On Schur multiple zeta functions

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Today's

Aim:

- To introduce a generalization of multiple zeta functions (what we call a Schur multiple zeta function) from the viewpoint of symmetric functions.
- To study its combinatorial and arithmetic properties.
- (Applications ?)

Main references:

[1] Maki Nakasuji, Ouamporn Phuksuwan and Y, On Schur multiple zeta functions: A combinatoric generalization of multiple zeta functions, *Adv. Math.*, **333** (2018), 570 – 619.

[2] Henrik Bachmann and Y, Checkerboard style Schur multiple zeta values and odd single zeta values, *Math. Z.*, **290** (2018) No. 3-4, 1173 – 1197.

Riemann zeta function

Definition (Riemann zeta function)

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots .$$

- The series converges absolutely for $\mathrm{Re}(s) > 1$.
- Analytic continuation, functional equation, Riemann hypothesis, $\ldots$.
- Special value at $2k \in 2\mathbb{N}$:

$$\zeta(2k) = (-1)^{k-1} \frac{2^{2k} B_{2k}}{2(2k)!} \pi^{2k} \in \mathbb{Q}\pi^{2k},$$

where $B_n \in \mathbb{Q}$ is the n-th Bernoulli number.

$$\zeta(2) = \frac{1}{6}\pi^2, \quad \zeta(4) = \frac{1}{90}\pi^4, \quad \zeta(6) = \frac{1}{945}\pi^6, \quad \ldots .$$

Multiple zeta functions & Multiple zeta-star functions

Definition (Multiple zeta and zeta-star functions (of Euler-Zagier type))

$$\zeta(s_1,\ldots,s_n) = \sum_{0<m_1<m_2<\cdots<m_n} \frac{1}{m_1^{s_1} m_2^{s_2} \cdots m_n^{s_n}},$$

$$\zeta^\star(s_1,\ldots,s_n) = \sum_{0<m_1\le m_2\le\cdots\le m_n} \frac{1}{m_1^{s_1} m_2^{s_2} \cdots m_n^{s_n}}.$$

- These converge absolutely for $\mathrm{Re}(s_1),\ldots,\mathrm{Re}(s_{n-1}) \geq 1$ and $\mathrm{Re}(s_n) > 1$.
- $n = 2$: Euler, General n: Zagier (1990's).
- Connection to:
  quantum field theory, knot theory, mixed Tate motive, quantum groups, $\ldots$.

Symmetric functions

Definition (Symmetric functions)

$\boldsymbol{x} = (x_1, x_2, \dots)$: (infinitely many) variables.

$f = f(\boldsymbol{x}) \in \mathbb{Q}[\![x_1, x_2, \dots]\!]$: symmetric function of degree n

$\overset{\text{def}}{\iff}$ f : invariant under permutation and homogeneous of degree n.

- Elementary symmetric function : $e_n = e_n(\boldsymbol{x}) = \displaystyle\sum_{0<m_1<\cdots<m_n} x_{m_1} \cdots x_{m_n}$
- Complete symmetric function : $h_n = h_n(\boldsymbol{x}) = \displaystyle\sum_{0<m_1\le\cdots\le m_n} x_{m_1} \cdots x_{m_n}$
- Power-sum symmetric function : $p_n = p_n(\boldsymbol{x}) = \displaystyle\sum_{m=1}^{\infty} x_m^n.$

Fact

$\{p_\mu = p_{\mu_1} \cdots p_{\mu_l} \mid \mu = (\mu_1, \dots, \mu_l) \vdash n\}$ forms a base of the $\mathbb{Q}$-vector space of symmetric functions of degree n.

$$e_n = \sum_{\mu \vdash n} \frac{(-1)^{n-\ell(\mu)}}{z_\mu} p_\mu, \qquad h_n = \sum_{\mu \vdash n} \frac{1}{z_\mu} p_\mu,$$

where $z_\mu = \prod_{i \ge 1} i^{m_i(\mu)} m_i(\mu)! \in \mathbb{N}$ when we write $\mu = (1^{m_1(\mu)}, 2^{m_2(\mu)}, \dots)$.

Specialization of symmetric functions

For $s \in \mathbb{C}$, let

$$\boldsymbol{m}^{-s} = \left(1^{-s}, 2^{-s}, 3^{-s}, \ldots\right).$$

Then, the specialization $\boldsymbol{x} = \boldsymbol{m}^{-s}$ yields the multiple zeta functions;

$$e_n(\boldsymbol{m}^{-s}) = \sum_{0<m_1<\cdots<m_n} \frac{1}{m_1^s \cdots m_n^s} = \zeta(s, \ldots, s) = \zeta(\{s\}^n),$$

$$h_n(\boldsymbol{m}^{-s}) = \sum_{0<m_1\leq\cdots\leq m_n} \frac{1}{m_1^s \cdots m_n^s} = \zeta^{\star}(s, \ldots, s) = \zeta^{\star}(\{s\}^n),$$

$$p_n(\boldsymbol{m}^{-s}) = \sum_{m=1}^{\infty} \frac{1}{m^{ns}} = \zeta(ns).$$

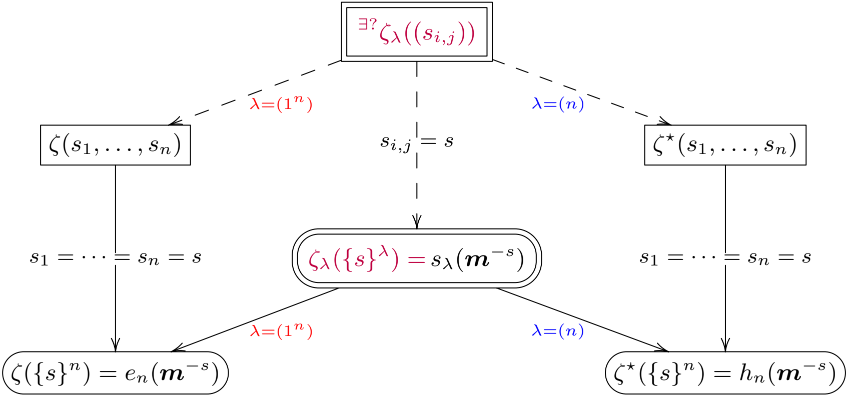
Remark. General $\zeta(s_1, \ldots, s_n)$ and $\zeta^{\star}(s_1, \ldots, s_n)$ may not be obtained as some other specializations of e_n and h_n, respectively.

What we want

- We have the Schur functions s_λ, which is a symmetric function satisfying

$$s_{(1^n)} = e_n \quad (\leftrightarrow \zeta), \qquad s_{(n)} = h_n \quad (\leftrightarrow \zeta^\star)$$

- Can we construct a zeta function ζ_λ satisfying the following diagram ?



Answer

Schur function (tableau expression)

$$s_{\begin{array}{|c|c|c|c|}\hline \end{array}}(\boldsymbol{x})$$

$$s_{\scriptsize\begin{array}{cccc} & & & \square \\ & \square & \square & \square \\ \square & \square & \square & \end{array}}(\boldsymbol{x}) = \sum_{\substack{\begin{array}{ccccc} & & & & m_{1,4} \\ & & & & \wedge \\ & & m_{2,2} \le m_{2,3} & \le & m_{2,4} \\ & & \wedge \quad\quad \wedge & & \\ m_{3,1} & \le & m_{3,2} \le m_{3,3} & & \end{array}}} x_{m_{1,4}} \cdot x_{m_{2,2}} x_{m_{2,3}} x_{m_{2,4}} \cdot x_{m_{3,1}} x_{m_{3,2}} x_{m_{3,3}}$$

Schur multiple zeta function

$$\zeta\left(\begin{array}{cccc} & & & \boxed{a} \\ & \boxed{b} & \boxed{c} & \boxed{d} \\ \boxed{e} & \boxed{f} & \boxed{g} & \end{array}\right) = \sum_{\substack{\begin{array}{ccccc} & & & & m_{1,4} \\ & & & & \wedge \\ & & m_{2,2} \le m_{2,3} & \le & m_{2,4} \\ & & \wedge \quad\quad \wedge & & \\ m_{3,1} & \le & m_{3,2} \le m_{3,3} & & \end{array}}} \frac{1}{m_{1,4}^{a} \cdot m_{2,2}^{b} m_{2,3}^{c} m_{2,4}^{d} \cdot m_{3,1}^{e} m_{3,2}^{f} m_{3,3}^{g}}$$

Notations

$\lambda = (\lambda_1, \dots, \lambda_l), \mu = (\mu_1, \dots, \mu_l)$: partitions such that $\mu \subset \lambda$.

