THE COMPETITION NUMBERS OF JOHNSON GRAPHS

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Abstract

The competition graph of a digraph $D$ is a graph which has the same vertex set as $D$ and has an edge between two distinct vertices $x$ and $y$ if and only if there exists a vertex $v$ in $D$ such that $(x, v)$ and $(y, v)$ are arcs of $D$. For any graph $G$, $G$ together with sufficiently many isolated vertices is the competition graph of some acyclic digraph. The competition number $k(G)$ of a graph $G$ is defined to be the smallest number of such isolated vertices. In general, it is hard to compute the competition number $k(G)$ for a graph $G$ and to characterize all graphs with given competition number $k$ has been one of the important research problems in the study of competition graphs.

The Johnson graph $J(n, d)$ has the vertex set $\{v_X \mid X \in \binom{[n]}{d}\}$, where $\binom{[n]}{d}$ denotes the set of all $d$-subsets of an $n$-set $[n] = \{1, \ldots, n\}$, and two vertices $v_X$ and $v_Y$ are adjacent if and only if $|X \cap Y| = d - 1$. In this paper, we study the edge clique number and the competition number of $J(n, d)$. Especially we give the exact competition numbers of $J(n, 2)$ and $J(n, 3)$.

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1. Introduction

The competition graph $C(D)$ of a digraph $D$ is a simple undirected graph which has the same vertex set as $D$ and has an edge between two distinct vertices $x$ and $y$ if and only if there is a vertex $v$ in $D$ such that $(x,v)$ and $(y,v)$ are arcs of $D$. The notion of a competition graph was introduced by Cohen [3] as a means of determining the smallest dimension of ecological phase space (see also [4]). Since then, various variations have been defined and studied by many authors (see [11, 15] for surveys and [1, 2, 7, 8, 9, 10, 12, 14, 19, 20] for some recent results). Besides an application to ecology, the concept of competition graph can be applied to a variety of fields, as summarized in [17].

Roberts [18] observed that, for a graph $G$, $G$ together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. Then he defined the competition number $k(G)$ of a graph $G$ to be the smallest number $k$ such that $G$ together with $k$ isolated vertices is the competition graph of an acyclic digraph.

A subset $S$ of the vertex set of a graph $G$ is called a clique of $G$ if the subgraph of $G$ induced by $S$ is a complete graph. For a clique $S$ of a graph $G$ and an edge $e$ of $G$, we say $e$ is covered by $S$ if both of the endpoints of $e$ are contained in $S$. An edge clique cover of a graph $G$ is a family of cliques such that each edge of $G$ is covered by some clique in the family. The edge clique cover number $\theta_E(G)$ of a graph $G$ is the minimum size of an edge clique cover of $G$. We call an edge clique cover of $G$ with the minimum size $\theta_E(G)$ a minimum edge clique cover of $G$. A vertex clique cover of a graph $G$ is a family of cliques such that each vertex of $G$ is contained in some clique in the family. The vertex clique cover number $\theta_V(G)$ of a graph $G$ is the minimum size of a vertex clique cover of $G$. Dutton and Brigham [5] characterized the competition graphs of acyclic digraphs using edge clique covers of graphs.

Roberts [18] observed that the characterization of competition graphs is equivalent to the computation of competition numbers. It does not seem to be easy in general to compute $k(G)$ for a graph $G$, as Opsut [16] showed
that the computation of the competition number of a graph is an NP-hard problem (see [11, 13] for graphs whose competition numbers are known). For some special graph families, we have explicit formulae for computing competition numbers. For example, if $G$ is a chordal graph without isolated vertices then $k(G) = 1$, and if $G$ is a nontrivial triangle-free connected graph then $k(G) = |E(G)| - |V(G)| + 2$ (see [18]).

