcal{A}$-module by Lemma 2.39. Since we can regard M as a $\mathcal{A}$-submodule of $M \otimes_{\mathcal{A}} \mathcal{B}$, then M is also flat by Remark 2.10.

Let $M \otimes_{\mathcal{A}} \mathcal{B}$ be finitely generated projective. Then it is equivalent to being finitely presented and flat. Hence M is finitely presented and flat, that is, finitely projective.

Let $M \otimes_{\mathcal{A}} \mathcal{B}$ be locally projective. We take a finitely generated submodule $K \subset M$. Then since $K \otimes_{\mathcal{A}} \mathcal{B} \subset M \otimes_{\mathcal{A}} \mathcal{B}$ is finitely generated, $K \otimes_{\mathcal{A}} \mathcal{B}$ is finitely generated projective. Hence $K \otimes_{\mathcal{A}} \mathcal{B}$ is finitely presented and flat. Hence K is finitely presented and flat, that is, finitely generated projective.

If M has a flat quotient, then $M \otimes_{\mathcal{A}} \mathcal{B}$ clearly has a flat quotient. If $M \otimes_{\mathcal{A}} \mathcal{B}$ has a flat quotient F, then M has a flat quotient since the restricted map on M to F is non-trivial and F can be regarded as a flat right $\mathcal{A}$-module by Lemma 2.39. □

Theorem 2.41. *Two short exact sequences:*

$$0 \to t(M \otimes_{\mathcal{A}} \mathcal{B}) \to M \otimes_{\mathcal{A}} \mathcal{B} \to p(M \otimes_{\mathcal{A}} \mathcal{B}) \to 0$$

and

$$0 \to (tM) \otimes_{\mathcal{A}} \mathcal{B} \to M \otimes_{\mathcal{A}} \mathcal{B} \to (pM) \otimes_{\mathcal{A}} \mathcal{B} \to 0$$

are $\mathcal{B}$-isomorphic for any right $\mathcal{A}$-module M.

Proof. We have $tM \otimes_{\mathcal{A}} \mathcal{B} \supset t(M \otimes_{\mathcal{A}} \mathcal{B})$. Indeed since pM is a flat right $\mathcal{A}$-module, $pM \otimes_{\mathcal{A}} \mathcal{B}$ is a flat right $\mathcal{B}$-module by Proposition 2.40. Also we have $tM \otimes_{\mathcal{A}} \mathcal{B} \subset t(M \otimes_{\mathcal{A}} \mathcal{B})$. Actually when we assume $\emptyset \neq tM \otimes_{\mathcal{A}} \mathcal{B} \setminus t(M \otimes_{\mathcal{A}} \mathcal{B})$, then there exist a flat right $\mathcal{B}$-module F, a $\mathcal{B}$-surjection $f : M \otimes_{\mathcal{A}} \mathcal{B} \to F$ and an $m_0 \in tM$ such that $f(m_0 \otimes 1) \neq 0$. $f' : M \ni m \mapsto f(m \otimes 1) \in F$ is regarded as a non-trivial $\mathcal{A}$-homomorphism such that $f'(m_0) \neq 0$ and F is regarded as a flat right $\mathcal{A}$-module by Lemma 2.39. It contradicts that tM has no non-trivial flat quotients. □

2.4. **Modules over finite von Neumann algebras.** Let $\mathcal{A}$ be a von Neumann algebras with a finite trace $\mathrm{tr}_{\mathcal{A}}$ through this subsection.

Lemma 2.42. *For any finitely generated right $\mathcal{A}$-module M, there exists a non-trivial finitely generated projective submodule of M if and only if there exists a non-trivial finitely generated projective quotient of M.*

Proof. We remark that for any finitely generated right $\mathcal{A}$-module M, M has a non-trivial finitely generated projective quotient if and only if PM is not trivial.

We use some properties of $\dim_{\mathcal{A}}$ in [16, Chapter 6]. The part of 'if' is trivial. Hence we prove the part of 'only if'. Let $Q \subset M$ be a non-trivial finitely generated projective submodule. Then we have $\dim_{\mathcal{A}} Q > 0$ by faithfulness of $\dim_{\mathcal{A}}$. When we take a positive integer m and an $\mathcal{A}$-surjection $\pi : \mathcal{A}^m \to M$, we have $\mathcal{A}^m \supset \pi^{-1}(Q) \cong \pi^{-1}(0) \oplus Q$. Thus we have $m = \dim_{\mathcal{A}} \mathcal{A}^m \geq \dim_{\mathcal{A}} \pi^{-1}(Q) = \dim_{\mathcal{A}} \pi^{-1}(0) + \dim_{\mathcal{A}} Q$ by additivity of $\dim_{\mathcal{A}}$. We assume $PM = 0$, namely $TM = M$. Since we have $\overline{\pi^{-1}(0)} = \pi^{-1}(\overline{0}) = \pi^{-1}(TM) = \pi^{-1}(M) = \mathcal{A}^m$, we have $\dim_{\mathcal{A}} \overline{\pi^{-1}(0)} = m$. Since we have $\dim_{\mathcal{A}} \pi^{-1}(0) = \dim_{\mathcal{A}} \overline{\pi^{-1}(0)} = m$ by normality of $\dim_{\mathcal{A}}$, we have $\dim_{\mathcal{A}} Q \leq 0$ by $m \geq \dim_{\mathcal{A}} \pi^{-1}(0) + \dim_{\mathcal{A}} Q$. It contradicts $\dim_{\mathcal{A}} Q > 0$. □

By the above, for any finitely generated right $\mathcal{A}$-module M, TM has no non-trivial projective submodules.

Lemma 2.43. *Let $\phi : M_1 \hookrightarrow M_2$ be an $\mathcal{A}$-injection between two finitely generated right $\mathcal{A}$-modules. Then its induced $\mathcal{A}$-homomorphism $\phi_P : PM_1 \to PM_2$ is also injective.*

Proof. When we take an $\mathcal{A}$-section s_1 of the quotient map $\pi_1 : M_1 \to PM_1$, then we have $\mathrm{Ker}\phi_P \cong \phi \circ s_1(\mathrm{Ker}\phi_P) \subset TM_2$. Since $\mathrm{Ker}\phi_P$ is locally projective and TM_2 has no non-trivial projective submodules, $\mathrm{Ker}\phi_P$ is trivial. □

Proposition 2.44. *For any right $\mathcal{A}$-module M, P_uM is locally projective and T_uM has no non-trivial projective submodules.*

Proof. When we write $M = \bigcup_{i\in I} M_i$ by a directed union of all finitely generated submodules, then we have $T_uM = \bigcup_{i\in I} TM_i$ and $P_uM = \varinjlim_{i\in I} PM_i$. Moreover by Lemma 2.43 we have $P_uM = \varinjlim_{i\in I} PM_i = \bigcup_{i\in I} PM_i$. Hence P_uM is locally projective. Also T_uM has no non-trivial projective submodules by Lemma 2.42. □

Corollary 2.45. *Let M be a right $\mathcal{A}$-module and Q be a finitely generated projective right $\mathcal{A}$-module. There exists an $\mathcal{A}$-injection from Q to M if and only if there exists an $\mathcal{A}$-injection from Q to P_uM.*

Theorem 2.46. *Let M be a right $\mathcal{A}$-module. Then we have $P_uM = PM$ and $T_uM = TM$. In particular PM is the universal locally projective quotient of M.*

Proof. We have $T_uM \subset TM$ by Lemma 2.29. Hence we have the $\mathcal{A}$-surjection $P_uM \to PM$. On the other hand because P_uM is locally projective, $\mathcal{A}$-surjection $M \to P_uM$ must factor through PM by the definition of PM. Hence we have $P_uM = PM$. □

We have the following by Lemma 2.20 and Theorem 2.46.

Corollary 2.47. *Let M be a right $\mathcal{A}$-module. Then we have $qM = 0$ if and only if pM is locally projective.*

The following is clear.

Corollary 2.48. *Let M be a right $\mathcal{A}$-module. Then we have $qM = q_uM$.*

Remark 2.49. Let M be a right $\mathcal{A}$-module. Its von Neumann dimension $\dim_{\mathcal{A}}(M)$ and its Novikov-Shubin invariant $\alpha_{\mathcal{A}}(M)$ are defined by using $\mathrm{tr}_{\mathcal{A}}$ in [14] and [17], respectively (refer to [16, Chapter 6]). We can confirm $\dim_{\mathcal{A}} M = \dim_{\mathcal{A}} P_uM$, that $\dim_{\mathcal{A}} M = 0$ if and only if $P_uM = 0$, $\alpha_{\mathcal{A}}(M) = \alpha_{\mathcal{A}}(t_uM)$ and that $\alpha_{\mathcal{A}}(M) = \infty^+$ if and only if $t_uM = 0$ by applying our notations and Corollary 2.45 to their definitions of $\dim_{\mathcal{A}}(M)$ and $\alpha_{\mathcal{A}}(M)$. Thus we have $\dim_{\mathcal{A}}(M) = \dim_{\mathcal{A}}(PM)$, that $\dim_{\mathcal{A}}(M) = 0$ if and only if $PM = 0$, $\alpha_{\mathcal{A}}(M) = \alpha_{\mathcal{A}}(tM)$ and that $\alpha_{\mathcal{A}}(M) = \infty^+$ if and only if $tM = 0$ by using Theorem 2.46 and Theorem 2.27. We can say that $\dim_{\mathcal{A}}$ and $\alpha_{\mathcal{A}}$ faithfully measures PM and tM, respectively.

Remark 2.50. Let $\mathcal{A}$ and $\mathcal{B}$ be von Neumann algebras with finite traces, respectively and $\iota : \mathcal{A} \hookrightarrow \mathcal{B}$ be an injective $*$-homomorphism preserving traces. Then for any right $\mathcal{A}$-module M, we have $\alpha_{\mathcal{A}}(M) = \alpha_{\mathcal{B}}(M \otimes_{\mathcal{A}} \mathcal{B})$ by Theorem 2.41, Remark 2.49 and [17, Lemma 2.12(1)]. This fact is known by Vaš ([29, Theorem 7.1]).

3. Some algebras for cocompact étale groupoids and group actions

In this section we give definitions and properties with respect to cocompact étale groupoids and their algebras.

Now we recall group algebras and group von Neumann algebras. Let G be a discrete group and R be a unital ring. Then its group algebra RG is the set of all R-valued functions on G with finite supports. We define $U_g \in RG$ by $U_g(h) = 1$ if $g = h$ and $U_g(h) = 0$ if $g \neq h$ for any $h \in G$. Its element of RG is written by $\sum_{i\in I} r_i U_{g_i}$ by using $r_i \in R$ and $g_i \in G$, where $(\sum_{i\in I} r_i U_{g_i})(g) := \sum_{i\in I} r_i U_{g_i}(g)$ for any $g \in G$. For $\sum_{i\in I} r_i U_{g_i}, \sum_{j\in J} r_j U_{h_j} \in RG$, their multiplication is defined by $(\sum_{i\in I} r_i U_{g_i})(\sum_{j\in J} r_j U_{h_j}) := \sum_{i\in I, j\in J} r_i r_j U_{g_i h_j}$. For $\mathbb{C}G$, we can define $*$ by $f^*(g) := \overline{f(g^{-1})}$ for any $f \in \mathbb{C}G$ and any $g \in G$. When we define a hermitian inner product $\langle -, - \rangle : \mathbb{C}G \times \mathbb{C}G \ni (f, h) \mapsto (f^*h)(e) \in \mathbb{C}$, then the left multiplication of f on $\mathbb{C}G$ is bounded for any $f \in \mathbb{C}G$. Hence we have the completion l^2G and the left regular representation $L : \mathbb{C}G \to B(l^2G)$. Now a group von Neumann algebra $\mathcal{N}G$ is defined by the double commutant of $L(\mathbb{C}G) \subset B(l^2G)$. Then $\mathcal{N}G$ has a finite trace tr_G which is defined by $\mathrm{tr}_G(a) := \langle U_e, a(U_e)\rangle$ for any $a \in \mathcal{N}G$. In particular we have $\mathrm{tr}_G(L(f)) = f(e) \in \mathbb{C}$ for any $f \in \mathbb{C}G$.

3.1. **Cocompact étale groupoids and Morita equivalence.**

In this subsection we give some definitions and properties related to étale groupoids (refer to [19] and [2]).