- $D(\lambda/\mu) = \{(i,j) \in \mathbb{Z}^2 \mid \mu_i < j \leq \lambda_i \ (1 \leq i \leq l)\}$: Young diagram of λ/μ.
- $T(X, \lambda/\mu) = \{(t_{i,j})_{(i,j) \in D(\lambda/\mu)} \mid t_{i,j} \in X\}$
  : set of all Young tableaux of shape λ/μ with entries in X.
- $\mathrm{SSYT}(\lambda/\mu) = \{(m_{i,j}) \in T(\mathbb{N}, \lambda/\mu) \mid m_{i,j} \leq m_{i,j+1},\ m_{i,j} < m_{i+1,j}\}$
  : set of all semi-standard Young tableaux of shape λ/μ.

Example. $\lambda/\mu = (4,4,3)/(3,1)$.

$$M = (m_{i,j}) = \begin{array}{|c|c|c|c|} \hline & & & m_{1,4} \\ \hline & m_{2,2} & m_{2,3} & m_{2,4} \\ \hline m_{3,1} & m_{3,2} & m_{3,3} & \\ \hline \end{array} \in \mathrm{SSYT}((4,4,3)/(3,1))$$

if and only if

$$\begin{array}{ccccccc} & & & & & & m_{1,4} \\ & & & & & & \wedge \\ & & m_{2,2} & \leq & m_{2,3} & \leq & m_{2,4} \\ & & \wedge & & \wedge & & \\ m_{3,1} & \leq & m_{3,2} & \leq & m_{3,3} & & \end{array}$$

Remark. We simply write $\lambda/\mu = \lambda$ if $\mu = \emptyset$.

(Skew) Schur functions

Definition ((Skew) Schur functions)

The (skew) Schur function $s_{\lambda/\mu}$ of shape λ/μ is defined by

$$s_{\lambda/\mu} = s_{\lambda/\mu}(\boldsymbol{x}) = \sum_{M=(m_{i,j})\in \mathrm{SSYT}(\lambda/\mu)} \prod_{(i,j)\in D(\lambda/\mu)} x_{m_{i,j}}.$$

- $s_{\lambda/\mu}$ is a symmetric function of degree $|\lambda/\mu|$.

Example. $\lambda/\mu = (3,2)/(1)$.

$$\mathrm{SSYT}(\lambda/\mu) = \left\{ \begin{array}{ccc} & 1 & 1 \\ 1 & 2 & \end{array}, \begin{array}{ccc} & 1 & 2 \\ 1 & 2 & \end{array}, \begin{array}{ccc} & 1 & 3 \\ 1 & 2 & \end{array}, \dots, \begin{array}{ccc} & 1 & 1 \\ 1 & 3 & \end{array}, \begin{array}{ccc} & 1 & 2 \\ 1 & 3 & \end{array}, \begin{array}{ccc} & 1 & 3 \\ 1 & 3 & \end{array}, \dots \right.$$
$$\left. \begin{array}{ccc} & 1 & 1 \\ 2 & 2 & \end{array}, \begin{array}{ccc} & 1 & 2 \\ 2 & 2 & \end{array}, \begin{array}{ccc} & 1 & 3 \\ 2 & 2 & \end{array}, \dots, \begin{array}{ccc} & 1 & 1 \\ 2 & 3 & \end{array}, \begin{array}{ccc} & 1 & 2 \\ 2 & 3 & \end{array}, \begin{array}{ccc} & 1 & 3 \\ 2 & 3 & \end{array}, \dots \right\}$$

$$s_{\lambda/\mu} = x_1x_1x_1x_2 + x_1x_2x_1x_2 + x_1x_3x_1x_2 + \cdots + x_1x_1x_1x_3 + x_1x_2x_1x_3 + x_1x_3x_1x_3 + \cdots$$
$$+ x_1x_1x_2x_2 + x_1x_2x_2x_2 + x_1x_3x_2x_2 + \cdots + x_1x_1x_2x_3 + x_1x_2x_2x_3 + x_1x_3x_2x_3 + \cdots .$$

(Skew) Schur multiple zeta functions

Definition ((Skew) Schur multiple zeta functions)

For $\boldsymbol{s} = (s_{i,j}) \in T(\mathbb{C}, \lambda/\mu)$, the (skew) Schur multiple zeta function $\zeta(\boldsymbol{s}) = \zeta_{\lambda/\mu}(\boldsymbol{s})$ of shape λ/μ is defined by

$$\zeta(\boldsymbol{s}) = \zeta_{\lambda/\mu}(\boldsymbol{s}) = \sum_{M=(m_{i,j})\in \mathrm{SSYT}(\lambda/\mu)} \prod_{(i,j)\in D(\lambda/\mu)} \frac{1}{m_{i,j}^{s_{i,j}}}.$$

Example. $\lambda/\mu = (3,2)/(1)$.

$$\mathrm{SSYT}(\lambda/\mu) = \left\{ \begin{array}{ccc} & 1 & 1 \\ 1 & 2 & \end{array}, \begin{array}{ccc} & 1 & 2 \\ 1 & 2 & \end{array}, \begin{array}{ccc} & 1 & 3 \\ 1 & 2 & \end{array}, \ldots, \begin{array}{ccc} & 1 & 1 \\ 1 & 3 & \end{array}, \begin{array}{ccc} & 1 & 2 \\ 1 & 3 & \end{array}, \begin{array}{ccc} & 1 & 3 \\ 1 & 3 & \end{array}, \ldots \right.$$

$$\left. \begin{array}{ccc} & 1 & 1 \\ 2 & 2 & \end{array}, \begin{array}{ccc} & 1 & 2 \\ 2 & 2 & \end{array}, \begin{array}{ccc} & 1 & 3 \\ 2 & 2 & \end{array}, \ldots, \begin{array}{ccc} & 1 & 1 \\ 2 & 3 & \end{array}, \begin{array}{ccc} & 1 & 2 \\ 2 & 3 & \end{array}, \begin{array}{ccc} & 1 & 3 \\ 2 & 3 & \end{array}, \ldots \right\}$$

$$\zeta\left(\begin{array}{ccc} & a & b \\ c & d & \end{array} \right) = \frac{1}{1^a 1^b 1^c 2^d} + \frac{1}{1^a 2^b 1^c 2^d} + \frac{1}{1^a 3^b 1^c 2^d} + \cdots + \frac{1}{1^a 1^b 1^c 3^d} + \frac{1}{1^a 2^b 1^c 3^d} + \frac{1}{1^a 3^b 1^c 3^d} +$$

$$+ \frac{1}{1^a 1^b 2^c 2^d} + \frac{1}{1^a 2^b 2^c 2^d} + \frac{1}{1^a 3^b 2^c 2^d} + \cdots + \frac{1}{1^a 1^b 2^c 3^d} + \frac{1}{1^a 2^b 2^c 3^d} + \frac{1}{1^a 3^b 2^c 3^d}$$

Remarks

- $\zeta_{\lambda/\mu}$ is actually a generalization of both ζ and $\zeta^\star$;

$$\zeta_{(1^n)}\left(\begin{array}{|c|}\hline s_1 \\ \hline \vdots \\ \hline s_n \\ \hline\end{array}\right) = \sum_{m_1 > \cdots > m_n} \frac{1}{m_1^{s_1}\cdots m_n^{s_n}} = \zeta(s_1,\ldots,s_n),$$

$$\zeta_{(n)}\left(\begin{array}{|c|c|c|}\hline s_1 & \cdots & s_n \\ \hline\end{array}\right) = \sum_{m_1 \leq \cdots \leq m_n} \frac{1}{m_1^{s_1}\cdots m_n^{s_n}} = \zeta^\star(s_1,\ldots,s_n).$$

- $\zeta_{\lambda/\mu}(\boldsymbol{s})$ converges absolutely for

$$\boldsymbol{s} \in W_{\lambda/\mu} = \left\{ \boldsymbol{s} = (s_{i,j}) \in T(\mathbb{C},\lambda/\mu) \;\middle|\; \begin{array}{l} \mathrm{Re}(s_{i,j}) \geq 1 \text{ for } (i,j) \notin C(\lambda/\mu) \\ \mathrm{Re}(s_{i,j}) > 1 \text{ for } (i,j) \in C(\lambda/\mu) \end{array} \right\},$$

where $C(\lambda/\mu)$ is the set of all corners of $D(\lambda/\mu)$.