In this paper, we study the competition numbers of Johnson graphs. We denote an $n$-set $\{1, \ldots, n\}$ by $[n]$ and the set of all $d$-subsets of an $n$-set by $\binom{[n]}{d}$. The Johnson graph $J(n, d)$ has the vertex set $\{v_X \mid X \in \binom{[n]}{d}\}$, and two vertices $v_{X_1}$ and $v_{X_2}$ are adjacent if and only if $|X_1 \cap X_2| = d - 1$ (for reference, see [6]). For example, the Johnson graph $J(5, 2)$ is given in Figure 1.

![Figure 1. The Johnson graph $J(5, 2)$.](image)

As it is known that $J(n, d) \cong J(n, n-d)$, we assume that $n \geq 2d$. Our main results are the following.

**Theorem 1.** For $n \geq 4$, we have $k(J(n, 2)) = 2$.

**Theorem 2.** For $n \geq 6$, we have $k(J(n, 3)) = 4$.

We use the following notation and terminology in this paper. For a digraph $D$, an ordering $v_1, v_2, \ldots, v_n$ of the vertices of $D$ is called an acyclic ordering of $D$ if $(v_i, v_j) \in A(D)$ implies $i < j$. It is well-known that a digraph $D$ is acyclic if and only if there exists an acyclic ordering of $D$. For a digraph $D$ and a vertex $v$ of $D$, the out-neighborhood of $v$ in $D$ is the set \{w \in V(D) \mid (v, w) \in A(D)\}. A vertex in the out-neighborhood of a vertex $v$ in a digraph $D$ is called a prey of $v$ in $D$. For simplicity, we denote the
out-neighborhood of a vertex \( v \) in a digraph \( D \) by \( P_D(v) \) instead of usual notation \( N^+_D(v) \). For a graph \( G \) and a vertex \( v \) of \( G \), we define the (open) neighborhood \( N_G(v) \) of \( v \) in \( G \) to be the set \( \{ u \in V(G) \mid uv \in E(G) \} \). We sometimes also use \( N_G(v) \) to stand for the subgraph induced by its vertices.

2. A Lower Bound for the Competition Number of \( J(n,d) \)

In this section, we give lower bounds for the competition number of the Johnson graph \( J(n,d) \).

**Lemma 3.** Let \( n \) and \( d \) be positive integers with \( n \geq 2d \). For any vertex \( x \) of the Johnson graph \( J(n,d) \), we have \( \theta_V(N_{J(n,d)}(x)) = d \).

**Proof.** If \( d = 1 \), then \( J(n,d) \) is a complete graph and the lemma is trivially true. Assume that \( d \geq 2 \). Take any vertex \( x \) in \( J(n,d) \). Then \( x = v_A \) for some \( A \in \binom{[n]}{d} \). For any vertex \( v_A \) in \( J(n,d) \), the set

\[
S_i(v_A) := \{ v_B \mid B = (A \setminus \{ i \}) \cup \{ j \} \text{ for some } j \in [n] \setminus A \}
\]

forms a clique of \( J(n,d) \) for each \( i \in A \). To see why, take two distinct vertices \( v_B \) and \( v_C \) in \( S_i(v_A) \). Then \( B = (A \setminus \{ i \}) \cup \{ j \} \) and \( C = (A \setminus \{ i \}) \cup \{ k \} \) for some distinct \( j, k \in [n] \setminus A \). Clearly \( |B \cap C| = d - 1 \), and so \( v_B \) and \( v_C \) are adjacent in \( J(n,d) \).

Take a vertex \( v_B \) in \( N_{J(n,d)}(v_A) \). Then \( B = (A \setminus \{ i \}) \cup \{ j \} \) for some \( i \in A \) and \( j \in [n] \setminus A \) and so \( v_B \in S_i(v_A) \). Thus \( \{ S_i(v_A) \mid i \in A \} \) is a vertex clique cover of \( N_{J(n,d)}(v_A) \). Thus \( \theta_V(N_{J(n,d)}(v_A)) \leq d \). On the other hand,

\[
|(A \setminus \{ i \}) \cup \{ j \}) \cap ((A \setminus \{ i' \}) \cup \{ j' \})| = d - 2
\]

if \( i, i' \in A \) and \( j, j' \in [n] \setminus A \) satisfy \( i \neq i' \) and \( j \neq j' \) (such \( i, i', j, j' \) exist since \( n \geq 2d \geq 4 \)). This implies that \( \theta_V(N_{J(n,d)}(v_A)) \geq d \). Hence \( \theta_V(N_{J(n,d)}(v_A)) = d \).