Let $\mathcal{G}$ be a groupoid, that is, a category whose arrows are invertible. We write the set of arrows $\mathcal{G}^{(1)}$ and also the set of objects $\mathcal{G}^{(0)}$, but we always write the set of arrows $\mathcal{G}$ instead of $\mathcal{G}^{(1)}$. We have five structure maps, the range map $r : \mathcal{G} \to \mathcal{G}^{(0)}$, the state map $s : \mathcal{G} \to \mathcal{G}^{(0)}$, the multiplication map $\mathcal{G}^{(2)} := \{(g_1, g_2) \in \mathcal{G} \times \mathcal{G} |\ s(g_1) = r(g_2)\} \ni (g_1, g_2) \mapsto g_1 g_2 \in \mathcal{G}$, the unit map $\mathcal{G}^{(0)} \ni x \mapsto 1_x \in \mathcal{G}$ and the inverse map $\mathcal{G} \ni g \mapsto g^{-1} \in \mathcal{G}$. Occasionally we regard $\mathcal{G}^{(0)}$ as the subset of $\mathcal{G}$ without writing the unit map. Let V, W be subsets of $\mathcal{G}^{(0)}$, $\alpha, \alpha_1, \alpha_2$ be subsets of $\mathcal{G}$, x be an element of $\mathcal{G}^{(0)}$. We define $\alpha_V^W := \{g \in \alpha |\ r(g) \in W, s(g) \in V\}$, $\alpha_V^x := \alpha_V^{\{x\}}$, $\alpha_x^V := \alpha_{\{x\}}^V$, $\alpha_U := \alpha_U^{\mathcal{G}^{(0)}}$ and $\alpha^U := \alpha^U_{\mathcal{G}^{(0)}}$. In particular we call $\mathcal{G}^x$ the fiver at x by r and $\mathcal{G}_x$ the fiber at x by s. Also we define $1_V := \{1_x \in \mathcal{G} |\ x \in V\}$, $\alpha^{-1} := \{g \in \mathcal{G} \mid g^{-1} \in \alpha\}$, $\alpha_1\alpha_2 := \{g \in \mathcal{G} |\ g = g_1 g_2,\ g_1 \in \alpha_1, g_2 \in \alpha_2, (g_1, g_2) \in \mathcal{G}^{(2)}\}$ and $\alpha(V) := r(\alpha_V)$. We remark that $\mathcal{G}_V^V$ is the restricted groupoid of $\mathcal{G}$ on V. A functor between two groupoids is called a homomorphism.

We define the orbit space $\mathcal{G}^{(0)}/\mathcal{G}$ by the set of all isomorphic classes of $\mathcal{G}^{(0)}$ when we regard $\mathcal{G}$ as a category. We call $X \subset \mathcal{G}^{(0)}$ a complete transversal if the orbit space $X/\mathcal{G}_X^X$ of the restricted groupoid $\mathcal{G}_X^X$ is bijective to the orbit space $\mathcal{G}^{(0)}/\mathcal{G}$ of $\mathcal{G}$ by the injection $\mathcal{G}_X^X \subset \mathcal{G}$.

We call $\mathcal{G}$ an étale groupoid if $\mathcal{G}^{(0)}$ is a locally compact Hausdorff space, $\mathcal{G}$ is a locally compact space which is not necessarily Hausdorff, five structure maps are continuous, r and s are étale maps, namely local homeomorphisms. In this paper we suppose that each fiber by r and s is a countable set.

Let $\mathcal{G}$ be an étale groupoid. We define $\Gamma(\mathcal{G})$ by the set of all open bisections of $\mathcal{G}$, where an open bisection γ is an open subset of $\mathcal{G}$ such that the restricted state map and the restricted range map on γ are embedding in $\mathcal{G}^0$. If $\Gamma(\mathcal{G})$ is a pseudogroup, then $\mathcal{G}$ is called an effective étale groupoid. In the case, $\mathcal{G}$ is regarded as the induced groupoid by taking germs of $\Gamma(\mathcal{G})$. We remark that étale groupoids in this paper are not necessarily effective.

Definition 3.1. Let $\mathcal{G}$ be an étale groupoid. $\gamma \in \Gamma(\mathcal{G})$ is an extendable bisection if the closure $\overline{\gamma}$ of γ is compact and there exists $\gamma' \in \Gamma(\mathcal{G})$ such that $\overline{\gamma} \subset \gamma'$. We define $\widehat{\Gamma}(\mathcal{G})$ by the set of all extendable bisections. Also we define $\widehat{\Gamma}(\mathcal{G}, X) := \widehat{\Gamma}(\mathcal{G}) \cap \Gamma(\mathcal{G}_X^X)$ for any relatively compact open set $X \subset \mathcal{G}^{(0)}$.

We remark that if γ_1 and γ_2 are extendable, then $\gamma_1\gamma_2$ and $(\gamma_1)^{-1}$ are also extendable.

Let $\mathcal{G}$ be an étale groupoid and $X \subset \mathcal{G}^{(0)}$ be a relatively compact open set. A probability measure μ on X is invariant by $\mathcal{G}$ if we have $\mu(\gamma^{-1}(B)) = \mu(B)$ for any $\gamma \in \Gamma(\mathcal{G})$ and any Borel set $B \subset X$ such that $\gamma^{-1}(B) \subset X$. A probability measure μ on X is quasi-invariant by $\mathcal{G}$ if we have that $\mu(\gamma^{-1}(B)) = 0$ if and only if $\mu(B) = 0$ for any $\gamma \in \Gamma(\mathcal{G})$ and any Borel set $B \subset X$ such that $\gamma^{-1}(B) \subset X$. In other words, μ is quasi-invariant by $\mathcal{G}$ if the measure class of μ is preserved by the action of $\mathcal{G}$. Hence measure classes are more important than measures when we consider quasi-invariant measures. We remark that we do not necessarily have an invariant probability measure on X, but we always have a quasi-invariant probability measure on X because we suppose that each fiber of an étale groupoid is countable in this paper.

Definition 3.2. An étale groupoid $\mathcal{G}$ is a cocompact étale groupoid if there exists a relatively compact open complete transversal.

Definition 3.3. Let $\mathcal{G}$ be a cocompact étale groupoid. We define $T(\mathcal{G})$ by the set of all relatively compact open complete transvarsals of $\mathcal{G}$, $TP(\mathcal{G})$ as the set of all pairs of a relatively compact open complete transversal of $\mathcal{G}$ and a quasi-invariant probability measure on it by $\mathcal{G}$. Also when we take a relatively compact open complete transversal X, then we define $P(\mathcal{G}, X)$ by the set of all quasi-invariant probability measures on X by $\mathcal{G}$.

Remark 3.4. Let $\mathcal{G}$ be a cocompact étale groupoid and X be a relatively compact open complete transversal. Suppose that we have a $(Y, \mu_Y) \in TP(\mathcal{G})$. By pulling back μ_Y on $Z := X \cup Y$ by the state map s and normalizing it, then we have a $\mu_Z \in P(\mathcal{G}, Z)$. Also by restricting μ_Z on X and normalizing it, then we have a $\mu_X \in P(\mathcal{G}, X)$. Then we call μ_X and μ_Z compatible to μ_Y.

Now we give a definition of Morita equivalence for étale groupoids, which is based on [19, Chapter 5]. In [19, Chapter 5], they deal with more general topological groupoids, but we consider the only case of étale groupoids.

Definition 3.5. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two étale groupoids and $\phi : \mathcal{G}_2 \to \mathcal{G}_1$ be an étale homomorphism. ϕ is essentially surjective if $\mathcal{G}_1 \times_{\mathcal{G}_1^{(0)}} \mathcal{G}_2^{(0)} := \{(g_1, y) \in \mathcal{G}_1 \times$

$\mathcal{G}_2^{(0)} \mid s(g_1) = \phi(y)\} \ni (g_1, y) \mapsto r(g_1) \in \mathcal{G}_1^{(0)}$ is an étale surjection. ϕ is full and faithful if a commutative diagram

$$\begin{array}{ccc} \mathcal{G}_2 & \xrightarrow{\phi} & \mathcal{G}_1 \\ {\scriptstyle (r,s)}\downarrow & & \downarrow{\scriptstyle (r,s)} \\ \mathcal{G}_2^{(0)} \times \mathcal{G}_2^{(0)} & \xrightarrow{\phi\times\phi} & \mathcal{G}_1^{(0)} \times \mathcal{G}_1^{(0)} \end{array}$$

is a fibered product of topological spaces, where $(r, s) : \mathcal{G}_2 \ni g_2 \mapsto (r(g_2), s(g_1)) \in \mathcal{G}_2^{(0)} \times \mathcal{G}_2^{(0)}$ and $(r, s) : \mathcal{G}_1 \ni g_1 \mapsto (r(g_1), s(g_1)) \in \mathcal{G}_1^{(0)} \times \mathcal{G}_1^{(0)}$. ϕ is a Morita equivalence map (or a weak equivalence map) if it is essentially surjective, full and faithful.

Two étale groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ are Morita equivalent (or weakly equivalent) if there exist an étale groupoid $\mathcal{G}$, a Morita equivalence map $\phi_1 : \mathcal{G} \to \mathcal{G}_1$ and a Morita equivalence map $\phi_2 : \mathcal{G} \to \mathcal{G}_2$.

If two étale groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ are Morita equivalent, then we can take an étale groupoid $\mathcal{G}$, a Morita equivalence map $p_1 : \mathcal{G} \to \mathcal{G}_1$ and a Morita equivalence map $p_2 : \mathcal{G} \to \mathcal{G}_2$ such that p_1 and p_2 are surjective ([19, Exercises 5.22 (2)]).

Remark 3.6. Let G_1 and G_2 be discrete groups. In section 1, we claim that when we have a compact Hausdorff space X_1 with a G_1-action and a compact Hausdorff space X_2 with a G_2-action such that transformation groupoids $\mathcal{G}_1 := G_1 \ltimes X_1$ and $\mathcal{G}_2 := G_2 \ltimes X_2$ are Morita equivalent, then we have a topological coupling Ω of G_1 and G_2, a G_1-homeomorphism $X_1 \cong G_2\backslash\Omega$ and a G_2-homeomorphism $X_2 \cong G_1\backslash\Omega$. We give a rough proof. We take étale groupoid $\mathcal{G}$, a Morita equivalence surjection $p_1 : \mathcal{G} \to \mathcal{G}_1$ and a Morita equivalence surjection $p_2 : \mathcal{G} \to \mathcal{G}_2$. When we take $x, x' \in p_i^{-1}(x_i)$ for $x_i \in X_i$, then we have a unique $\widetilde{1_{x_i}} \in \mathcal{G}_x^{x'}$ such that $p_i(\widetilde{1_{x_i}}) = 1_{x_i} \in (\mathcal{G}_i)_{x_i}^{x_i}$ since p_i is a Morita equivalence map for each $i = 1, 2$. We define a equivalence relation $\sim$ on $\mathcal{G}$ as follows: For $g, g' \in \mathcal{G}$, $g \sim g'$ if $p_1 \circ r(g) = p_1 \circ r(g')$, $p_2 \circ s(g) = p_2 \circ s(g')$ and $\widetilde{1_{p_1\circ r(g)}}g = g'\widetilde{1_{p_2\circ s(g)}}$, where $\widetilde{1_{p_1\circ r(g)}} \in \mathcal{G}_{r(g)}^{r(g')}$ is the lift of $1_{p_1\circ r(g)} \in (\mathcal{G}_1)_{p_1\circ r(g)}^{p_1\circ r(g)}$ and $\widetilde{1_{p_2\circ s(g)}} \in \mathcal{G}_{s(g)}^{s(g')}$ is the lift of $1_{p_2\circ s(g)} \in (\mathcal{G}_2)_{p_2\circ s(g)}^{p_2\circ s(g)}$. Then we can naturally define the left G_1-action and the right G_2-action on Ω, which is a topological coupling of G_1 and G_2. Also we can confirm that p_1 and p_2 induce surjections $\Omega \to X_1$ and $\Omega \to X_2$, which give a G_1-homeomorphism $X_1 \cong G_2\backslash\Omega$ and a G_2-homeomorphism $X_2 \cong G_1\backslash\Omega$.

The following is well-known for the case of holonomy groupoids on complete transvarsals of closed foliated manifolds.