- $\zeta_{\lambda/\mu}(\boldsymbol{s})$ can be expressed as a linear combination of ζ or $\zeta^\star$:

$$\zeta\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & \\ \cline{1-1}\end{array}\right) = \sum_{\substack{k \leq m \\ \wedge \\ n}} \frac{1}{k^a m^b n^c} = \zeta(a,b,c)+\zeta(a,c,b)+\zeta(a,b+c)+\zeta(a+b,c)$$

$$= \zeta^\star(a,b,c)+\zeta^\star(a,c,b)-\zeta^\star(a,b+c)-\zeta^\star(a+c,b).$$

$\zeta(\{2k\}^{\lambda/\mu})$

- Define $\{s\}^{\lambda/\mu} = (s_{i,j}) \in T(\mathbb{C}, \lambda/\mu)$ by $s_{i,j} = s$ for all $(i,j) \in D(\lambda/\mu)$.

Proposition

For any $k \in \mathbb{N}$,

$$\zeta(\{2k\}^{\lambda/\mu}) \in \mathbb{Q}\pi^{2k|\lambda/\mu|}.$$

In particular, $\zeta(\{2k\}^n), \zeta^\star(\{2k\}^n) \in \mathbb{Q}\pi^{2kn}$.

Example (with a proof). $\lambda/\mu = (3,2)/(1)$.

$$s_{(3,2)/(1)} = -\frac{1}{4}p_4 - \frac{1}{3}p_3p_1 + \frac{1}{8}p_2^2 + \frac{1}{4}p_2p_1^2 + \frac{5}{24}p_1^4$$

$$\downarrow \ \boldsymbol{x} = \boldsymbol{m}^{-2k} = (1^{-2k}, 2^{-2k}, 3^{-2k}, \ldots)$$

$$\zeta\left(\begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 2k & 2k \\ \hline 2k & 2k & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}\right) = -\frac{1}{4}\zeta(8k) - \frac{1}{3}\zeta(6k)\zeta(2k) + \frac{1}{8}\zeta(4k)^2 + \frac{1}{4}\zeta(4k)\zeta(2k)^2 + \frac{5}{24}\zeta(2k)^4 \in \mathbb{Q}\pi^{8k}.$$

$$\zeta\left(\begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 2 & 2 \\ \hline 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}\right) = \frac{61}{362880}\pi^8,$$

$$\zeta\left(\begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 4 & 4 \\ \hline 4 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}\right) = \frac{667}{631547280000}\pi^{16},$$

$$\zeta\left(\begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 6 & 6 \\ \hline 6 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}\right) = \frac{9077644}{4326847970651925468 75}\pi^{24}, \quad \ldots.$$

What is a meaning of the rational part ?

A combinatorial application

λ/μ : partitions such that $n = |\lambda/\mu|$.

- $(m_{i,j}) \in \mathrm{SSYT}(\lambda/\mu)$: standard Young tableau of shape λ/μ
  $\overset{\text{def}}{\Longleftrightarrow} \{m_{i,j} \mid (i,j) \in D(\lambda/\mu)\} = \{1, 2, \ldots, n\}$
- $f^{\lambda/\mu} = \#\{$standard Young tableaux of shape $\lambda/\mu\}$.
  - $f^{\lambda/\mu}$ is equal to a dimension of (reducible) representation of $\mathfrak{S}_n$.

Example.

$f^{(3,2)/(1)} = 5$;

$$\begin{array}{ccc} & \boxed{1} & \boxed{2} \\ \boxed{3} & \boxed{4} & \end{array},\ \begin{array}{ccc} & \boxed{1} & \boxed{3} \\ \boxed{2} & \boxed{4} & \end{array},\ \begin{array}{ccc} & \boxed{1} & \boxed{4} \\ \boxed{2} & \boxed{3} & \end{array},\ \begin{array}{ccc} & \boxed{2} & \boxed{3} \\ \boxed{1} & \boxed{4} & \end{array},\ \begin{array}{ccc} & \boxed{2} & \boxed{4} \\ \boxed{1} & \boxed{3} & \end{array}.$$

$f^{(3,2,1)/(1)} = 16$;

$$\begin{array}{ccc} & \boxed{1} & \boxed{2} \\ \boxed{3} & \boxed{4} & \\ \boxed{5} & & \end{array},\ \begin{array}{ccc} & \boxed{1} & \boxed{2} \\ \boxed{3} & \boxed{5} & \\ \boxed{4} & & \end{array},\ \begin{array}{ccc} & \boxed{1} & \boxed{3} \\ \boxed{2} & \boxed{4} & \\ \boxed{5} & & \end{array},\ \begin{array}{ccc} & \boxed{1} & \boxed{3} \\ \boxed{2} & \boxed{5} & \\ \boxed{4} & & \end{array},\ \begin{array}{ccc} & \boxed{1} & \boxed{4} \\ \boxed{2} & \boxed{3} & \\ \boxed{5} & & \end{array},\ \begin{array}{ccc} & \boxed{1} & \boxed{4} \\ \boxed{2} & \boxed{5} & \\ \boxed{3} & & \end{array},\ \begin{array}{ccc} & \boxed{1} & \boxed{5} \\ \boxed{2} & \boxed{3} & \\ \boxed{4} & & \end{array},\ \begin{array}{ccc} & \boxed{1} & \boxed{5} \\ \boxed{2} & \boxed{4} & \\ \boxed{3} & & \end{array},$$

$$\begin{array}{ccc} & \boxed{2} & \boxed{3} \\ \boxed{1} & \boxed{4} & \\ \boxed{5} & & \end{array},\ \begin{array}{ccc} & \boxed{2} & \boxed{3} \\ \boxed{1} & \boxed{5} & \\ \boxed{4} & & \end{array},\ \begin{array}{ccc} & \boxed{2} & \boxed{4} \\ \boxed{1} & \boxed{3} & \\ \boxed{5} & & \end{array},\ \begin{array}{ccc} & \boxed{2} & \boxed{4} \\ \boxed{1} & \boxed{5} & \\ \boxed{3} & & \end{array},\ \begin{array}{ccc} & \boxed{2} & \boxed{5} \\ \boxed{1} & \boxed{3} & \\ \boxed{4} & & \end{array},\ \begin{array}{ccc} & \boxed{2} & \boxed{5} \\ \boxed{1} & \boxed{4} & \\ \boxed{3} & & \end{array},\ \begin{array}{ccc} & \boxed{3} & \boxed{4} \\ \boxed{1} & \boxed{5} & \\ \boxed{2} & & \end{array},\ \begin{array}{ccc} & \boxed{3} & \boxed{5} \\ \boxed{1} & \boxed{4} & \\ \boxed{2} & & \end{array}.$$

A combinatorial application

Theorem (Stanley '11)

For any λ/μ *satisfying* $\lambda_1 \le m$ *and* $|\lambda/\mu| = n$, *we have*

$$\zeta(\{2\}^{\lambda/\mu}) = \frac{f^{(2\lambda'+\gamma_m+\delta_m)/(2\mu'+\delta_m)}}{(2n+m)!}\pi^{2n},$$

$$\zeta(\{4\}^{\lambda/\mu}) = \frac{2^{m+2n} f^{(4\lambda'+2\gamma_m+3\delta_m)/(4\mu'+3\delta_m)}}{(4n+2m)!}\pi^{4n},$$

$$\zeta(\{6\}^{\lambda/\mu}) = \frac{6^m 2^{6n} f^{(6\lambda'+3\gamma_m+5\delta_m)/(6\mu'+5\delta_m)}}{(6n+3m)!}\pi^{6n},$$

where $\gamma_m = (1^m)$ *and* $\delta_m = (m-1, m-2, \ldots, 2, 1, 0)$.

Example. $\lambda/\mu = (3,2)/(1)$, $m = \lambda_1 = 3$, $n = 4$.

$$\frac{61\pi^8}{362880} = \zeta\left(\begin{array}{|c|c|c|}\hline & 2 & 2 \\ \hline 2 & 2 & \\ \hline\end{array}\right) = \frac{f^{(7,6,3)/(4,1)}\pi^8}{11!} \longrightarrow f^{(7,6,3)/(4,1)} = 6710,$$

$$\frac{667\pi^{16}}{631547280000} = \zeta\left(\begin{array}{|c|c|c|}\hline & 4 & 4 \\ \hline 4 & 4 & \\ \hline\end{array}\right) = \frac{2^{11} f^{(16,13,6)/(10,3)}\pi^{16}}{22!} \longrightarrow f^{(16,13,6)/(10,3)} = 579637674,$$

$$\frac{9077644\pi^{24}}{43268479706519254687 5} = \zeta\left(\begin{array}{|c|c|c|}\hline & 6 & 6 \\ \hline 6 & 6 & \\ \hline\end{array}\right) = \frac{6^3 2^{24} f^{(25,20,9)/(16,5)}\pi^{24}}{33!} \longrightarrow f^{(25,20,9)/(16,5)} = 50270540048960.$$

Study on Schur multiple zeta functions

As and analogue of the Schur functions;

- Jacobi-Trudi formula
- Giambelli formula
- Dual Cauchy formula
- Pieri rule ? Littlewood-Richardson rule ?
- $\cdots$

As a generalization of the multiple zeta and zeta-star functions;

- Relation
- Explicit expression (e.g., $\zeta(\{2k\}^{\lambda/\mu})$, 1-3 formula)
- Integral representation
- Sum formula ? duality ? Ohno relation ?
- $\cdots$

Some more notations

λ, μ : partitions such that $\mu \subset \lambda$.