Opsut [16] gave a lower bound for the competition number of a graph \( G \) as follows:

\[
k(G) \geq \min \{ \theta_V(N_G(v)) \mid v \in V(G) \}.
\]

Together with Lemma 3, we have \( k(J(n,d)) \geq d \) for positive integers \( n \) and \( d \) satisfying \( n \geq 2d \). The following theorem gives a better lower bound for \( k(J(n,d)) \) if \( d \geq 2 \).
Theorem 4. For \( n \geq 2d \geq 4 \), we have \( k(J(n, d)) \geq 2d - 2 \).

**Proof.** Put \( k := k(J(n, d)) \). Then there exists an acyclic digraph \( D \) such that \( C(D) = J(n, d) \cup I_k \), where \( I_k = \{z_1, z_2, \ldots, z_k\} \) is a set of isolated vertices. Let \( x_1, x_2, \ldots, x_{\binom{n}{d}}, z_1, z_2, \ldots, z_k \) be an acyclic ordering of \( D \). Let \( v_1 := x_{\binom{n}{d}} \) and \( v_2 := x_{\binom{n}{d}-1} \). By Lemma 3, we have \( \theta_V(N_{J(n, d)}(x_i)) = d \) for \( i = 1, \ldots, \binom{n}{d} \). Thus \( v_i \) has at least \( d \) distinct prey in \( D \), that is,

\[
|PD(v_1)| \geq d.
\]

Since \( x_1, x_2, \ldots, x_{\binom{n}{d}}, z_1, z_2, \ldots, z_k \) is an acyclic ordering of \( D \), we have

\[
PD(v_1) \cup PD(v_2) \subset I_k \cup \{v_1\}.
\]

Moreover, we may claim the following:

**Claim.** For any two adjacent vertices \( v_{X_1} \) and \( v_{X_2} \) of \( J(n, d) \), we have \( |PD(v_{X_1}) \setminus PD(v_{X_2})| \geq d - 1 \).

**Proof of Claim.** Suppose that \( v_{X_1} \) and \( v_{X_2} \) are adjacent in \( J(n, d) \). Then \( |X_1 \cap X_2| = d - 1 \) and

\[
|\{y \in n \setminus (X_1 \cup X_2) : y \not\in X_1 \setminus X_2\}| \geq 2d - |X_1| - |X_2| + |X_1 \cap X_2| = d - 1.
\]

We take \( d - 1 \) elements from \( n \setminus (X_1 \cup X_2) \), say \( z_1, z_2, \ldots, z_{d-1} \), and put \( X_1 \cap X_2 := \{y_1, y_2, \ldots, y_{d-1}\} \).

For each \( 1 \leq j \leq d-1 \), we put \( Z_j := X_1 \cup \{z_j\} \setminus \{y_j\} \). Then \( |Z_j| = d \) and so \( v_{Z_j} \) is a vertex in \( J(n, d) \). Note that \( |Z_j \cap X_1| = d-1 \) and \( |Z_j \cap X_2| = d-2 \). Thus \( v_{Z_j} \) is adjacent to \( v_{X_1} \) while it is not adjacent to \( v_{X_2} \). Therefore

\[
PD(v_{X_1}) \cap PD(v_{Z_j}) \neq \emptyset \quad \text{and} \quad PD(v_{X_2}) \cap PD(v_{Z_j}) = \emptyset.
\]