Proposition 3.7. *If $\mathcal{G}$ is a cocompact étale groupoid, then any étale groupoid which is Morita equivalent to $\mathcal{G}$ is also a cocompact étale groupoid.*

Proof. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be étale groupoids and $p : \mathcal{G}_2 \to \mathcal{G}_1$ be a Morita equivalence surjection. If $\mathcal{G}_2$ is a cocompact étale groupoid, then $\mathcal{G}_1$ is also a cocompact étale groupoid. Indeed when we take a relatively compact open complete transversal X_2 of $\mathcal{G}_2$, then $p(X_2)$ is a relatively compact open complete transversal of $\mathcal{G}_1$. Next we prove that if $\mathcal{G}_1$ is a cocompact étale groupoid, then $\mathcal{G}_2$ is also a cocompact étale groupoid. Let X_1 be a relatively compact open complete transversal of $\mathcal{G}_1$. We can take a subset $B_2 \subset \mathcal{G}_2^{(0)}$ which is bijective to $\overline{X_1}$ by p. Then for any $x_2 \in B_2$ and $x_1 := p(x_2) \in \overline{X_1}$ we can take open neighborhoods U_{x_1}, V_{x_1} of x_1 and U_{x_2}, V_{x_2} of x_2 such that $x_1 \in V_{x_1} \subset \overline{V_{x_1}} \subset U_{x_1} \subset \mathcal{G}_1^{(0)}$ and $x_2 \in V_{x_2} \subset \overline{V_{x_2}} \subset U_{x_2} \subset \mathcal{G}_2^{(0)}$ where U_{x_2} is homeomorphic to U_{x_1} and V_{x_2} is homeomorphic to V_{x_1} by p. Then we have a finite subset $I_1 \subset \overline{X_1}$ such that $\{V_{x_1}\}_{x_1 \in I_1}$ is an open cover of $\overline{X_1}$ because $\overline{X_1}$ is compact. We define a finite set $I_2 := p^{-1}(I_1) \cap B_2$. When we define $W_{x_1} := X_1 \cap V_{x_1}$ for any $x_1 \in I_1$ and $W_{x_2} := X_2 \cap V_{x_2}$ for any $x_2 \in I_2$, then we have $X_1 = \bigcup_{x_1 \in I_1} W_{x_1}$. Hence when we define an open set $X_2 := \bigcup_{x_2 \in I_2} W_{x_2}$, then it is an open complete transversal because p is a Morita equivalence map and X_1 is a complete transversal of $\mathcal{G}_1$. Moreover since W_{x_2} is relatively compact and I_2 is a finite set, then X_2 is also relatively compact. Hence X_2 is a relatively compact open transversal of $\mathcal{G}_2$. □

Corollary 3.8. *Let two cocompact étale groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ be Morita equivalent and have relatively compact open complete transvarsals X_1 and X_2, respectively. Suppose that an étale groupoid $\mathcal{G}$, a Morita equivalence surjection $p_1 : \mathcal{G} \to \mathcal{G}_1$ and a Morita equivalence surjection $p_2 : \mathcal{G} \to \mathcal{G}_2$ give Morita equivalence between $\mathcal{G}_1$ and $\mathcal{G}_2$. Then there exists a relatively compact open complete transversal X of $\mathcal{G}$ such that $p_1(X) \supset \overline{X_1}$ and $p_2(X) \supset \overline{X_2}$.*

Proof. We take a relatively compact open set X_i' such that $\overline{X_i} \subset X_i' \subset \mathcal{G}_i^{(0)}$ for each $i = 1, 2$. Clearly X_i' is also a relatively compact open complete transversal $\mathcal{G}_i$ for each $i = 1, 2$. Then by the latter part of the proof of the above proposition, we can take a relatively compact open complete transversal Y_i of $\mathcal{G}$ such that $p_i(Y_i) = X_i'$ for each $i = 1, 2$. When we define $X := Y_1 \cup Y_2$, then it is also a relatively compact open complete transversal of $\mathcal{G}$. Clearly we have $p_i(X) \supset X_i' \supset \overline{X_i}$ for each $i = 1, 2$. □

Remark 3.9. We consider the situation in Corollary 3.8. Suppose that we have a $(X_1, \mu_1) \in TP(\mathcal{G}_1)$. Then we have a $(p_1(X), \mu_{p_1(X)}) \in TP(\mathcal{G}_1)$ compatible to (X_1, μ_1) by Remark 3.4. By pulling back $\mu_{p_1(X)}$ on X by p_1 and normalizing it, then we have a $\mu_X \in P(\mathcal{G}, X)$. We can take a $\mu_2 \in P(\mathcal{G}_2, X_2)$ whose normalized pulling back on X by p_2 is compatible to μ_X (Remark 3.4). Then we call μ_X and μ_2 compatible to μ_1.

3.2. **Groupoid algebras.** Let R be a unital ring, for example $\mathbb{Z}$ and $\mathbb{C}$ through this subsection.

Definition 3.10. Let X be a locally compact Hausdorff space. $\mathcal{B}_f(X)$ is defined by a finitely additive class generated by all open sets of X. $B_f(X, R)$ is the set of all R-valued simple functions for $\mathcal{B}_f(X)$. $B_f(X, R)$ is an R-algebra with the unit χ_X by pointwise multiplications, where χ_X is the characteristic function of X.

For a function f on X, f is an R-valued simple function for $\mathcal{B}_f(X)$ if there exists $\{B_i\}_{i \in I} \subset \mathcal{B}_f(X)$ and $\{r_i\}_{i \in I} \subset R$ such that I is a finite index set and $f = \sum_{i \in I} r_i \chi_{B_i}$. $B_f(X, R)$ can be generated by all characteristic functions for the set of all open subsets of X as a subalgebra in the set of all R-valued functions on X. Also $B_f(X, R)$ can be generated by all characteristic functions for $\mathcal{B}_f(X)$ as a submodule in the set of all R-valued functions on X.

Lemma 3.11. *Let X be a locally compact Hausdorff space. $B_f(X,R)$ is faithfully flat as a left (resp. right) R-module.*

Proof. We take a family $\{V_i\}_{i\in I} \subset \mathcal{B}_f(X)$ of mutually disjoint subsets in X such that I is a finite index set. Also we take a similar family $\{W_j\}_{j\in J} \subset \mathcal{B}_f(X)$. Then $\{V_i, W_j \setminus (\bigcup_I V_i)\}_{I\sqcup J}$ is also a similar family. Hence such families make a directed set. We have $\bigoplus_I R\chi_{V_i} \subset B_f(X,R)$ and $\bigoplus_J R\chi_{W_j} \subset B_f(X,R)$, which are finitely generated free left R-modules. Also we have $(\bigoplus_I R\chi_{V_i}) + (\bigoplus_J R\chi_{W_j}) = (\bigoplus_I R\chi_{V_i}\oplus) \oplus (\bigoplus_J R\chi_{W_j\setminus(\bigcup_I V_i)}) \subset B_f(X,R)$. Hence $B_f(X,R)$ is an inductive limit of all submodules like $\bigoplus_I R\chi_{V_i}$. Hence $B_f(X,R)$ is a flat R-module because it is an inductive limit of free left R-modules.

We fix $x_0 \in X$, then we have a submodule $\{f \in B_f(X,R) \mid f(x_0) = 0\}$ of $B_f(X,R)$. Then we have $B_f(X,R) \cong R \oplus \{f \in B_f(X,R) \mid f(x_0) = 0\}$ as left R-modules. Hence $B_f(X,R)$ is a faithfully flat left R-module. □

We consider a cocompact étale groupoid $\mathcal{G}$. In this subsection we define a groupoid algebra $R(\mathcal{G},X)$ of a pair $(\mathcal{G},X)$ and also its trivial action on $B_f(X,R)$ for any relatively compact open complete transversal X.

The set of all R-valued functions of $\mathcal{G}$ is an R-module by pointwise sums and pointwise scalar products. We define an R-algebra structure for its certain submodule by introducing convolution products. We call it a groupoid algebra of $\mathcal{G}$. For any $\gamma \in \Gamma(\mathcal{G})$, we write its characteristic function U_γ, that is, $U_\gamma(g) := 1$ if $g \in \gamma$ and $U_\gamma(g) := 0$ if $g \notin \gamma$. In particular for any open set $V \subset \mathcal{G}^{(0)}$, we write $\chi_V := U_{1_V}$. For example, we have $\chi_{r(\gamma)} = \chi_{s(\gamma)} = U_\gamma$ if $\gamma \subset 1_{\mathcal{G}^{(0)}} \subset \mathcal{G}$. For any $\gamma_1, \gamma_2 \in \Gamma(\mathcal{G})$, we define the convolution product of U_{γ_1} and U_{γ_2} by $U_{\gamma_1}U_{\gamma_2} := U_{\gamma_1\gamma_2}$ if $\gamma_1\gamma_2 \neq \emptyset$ and $U_{\gamma_1}U_{\gamma_2} := 0$ if $\gamma_1\gamma_2 = \emptyset$. In particular we have $U_\gamma = \chi_{r(\gamma)}U_\gamma = U_\gamma\chi_{s(\gamma)}$ for any $\gamma \in \Gamma(\mathcal{G})$. Also we define $U_\gamma^{-1} := U_{\gamma^{-1}}$. For any $B \in \mathcal{B}_f(\mathcal{G}^{(0)})$ and any $\gamma \in \Gamma(\mathcal{G})$, we define $(\chi_B U_\gamma)(g) := \chi_B(r(g))U_\gamma(g)$ and $U_\gamma\chi_B := \chi_{\gamma(B)}U_\gamma$.

Definition 3.12. $R\mathcal{G}$ is the set of all R-valued functions on $\mathcal{G}$ given by combining sums and scalar products and convolutions of any finite elements of $\{\chi_B\}_{B\in\mathcal{B}_f(\mathcal{G}^{(0)})} \cup \{U_\gamma\}_{\gamma\in\widehat{\Gamma}(\mathcal{G})}$. In other words, an R-valued function f on $\mathcal{G}$ is an element of $R\mathcal{G}$ if there exists a finite set $\{(r_i, B_i, \gamma_i) \in R \times \mathcal{B}_f(\mathcal{G}^{(0)}) \times \widehat{\Gamma}(\mathcal{G})\}_{i\in I}$ such that $f = \sum_{i\in I} r_i\chi_{B_i}U_{\gamma_i}$, namely $f(g) = \sum_{i\in I} r_i\chi_{B_i}(r(g))U_{\gamma_i}(g)$ for any $g \in \mathcal{G}$. $R(\mathcal{G},X)$ is the set of all R-valued functions on $\mathcal{G}_X^X$ given by combining sums and scalar products and convolutions of any finite elements of $\{\chi_B\}_{B\in\mathcal{B}_f(X)} \cup \{U_\gamma\}_{\gamma\in\widehat{\Gamma}(\mathcal{G},X)}$.

When for $f_1, f_2 \in R\mathcal{G}$ their convolution product f_1f_2 is defined by $(f_1f_2)(g) := \sum_{g=g_1g_2} f_1(g_1)f_2(g_2)$ for any $g \in \mathcal{G}$, then $R\mathcal{G}$ is an R-algebra which does not necessarily have a unit. Since we can regard $f \in R(\mathcal{G},X)$ as a function on $\mathcal{G}$ by defining $f(g) := 0$ if $g \in \mathcal{G} \setminus \mathcal{G}_X^X$, then $R(\mathcal{G},X)$ has an R-algebra structure with a unit χ_X. We call $R\mathcal{G}$ a groupoid algebra of $\mathcal{G}$ and $R(\mathcal{G},X)$ a groupoid algebra of $(\mathcal{G},X)$. We remark that $R\mathcal{G}$ has the unit $\chi_{\mathcal{G}^{(0)}}$ if and only if $\mathcal{G}^{(0)}$ is compact. We define a subalgebra $\chi_A R\mathcal{G}\chi_A := \{f \in R\mathcal{G} \mid f = \chi_A f\chi_A\}$ for any $A \in \mathcal{B}_f(\mathcal{G}^{(0)})$. In the similar way we can define a subalgebra $\chi_A R(\mathcal{G},X)\chi_A$ with the unit χ_A for any $A \in \mathcal{B}_f(X)$. In particular we have an algebraic isomorphism $R(\mathcal{G},X) \cong \chi_X R\mathcal{G}\chi_X$. We remark that

$R(\mathcal{G},X)$ can be regarded as the set of all R-valued functions on $\mathcal{G}_X^X$ given by combining sums and scalar products of any finite elements of $\{\chi_B U_\gamma\}_{(B,\gamma)\in\mathcal{B}_f(X)\times\widehat{\Gamma}(\mathcal{G},X)}$.

Proposition 3.13. *$R(\mathcal{G},X)$ is a faithfully flat left (resp. right) $B_f(X,R)$-module.*

Proof. We take a finite family $\{(B_i,\gamma_i)\}_{i\in I}\subset\mathcal{B}_f(X)\times\widehat{\Gamma}(\mathcal{G},X)$ such that $\{(\gamma_i)^{B_i}\}_{i\in I}$ are mutually disjoint. All families like the above make a directed set. We have $\bigoplus_I R\chi_{B_i}U_{\gamma_i}\subset R(\mathcal{G},X)$, which is a finitely generated projective left $B_f(X,R)$-module. $R(\mathcal{G},X)$ is an inductive limit of all left $B_f(X,R)$-modules like $\bigoplus_I R\chi_{B_i}U_{\gamma_i}$. Hence $R(\mathcal{G},X)$ is a flat left $B_f(X,R)$-module because it is an inductive limit of projective left $B_f(X,R)$-modules.

We have a submodule $\{f\in R(\mathcal{G},X)\mid f|_{1_{\mathcal{G}^{(0)}}}=0\}$ of $R(\mathcal{G},X)$ as a left $B_f(X,R)$-module. Then we have $R(\mathcal{G},X)\cong B_f(X,R)\oplus\{f\in R(\mathcal{G},X)\mid f|_{1_{\mathcal{G}^{(0)}}}=0\}$ as a left $B_f(X,R)$-module. Hence $R(\mathcal{G},X)$ is a faithfully flat left $B_f(X,R)$-module. $\square$

Definition 3.14. $\epsilon: B_f(X,R)\times R(\mathcal{G},X)\to B_f(X,R)$ is defined by $\epsilon(\xi,f)(x):=\sum_{s(g)=x}\xi(r(g))f(g)$ for $f\in R(\mathcal{G},X)$ and $\xi\in B_f(X,R)$. We call ϵ the right trivial action of $R(\mathcal{G},X)$ on $B_f(X,R)$.