- $c(i,j) = j - i \in \mathbb{Z}$: content of $(i,j) \in D(\lambda/\mu)$.
- $T^{\text{diag}}(X, \lambda/\mu) = \{(t_{i,j}) \in T(X, \lambda/\mu) \,|\, t_{i,j} = t_{i',j'} \text{ if } c(i,j) = c(i',j')\}$
  - For $T = (t_{i,j}) \in T^{\text{diag}}(X, \lambda/\mu)$, we also write $T = (t_{c(i,j)}) = (t_{j-i})$.
- $W^{\text{diag}}_{\lambda/\mu} = W_{\lambda/\mu} \cap T^{\text{diag}}(\mathbb{C}, \lambda/\mu)$

$$= \left\{ (s_{i,j}) \in T^{\text{diag}}(\mathbb{C}, \lambda/\mu) \;\middle|\; \begin{array}{l} \mathrm{Re}(s_{i,j}) \geq 1 \text{ for } (i,j) \notin C(\lambda/\mu) \\ \mathrm{Re}(s_{i,j}) > 1 \text{ for } (i,j) \in C(\lambda/\mu) \end{array} \right\}.$$

Example.

s_0	s_1	s_2	s_3	$\boldsymbol{s_4}$
s_{-1}	s_0	s_1	s_2	
s_{-2}	s_{-1}	s_0		
s_{-3}	s_{-2}	s_{-1}		

$\in W^{\text{diag}}_{(5,4,3,3)}$,

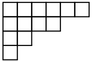
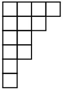
if $\mathrm{Re}(\boldsymbol{s_4}), \mathrm{Re}(s_2), \mathrm{Re}(s_{-1}) > 1$

			s_3	$\boldsymbol{s_4}$
		s_1	s_2	
s_{-2}	s_{-1}	s_0		
s_{-3}				

$\in W^{\text{diag}}_{(5,4,3,1)/(3,2)}$.

if $\mathrm{Re}(\boldsymbol{s_4}), \mathrm{Re}(s_2), \mathrm{Re}(s_0), \mathrm{Re}(s_{-3}) > 1$

Jacobi-Trudi formulas

- For $\lambda =$  $= (\lambda_1, \ldots, \lambda_p)$, put $\lambda' =$  $= (\lambda'_1, \ldots, \lambda'_{p'})$.

Theorem (Jacobi-Trudi formulas)

(1) $s_{\lambda/\mu} = \det\left[h_{\lambda_i - \mu_j - i + j}\right]_{1 \le i,j \le p}$.

(2) $s_{\lambda/\mu} = \det\left[e_{\lambda'_i - \mu'_j - i + j}\right]_{1 \le i,j \le p'}$.

Theorem (Nakasuji-Phuksuwan-Y '18)

Let $\boldsymbol{s} = (s_{i,j}) \in W^{\mathrm{diag}}_{\lambda/\mu}$ *and write* $s_{i,j} = s_{c(i,j)} (= s_{j-i})$.

(1) *Assume further that* $\mathrm{Re}(s_{i,\lambda_i}) > 1$ *for all* $1 \le i \le p$. *Then, we have*

$$\zeta_{\lambda/\mu}(\boldsymbol{s}) = \det\left[\zeta^{\star}(\underbrace{s_{\mu_j - j + 1}, s_{\mu_j - j + 2}, \ldots, s_{\mu_j - j + (\lambda_i - \mu_j - i + j)}}_{\lambda_i - \mu_j - i + j})\right]_{1 \le i,j \le p}.$$

(2) *Assume further that* $\mathrm{Re}(s_{\lambda'_i,i}) > 1$ *for all* $1 \le i \le p'$. *Then, we have*

$$\zeta_{\lambda/\mu}(\boldsymbol{s}) = \det\left[\zeta(\underbrace{s_{-\mu'_j + j - 1}, s_{-\mu'_j + j - 2}, \ldots, s_{-\mu'_j + j - (\lambda'_i - \mu'_j - i + j)}}_{\lambda'_i - \mu'_j - i + j})\right]_{1 \le i,j \le p'}.$$

Example

Example. $\lambda = (2,2,1)$.

(1) For $\mathrm{Re}(s_{-1}) \geq 1$ and $\mathrm{Re}(s_1), \mathrm{Re}(s_0), \mathrm{Re}(s_{-2}) > 1$,

$$\zeta\left(\begin{array}{|c|c|}\hline s_0 & s_1 \\ \hline s_{-1} & s_0 \\ \hline s_{-2} & \\ \cline{1-1}\end{array}\right) = \begin{vmatrix} \zeta^\star(s_0, s_1) & \zeta^\star(s_{-1}, s_0, s_1) & \zeta^\star(s_{-2}, s_{-1}, s_0, s_1) \\ \zeta^\star(s_0) & \zeta^\star(s_{-1}, s_0) & \zeta^\star(s_{-2}, s_{-1}, s_0) \\ 0 & 1 & \zeta^\star(s_{-2}) \end{vmatrix}.$$

$$= \zeta^\star(s_0, s_1)\zeta^\star(s_{-1}, s_0)\zeta^\star(s_{-2}) + \zeta^\star(s_0)\zeta^\star(s_{-2}, s_{-1}, s_0, s_1)$$
$$- \zeta^\star(s_0)\zeta^\star(s_{-1}, s_0, s_1)\zeta^\star(s_{-2}) - \zeta^\star(s_0, s_1)\zeta^\star(s_{-2}, s_{-1}, s_0).$$

(2) For $\mathrm{Re}(s_1), \mathrm{Re}(s_{-1}) \geq 1$ and $\mathrm{Re}(s_0), \mathrm{Re}(s_{-2}) > 1$,

$$\zeta\left(\begin{array}{|c|c|}\hline s_0 & s_1 \\ \hline s_{-1} & s_0 \\ \hline s_{-2} & \\ \cline{1-1}\end{array}\right) = \begin{vmatrix} \zeta(s_0, s_{-1}, s_{-2}) & \zeta(s_1, s_0, s_{-1}, s_{-2}) \\ \zeta(s_0) & \zeta(s_1, s_0) \end{vmatrix}$$

$$= \zeta(s_0, s_{-1}, s_{-2})\zeta(s_1, s_0) - \zeta(s_1, s_0, s_{-1}, s_{-2})\zeta(s_0).$$

Combining these equations, we get an algebraic relation among ζ and $\zeta^\star$!

Proof and Remark

(Proof) We have two proofs for our Jacobi-Trudi formulas;

If $\boldsymbol{s} \in W^{\text{diag}}_{\lambda/\mu}$, then

(1) we can establish an analogue of Lindström-Gessel-Viennot Lemma.

(2) we see that $\zeta_{\lambda/\mu}(\boldsymbol{s})$ is obtained as a specialization of Macdonald's ninth variation of Schur functions studied by Nakagawa, Noumi, Shirakawa and Yamada (2001).

□

What happens if $\boldsymbol{s} \notin W^{\text{diag}}_{\lambda/\mu}$?

$\rightarrow$ In general, we have **"error terms"** (which vanish when $\boldsymbol{s} \in W^{\text{diag}}_{\lambda/\mu}$).

Example. $\lambda = (2,2)$.

$$\zeta\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}\right) = \begin{vmatrix} \zeta^\star(a,b) & \zeta^\star(c,d,b) \\ \zeta^\star(a) & \zeta^\star(c,d) \end{vmatrix}$$

$$+\zeta^\star(c,\underline{d},b,\underline{a}) - \zeta^\star(c,\underline{a},b,\underline{d}) + \zeta^\star(c,\underline{a},b+\underline{d}) - \zeta^\star(c,\underline{d},b+\underline{a}),$$

$$\zeta\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}\right) = \begin{vmatrix} \zeta(a,c) & \zeta(b,d,c) \\ \zeta(a) & \zeta(b,d) \end{vmatrix}$$

$$+\zeta(b,\underline{d},c,\underline{a}) - \zeta(b,\underline{a},c,\underline{d}) + \zeta(b,\underline{d},c+\underline{a}) - \zeta(b,\underline{a},c+\underline{d}).$$

Giambelli formula for s_λ

$\lambda = (p_1 - 1, \dots, p_t - 1 \,|\, q_1, \dots, q_t)$: Frobenius notation of λ.