This implies

\[
PD(v_{X_1}) \setminus PD(v_{X_2}) \supseteq \bigcup_{j=1}^{d-1} (PD(v_{X_1}) \cap PD(v_{Z_j}))
\]

and, trivially, for each \( j \in \{1, \ldots, d-1\} \),

\[
|PD(v_{X_1}) \cap PD(v_{Z_j})| \geq 1.
\]
Note that \(|Z_j \cap Z_i| = d - 2\) for \(i \neq j\). Therefore \(v_{Z_i}\) and \(v_{Z_j}\) are not adjacent and so \(P_D(v_{Z_i}) \cap P_D(v_{Z_j}) = \emptyset\). Thus, for \(i \neq j\),

\[
\text{(2.5) } (P_D(v_{X_1}) \cap P_D(v_{Z_j})) \cap (P_D(v_{X_1}) \cap P_D(v_{Z_j})) = \emptyset.
\]

From (2.3), (2.4), and (2.5), it follows that

\[
|P_D(v_{X_1}) \setminus P_D(v_{X_2})| \geq \sum_{j=1}^{d-1} |P_D(v_{X_1}) \cap P_D(v_{Z_j})| \geq d - 1.
\]

This completes the proof of the claim. \(\square\)

Now suppose that \(v_1\) and \(v_2\) are not adjacent in \(J(n,d)\). Then \(v_1\) and \(v_2\) do not have a common prey in \(D\), that is,

\[
\text{(2.6) } P_D(v_1) \cap P_D(v_2) = \emptyset.
\]

By (2.1), (2.2) and (2.6), we have

\[
k + 1 \geq |P_D(v_1) \cup P_D(v_2)| = |P_D(v_1)| + |P_D(v_2)| \geq 2d.
\]

Hence \(k \geq 2d - 1 > 2d - 2\).

Next suppose that \(v_1\) and \(v_2\) are adjacent in \(J(n,d)\). Then \(v_1\) and \(v_2\) have at least one common prey in \(D\), that is,

\[
|P_D(v_1) \cap P_D(v_2)| \geq 1.
\]

By the above claim,

\[
\text{(2.7) } |P_D(v_1) \setminus P_D(v_2)| \geq d - 1 \quad \text{and} \quad |P_D(v_2) \setminus P_D(v_1)| \geq d - 1.
\]

Then

\[
k + 1 \geq |P_D(v_1) \cup P_D(v_2)| \quad \text{(by (2.2))}
\]

\[
= |P_D(v_1) \setminus P_D(v_2)| + |P_D(v_2) \setminus P_D(v_1)| + |P_D(v_1) \cap P_D(v_2)|
\]

\[
\geq (d - 1) + (d - 1) + 1 \quad \text{(by (2.7) and (2.8))}
\]

\[
= 2d - 1.
\]

Hence it holds that \(k \geq 2d - 2\). \(\blacksquare\)
3. A Minimum Edge Clique Cover of \( J(n, d) \)

In this section, we build a minimum edge clique cover of \( J(n, d) \).

Given a Johnson graph \( J(n, d) \), we define a family \( \mathcal{F}_d^n \) of cliques of \( J(n, d) \) as follows. For each \( Y \in \binom{[n]}{d-1} \), we put

\[
S_Y := \{ v_X \mid X = Y \cup \{ j \} \text{ for } j \in [n] - Y \}.
\]

Note that \( S_Y \) is a clique of \( J(n, d) \) with size \( n - d + 1 \). We let

\[
(3.1) \quad \mathcal{F}_d^n := \{ S_Y \mid Y \in \binom{[n]}{d-1} \}.
\]

Then it is not difficult to show that \( \mathcal{F}_d^n \) is the collection of cliques of maximum size. Moreover the family \( \mathcal{F}_d^n \) is an edge clique cover of \( J(n, d) \). To see why, take any edge \( v_{X_1}v_{X_2} \) of \( J(n, d) \). Then \( |X_1 \cap X_2| = d - 1 \) and both of \( v_{X_1} \) and \( v_{X_2} \) belong to the clique \( S_{X_1 \cap X_2} \in \mathcal{F}_d^n \). Thus \( \mathcal{F}_d^n \) is an edge clique cover of \( J(n, d) \).