Now we prove that if cocompact étale groupoids are Morita equivalent, then their groupoid algebras are also Morita equivalent.

Lemma 3.15. *Let $\mathcal{G}$ be a cocompact étale groupoid with relatively compact open complete transvarsals X and Y such that $X\subset Y$. Then we have an algebraic isomorphism $R(\mathcal{G},X)\cong\chi_X R(\mathcal{G},Y)\chi_X$. Moreover χ_X is a full idempotent of $R(\mathcal{G},Y)$. Also we have $B_f(X,R)\otimes_{R(\mathcal{G},X)}\chi_X R(\mathcal{G},Y)\cong B_f(Y,R)$ and $B_f(Y,R)\otimes_{R(\mathcal{G},Y)}R(\mathcal{G},Y)\chi_X\cong B_f(X,R)$.*

Proof. We have $R(\mathcal{G},X)\cong\chi_X R\mathcal{G}\chi_X=\chi_X\chi_Y R\mathcal{G}\chi_Y\chi_X\cong\chi_X R(\mathcal{G},Y)\chi_X$.

We can take a finite set $\{\gamma_i\}_{i\in I:=\{1,\dots,n\}}\subset\widehat{\Gamma}(\mathcal{G})$ such that $\gamma_1=1_X$, $\overline{Y}\subset\bigcup_{i\in I}r(\gamma_i)$ and $s(\gamma_i)\subset X$ for any $i\in I$ because X is complete transverse and $\overline{Y}$ is compact. Hence we have $Y=\bigcup_{i\in I}r(\gamma_i^Y)$ and $X=s(1_X)=\bigcup_{i\in I}s(\gamma_i^Y)$. Because we have

$$Y=\bigsqcup_{i\in I}r(\gamma_i^Y)\setminus r(\gamma_1^Y)\setminus\cdots\setminus r(\gamma_{i-1}^Y),$$

then

$$\begin{aligned}\chi_Y&=\sum_{i\in I}\chi_{r(\gamma_i^Y)\setminus r(\gamma_1^Y)\setminus\cdots\setminus r(\gamma_{i-1}^Y)}=\sum_{i\in I}\chi_{r(\gamma_i^Y)}\chi_{r(\gamma_i^Y)\setminus r(\gamma_1^Y)\setminus\cdots\setminus r(\gamma_{i-1}^Y)}\\&=\sum_{i\in I}U_{\gamma_i^Y}\chi_X U_{(\gamma_i^Y)^{-1}}\chi_{r(\gamma_i^Y)\setminus r(\gamma_1^Y)\setminus\cdots\setminus r(\gamma_{i-1}^Y)}.\end{aligned}$$

Hence χ_X is a full idempotent of $R(\mathcal{G},Y)$.

We can confirm that

$$B_f(X,R)\otimes_{R(\mathcal{G},X)}\chi_X R(\mathcal{G},Y)\ni\xi\otimes\chi_X f\mapsto\epsilon(\xi,\chi_X f)\in B_f(Y,R)$$

and

$$B_f(Y,R)\ni\eta\mapsto\sum_{i\in I}\chi_X\otimes U_{(\gamma_i^Y)^{-1}}\chi_{r(\gamma_i^Y)\setminus r(\gamma_1^Y)\setminus\cdots\setminus r(\gamma_{i-1}^Y)}\eta\in B_f(X,R)\otimes_{R(\mathcal{G},X)}\chi_X R(\mathcal{G},Y)$$

are mutual inverses. Also we can confirm that

$$B_f(Y,R)\otimes_{R(\mathcal{G},Y)} R(\mathcal{G},Y)\chi_X \ni \eta\otimes f\chi_X \mapsto \epsilon(\eta, f\chi_X)\in B_f(X,R)$$

and

$$B_f(X,R)\ni \xi\mapsto \xi\otimes\chi_X\in B_f(Y,R)\otimes_{R(\mathcal{G},Y)} R(\mathcal{G},Y)\chi_X$$

are mutual inverses. □

Theorem 3.16. *Let two cocompact étale groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ be Morita equivalent and have relatively compact open complete transvarsals X_1 and X_2, respectively. Suppose that a cocompact étale groupoid $\mathcal{G}$ with a relatively compact open complete transversal X, a Morita equivalence surjection $p_1 : \mathcal{G}\to\mathcal{G}_1$ such that $p_1(X)\supset\overline{X_1}$ and a Morita equivalence surjection $p_2:\mathcal{G}\to\mathcal{G}_2$ such that $p_2(X)\supset\overline{X_2}$ give Morita equivalence between $\mathcal{G}_1$ and $\mathcal{G}_2$. Then there exists $A_i\in\mathcal{B}_f(X)$ which is bijective to X_i by p_i for each $i=1,2$. Also χ_{A_1} and χ_{A_2} are full idempotents of $R(\mathcal{G},X)$ and we have an algebra isomorphism $R(\mathcal{G}_i,X_i)\cong\chi_{A_i}R(\mathcal{G},X)\chi_{A_i}$ by p_i for each $i=1,2$. Moreover we have $B_f(X_i,R)\cong B_f(X,R)\otimes_{R(\mathcal{G},X)}R(\mathcal{G},X)\chi_{A_i}$, and $B_f(X,R)\cong B_f(X_i,R)\otimes_{R(\mathcal{G}_i,X_i)}\chi_{A_i}R(\mathcal{G},X)$ for each $i=1,2$.*

Proof. Since $p_i : X\to\mathcal{G}_i^{(0)}$ is locally homeomorphism and $\overline{X_i}$ is compact, we can take a finite set of open subsets $\{U_{i,k}\subset X\}_{k\in K:=\{1,\dots,n\}}$ such that $p_i|_{U_{i,k}}$ is a homeomorphism into $\mathcal{G}_i^{(0)}$ and $\overline{X_i}\subset\bigcup_{k\in K}p_i(U_{i,k})$. $\{V_{i.k}:=(p_i|_{U_{i,k}})^{-1}(p_i(U_{i,k})\cap X_i)\}_{k\in K}$ is a finite set of open subsets such that $p_i|_{V_{i,k}}$ is a homeomorphism into X_i and $X_i=\bigcup_{k\in K}p_i(V_{i,k})$. Since we have

$$X_i=\bigsqcup_{k\in K}p_i(V_{i,k})\setminus p_i(V_{i,k-1})\setminus\cdots\setminus p_i(V_{i,1}),$$

we define

$$A_i=\bigsqcup_{k\in K}(p_i|_{V_{i,k}})^{-1}(p_i(V_{i,k})\setminus p_i(V_{i,k-1})\setminus\cdots\setminus p_i(V_{i,1})).$$

Then we have $A_i\in\mathcal{B}_f(X)$ and $p_i|_{A_i}:A_i\to X_i$ is bijective. Also $p_i|_{\mathcal{G}_{A_i}^{A_i}}$ is a bijection from $\mathcal{G}_{A_i}^{A_i}$ to $(\mathcal{G}_i)_{X_i}^{X_i}$. Then

$$R(\mathcal{G}_i,X_i)\ni f\to f\circ p_i|_{\mathcal{G}_{A_i}^{A_i}}\in\chi_{A_i}R(\mathcal{G},X)\chi_{A_i}$$

and

$$\chi_{A_i}R(\mathcal{G},X)\chi_{A_i}\ni f\to f\circ(p_i|_{\mathcal{G}_{A_i}^{A_i}})^{-1}\in R(\mathcal{G}_i,X_i)$$

are mutual inverses.

Others are proved by similar arguments in the proof of Lemma 3.15. □

3.3. **Groupoid von Neumann algebras.** Let $\mathcal{G}$ be a cocompact étale groupoid and X be a relatively compact open complete transversal. We consider a groupoid algebra $\mathbb{C}(\mathcal{G},X)$ with $*$ which is defined by $f^*(g):=\overline{f(g^{-1})}$ for any $f\in\mathbb{C}(\mathcal{G},X)$ and any $g\in\mathcal{G}_X^X$. Let μ be a quasi-invariant probability measure on X. We have a measure μ' on $\mathcal{G}_X^X$ induced by μ, which is unique as a measure class. When $f_1,f_2\in\mathbb{C}(\mathcal{G},X)$ is called μ-equivalent if $\mu'(\{g\in\mathcal{G}_X^X|\ f_1(g)\neq f_2(g)\})=0$, then we define $\mathbb{C}_\mu(\mathcal{G},X)$ by the set of all μ-equivalent classes. $\mathbb{C}_\mu(\mathcal{G},X)$ is a $\mathbb{C}$-algebra with $*$ because convolution products and so on are well-defined. The projection $p_\mu : \mathbb{C}(\mathcal{G}, X) \to \mathbb{C}_\mu(\mathcal{G}, X)$ is an algebra homomorphism preserving units and $*$. When we define a hermitian inner product $\langle -, - \rangle_\mu : \mathbb{C}_\mu(\mathcal{G}, X) \times \mathbb{C}_\mu(\mathcal{G}, X) \ni (f, h) \mapsto \int_X (f^* h)(1_x) d\mu(x) \in \mathbb{C}$, then the left multiplication of $p_\mu(f)$ on $\mathbb{C}_\mu(\mathcal{G}, X)$ is bounded for any $f \in \mathbb{C}(\mathcal{G}, X)$. Hence we have the completion $L^2_\mu(\mathcal{G}, X)$ and the left regular representation $L_\mu : \mathbb{C}(\mathcal{G}, X) \to B(L^2_\mu(\mathcal{G}, X))$. Now a groupoid von Neumann algebra $\mathcal{N}_\mu(\mathcal{G}, X)$ of $(\mathcal{G}, X, \mu)$ is defined by the double commutant of $L_\mu(\mathbb{C}(\mathcal{G}, X)) \subset B(L^2_\mu(\mathcal{G}, X))$. Then $\mathcal{N}_\mu(\mathcal{G}, X)$ has a faithful normal state ψ_μ which is defined by $\psi_\mu(a) := \langle \chi_X, a(\chi_X) \rangle_\mu$ for any $a \in \mathcal{N}_\mu(\mathcal{G}, X)$. In particular we have $\psi_\mu(L_\mu(f)) = \int_X f(1_x) d\mu(x) \in \mathbb{C}$ for any $f \in \mathbb{C}(\mathcal{G}, X)$.

We can confirm the following by extending Lemma 3.15 and Theorem 3.16 on weak closures, that is, von Neumann algebras.

Lemma 3.17. *Let $\mathcal{G}$ be a cocompact étale groupoid with relatively compact open complete transvarsals X and Y such that $X \subset Y$. Let $\mu_X \in P(\mathcal{G}, X)$ and $\mu_Y \in P(\mathcal{G}, Y)$ be compatible. Then we have an algebraic isomorphism $\mathcal{N}_{\mu_X}(\mathcal{G}, X) \cong \chi_X \mathcal{N}_{\mu_Y}(\mathcal{G}, Y) \chi_X$. Moreover χ_X is a full projection of $\mathcal{N}_\mu(\mathcal{G}, Y)$.*

Theorem 3.18. *Let two cocompact étale groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ be Morita equivalent and have relatively compact open complete transversals X_1 and X_2, respectively. Suppose that a cocompact étale groupoid $\mathcal{G}$ with a relatively compact open complete transversal X, a Morita equivalence surjection $p_1 : \mathcal{G} \to \mathcal{G}_1$ such that $p_1(X) \supset \overline{X_1}$ and a Morita equivalence surjection $p_2 : \mathcal{G} \to \mathcal{G}_2$ such that $p_2(X) \supset \overline{X_2}$ give Morita equivalence between $\mathcal{G}_1$ and $\mathcal{G}_2$. Let $\mu_1 \in P(\mathcal{G}, X_1)$, $\mu_2 \in P(\mathcal{G}, X_2)$ and $\mu \in P(\mathcal{G}, X)$ be compatible. Then there exists $A_i \in \mathcal{B}_f(X)$ which is bijective to X_i by p_i for each $i = 1, 2$. Also χ_{A_1} and χ_{A_2} are full projections of $\mathcal{N}_\mu(\mathcal{G}, X)$ and we have an algebra isomorphism $\mathcal{N}_{\mu_i}(\mathcal{G}_i, X_i) \cong \chi_{A_i} \mathcal{N}_\mu(\mathcal{G}, X) \chi_{A_i}$ by p_i for each $i = 1, 2$.*

3.4. **The case of group actions.** In this subsection we consider the so-called transformation groupoids and compare groupoid algebras with group algebras. Let X be a compact Hausdorff space with a left action of G through this subsection. Its transformation groupoid $G \ltimes X$ is an étale groupoid which is $G \times X$ as a set. Five structure maps are following: the range map $r : G \ltimes X \ni (g, x) \mapsto gx \in X$, the state map $s : G \ltimes X \ni (g, x) \mapsto x \in X$, the multiplication map $(G \ltimes X)^{(2)} \ni (g_1, x_1, g_2, x_2) \mapsto (g_1 g_2, x_2) \in G \ltimes X$, the unit map $X \ni x \mapsto (e, x) \in G \ltimes X$ and the inverse map $G \ltimes X \ni (g, x) \mapsto (g^{-1}, gx) \in G \ltimes X$. Let R be a unital ring through this subsection. Since $G \ltimes X$ is a cocompact étale groupoid and X is a compact open transversal, then we define its groupoid algebra $R(G \ltimes X) := R(G \ltimes X, X)$. We have a natural injection $RG \ni rU_g \mapsto rU_{\{g\} \times X} \in R(G \ltimes X)$.