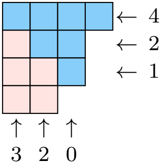
- t : number of the principal diagonal entries of λ.
- $p_i = \lambda_i - i + 1$, $q_i = \lambda'_i - i$.

Theorem (Giambelli formula)

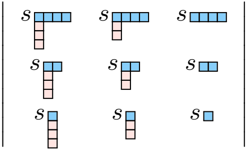
$$s_\lambda = \det \left[s_{(p_i, 1^{q_j})} \right]_{1 \le i,j \le t}.$$

Example. $\lambda = (4,3,3,2) = (3,1,0 \,|\, 3,2,0)$

$\to t = 3$, $(p_1, p_2, p_3) = (4,2,1)$, $(q_1, q_2, q_3) = (3,2,0)$.



$$s_\lambda = \begin{vmatrix} s_{(4,1^3)} & s_{(4,1^2)} & s_{(4,1^0)} \\ s_{(2,1^3)} & s_{(2,1^2)} & s_{(2,1^0)} \\ s_{(1,1^3)} & s_{(1,1^2)} & s_{(1,1^0)} \end{vmatrix} =$$



Giambelli formula for ζ_λ

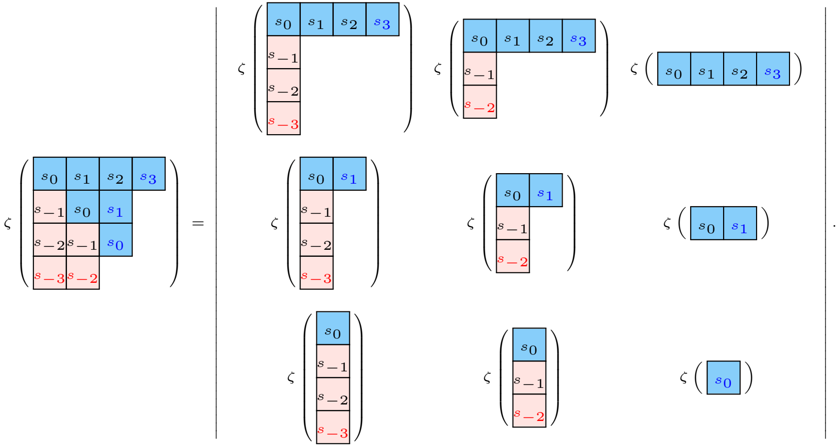
Theorem (Nakasuji-Phuksuwan-Y '18)

Let $\boldsymbol{s} = (s_{i,j}) \in W_\lambda^{\mathrm{diag}}$ *and write* $s_{i,j} = s_{c(i,j)}\ (= s_{j-i})$. *Assume that* $\mathrm{Re}(s_{p_i-1}) > 1$ *and* $\mathrm{Re}(s_{-q_i}) > 1$ *for* $1 \le i \le t$. *Then, we have*

$$\zeta_\lambda(\boldsymbol{s}) = \det\left[\zeta_{(p_i,1^{q_j})}\left(\begin{array}{|c|c|c|c|c|}\hline s_0 & s_1 & s_2 & \cdots & s_{p_i-1} \\ \hline s_{-1} & & & & \\ \cline{1-1} \vdots & & & & \\ \cline{1-1} s_{-q_j} & & & & \\ \hline \end{array}\right)\right]_{1\le i,j\le t}.$$

Example

Example. $\lambda = (4, 3, 3, 2) = (3, 1, 0 \,|\, 3, 2, 0)$.

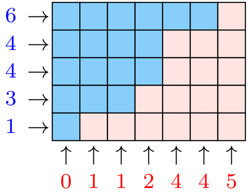


Dual Cauchy formula for s_λ

- $\lambda = (\lambda_1, \ldots, \lambda_r) \subset (s^r)$

  $\rightarrow \ \lambda^* = (r - \lambda'_s, \ldots, r - \lambda'_1) \subset (r^s)$: complementary partition of λ.

Example. $r = 5$, $s = 7$, $\lambda = (6, 4, 4, 3, 1) \ \rightarrow \ \lambda^* = (5, 4, 4, 2, 1, 1, 0)$.



Theorem (Dual Cauchy formula)

For variables $\boldsymbol{x} = (x_1, x_2, \ldots, x_r)$ *and* $\boldsymbol{y} = (y_1, y_2, \ldots, y_s)$, *we have*

$$\sum_{\lambda \subset (s^r)} s_\lambda(\boldsymbol{x}) s_{\lambda^*}(\boldsymbol{y}) = \prod_{i=1}^{r} \prod_{j=1}^{s} (x_i + y_j)$$

Dual Cauchy formula for ζ_λ

Theorem (Nakasuji-Phuksuwan-Y '18)

Let $\boldsymbol{s} = (s_{i,j}) \in W^{\mathrm{diag}}_{(s^r)}$ *and* $\boldsymbol{t} = (t_{i,j}) \in W^{\mathrm{diag}}_{(r^s)}$ *and write* $s_{i,j} = s_{c(i,j)}\ (= s_{j-i})$ *and* $t_{i,j} = t_{c(i,j)}\ (= t_{j-i})$. *Assume that* $\mathrm{Re}(s_i) > 1$ *and* $\mathrm{Re}(t_i) > 1$ *for all* $i \in \mathbb{Z}$. *Put* $\eta = r + s$. *Then, we have*

$$
\sum_{\lambda \subset (s^r)} (-1)^{|\lambda|} \zeta_\lambda(\boldsymbol{s}|_\lambda)\, \zeta_{\lambda^*}(\boldsymbol{t}|_{\lambda^*})
$$

$$
= \det \left[\begin{array}{ccccccc}
1 & \zeta^\star(s_{1-r}) & \zeta^\star(s_{1-r}, s_{2-r}) & \cdots & \zeta^\star(s_{1-r}, \ldots, s_0) & \cdots & \zeta^\star(s_{1-r}, \ldots, s_{\eta-1-r}) \\
0 & 1 & \zeta^\star(s_{2-r}) & \cdots & \zeta^\star(s_{2-r}, \ldots, s_0) & \cdots & \zeta^\star(s_{2-r}, \ldots, s_{\eta-1-r}) \\
\vdots & \ddots & \ddots & \ddots & \vdots & & \vdots \\
0 & \cdots & 0 & 1 & \zeta^\star(s_0) & \cdots & \zeta^\star(s_0, \ldots, s_{\eta-1-r}) \\
\hline
1 & \zeta^\star(t_{1-s}) & \zeta^\star(t_{1-s}, t_{2-s}) & \cdots & \zeta^\star(t_{1-s}, \ldots, t_0) & \cdots & \zeta^\star(t_{1-s}, \ldots, t_{\eta-1-s}) \\
0 & 1 & \zeta^\star(t_{2-s}) & \cdots & \zeta^\star(t_{2-s}, \ldots, t_0) & \cdots & \zeta^\star(t_{2-s}, \ldots, t_{\eta-1-s}) \\
\vdots & \ddots & \ddots & \ddots & \vdots & & \vdots \\
0 & \cdots & 0 & 1 & \zeta^\star(t_0) & \cdots & \zeta^\star(t_0, \ldots, t_{\eta-1-s})
\end{array}\right].
$$

Here, $\boldsymbol{s}|_\lambda \in W^{\mathrm{diag}}_\lambda$ *and* $\boldsymbol{t}|_{\lambda^*} \in W^{\mathrm{diag}}_{\lambda^*}$ *are the shape restriction of* $\boldsymbol{s}$ *and* $\boldsymbol{t}$ *to* λ *and* λ^**, respectively.*