We will show that \( \mathcal{F}_d^n \) is a minimum edge clique cover of \( J(n, d) \). Prior to that, we present the following theorem. For two distinct cliques \( S \) and \( S' \) of a graph \( G \), we say \( S \) and \( S' \) are edge disjoint if \( |S \cap S'| \leq 1 \).

**Theorem 5.** \( \theta_E(J(n, d)) = \binom{n}{d-1} \) and any minimum edge clique cover of \( J(n, d) \) consists of edge disjoint maximum cliques.

**Proof.** Let \( \mathcal{E} \) be a minimum edge clique cover for \( J(n, d) \), that is, \( \theta_E(J(n, d)) = |\mathcal{E}| \). Since \( \mathcal{F}_d^n \) is an edge clique cover with \( |\mathcal{F}_d^n| = \binom{n}{d-1} \), we have \( \theta_E(J(n, d)) \leq \binom{n}{d-1} \).

Now we show that \( |\mathcal{E}| \geq \binom{n}{d-1} \). Since the size of a maximum clique is \( n - d + 1 \), we have \( |E(S)| \leq \binom{n-d+1}{2} \) for each \( S \in \mathcal{E} \) where \( E(S) = \binom{S}{2} \). Therefore,

\[
(3.2) \quad |E(J(n, d))| \leq \sum_{S \in \mathcal{E}} |E(S)| \leq \binom{n-d+1}{2} \times |\mathcal{E}|,
\]

and the first equality holds if and only if none of two distinct cliques in \( \mathcal{E} \) have a common edge, and the second equality holds if and only if any element of \( \mathcal{E} \) is a maximum clique in \( J(n, d) \).
Since the Johnson graph $J(n,d)$ is a $d(n-d)$-regular graph and the number of vertices of $J(n,d)$ is $\binom{n}{d}$,

$$|E(J(n,d))| = \frac{1}{2}d(n-d) \times \binom{n}{d} = \left(\frac{n-d+1}{2}\right) \times \binom{n}{d-1}.$$  

From (3.2) and (3.3), it follows that $$\binom{n}{d-1} \leq |E|.$$ Hence we can conclude that $\theta_E(J(n,d)) = \binom{n}{d-1}$.

Furthermore, two equalities in (3.2) must hold, and therefore any minimum edge clique cover of $J(n,d)$ consists of edge disjoint maximum cliques.

Since $|\mathcal{F}_d^n| = \binom{n}{d-1}$, the following corollary is an immediate consequence of Theorem 5:

**Corollary 6.** The edge clique cover $\mathcal{F}_d^n$ of $J(n,d)$ defined in (3.1) is a minimum edge clique cover of $J(n,d)$.

4. Proofs of Theorems 1 and 2

First, we define an order $\prec$ on the set $\binom{[n]}{d}$ as follows. Take two distinct elements $X_1$ and $X_2$ in $\binom{[n]}{d}$. Let $X_1 = \{i_1, i_2, \ldots, i_d\}$ and $X_2 = \{j_1, j_2, \ldots, j_d\}$ where $i_1 < \cdots < i_d$ and $j_1 < \cdots < j_d$. Then we define $X_1 \prec X_2$ if there exists $t \in \{1, \ldots, d\}$ such that $i_s = j_s$ for $1 \leq s \leq t-1$ and $i_t < j_t$. It is easy to see that $\prec$ is a total order.

Now we prove Theorem 1.