First we describe extendable bisections of $G \ltimes X$.

Theorem 3.19. *For $\gamma \in \Gamma(G \ltimes X)$, $\gamma \in \widehat{\Gamma}(G \ltimes X)$ if and only if there exists a unique finite set of pairs $\{(g_i, U_i) \in G \times \mathcal{O}(X)\}_{i \in I}$ such that $\gamma = \bigsqcup_i \{g_i\} \times U_i \subset G \ltimes X$ where $\{g_i\}_{i \in I}$ are mutually different, $\{\overline{U_i}\}_{i \in I}$ are mutually disjoint and $\{\overline{g_i(U_i)}\}$ are mutually disjoint. Here $\mathcal{O}(X)$ is the set of all open sets of X.*

Proof. We define $\pi : G \ltimes X \ni (g, x) \mapsto g \in G$. For any $\gamma \in \Gamma(G \ltimes X)$, we have $\gamma = \bigsqcup_{g \in G} \{g\} \times (\pi|_\gamma)^{-1}(g)$. Also $\{(\pi|_\gamma)^{-1}(g)\}_{g \in G}$ are mutually disjoint and also

$\{g((\pi|_\gamma)^{-1}(g))\}_{g\in G}$ are mutually disjoint since γ is an open bisection. Hence we have $s(\gamma)=\bigsqcup_{g\in G}(\pi|_\gamma)^{-1}(g)$ and $r(\gamma)=\bigsqcup_{g\in G} g((\pi|_\gamma)^{-1}(g))$. Now the part of 'if' is clear, thus we prove the part of 'only if'. For any $\gamma\in\widehat{\Gamma}(G\ltimes X)$ we have $\gamma'\in\Gamma(G\ltimes X)$ which satisfies $\overline{\gamma}\subset\gamma'$. Since we have $\gamma'=\bigsqcup_{g\in G}\{g\}\times(\pi|_{\gamma'})^{-1}(g)$ and $\overline{\gamma}$ is compact, then we have a finite subset $F\subset G$ such that $\overline{\gamma}\subset\bigsqcup_{g\in F}\{g\}\times(\pi|_{\gamma'})^{-1}(g)$. Hence we have $\gamma=\bigsqcup_{g\in F}\{g\}\times(\pi|_\gamma)^{-1}(g)$. Thus we have $(\pi|_\gamma)^{-1}(g)\subset\overline{(\pi|_\gamma)^{-1}(g)}\subset(\pi|_{\gamma'})^{-1}(g)$ and $g((\pi|_\gamma)^{-1}(g))\subset\overline{g((\pi|_\gamma)^{-1}(g))}\subset g((p|_{\gamma'})^{-1}(g))$ for any $g\in F$. □

Since $R(G\ltimes X)$ is generated as an R-algebra by $\{\chi_B\}_{B\in\mathcal{B}_f(X)}\cup\{U_\gamma\}_{\gamma\in\widehat{\Gamma}(G\ltimes X)}$, Theorem 3.19 proves the following.

Corollary 3.20. *$R(G\ltimes X)$ is generated as an R-algebra by $\{\chi_B\}_{B\in\mathcal{B}_f(X)}\cup\{U_{\{g\}\times X}\}_{g\in G}$.*

$B_f(X,R)\otimes_R RG$ can be regarded as an R-algebra by $(f\otimes\sum r_iU_{g_i})(f'\otimes\sum r_i'U_{g_i'}):=\sum_{i,j} fr_i(f'\circ(g_i)^{-1})r_j'\otimes U_{g_ig_j'}$ for any $f\otimes\sum r_iU_{g_i}, f'\otimes\sum r_i'U_{g_i'}\in B_f(X,R)\otimes_R RG$. Then we call it an algebraic crossed product of (G,X) and write it $B_f(X,R)\rtimes_{alg}G$. Clearly we have an algebra isomorphism

$$B_f(X,R)\rtimes_{alg}G\ni f\otimes\sum r_iU_{g_i}\mapsto\sum fr_iU_{\{g_i\}\times X}\in R(G\ltimes X).$$

Let M be a right RG-module. Then $M\otimes_{RG}R(G\ltimes X)$ is naturally a right $R(G\ltimes X)$-module. Also when we define $M\otimes_R B_f(X,R)$ by regarding M as a right R-module, then $M\otimes_R B_f(X,R)$ can be regarded as a right $R(G\ltimes X)$-module by defining $(m\otimes f)(\chi_BU_{\{g\}\times X}):=(mg)\otimes(\chi_B\circ g)$ for any $m\in M$, $f\in B_f(X,R)$, $B\in\mathcal{B}_f(X)$ and $g\in G$. Then we have $R(G\ltimes X)$-homomorphisms

$$M\otimes_R B_f(X,R)\ni m\otimes f\mapsto m\otimes f\in M\otimes_{RG}R(G\ltimes X)$$

and

$$M\otimes_{RG}R(G\ltimes X)\ni m\otimes\chi_BU_{\{g\}\times X}\mapsto(mg)\otimes(\chi_B\circ g)\in M\otimes_R B_f(X,R)$$

which are mutual inverses.

Lemma 3.21. *Let M be a right RG-module. Then we have $M\otimes_R B_f(X,R)\cong M\otimes_{RG}R(G\ltimes X)$ as right $R(G\ltimes X)$-modules. In particular we have $B_f(X,R)\cong R\otimes_{RG}R(G\ltimes X)$ as right $R(G\ltimes X)$-modules.*

Proof. The former is already proved. If we take the trivial right RG-module R as M, the latter is clear. □

Lemma 3.22. *$R(G\ltimes X)$ is a faithfully flat left RG-module.*

Proof. Since we have $M\otimes_R B_f(X,R)\cong M\otimes_{RG}R(G\ltimes X)$ for any right RG-module M by Lemma 3.21 and $B_f(X,R)$ is a faithfully flat left R-module by Lemma 3.11, $R(G\ltimes X)$ is a faithfully flat left RG-module. □

When we take a quasi-invariant probability measure μ on X, then we define $\mathbb{C}_\mu(G\ltimes X):=\mathbb{C}_\mu(G\ltimes X,X)$ and $\mathcal{N}_\mu(G\ltimes X):=\mathcal{N}_\mu(G\ltimes X,X)$. Then an injection $\iota:\mathbb{C}G\to\mathbb{C}_\mu(G\ltimes X)$ satisfies $\mathrm{tr}_G(L(f))=\psi_\mu(L_\mu\circ\iota(f))$ for any $f\in\mathbb{C}G$. We can confirm that this is extended on weak closures, that is, von Neumann algebras.

Proposition 3.23. *$\iota : \mathcal{N}G \to \mathcal{N}_\mu(G \ltimes X)$ is an injective $*$-homomorphism preserving states.*

Remark 3.24. Now we briefly compare groupoid algebras of a transformation groupoid coming from a compact Hausdorff space X with an action of a discrete group G in [27] and ours. Here this action is continuous (cut-and-paste continuous actions are considered in [27]). We define $\mathcal{S}_f(X)$ by a finitely additive class generated by all clopen sets of X. If we practice all arguments of this subsection by using a finitely additive class $\mathcal{S}_f(X)$ instead of $\mathcal{B}_f(X)$, then we can get a groupoid algebra in [27]. Generally our groupoid algebras are bigger than or equal to his because we have $\mathcal{B}_f(X) \supset \mathcal{S}_f(X)$. For example if X is connected, then his groupoid algebra is just the group algebra because of $\mathcal{S}_f(X) = \{\emptyset, X\}$, but ours is bigger than the group algebra if the topology of X is not trivial. Since the paper [27] mainly deals with a subspace of a space of maps between discrete sets with compact open topology as X, then $\mathcal{S}_f(X)$ is rich and his groupoid algebra has much information. Now let us consider a Lie group $\boldsymbol{G}$ and their two uniform lattices G_1 and G_2. Then we have a natural topological coupling $\boldsymbol{G}$ for them. Hence it is natural to consider transformation groupoids $G_1 \ltimes (\boldsymbol{G}/G_2)$ and $(G_1\backslash\boldsymbol{G}) \rtimes G_2$. In the case our groupoid algebras of them can be useful (his groupoid algebras of them are just group algebras). We remark that he use another algebra for a group action in [25, Chapter 4] (also see [27, Theorem 6.11])

4. L^2-invariants of cocompact étale groupoids and proofs of main theorems

In this section we give definitions of L^2-invariant of cocompact étale groupoids and prove main theorems Theorem 1.2 (Theorem 4.5), Theorem 1.3 (Theorem 4.7) and Theorem 1.1 (Theorem 4.8).

4.1. L^2-invariants of discrete groups.

We mention definitions of L^2-invariants of discrete groups in Section 1. We only give a definition and a lemma to connect this subsection to the next.

Definition 4.1. Let G be a discrete group and k be a non-negative integer. Then we define the k-th L^2-homology of G, its Novikov-Shubin part, flat part, non-measurable part and L^2-Betti part by

$$\begin{aligned} H_k(G, \mathcal{N}G) &:= \operatorname{Tor}_k^{\mathbb{Z}G}(\mathbb{Z}, \mathcal{N}G), \\ tH_k(G, \mathcal{N}G) &:= t(H_k(G, \mathcal{N}G)), \\ pH_k(G, \mathcal{N}G) &:= p(H_k(G, \mathcal{N}G)), \\ qH_k(G, \mathcal{N}G) &:= q(H_k(G, \mathcal{N}G)) \end{aligned}$$

and

$$PH_k(G, \mathcal{N}G) := P(H_k(G, \mathcal{N}G)),$$

respectively.

Let G be a discrete group. Then the k-th L^2-Betti number $b_k^{(2)}(G)$ is equal to $\dim_{\mathcal{N}G}(P_u(H_k(G,\mathcal{N}G)))$ (refer to [14] and Remark 2.49). Hence it is equal to $\dim_{\mathcal{N}G}(PH_k(G,\mathcal{N}G)))$ by Remark 2.49. Also the $(k+1)$-st Novikov-Shubin invariant of G is equal to $\alpha_{\mathcal{N}G}(t_u(H_k(G,\mathcal{N}G)))$ (refer to [17] and Remark 2.49). Hence it is equal to $\alpha_{\mathcal{N}G}(tH_k(G,\mathcal{N}G)))$ by Remark 2.49. Thus we have the following by using Remark 2.49.

Lemma 4.2. *Let G be a discrete group and k be a non-negative integer. Then we have the following:*
(1) $\alpha_{k+1}(G) = \alpha_{\mathcal{N}G}(tH_k(G,\mathcal{N}G))$ *and also that the $(k+1)$-st Novikov-Shubin invariant of G is trivial, that is, $\alpha_{k+1}(G) = \infty^+$ if and only if the Novikov-Shubin part of the k-th L^2-homology of G is trivial, that is, $tH_k(G,\mathcal{N}G) = 0$;*
(2) $b_k^{(2)}(G) = \dim_{\mathcal{N}G}(PH_k(G,\mathcal{N}G))$ *and also that the k-th L^2-Betti number of G is trivial, that is, $b_k^{(2)}(G) = 0$ if and only if the L^2-Betti part of the k-th L^2-homology of G is trivial, that is, $PH_k(G,\mathcal{N}G) = 0$.*

4.2. **L^2-invariants of cocompact étale groupoids.** In this subsection we give definitions of L^2-invariant of cocompact étale groupoids and prove Theorem 4.5 (Theorem 1.2).