Example

Example. $r = 2,\ s = 3.$

$$
\begin{aligned}
(\mathrm{LHS}) = {} & -\zeta\left(\begin{array}{|c|c|c|}\hline s_0 & s_1 & s_2 \\ \hline s_{-1} & s_0 & s_1 \\ \hline\end{array}\right) - \zeta\left(\begin{array}{|c|c|c}\hline s_0 & s_1 & \multicolumn{1}{c|}{s_2} \\ \hline s_{-1} & \multicolumn{1}{c|}{s_0} & \\ \cline{1-2}\end{array}\right)\zeta\left(\begin{array}{|c|}\hline t_0 \\ \hline\end{array}\right) + \zeta\left(\begin{array}{|c|c|c}\hline s_0 & s_1 & \multicolumn{1}{c|}{s_2} \\ \hline \multicolumn{1}{|c|}{s_{-1}} & & \\ \cline{1-1}\end{array}\right)\zeta\left(\begin{array}{|c|}\hline t_0 \\ \hline t_{-1} \\ \hline\end{array}\right) \\
& -\zeta\left(\begin{array}{|c|c|c|}\hline s_0 & s_1 & s_2 \\ \hline\end{array}\right)\zeta\left(\begin{array}{|c|}\hline t_0 \\ \hline t_{-1} \\ \hline t_{-2} \\ \hline\end{array}\right) + \zeta\left(\begin{array}{|c|c|}\hline s_0 & s_1 \\ \hline s_{-1} & s_0 \\ \hline\end{array}\right)\zeta\left(\begin{array}{|c|c|}\hline t_0 & t_1 \\ \hline\end{array}\right) - \zeta\left(\begin{array}{|c|c}\hline s_0 & \multicolumn{1}{c|}{s_1} \\ \hline \multicolumn{1}{|c|}{s_{-1}} & \\ \cline{1-1}\end{array}\right)\zeta\left(\begin{array}{|c|c}\hline t_0 & \multicolumn{1}{c|}{t_1} \\ \hline \multicolumn{1}{|c|}{t_{-1}} & \\ \cline{1-1}\end{array}\right) \\
& +\zeta\left(\begin{array}{|c|c|}\hline s_0 & s_1 \\ \hline\end{array}\right)\zeta\left(\begin{array}{|c|c}\hline t_0 & \multicolumn{1}{c|}{t_1} \\ \hline \multicolumn{1}{|c|}{t_{-1}} & \\ \cline{1-1} \multicolumn{1}{|c|}{t_{-2}} & \\ \cline{1-1}\end{array}\right) + \zeta\left(\begin{array}{|c|}\hline s_0 \\ \hline s_{-1} \\ \hline\end{array}\right)\zeta\left(\begin{array}{|c|c|}\hline t_0 & t_1 \\ \hline t_{-1} & t_0 \\ \hline\end{array}\right) - \zeta\left(\begin{array}{|c|}\hline s_0 \\ \hline\end{array}\right)\zeta\left(\begin{array}{|c|c}\hline t_0 & \multicolumn{1}{c|}{t_1} \\ \hline t_{-1} & \multicolumn{1}{c|}{t_0} \\ \hline \multicolumn{1}{|c|}{t_{-2}} & \\ \cline{1-1}\end{array}\right) + \zeta\left(\begin{array}{|c|c|}\hline t_0 & t_1 \\ \hline t_{-1} & t_0 \\ \hline t_{-2} & t_{-1} \\ \hline\end{array}\right),
\end{aligned}
$$

$$
(\mathrm{RHS}) = \left|\begin{array}{ccccc}
1 & \zeta^\star(s_{-1}) & \zeta^\star(s_{-1}, s_0) & \zeta^\star(s_{-1}, s_0, s_1) & \zeta^\star(s_{-1}, s_0, s_1, s_2) \\
0 & 1 & \zeta^\star(s_0) & \zeta^\star(s_0, s_1) & \zeta^\star(s_0, s_1, s_2) \\
\hline
1 & \zeta^\star(t_{-2}) & \zeta^\star(t_{-2}, t_{-1}) & \zeta^\star(t_{-2}, t_{-1}, t_0) & \zeta^\star(t_{-2}, t_{-1}, t_0, t_1) \\
0 & 1 & \zeta^\star(t_{-1}) & \zeta^\star(t_{-1}, t_0) & \zeta^\star(t_{-1}, t_0, t_1) \\
0 & 0 & 1 & \zeta^\star(t_0) & \zeta^\star(t_0, t_1)
\end{array}\right|.
$$

1-3 formulas for multiple zeta and zeta-star values

Theorem (1-3 formulas)

(1) *(Borwein-Bradley-Broadhurst '98, Bowman-Bradley '03)*

$$\zeta(\{1,3\}^n) = \frac{1}{4^n}\zeta(\{4\}^n) = \frac{2}{(4n+2)!}\pi^{4n} \in \mathbb{Q}\pi^{4n},$$

$$\zeta(3,\{1,3\}^n) = \sum_{k=0}^{n}\left(-\frac{1}{4}\right)^k \zeta(4k+3)\zeta(\{1,3\}^{n-k}) \in \mathbb{Q}[\pi^4,\zeta(3),\zeta(7),\ldots].$$

(2) *(Muneta '08)*

$$\zeta^\star(\{1,3\}^n) = \sum_{k=0}^{n}\zeta(\{1,3\}^k)\zeta^\star(\{4\}^{n-k}) \in \mathbb{Q}\pi^{4n},$$

$$\zeta^\star(3,\{1,3\}^n) = \sum_{k=0}^{n}\left(-\frac{1}{4}\right)^k \zeta(4k+3)\zeta^\star(\{1,3\}^{n-k}) \in \mathbb{Q}[\pi^4,\zeta(3),\zeta(7),\ldots].$$

Example.

$$\zeta(1,3,1,3) = \frac{\pi^8}{1814400}, \qquad \zeta(3,1,3,1,3) = \frac{\pi^8\zeta(3)}{1814400} - \frac{\pi^4\zeta(7)}{1440} + \frac{\zeta(11)}{16},$$

$$\zeta^\star(1,3,1,3) = \frac{13\pi^8}{113400}, \qquad \zeta^\star(3,1,3,1,3) = \frac{53\pi^8\zeta(3)}{362880} - \frac{\pi^4\zeta(7)}{288} + \frac{\zeta(11)}{16}.$$

Checkerboard style Schur multiple zeta values

$$\zeta(\{1,3\}^n)=\zeta\left(\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline \vdots\\ \hline 1\\ \hline 3\\ \hline\end{array}\right),\qquad \zeta(3,\{1,3\}^n)=\zeta\left(\begin{array}{|c|}\hline 3\\ \hline 1\\ \hline \\ \hline \vdots\\ \hline 1\\ \hline 3\\ \hline\end{array}\right),$$

$$\zeta^\star(\{1,3\}^n)=\zeta\left(\begin{array}{|c|c|c|c|c|}\hline 1&3&\cdots&1&3\\ \hline\end{array}\right),\quad \zeta^\star(3,\{1,3\}^n)=\zeta\left(\begin{array}{|c|c|c|c|c|c|}\hline 3&1&3&\cdots&1&3\\ \hline\end{array}\right).$$

Evaluate Checkerboard style Schur multiple zeta values with entries $1,3$;

$$\zeta\left(\begin{array}{cccc} 3&1&3&1\\ 1&3&1&3\\ 3&1&3&1\\ 1&3&1&3\end{array}\right),\qquad \zeta\left(\begin{array}{ccccc} &&&1&3\\ &3&1&3&\\ &1&&&\\ 1&3&&&\\ 3&&&&\end{array}\right),\qquad \zeta\left(\begin{array}{ccccc} 3&1&3&1&3\\ 1&3&1&3&\\ 3&1&3&&\\ 1&3&&&\\ 3&&&&\end{array}\right).$$

Square Ribbon Stair

Remark. The entries in the corners of λ/μ should be 3.

1-3 formulas for primitive ribbons

Theorem (Bachmann-Y '18)

$$(1)\quad \zeta\left(\begin{array}{ccc} & & 1\\ & \cdot^{\cdot^{\cdot}} & 3\\ 1 & \cdot^{\cdot^{\cdot}} & \\ 3 & & \end{array}\right) = \frac{1}{4^n}\zeta^\star(\{4\}^n), \qquad (2)\quad \zeta\left(\begin{array}{cccc} & & 1 & 3\\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ 1 & 3 & & \end{array}\right) = \sum_{k=0}^{n}\frac{1}{4^k}\zeta^\star(\{4\}^k)\zeta(\{4\}^{n-k}),$$

$$(3)\quad \zeta\left(\begin{array}{cccc} & & & 1\\ & & \cdot^{\cdot^{\cdot}} & 3\\ & 1 & \cdot^{\cdot^{\cdot}} & \\ \color{red}{1} & 3 & & \end{array}\right) = \frac{2}{4^n}\zeta(4n+1), \qquad (4)\quad \zeta\left(\begin{array}{cccc} & & 1 & 3\\ & \cdot^{\cdot^{\cdot}} & 3 & \\ 1 & \cdot^{\cdot^{\cdot}} & & \\ \color{red}{3} & & & \end{array}\right) = \frac{1}{4^n}\zeta(4n+3),$$

where n *denotes the number of* $\begin{array}{c}1\\3\end{array}$ *or* $\begin{array}{cc}1 & 3\end{array}$.