**Proof of Theorem 1.** As $k(J(n,2)) \geq 2$ by Theorem 4, it remains to show $k(J(n,2)) \leq 2$. We define a digraph $D$ as follows:
\[
V(D) = V(J(n,2)) \cup I_2
\]
where $I_2 = \{z_1, z_2\}$, and
\[
A(D) = \bigcup_{i=1}^{n-2} \{ (x, v_{i+1, i+2}) \mid x \in S_{\{i\}} \in \mathcal{F}_2^{n-1} \}
\]
\[
\cup \bigcup_{i=1}^{n-2} \{ (x, z_i) \mid x \in S_{\{n-2+i\}} \in \mathcal{F}_2^{n-1} \}.
\]
Since the vertices of each clique in the edge clique cover $\mathcal{F}_2^n$ has a common prey in $D$, it holds that $C(D) = J(n, 2) \cup I_2$. Each vertex in $S_{\{i\}}$ is denoted by $v_X$ for some $X \in \binom{[n]}{2}$ which contains $i$. Then by the definition of $\prec$, $v_X \prec v_{\{i+1,i+2\}}$ for $i = 1, \ldots, n - 2$. Thus, there exists an arc from a vertex $x$ to a vertex $y$ in $D$ if and only if either $x = v_X$ and $y = v_Y$ with $X \prec Y$, or $x = v_X$ and $y = z_i$ with $X \in S_{\{n-1\}} \cup S_{\{n\}}$ and $i \in \{1, 2\}$. Therefore $D$ is acyclic. Thus we have $k(J(n, 2)) \leq 2$ and this completes the proof.

**Proof of Theorem 2.** By Theorem 4, we have $k(J(n, 3)) \geq 4$. It remains to show $k(J(n, 3)) \leq 4$. We define a digraph $D$ as follows:

$$V(D) = V(J(n, 3)) \cup I_4$$

where $I_4 = \{z_1, z_2, z_3, z_4\}$, and

$$A(D) = \bigcup_{i=1}^{n-3} \bigcup_{j=i+1}^{n-2} \{(x, v_{\{i,j+1,j+2\}}) \mid x \in S_{\{i,j\}} \in \mathcal{F}_3^n\}$$

$$\cup \bigcup_{i=1}^{n-3} \{(x, v_{\{i+1,i+2,i+3\}}) \mid x \in S_{\{i,n-1\}} \in \mathcal{F}_3^n\}$$

$$\cup \bigcup_{i=1}^{n-4} \{(x, v_{\{i+1,i+2,i+3\}}) \mid x \in S_{\{i,n\}} \in \mathcal{F}_3^n\}$$

$$\cup \bigcup_{i=1}^{3} \{(x, z_i) \mid x \in S_{\{n-4+i,n\}} \in \mathcal{F}_3^n\}$$

$$\cup \{(x, z_4) \mid x \in S_{\{n-2,n-1\}} \in \mathcal{F}_3^n\}.$$  

It is easy to check that

$$\mathcal{F}_3^n = \{S_{\{i,j\}} \mid i = 1, \ldots, n-3; j = i+1, \ldots, n-2\}$$

$$\cup \{S_{\{i,n-1\}} \mid i = 1, \ldots, n-3\} \cup \{S_{\{i,n\}} \mid i = 1, \ldots, n-4\}$$

$$\cup \{S_{\{n-3,n\}}, S_{\{n-2,n\}}, S_{\{n-1,n\}}\} \cup \{S_{\{n-2,n-1\}}\}.$$  

Thus $C(D) = J(n, 3) \cup I_4$. Moreover, any vertex $x \in S_{\{i,j\}}$ is denoted by $v_X$ for some $X \in \binom{[n]}{3}$ which contains $i$ and $j$. By the definition of $\prec$, 


$X \prec \{i, j+1, j+2\}$. In a similar manner, for $x$ in other cliques in $F^3_n$, we may show that $(x, y) \in A(D)$ if and only if either $x = v_X$ and $y = v_Y$ with $X \prec Y$, or $x = v_X$ and $y = z_i$ with $X \in S_{n-3,n} \cup S_{n-2,n} \cup S_{n-1,n} \cup S_{n-2,n-1}$ and $i \in \{1, 2, 3, 4\}$. Thus $D$ is acyclic. Hence $k(J(n, 3)) \leq 4$. ■

5. Concluding Remarks

In this paper, we gave some lower bounds for the competition numbers of Johnson graphs, and computed the competition numbers of Johnson graphs $J(n, 2)$ and $J(n, 3)$. It would be natural to ask: What is the exact value of the competition number of a Johnson graph $J(n, 4)$ for $n \geq 8$? Eventually, what are the exact values of the competition numbers of the Johnson graphs $J(n, q)$ for $q \geq 5$?

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