Definition 4.3. Let $\mathcal{G}$ be a cocompact étale groupoid and k be a non-negative integer. When we take $(X,\mu) \in TP(\mathcal{G})$, then we define the k-th L^2-homology of $(\mathcal{G},X,\mu)$, its Novikov-Shubin part, flat part, non-measurable part and L^2-Betti part by

$$\begin{aligned}
H_k((\mathcal{G},X),\mathcal{N}_\mu(\mathcal{G},X)) &:= \mathrm{Tor}_k^{\mathbb{Z}(\mathcal{G},X)}(B_f(X,\mathbb{Z}),\mathcal{N}_\mu(\mathcal{G},X)),\\
tH_k((\mathcal{G},X),\mathcal{N}_\mu(\mathcal{G},X)) &:= t(H_k((\mathcal{G},X),\mathcal{N}_\mu(\mathcal{G},X))),\\
pH_k((\mathcal{G},X),\mathcal{N}_\mu(\mathcal{G},X)) &:= p(H_k((\mathcal{G},X),\mathcal{N}_\mu(\mathcal{G},X))),\\
qH_k((\mathcal{G},X),\mathcal{N}_\mu(\mathcal{G},X)) &:= q(H_k((\mathcal{G},X),\mathcal{N}_\mu(\mathcal{G},X)))
\end{aligned}$$

and

$$PH_k((\mathcal{G},X),\mathcal{N}_\mu(\mathcal{G},X)) := P(H_k((\mathcal{G},X),\mathcal{N}_\mu(\mathcal{G},X))),$$

respectively. Also when we take $X \in T(\mathcal{G})$, then we call

$$\begin{gathered}
\{H_k((\mathcal{G},X),\mathcal{N}_\mu(\mathcal{G},X))\}_{(X,\mu)\in P(\mathcal{G},X)},\\
\{tH_k((\mathcal{G},X),\mathcal{N}_\mu(\mathcal{G},X))\}_{(X,\mu)\in P(\mathcal{G},X)},\\
\{pH_k((\mathcal{G},X),\mathcal{N}_\mu(\mathcal{G},X))\}_{(X,\mu)\in P(\mathcal{G},X)},\\
\{qH_k((\mathcal{G},X),\mathcal{N}_\mu(\mathcal{G},X))\}_{(X,\mu)\in P(\mathcal{G},X)}
\end{gathered}$$

and

$$\{PH_k((\mathcal{G},X),\mathcal{N}_\mu(\mathcal{G},X))\}_{(X,\mu)\in P(\mathcal{G},X)}$$

the k-th L^2-homology of $(\mathcal{G}, X)$, its Novikov-Shubin part, flat part, non-measurable part and L^2-Betti part, respectively. Moreover we call respectively

$$\{H_k((\mathcal{G}, X), \mathcal{N}_\mu(\mathcal{G}, X))\}_{(X,\mu)\in TP(\mathcal{G})},$$
$$\{tH_k((\mathcal{G}, X), \mathcal{N}_\mu(\mathcal{G}, X))\}_{(X,\mu)\in TP(\mathcal{G})},$$
$$\{pH_k((\mathcal{G}, X), \mathcal{N}_\mu(\mathcal{G}, X))\}_{(X,\mu)\in pP(\mathcal{G})},$$
$$\{qH_k((\mathcal{G}, X), \mathcal{N}_\mu(\mathcal{G}, X))\}_{(X,\mu)\in TP(\mathcal{G})}$$

and

$$\{PH_k((\mathcal{G}, X), \mathcal{N}_\mu(\mathcal{G}, X))\}_{(X,\mu)\in TP(\mathcal{G})}$$

the k-th L^2-homology of $\mathcal{G}$, its Novikov-Shubin part, flat part, non-measurable part and L^2-Betti part.

Proposition 4.4. *Let $\mathcal{G}$ be a cocompact étale groupoid with $X \in T(\mathcal{G})$ and k be a non-negative integer. Then we have the following:*
(1) *The k-th L^2-homology of $\mathcal{G}$ is trivial if and only if the k-th L^2-homology of $(\mathcal{G}, X)$ is trivial;*
(2) *The Novikov-Shubin part of the k-th L^2-homology of $\mathcal{G}$ is trivial if and only if the Novikov-Shubin part of the k-th L^2-homology of $(\mathcal{G}, X)$ is trivial;*
(3) *The flat part of the k-th L^2-homology of $\mathcal{G}$ is trivial if and only if the flat part of the k-th L^2-homology of $(\mathcal{G}, X)$ is trivial;*
(4) *The flat part of the k-th L^2-homology of $\mathcal{G}$ is locally projective if and only if the flat part of the k-th L^2-homology of $(\mathcal{G}, X)$ is locally projective;*
(5) *The L^2-Betti part of the k-th L^2-homology of $\mathcal{G}$ is trivial if and only if the L^2-Betti part of the k-th L^2-homology of $(\mathcal{G}, X)$ is trivial;*
(6) *The non-measurable part of the k-th L^2-homology of $\mathcal{G}$ is trivial if and only if the non-measurable part of the k-th L^2-homology of $(\mathcal{G}, X)$ is trivial.*

Proof. When we take any $(Y, \mu_Y) \in TP(\mathcal{G})$ and we define $Z := X \cup Y$, then we have quasi-invariant probability measures $\mu_Z \in P(\mathcal{G}, Z)$ and $\mu_X \in P(\mathcal{G}, X)$ compatible to (Y, μ_Y) (Remark 3.4). We have

$$\begin{aligned}
&H_k((\mathcal{G}, X), \mathcal{N}_{\mu_X}(\mathcal{G}, X)) \otimes_{\mathcal{N}_{\mu_X}(\mathcal{G},X)} \chi_X \mathcal{N}_{\mu_Z}(\mathcal{G}, Z) \\
&= \mathrm{Tor}_k^{\mathbb{Z}(\mathcal{G},X)}(B_f(X, \mathbb{Z}), \mathcal{N}_{\mu_X}(\mathcal{G}, X)) \otimes_{\mathcal{N}_{\mu_X}(\mathcal{G},X)} \chi_X \mathcal{N}_{\mu_Z}(\mathcal{G}, Z) \\
&\cong \mathrm{Tor}_k^{\mathbb{Z}(\mathcal{G},X)}(B_f(X, \mathbb{Z}), \mathcal{N}_{\mu_X}(\mathcal{G}, X) \otimes_{\mathcal{N}_{\mu_X}(\mathcal{G},X)} \chi_X \mathcal{N}_{\mu_Z}(\mathcal{G}, Z)) \\
&\cong \mathrm{Tor}_k^{\mathbb{Z}(\mathcal{G},X)}(B_f(X, \mathbb{Z}), \mathcal{N}_{\mu_Z}(\mathcal{G}, Z)) \\
&\cong \mathrm{Tor}_k^{\mathbb{Z}(\mathcal{G},X)}(B_f(X, \mathbb{Z}), \mathbb{Z}(\mathcal{G}, Z)\chi_X \otimes_{\mathbb{Z}(\mathcal{G},X)} \chi_X \mathcal{N}_{\mu_Z}(\mathcal{G}, Z)) \\
&\cong \mathrm{Tor}_k^{\mathbb{Z}(\mathcal{G},Z)}(B_f(X, \mathbb{Z}) \otimes_{\mathbb{Z}(\mathcal{G},X)} \chi_X \mathbb{Z}(\mathcal{G}, Z), \mathcal{N}_{\mu_Z}(\mathcal{G}, Z)) \\
&\cong \mathrm{Tor}_k^{\mathbb{Z}(\mathcal{G},Z)}(B_f(Z, \mathbb{Z}), \mathcal{N}_{\mu_Z}(\mathcal{G}, Z)) \\
&= H_k((\mathcal{G}, Z), \mathcal{N}_{\mu_Z}(\mathcal{G}, Z))
\end{aligned} \tag{1}$$

by Lemma 3.17, Lemma 3.15 and Theorem 2.32. Also we have

$$H_k((\mathcal{G},Y),\mathcal{N}_{\mu_Y}(\mathcal{G},Y))\otimes_{\mathcal{N}_{\mu_Y}(\mathcal{G},Y)}\chi_Y\mathcal{N}_{\mu_Z}(\mathcal{G},Z)\cong H_k((\mathcal{G},Z),\mathcal{N}_{\mu_Z}(\mathcal{G},Z)) \tag{2}$$

by the same argument. Then Lemma 3.17 and Theorem 2.32 follow (1); Lemma 3.17, Theorem 2.32 and Proposition 2.33 follow (2), (3) and (4); Lemma 3.17, Theorem 2.32 and Proposition 2.34 follow (5); and also Lemma 3.17, Theorem 2.32 and Corollary 2.35 follow (6). □

Theorem 4.5. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two cocompact étale groupoids which are Morita equivalent and k be a non-negative integer. Then we have the following:*
(1) *The k-th L^2-homology of $\mathcal{G}_1$ is trivial if and only if the k-th L^2-homology of $\mathcal{G}_2$ is trivial;*
(2) *The Novikov-Shubin part of the k-th L^2-homology of $\mathcal{G}_1$ is trivial if and only if the Novikov-Shubin part of the k-th L^2-homology of $\mathcal{G}_2$ is trivial;*
(3) *The flat part of the k-th L^2-homology of $\mathcal{G}_1$ is trivial if and only if the flat part of the k-th L^2-homology of $\mathcal{G}_2$ is trivial;*
(4) *The flat part of the k-th L^2-homology of $\mathcal{G}_1$ is locally projective if and only if the flat part of the k-th L^2-homology of $\mathcal{G}_2$ is locally projective;*
(5) *The L^2-Betti part of the k-th L^2-homology of $\mathcal{G}_1$ is trivial if and only if the L^2-Betti part of the k-th L^2-homology of $\mathcal{G}_2$ is trivial;*
(6) *The non-measurable part of the k-th L^2-homology of $\mathcal{G}_1$ is trivial if and only if the non-measurable part of the k-th L^2-homology of $\mathcal{G}_2$ is trivial.*

Proof. By Corollary 3.8, we have a cocompact étale groupoid $\mathcal{G}$ with a relatively compact open complete transversal X, a Morita equivalence surjection $p_1:\mathcal{G}\to\mathcal{G}_1$ such that $p_1(X)\supset\overline{X_1}$ and a Morita equivalence surjection $p_2:\mathcal{G}\to\mathcal{G}_2$ such that $p_2(X)\supset\overline{X_2}$. Also we have mutually compatible measures $\mu_1\in P(\mathcal{G}_1,X_1)$, $\mu_2\in P(\mathcal{G}_2,X_2)$ and $\mu\in P(\mathcal{G},X)$ even if which of them is firstly given (Remark 3.9). Moreover there exists $A_i\in\mathcal{B}_f(X)$ for each $i=1,2$ in Theorem 3.16. Then we have

$$\begin{aligned}
&H_k((\mathcal{G}_i,X_i),\mathcal{N}_{\mu_i}(\mathcal{G}_i,X_i))\otimes_{\mathcal{N}_{\mu_i}(\mathcal{G}_i,X_i)}\chi_{A_i}\mathcal{N}_\mu(\mathcal{G},X)\\
&=\mathrm{Tor}_k^{\mathbb{Z}(\mathcal{G}_i,X_i)}(B_f(X_i,\mathbb{Z}),\mathcal{N}_{\mu_i}(\mathcal{G}_i,X_i))\otimes_{\mathcal{N}_{\mu_i}(\mathcal{G}_i,X_i)}\chi_{A_i}\mathcal{N}_\mu(\mathcal{G},X)\\
&\cong\mathrm{Tor}_k^{\mathbb{Z}(\mathcal{G}_i,X_i)}(B_f(X_i,\mathbb{Z}),\mathcal{N}_{\mu_i}(\mathcal{G}_i,X_i)\otimes_{\mathcal{N}_{\mu_i}(\mathcal{G}_i,X_i)}\chi_{A_i}\mathcal{N}_\mu(\mathcal{G},X))\\
&\cong\mathrm{Tor}_k^{\mathbb{Z}(\mathcal{G}_i,X_i)}(B_f(X_i,\mathbb{Z}),\chi_{A_i}\mathcal{N}_\mu(\mathcal{G},X))\\
&\cong\mathrm{Tor}_k^{\mathbb{Z}(\mathcal{G}_i,X_i)}(B_f(X_i,\mathbb{Z}),\chi_{A_i}\mathbb{Z}(\mathcal{G},X)\otimes_{\mathbb{Z}(\mathcal{G},X)}\mathcal{N}_\mu(\mathcal{G},X))\\
&\cong\mathrm{Tor}_k^{\mathbb{Z}(\mathcal{G},X)}(B_f(X_i,\mathbb{Z})\otimes_{\mathbb{Z}(\mathcal{G}_i,X_i)}\chi_{A_i}\mathbb{Z}(\mathcal{G},X),\mathcal{N}_\mu(\mathcal{G},X))\\
&\cong\mathrm{Tor}_k^{\mathbb{Z}(\mathcal{G},X)}(B_f(X,\mathbb{Z}),\mathcal{N}_\mu(\mathcal{G},X))\\
&=H_k((\mathcal{G},X),\mathcal{N}_\mu(\mathcal{G},X)).
\end{aligned} \tag{3}$$

for each $i=1,2$ by Theorem 3.18, Theorem 3.16 and Theorem 2.32. Then Theorem 3.18 and Theorem 2.32 follow (1); Theorem 3.18, Theorem 2.32 and Proposition 2.33 follow (2), (3) and (4); Theorem 3.18, Theorem 2.32 and Proposition 2.34 follow (5); and also Theorem 3.18, Theorem 2.32 and Corollary 2.35 follow (6). □

Definition 4.6. Let $\mathcal{G}$ be a cocompact étale groupoid and k be a non-negative integer. When we have a relatively compact open complete transversal X and an invariant probability measure μ on X, then we define the $(k+1)$-st Novikov-Shubin invariant by

$$\alpha_{k+1}(\mathcal{G}, X, \mu) := \alpha_{\mathcal{N}_\mu(\mathcal{G},X)}(H_k((\mathcal{G}, X), \mathcal{N}_\mu(\mathcal{G}, X)))$$

and the k-th L^2-Betti number of $(\mathcal{G}, X, \mu)$ by

$$b_k^{(2)}(\mathcal{G}, X, \mu) := \dim_{\mathcal{N}_\mu(\mathcal{G},X)}(H_k((\mathcal{G}, X), \mathcal{N}_\mu(\mathcal{G}, X))).$$

If we fix the above $(X, \mu) \in TP(\mathcal{G})$, then we can consider $(Z, \mu_Z) \in TP(\mathcal{G})$ compatible to (X, μ) (Remark 3.4). Since μ_Z is also invariant, then we can define $\alpha_{k+1}(\mathcal{G}, Z, \mu_Z)$ and $b_k^{(2)}(\mathcal{G}, Z, \mu_Z)$. Then we have $\alpha_{k+1}(\mathcal{G}, Z, \mu_Z) = \alpha_{k+1}(\mathcal{G}, X, \mu)$ and also there exists a positive constant C independent of k such that $b_k^{(2)}(\mathcal{G}, Z, \mu_Z) = Cb_k^{(2)}(\mathcal{G}, X, \mu)$ by using Equation 3, [27, Theorem 6.9, Corollary 6.10] and [26, Theorem 2.9, Theorem 2.11]. For two Morita equivalent cocompact étale groupoids, if one has an invariant probability measure, then the other has an invariant probability measure compatible to the former (Remark 3.9). Hence their Novikov-Shubin invariants coincide and their L^2-Betti numbers proportionally coincide by the same argument of the above.