Example.

$$(1)\quad \zeta\left(\begin{array}{ccc} & & 1\\ & 1 & 3\\ 1 & 3 & \\ 3 & & \end{array}\right) = \frac{4009\pi^{12}}{217945728000}, \qquad (2)\quad \zeta\left(\begin{array}{cccc} & & 1 & 3\\ & 1 & 3 & \\ 1 & 3 & & \end{array}\right) = \frac{1247\pi^{12}}{10059033600},$$

$$(3)\quad \zeta\left(\begin{array}{cccc} & & & 1\\ & & 1 & 3\\ & 1 & 3 & \\ 1 & 3 & & \end{array}\right) = \frac{1}{32}\zeta(13), \qquad (4)\quad \zeta\left(\begin{array}{cccc} & & 1 & 3\\ & 1 & 3 & \\ 1 & 3 & & \\ 3 & & & \end{array}\right) = \frac{1}{64}\zeta(15).$$

1-3 formulas for ribbons

Theorem (Bachmann-Y '18)

Every Checkerboard style Schur multiple zeta value of ribbon type with entries $1, 3$ *is in* $\mathbb{Q}[\pi^4, \zeta(3), \zeta(5), \zeta(7), \ldots]$.

Example.

$$\zeta\left(\ytableaushort{\none\none3,\none\none1,313}\right) = \zeta\left(\ytableaushort{\none\none1,313}\right)\zeta\left(\ytableaushort{3}\right) - \zeta\left(\ytableaushort{\none\none13,313}\right)$$

$$= \left[\zeta\left(\ytableaushort{3}\right)\zeta\left(\ytableaushort{\none1,13}\right) - \zeta\left(\ytableaushort{\none1,13,3}\right)\right]\zeta\left(\ytableaushort{3}\right)$$

$$- \left[\zeta\left(\ytableaushort{3}\right)\zeta\left(\ytableaushort{\none13,13}\right) - \zeta\left(\ytableaushort{\none13,13,3}\right)\right] \quad \leftarrow \text{primitive ribbons !}$$

$$= \frac{1}{2}\zeta(3)^2\zeta(5) - \frac{7\pi^8}{129600}\zeta(3) + \frac{1}{16}\zeta(11).$$

Remark. We have used a harmonic product formula; e.g.,

$$\zeta\left(\ytableaushort{ab,cd}\right)\zeta\left(\ytableaushort{\none x,yz}\right) = \zeta\left(\ytableaushort{\none\none x,\none yz,ab,cd}\right) + \zeta\left(\ytableaushort{\none\none\none x,abyz,cd}\right).$$

1-3 formulas for stairs

Theorem (Bachmann-Y '18)

Every Checkerboard style Schur multiple zeta value of stair type with entries $1, 3$ is expressed as a Hankel determinant of a matrix whose entries are $\zeta(3), \zeta(7), \zeta(11), \ldots$.

Example.

$$\zeta\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \\ \cline{1-1}\end{array}\right) = \frac{1}{4}\left|\, \zeta(7) \,\right|, \qquad \zeta\left(\begin{array}{|c|c|c|}\hline 3 & 1 & 3 \\ \hline 1 & 3 & \\ \cline{1-2} 3 & & \\ \cline{1-1}\end{array}\right) = \frac{1}{4^2}\begin{vmatrix} \zeta(3) & \zeta(7) \\ \zeta(7) & \zeta(11) \end{vmatrix},$$

$$\zeta\left(\begin{array}{|c|c|c|c|}\hline 1 & 3 & 1 & 3 \\ \hline 3 & 1 & 3 & \\ \cline{1-3} 1 & 3 & & \\ \cline{1-2} 3 & & & \\ \cline{1-1}\end{array}\right) = \frac{1}{4^4}\begin{vmatrix} \zeta(7) & \zeta(11) \\ \zeta(11) & \zeta(15) \end{vmatrix}, \qquad \zeta\left(\begin{array}{|c|c|c|c|c|}\hline 3 & 1 & 3 & 1 & 3 \\ \hline 1 & 3 & 1 & 3 & \\ \cline{1-4} 3 & 1 & 3 & & \\ \cline{1-3} 1 & 3 & & & \\ \cline{1-2} 3 & & & & \\ \cline{1-1}\end{array}\right) = \frac{1}{4^6}\begin{vmatrix} \zeta(3) & \zeta(7) & \zeta(11) \\ \zeta(7) & \zeta(11) & \zeta(15) \\ \zeta(11) & \zeta(15) & \zeta(19) \end{vmatrix}.$$

(Proof) Use the Jacobi-Trudi formula and $\zeta\left(\begin{array}{cccc} & & 1 & 3 \\ & \iddots & 3 & \\ 1 & \iddots & & \\ {\color{red}3} & & & \end{array}\right) = \frac{1}{4^n}\zeta(4n+3)$.

Iterated integral representations of multiple zeta values

$$\zeta(k_1,\ldots,k_n)=\zeta\left(\begin{array}{|c|}\hline k_1\\ \hline \vdots \\ \hline k_n\\ \hline\end{array}\right)$$

$$=\int_{0<t_1<\cdots<t_k<1}\underbrace{\frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\cdots\frac{dt_{k_1}}{t_{k_1}}}_{k_1}\cdots\underbrace{\frac{dt_*}{1-t_*}\frac{dt_*}{t_*}\cdots\frac{dt_k}{t_k}}_{k_n}\quad(k=k_1+\cdots+k_n).$$

Example.

$$\zeta\left(\begin{array}{|c|}\hline 3\\ \hline 2\\ \hline\end{array}\right)=\int_{0<t_1<t_2<t_3<t_4<t_5<1}\underbrace{\frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}}_{3}\underbrace{\frac{dt_4}{1-t_4}\frac{dt_5}{t_5}}_{2}$$

[Change of variables $s_i=1-t_{4-i}$]

$$=\int_{0<s_1<s_2<s_3<s_4<s_5<1}\underbrace{\frac{ds_1}{1-s_1}\frac{ds_2}{s_2}}_{2}\underbrace{\frac{ds_3}{1-s_3}}_{1}\underbrace{\frac{ds_4}{1-s_4}\frac{ds_5}{s_5}}_{2}=\zeta\left(\begin{array}{|c|}\hline 2\\ \hline 1\\ \hline 2\\ \hline\end{array}\right)\quad \textbf{duality}$$

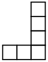
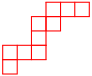
Remark. More generally, we have the **Ohno relations**;

$$\zeta\left(\begin{array}{|c|}\hline 4\\ \hline 2\\ \hline\end{array}\right)+\zeta\left(\begin{array}{|c|}\hline 3\\ \hline 3\\ \hline\end{array}\right)=\zeta\left(\begin{array}{|c|}\hline 3\\ \hline 1\\ \hline 2\\ \hline\end{array}\right)+\zeta\left(\begin{array}{|c|}\hline 2\\ \hline 2\\ \hline 2\\ \hline\end{array}\right)+\zeta\left(\begin{array}{|c|}\hline 2\\ \hline 1\\ \hline 3\\ \hline\end{array}\right),$$

$$\zeta\left(\begin{array}{|c|}\hline 5\\ \hline 2\\ \hline\end{array}\right)+\zeta\left(\begin{array}{|c|}\hline 4\\ \hline 3\\ \hline\end{array}\right)+\zeta\left(\begin{array}{|c|}\hline 3\\ \hline 4\\ \hline\end{array}\right)=\zeta\left(\begin{array}{|c|}\hline 4\\ \hline 1\\ \hline 2\\ \hline\end{array}\right)+\zeta\left(\begin{array}{|c|}\hline 3\\ \hline 2\\ \hline 2\\ \hline\end{array}\right)+\zeta\left(\begin{array}{|c|}\hline 3\\ \hline 1\\ \hline 3\\ \hline\end{array}\right)+\zeta\left(\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline 2\\ \hline\end{array}\right)+\zeta\left(\begin{array}{|c|}\hline 2\\ \hline 2\\ \hline 3\\ \hline\end{array}\right)+\zeta\left(\begin{array}{|c|}\hline 2\\ \hline 1\\ \hline 4\\ \hline\end{array}\right),$$

Iterated integral representation of $\zeta_{\lambda/\mu}$ of ribbon type

Iterated integral representation of $\zeta_{\lambda/\mu}$ is obtained for

- $\lambda/\mu =$  (i.e., $\zeta_{\lambda/\mu} = \zeta^{\star}$) [Yamamoto (to appear)]
- $\lambda/\mu =$  (anti-hook) [Kaneko-Yamamoto '18]
- $\lambda/\mu =$  (general ribbon) [Nakasuji-Phuksuwan-Y '18]
- the other cases $\cdots$ Not yet. e.g., $\lambda/\mu =$ 