4.3. **L^2-invariants of discrete groups and transformation groupoids.** In this subsection we prove Theorem 4.7 (Theorem 1.3).

Theorem 4.7. *Let G be a discrete group and k be a non-negative integer. Let (G, X) be an action on a compact Hausdorff space and μ be any element of $P(G \ltimes X, X)$. Then we have the following:*
(1) *The k-th L^2-homology of G is trivial if and only if the k-th L^2-homology of $(G \ltimes X, X, \mu)$ is trivial;*
(2) *The $(k+1)$-st Novikov-Shubin invariant of G is trivial if and only if the Novikov-Shubin part of the k-th L^2-homology of $(G \ltimes X, X, \mu)$ is trivial;*
(3) *The flat part of the k-th L^2-homology of G is trivial if and only if the flat part of the k-th L^2-homology of $(\mathcal{G}, X)$ is trivial;*
(4) *The non-measurable part of the k-th L^2-homology of G is trivial if and only if the flat part of the k-th L^2-homology of $(G \ltimes X, X, \mu)$ is locally projective;*
(5) *The k-th L^2-Betti number and the non-measurable part of the k-th L^2-homology of G are trivial if and only if the L^2-Betti part and the non-measurable part of the k-th L^2-homology of $(G \ltimes X, X, \mu)$ are trivial.*

Proof. We have

$$
\begin{aligned}
&H_k(G,\mathcal{N}G)\otimes_{\mathcal{N}G}\mathcal{N}_\mu(G\ltimes X)\\
&=\mathrm{Tor}_k^{\mathbb{Z}G}(\mathbb{Z},\mathcal{N}G)\otimes_{\mathcal{N}G}\mathcal{N}_\mu(G\ltimes X)\\
&\cong\mathrm{Tor}_k^{\mathbb{Z}G}(\mathbb{Z},\mathcal{N}G\otimes_{\mathcal{N}G}\mathcal{N}_\mu(G\ltimes X))\\
&\cong\mathrm{Tor}_k^{\mathbb{Z}G}(\mathbb{Z},\mathcal{N}_\mu(G\ltimes X))\\
&\cong\mathrm{Tor}_k^{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}(G\ltimes X)\otimes_{\mathbb{Z}(G\ltimes X)}\mathcal{N}_\mu(G\ltimes X))\\
&\cong\mathrm{Tor}_k^{\mathbb{Z}(G\ltimes X)}(\mathbb{Z}\otimes_{\mathbb{Z}G}\mathbb{Z}(G\ltimes X),\mathcal{N}_\mu(G\ltimes X))\\
&\cong\mathrm{Tor}_k^{\mathbb{Z}(G\ltimes X)}(B_f(X,\mathbb{Z}),\mathcal{N}_\mu(G\ltimes X))\\
&=H_k((G\ltimes X,X),\mathcal{N}_\mu(G\ltimes X))
\end{aligned}
\tag{4}
$$

by Proposition 3.23, Theorem 2.38, Lemma 3.22 and Lemma 3.21. Then Proposition 3.23 and Theorem 2.38 follow (1); Proposition 3.23, Theorem 2.38 and Theorem 2.41 follow (2) and (3); Proposition 3.23, Theorem 2.38, Proposition 2.40 and Corollary 2.47 follow (4); and also (3) and Lemma 2.20 follow (5). □

4.4. **L^2-invariants of discrete groups under coarse equivalence.** In this subsection we prove Theorem 4.8 (Theorem 1.1). 'A mild condition' for a discrete group G and a non-negative integer k in Section 1 means that the non-measurable part of the k-th L^2-homology of G is trivial.

Theorem 4.8. *Let G_1 and G_2 be two discrete groups which are coarsely equivalent and k be a non-negative integer. Then we have the following:*
(1) *The k-th L^2-homology of G_1 is trivial if and only if the k-th L^2-homology of G_2 is trivial;*
(2) *The $(k+1)$-st Novikov-Shubin invariant of G_1 is trivial if and only if the $(k+1)$-st Novikov-Shubin invariant of G_2 is trivial;*
(3) *The flat part of the k-th L^2-homology of G_1 is trivial if and only if the flat part of the k-th L^2-homology of G_2 is trivial;*
(4) *The non-measurable part of the k-th L^2-homology of G_1 is trivial if and only if the non-measurable part of the k-th L^2-homology of G_2 is trivial;*
(5) *The k-th L^2-Betti number and the non-measurable part of the k-th L^2-homology of G_1 are trivial if and only if the k-th L^2-Betti number and the non-measurable part of the k-th L^2-homology of G_2 are trivial.*

Proof. Let two discrete groups G_1 and G_2 be coarsely equivalent. When we take a topological coupling Ω of G_1 and G_2, then we have an étale G_1-surjection $p_1:\Omega\to X_1:=\Omega/G_2$ and an étale G_2-surjection $p_2:\Omega\to X_2:=\Omega/G_1$. When we define three transformation groupoids $\mathcal{G}_1:=G_1\ltimes X_1$, $\mathcal{G}_2:=G_2\ltimes X_2$ and $\mathcal{G}:=(G_1\times G_2)\ltimes\Omega$, then we have a Morita equivalence surjection $p_i:\mathcal{G}\to\mathcal{G}_i$ for each $i=1,2$. By Corollary 3.8, we can take a relatively compact open subset X of Ω such that $p_1|_X$ and $p_2|_X$ are still surjective because X_1 and X_2 are compact. Moreover we take $A_1,A_2\in\mathcal{B}_f(X)$ in Theorem 3.16. Also we fix compatible quasi-invariant probability measures μ_1 on

X_1, μ_2 on X_2 and μ on X. We have

$$
\begin{aligned}
&H_k(G_i, \mathcal{N}G_i) \otimes_{\mathcal{N}G_i} \chi_{A_i}\mathcal{N}_\mu(\mathcal{G}, X) \\
\cong\ & H_k(G_i, \mathcal{N}G_i) \otimes_{\mathcal{N}G_i} \mathcal{N}_{\mu_i}(\mathcal{G}_i, X_i) \otimes_{\mathcal{N}_{\mu_i}(\mathcal{G}_i, X_i)} \chi_{A_i}\mathcal{N}_\mu(\mathcal{G}, X) \\
\cong\ & H_k((\mathcal{G}_i, X_i), \mathcal{N}(\mathcal{G}_i, X_i)) \otimes_{\mathcal{N}_{\mu_i}(\mathcal{G}_i, X_i)} \chi_{A_i}\mathcal{N}_\mu(\mathcal{G}, X) \\
\cong\ & H_k((\mathcal{G}, X), \mathcal{N}_\mu(\mathcal{G}, X))
\end{aligned}
\tag{5}
$$

for each $i = 1, 2$ by Theorem 4.5 and Theorem 4.7. Hence Theorem 4.5 and Theorem 4.7 complete this proof. □

Lemma 4.9. *Let G be a discrete group. If there exists a projective resolution P_* of the right $\mathbb{C}G$-module $\mathbb{C}$ such that P_{k+1} is finitely generated, then $qH_k(G, \mathcal{N}G) = 0$.*

Proof. Since P_{k+1} is finitely generated, then $H_k(G, \mathcal{N}G)$ is finitely related. Hence we have $qH_k(G, \mathcal{N}G) = 0$ by Lemma 2.26. □

Corollary 4.10. *Let G_1 and G_2 be two discrete groups which are coarsely equivalent and k be a non-negative integer. If there exists a projective resolution P_* of the right $\mathbb{C}G_1$-module $\mathbb{C}$ such that P_{k+1} is finitely generated, then the k-th L^2-Betti number of G_1 are trivial if and only if the k-th L^2-Betti number of G_2 are trivial.*

Appendix A. Extensions of finite von Neumann algebras

In this appendix (which is not used except for Section 1), a stronger claim than Theorem 2.38 is proved under an assumption, which is always satisfied when we consider only finite von Neumann algebras. Let $\iota : \mathcal{A} \hookrightarrow \mathcal{B}$ be an injective $*$-homomorphism preserving states, where $(\mathcal{A}, \psi_\mathcal{A})$ and $(\mathcal{B}, \psi_\mathcal{B})$ are two von Neumann algebras with faithful normal states in this appendix. Moreover we assume that there exists an $\mathcal{A}$-surjection $E : \mathcal{B} \to \mathcal{A}$ when we regard $\mathcal{B}$ as a right $\mathcal{A}$-module by using ι such that $E \circ \iota = id_\mathcal{A}$, $E(|x|) = |E(|x|)|$ and $\psi_\mathcal{A} \circ E(|x|) = \psi_\mathcal{B}(|x|)$ for any $x \in \mathcal{B}$ in this appendix. We remark that $E^* : \mathcal{B} \ni x \mapsto (E(x^*))^* \in \mathcal{A}$ is an $\mathcal{A}$-surjection when we regard $\mathcal{B}$ as a left $\mathcal{A}$-module by using ι such that $E^* \circ \iota = id_\mathcal{A}$, $E^*(|x|) = |E^*(|x|)|$ and $\psi_\mathcal{A} \circ E^*(|x|) = \psi_\mathcal{B}(|x|)$ for any $x \in \mathcal{B}$. This additional assumption is satisfied if ι has a conditional expectation. We note that if $\mathcal{A}$ and $\mathcal{B}$ are two von Neumann algebras with finite traces and $\iota : \mathcal{A} \hookrightarrow \mathcal{B}$ is an injective $*$-homomorphism preserving traces, then it always has a conditional expectation.

Theorem A.1. *$\mathcal{B}$ is a faithfully locally projective right (resp. left) $\mathcal{A}$-module when $\mathcal{B}$ is regarded as a right (resp. left) $\mathcal{A}$-module by using ι.*

Proof. We consider the case that $\mathcal{B}$ is regarded as a right $\mathcal{A}$-module by using ι. When we fix $x \in \mathcal{B} \setminus \{0\}$ and the polar decomposition $x = u_x|x|$, then we have an $\mathcal{A}$-homomorphism $L_{(u_x)^*} : \mathcal{B} \ni b \mapsto (u_x)^* b \in \mathcal{B}$. We have $\psi_\mathcal{A} \circ E \circ L_{(u_x)^*}(x) = \psi_\mathcal{A} \circ E(|x|) = \psi_\mathcal{B}(|x|) \neq 0$ because $\psi_\mathcal{B}$ is faithful and we have $|x| \neq 0$. Hence we have $E \circ L_{(u_x)^*}(x) \neq 0$. Thus for any $x \in \mathcal{B} \setminus \{0\}$, we have an $\mathcal{A}$-homomorphism $E \circ L_{(u_x)^*} : \mathcal{B} \to \mathcal{A}$ such that $E \circ L_{(u_x)^*}(x) \neq 0$. In particular any finitely generated submodule M of $\mathcal{B}$ as a left $\mathcal{A}$-module is projective because of $PM = M$ by the above. Hence $\mathcal{B}$ is locally projective as a left $\mathcal{A}$-module. In particular it is flat. Because we have $\mathcal{B} = \iota(\mathcal{A}) \oplus \mathrm{Ker}E \cong \mathcal{A} \oplus \mathrm{Ker}E$ as right $\mathcal{A}$-modules, $\mathcal{B}$ is faithfully flat as a right $\mathcal{A}$-module. By using E^* instead of E, we can deal with the case that $\mathcal{B}$ is regarded as a left $\mathcal{A}$-module by using ι, too. □

Lemma A.2. *If M is a locally projective right $\mathcal{B}$-module, then M is locally projective as a right $\mathcal{A}$-module by using ι.*

Proof. If K is finitely generated submodule of M, then K can be regarded as a directed factor of a finitely generated free right $\mathcal{B}$-module $\mathcal{B}^n$. Hence we have $K \subset \mathcal{B}^n$. $\mathcal{B}^n$ is locally projective as a right $\mathcal{A}$-module by Theorem A.1. Thus K is locally projective as a right $\mathcal{A}$-module. Hence M is locally projective as a right $\mathcal{A}$-module. □

Theorem A.3. *Two short exact sequences:*

$$0 \to T(M \otimes_{\mathcal{A}} \mathcal{B}) \to M \otimes_{\mathcal{A}} \mathcal{B} \to P(M \otimes_{\mathcal{A}} \mathcal{B}) \to 0$$

and

$$0 \to (TM) \otimes_{\mathcal{A}} \mathcal{B} \to M \otimes_{\mathcal{A}} \mathcal{B} \to (PM) \otimes_{\mathcal{A}} \mathcal{B} \to 0$$

are $\mathcal{B}$-isomorphic for any right $\mathcal{A}$-module M.

Proof. We prove $TM \otimes_{\mathcal{A}} \mathcal{B} \subset T(M \otimes_{\mathcal{A}} \mathcal{B})$. Let P be a locally projective $\mathcal{B}$-module and $f : M \otimes_{\mathcal{A}} \mathcal{B} \to P$ be a $\mathcal{B}$-homomorphism such that $f|_{TM\otimes_{\mathcal{A}}\mathcal{B}} : TM \otimes_{\mathcal{A}} \mathcal{B} \to P$ is non-trivial. Since P can be regarded as a locally projective right $\mathcal{A}$-module by Lemma A.2 and $f : TM \subset M \to M \otimes_{\mathcal{A}} \mathcal{B} \to P$ is non-trivial, it contradict to that TM has no non-trivial locally projective quotients. On the other hand we can confirm $TM \otimes_{\mathcal{A}} \mathcal{B} \supset T(M \otimes_{\mathcal{A}} \mathcal{B})$ because $Q \otimes_{\mathcal{A}} \mathcal{B}$ is a locally projective right $\mathcal{B}$-module for any locally projective right $\mathcal{A}$-module Q by Proposition 2.40. □

Theorem A.4. *Two short exact sequences:*

$$0 \to T_u(M \otimes_{\mathcal{A}} \mathcal{B}) \to M \otimes_{\mathcal{A}} \mathcal{B} \to P_u(M \otimes_{\mathcal{A}} \mathcal{B}) \to 0$$

and

$$0 \to (T_uM) \otimes_{\mathcal{A}} \mathcal{B} \to M \otimes_{\mathcal{A}} \mathcal{B} \to (P_uM) \otimes_{\mathcal{A}} \mathcal{B} \to 0$$

are $\mathcal{B}$-isomorphic for any right $\mathcal{A}$-module M.

Proof. If we take a finitely generated submodule $M_i \subset T_uM$ such that $M_i = TM_i$, then we have $TM_i \otimes_{\mathcal{A}} \mathcal{B} = T(M_i \otimes_{\mathcal{A}} \mathcal{B})$ by Theorem A.3. Hence we have $T_uM \otimes_{\mathcal{A}} \mathcal{B} \subset T_u(M \otimes_{\mathcal{A}} \mathcal{B})$. We prove $T_uM \otimes_{\mathcal{A}} \mathcal{B} \supset T_u(M \otimes_{\mathcal{A}} \mathcal{B})$. When we take a finitely generated submodule $M_i' \subset T_u(M \otimes_{\mathcal{A}} \mathcal{B})$ such that $M_i' = TM_i'$, then there exist a finitely generated submodule $M_i \subset M$ such that $M_i \otimes_{\mathcal{A}} \mathcal{B} \supset M_i'$. Hence we have $TM_i \otimes_{\mathcal{A}} \mathcal{B} = T(M_i \otimes_{\mathcal{A}} \mathcal{B}) \supset TM_i' = M_i'$ by Theorem A.3. Thus we have $T_uM \otimes_{\mathcal{A}} \mathcal{B} \supset T_u(M \otimes_{\mathcal{A}} \mathcal{B})$. □

Appendix B. Torsion modules over infinite von Neumann algebras by Makoto Yamashita

Note that when M is a finite von Neumann algebra, a finitely generated M-module has a nonzero projective submodule if and only if it has a nontrivial projective quotient (Lemma 2.42). Indeed, this property characterizes finite algebras among general von Neumann algebras.

Consider the separable infinite dimensional Hilbert space $\ell_2\mathbb{N}$ spanned by orthonormal unit vectors $(\delta_n)_{n\in\mathbb{N}}$. For each positive integer n, let e_n denote the rank 1 projection on the subspace $\mathbb{C}\delta_n$ of $\ell_2\mathbb{N}$ and put $p_n = \sum_{j=1}^{n} e_j$, $E_n = B(\ell_2\mathbb{N})p_n$. The left ideals $(E_n)_{n\in\mathbb{N}}$ of $B(\ell_2\mathbb{N})$ form an increasing sequence and its (set theoretic) union E_∞ is again a left ideal of $B(\ell_2\mathbb{N})$. The closure of E_∞ with respect to the operator norm topology is the ideal $K(\ell_2\mathbb{N})$ of the compact operators on $\ell_2\mathbb{N}$, and the one with respect to the σ-weak topology is the entire algebra $B(\ell_2\mathbb{N})$.

Proposition B.1. *The singly generated left $B(\ell_2\mathbb{N})$-module $F = B(\ell_2\mathbb{N})/E_\infty$ contains a non-zero projective module P_0, but does not admit any non-zero projective quotient.*

Proof. Let q be the rank one projection on the subspace spanned by the vector $\sum_{n\in\mathbb{N}} n^{-1}\delta_n$. We claim that the intersection of $B(\ell_2\mathbb{N})q$ and E_∞ is trivial. Indeed, for any element $x \in E_\infty$, there exists a positive integer k satisfying $x(1 - p_k) = 0$. On the other hand, we have $q(1 - p_n)q = \lambda_n q \neq 0$ for any integer n. Thus any nonzero element $y \in B(\ell_2\mathbb{N})q$ satisfies $y(1 - p_n) \neq 0$. Consequently the image of $B(\ell_2\mathbb{N})q$ in F is isomorphic to $B(\ell_2\mathbb{N})q$, which is projective over $B(\ell_2\mathbb{N})$.

For the latter part of the assertion, it is enough to show that any left $B(\ell_2\mathbb{N})$-module homomorphism ϕ of $B(\ell_2\mathbb{N})$ into another free module $B(\ell_2\mathbb{N})^{\oplus m}$ satisfying $\phi(E_\infty) = 0$ is actually trivial $\phi = 0$ on $B(\ell_2\mathbb{N})$. Given such ϕ, there exist m elements $T_1, \ldots, T_m$ of $B(\ell_2\mathbb{N})$ satisfying $\phi(x) = (xT_1, \ldots, xT_m)$. The condition $\phi(E_\infty) = 0$ implies that $e_n T_j = 0$ for any n and j, which is enough to conclude $T_j = 0$ for each j. This means $\phi = 0$. □

Proposition B.2. *Let M be a von Neumann algebra with a properly infinite component. There exists an M-module which contains a non-zero projective M-submodule yet admits no non-zero projective quotient.*

Proof. There exists a sequence of projections $(q_n)_{n\in\mathbb{N}}$ in M which are mutually orthogonal and equivalent. A choice of partial isometries $(v_{m,n})_{(m,n)\in\mathbb{N}^2}$ satisfying

$$v_{m,n}^* = v_{n,m}, \qquad v_{m,n}v_{m',n'} = v_{m+m',n+n'}, \qquad v_{m,m} = q_m$$

determines an inclusion of $B(\ell_2\mathbb{N})$ into M as a closed subalgebra in the σ-weak topology such that $e_n \in B(\ell_2\mathbb{N})$ is identified to q_n for each $n \in \mathbb{N}$.

Since M is flat as a right $B(\ell_2\mathbb{N})$-module, the extension of scalar from $B(\ell_2\mathbb{N})$ to M induces an inclusion $M \otimes_{B(\ell_2\mathbb{N})} P_0$ of a non-zero projective module into $M \otimes_{B(\ell_2\mathbb{N})} F$.

Recall that $M \otimes_{B(\ell_2\mathbb{N})} F$ is a quotient of $M1_{B(\ell_2\mathbb{N})}$ by $M \otimes_{B(\ell_2\mathbb{N})} E_\infty$. Any M-module homomorphism ϕ of $M1_{B(\ell_2\mathbb{N})}$ into a free module $M^{\oplus m}$ can be (uniquely) described by elements $y_1, \ldots, y_m$ of M satisfying $1_{B(\ell_2\mathbb{N})}y_j = y_j$ by the relation $\phi(x) = (xy_1, \ldots, xy_m)$. If ϕ vanishes on the submodule $M \otimes_{B(\ell_2\mathbb{N})} E_\infty$, it follows that $q_n y_j = 0$ for any n and j. Since $\sum_{n=1}^{N} q_n y_j$ converges to y_n as $N \to \infty$ in σ-weak topology, one has $y_j = 0$ for $j = 1, \ldots, m$ which shows $\phi = 0$. This shows that any M-module homomorphism from $M \otimes_{B(\ell_2\mathbb{N})} F$ into a free M-module is trivial, which proves our assertion. □

Appendix C. Type FP_n of groups under coarse equivalence

We give another proof to [1, Corollary 9 (1)].

Definition C.1. Let A be a unital ring. Let M be a right A-module and n be a non-negative integer. M belongs to FP_n if there exists its projective resolution P_* such that P_k is finitely generated for any $k \leq n$.

Lemma C.2. *Let A and B be unital rings. Suppose that $A \subset B$ are faithfully flat, where $A \subset B$ preserves their unit elements. Then M is a right A-module which belongs to FP_n if and only if $M \otimes_A B$ is a right B-module which belongs to FP_n.*

Proof. The part of 'only if' is trivial. Hence we will prove the part of 'if'. Let M be a right A-module. If $M \otimes_A B$ is finitely generated, then we have its finitely generating set $\{\sum_i m_i^j \otimes s_i^j \in M \otimes_A B\}_j$. Then we has a finitely generated A-submodule $N \subset M$ generated by $\{m_i^j\}_{i,j}$. Then we have $M \otimes_A B = N \otimes_A B$. Thus $M = N$ because $A \subset B$ is faithfully flat. We assume that for $n \geq 0$, M is a right A-module which belongs to FP_n if $M \otimes_A B$ is a right B-module which belongs to FP_n. If $M \otimes_A B$ is a right B-module which belongs to FP_{n+1} (in particular FP_n), then we can take a projective sub-resolution D_* of of M of the length n such that D_k is finitely generated for any $k \leq n$. We need to prove that the n-th differential d_n has a finitely generated kernel. Now $D'_* := D_* \otimes_A B$ is a projective sub-resolution of the length n for $M \otimes_A B$. Then $\mathrm{Ker} d'_n$ is finitely generated because $M \otimes_A B$ is type FP_{n+1}. Also $\mathrm{Ker} d'_n = \mathrm{Ker} d_n \otimes_A B$ because $A \subset B$ is flat. Here we take a finitely generating set $\{\sum_i k_i^j \otimes s_i^j \in \mathrm{Ker} d_n \otimes_A B\}_j$ for a right B-module $\mathrm{Ker} d_n \otimes_A B$. Then we has a finitely generated A-submodule $N \subset \mathrm{Ker} d_n$ generated by $\{k_i^j\}_{i,j}$. Then we have $\mathrm{Ker} d_n \otimes_A B = N \otimes_A B$. Thus $\mathrm{Ker} d_n = N$ because $A \subset B$ is faithfully flat. □

Let R be a unital ring. By using Lemmas C.2, 3.21 and 3.22, we have the following.

Proposition C.3. *Let a discrete group G act on a compact space X. Then the right $R(G \ltimes X)$-module $\mathcal{B}_f(X, R)$ belongs to FP_n if and only if the right RG-module R belongs to FP_n.*

[1, Corollary 9 (1)] claims that the following in the case of $R = \mathbb{Z}$ and $n > 1$.

Theorem C.4. *Let G_1 be coarsely equivalent to G_2 and n be a non-negative integer. The right RG_1-module R belongs to FP_n if and only if the right RG_2-module R belongs to FP_n. Also G_1 is finitely generated if and only if G_2 is finitely generated.*

Proof. When we take a topological coupling Ω of G_1 and G_2, then $G_1 \ltimes X_1$ and $G_2 \ltimes X_2$ are Morita equivalent, where $X_1 := G_2 \backslash \Omega$ and $X_2 := G_1 \backslash \Omega$. Here X_1 and X_2 are compact. By Proposition C.3, $R(G_i \ltimes X_i)$-module $\mathcal{B}_f(X_i, R)$ belongs to FP_n if and only if the right RG_i-module R belongs to FP_n for each $i = 1, 2$. Also $R(G_1 \ltimes X_1)$-module $\mathcal{B}_f(X_1, R)$ belongs to FP_n if and only if $R(G_2 \ltimes X_2)$-module $\mathcal{B}_f(X_2, R)$ belongs to FP_n by Theorems 3.16 and 2.32. It is well-known that the right $\mathbb{Z}G_i$-module $\mathbb{Z}$ belongs to FP_1 if and only if G_i is finitely generated for each $i = 1, 2$ (refer to [4]). □

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