Example. $\lambda/\mu = (3,3,2)/(2,1)$.

$$\zeta\left(\begin{array}{ccc} & & 1 \\ & 1 & 2 \\ 2 & 2 & \end{array}\right) = \int_{\substack{t_1<t_2<t_3>t_4<t_5<t_6>t_7<t_8 \\ 0<t_1,\dots,t_8<1}} \underbrace{\frac{dt_1}{1-t_1}}_{1} \underbrace{\frac{dt_2}{1-t_2}\frac{dt_3}{t_3}}_{2} \underbrace{\frac{dt_4}{1-t_4}}_{1} \underbrace{\frac{dt_5}{1-t_5}\frac{dt_6}{t_6}}_{2} \underbrace{\frac{dt_7}{1-t_7}\frac{dt_8}{t_8}}_{2}$$

[Change of variables $s_i = 1 - t_{9-i}$]

$$= \int_{\substack{s_1<s_2>s_3<s_4<s_5>s_6<s_7<s_8 \\ 0<s_1,\dots,s_8<1}} \underbrace{\frac{ds_1}{1-s_1}\frac{ds_2}{s_2}}_{2} \underbrace{\frac{ds_3}{1-s_3}\frac{ds_4}{s_4}\frac{ds_5}{s_5}}_{3} \underbrace{\frac{ds_6}{1-s_6}\frac{ds_7}{s_7}\frac{ds_8}{s_8}}_{3}$$

$$= \zeta\left(\begin{array}{|c|c|c|}\hline 3 & 3 & 2 \\ \hline\end{array}\right) = \zeta^{\star}(3,3,2)$$ **duality for MZSV !**

Remark on the duality

Moreover, in this case, we have an Ohno relation;

$$\zeta\left(\begin{ytableau} \none & \none & 1 \\ \none & 1 & 2 \\ 2 & 2 \end{ytableau}\right) = \zeta\left(\begin{ytableau} 3 & 3 & 2 \end{ytableau}\right), \qquad \text{(duality)}$$

$$\zeta\left(\begin{ytableau} \none & \none & \color{red}2 \\ \none & 1 & 2 \\ 2 & 2 \end{ytableau}\right) + \zeta\left(\begin{ytableau} \none & \none & 1 \\ \none & 1 & \color{red}3 \\ 2 & 2 \end{ytableau}\right) + \zeta\left(\begin{ytableau} \none & \none & 1 \\ \none & \color{red}2 & 2 \\ 2 & 2 \end{ytableau}\right) + \zeta\left(\begin{ytableau} \none & \none & 1 \\ \none & 1 & 2 \\ 2 & \color{red}3 \end{ytableau}\right) + \zeta\left(\begin{ytableau} \none & \none & 1 \\ \none & 1 & 2 \\ \color{red}3 & 2 \end{ytableau}\right)$$
$$= \zeta\left(\begin{ytableau} \color{blue}4 & 3 & 2 \end{ytableau}\right) + \zeta\left(\begin{ytableau} 3 & \color{blue}4 & 2 \end{ytableau}\right) + \zeta\left(\begin{ytableau} 3 & 3 & \color{blue}3 \end{ytableau}\right), \ldots$$

Remark.

- In general, Schur multiple zeta values of ribbon types are **not** closed under the dual (e.g., $\zeta^{\star}(k_1,\ldots,k_n) = \zeta\left(\begin{ytableau} k_1 & \cdots & k_n \end{ytableau}\right)$ with $\underline{k_i = 1}$ for some i).
- If the dual of $\zeta(\boldsymbol{k})$ is again a Schur multiple zeta value, then one can show the Ohno relation.

Zagier's conjecture

$$\mathcal{Z}_0 = \mathbb{Q}, \quad \mathcal{Z}_1 = \{0\}, \quad \mathcal{Z}_k = \sum_{\substack{n \geq 1 \\ k_1, \ldots, k_{n-1} \geq 1, k_n \geq 2 \\ k_1 + \cdots + k_n = k}} \mathbb{Q}\zeta(k_1, \ldots, k_{n-1}, k_n) \quad (k \geq 2).$$

- $\zeta(1,2) = \zeta(3)$. $\quad \therefore \dim_{\mathbb{Q}} \mathcal{Z}_3 = 1$.
- $\zeta(1,3) = \frac{1}{4}\zeta(4)$, $\zeta(2,2) = \frac{3}{4}\zeta(4)$, $\zeta(1,1,2) = \zeta(4)$. $\quad \therefore \dim_{\mathbb{Q}} \mathcal{Z}_4 = 1$.
- $\zeta(1,4) = \zeta(1,1,3) = 2\zeta(5) - \zeta(3)\zeta(2)$,
  $\zeta(2,3) = \zeta(1,2,2) = -\frac{11}{2}\zeta(5) + 3\zeta(3)\zeta(2)$,
  $\zeta(3,2) = \zeta(2,1,2) = \frac{9}{2}\zeta(5) - 2\zeta(3)\zeta(2)$, $\zeta(1,1,1,2) = \zeta(5)$. $\quad \therefore \dim_{\mathbb{Q}} \mathcal{Z}_5 \leq 2$.

Conjecture (Zagier)

Define $\{d_k\}_{k \geq 0}$ *by* $d_0 = 1,\ d_1 = 0,\ d_2 = 1$ *and* $d_k = d_{k-2} + d_{k-3}$ $(k \geq 3)$. *Then,*

$$\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k.$$

k	0	1	2	3	4	5	6	7	8	9	10
#{MZVs of wteight k}	–	–	1	2	4	8	16	32	64	128	256
d_k	1	0	1	1	1	2	2	3	4	5	7

- "$\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k$" was proved by Deligne, Goncharov and Terasoma.
- "$\dim_{\mathbb{Q}} \mathcal{Z}_k \geq d_k$" seems to be difficult.

Kaneko-Yamamoto's conjecture

Conjecture (Kaneko-Yamamoto $'18$)

Any linear dependency of MZVs over $\mathbb{Q}$ can be deduced from the iterated integral representations of Schur multiple zeta values of anti-hook types.

Example. $\lambda/\mu = (2,2)/(1)$, $\boldsymbol{k} = \begin{array}{|c|c|}\hline & 1 \\ \hline 1 & 2 \\ \hline \end{array}$

$$\sum_{\substack{l \le n \\ m < n}} \frac{1}{lmn^2} \overset{\text{series}}{=} \zeta\left(\begin{array}{|c|c|}\hline & 1 \\ \hline 1 & 2 \\ \hline \end{array}\right) \overset{\text{integral}}{=} \int_{\substack{s<t<u>v \\ 0<s,t,u,v<1}} \frac{ds}{1-s}\frac{dt}{1-t}\frac{du}{u}\frac{dv}{1-v}.$$

- $(\text{LHS}) = \sum_{l<m<n} + \sum_{l=m<n} + \sum_{m<l<n} + \sum_{m<l=n} = 2\zeta(1,1,2) + \zeta(2,2) + \zeta(1,3),$
- $(\text{RHS}) = \int_{v<s<t<u} + \int_{s<v<t<u} + \int_{s<t<v<u} = 3\zeta(1,1,2).$

$\longrightarrow \quad \zeta(2,2) + \zeta(1,3) = \zeta(1,1,2).$

Relation among Schur MZVs may yield <u>many</u> relation among MZVs ?!

Thank you for your attention !

$$\frac{1}{32}\zeta(13) = \zeta\left(\begin{array}{cccc} & & & \boxed{1} \\ & & \boxed{1} & \boxed{3} \\ & \boxed{1} & \boxed{3} & \\ \boxed{1} & \boxed{3} & & \end{array}\right) = \sum_{\begin{smallmatrix} & & & g \\ & & & \wedge \\ & & e \leq & f \\ & & \wedge & \\ & c \leq & d & \\ & \wedge & & \\ a \leq & b & & \end{smallmatrix}} \frac{1}{ab^3cd^3ef^3g}$$

$$\frac{1}{64}\zeta(15) = \zeta\left(\begin{array}{cccc} & & \boxed{1} & \boxed{3} \\ & \boxed{1} & \boxed{3} & \\ \boxed{1} & \boxed{3} & & \\ \boxed{3} & & & \end{array}\right) = \sum_{\begin{smallmatrix} & & & f \leq g \\ & & & \wedge \\ & & d \leq & e \\ & & \wedge & \\ & b \leq & c & \\ & \wedge & & \\ & a & & \end{smallmatrix}} \frac{1}{a^3bc^3de^3fg^3}$$